On the Regularity Theory of Incompressible Flows

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Introduction

We consider the incompressible Navier-Stokes and Euler equations in $d \geq 2$ dimensions

$$\begin{aligned} &\partial_t u + u \cdot \nabla u + \nabla p - \nu \Delta u = 0, \\ &\nabla \cdot u = 0. \end{aligned} \tag{1}$$

where $u : \mathcal{D} \times [0, T] \to \mathbb{R}^d$ is the velocity field, $p : \mathcal{D} \times [0, T] \to \mathbb{R}$ is the pressure, and $\nu \geq 0$. In components

$$\begin{aligned} &\partial_t u_k + u_j \partial_j u_k + \partial_k p - \nu \Delta u_k = 0, \\ &\partial_j u_j = 0. \end{aligned} \tag{2}$$

- The pressure is such that the incompressibility condition is propagated by the flow

$$\Delta p + \partial_k(u_j \partial_j u_k) = 0 \implies p = R_j R_k(u_j u_k),$$

where R_j denotes the Riesz transform $R_j = \partial_j |\nabla|^{-1}$.

- Several possible frameworks: periodic domain $x \in \mathbb{T}^d$, Euclidean domain $x \in \mathbb{R}^d$, chanel $(x, y) \in \mathbb{T} \times \mathbb{R}$, bounded domains $\mathcal{D} \subseteq \mathbb{R}^d$ (with suitable boundary conditions).

- The case $\nu = 0$ corresponds to the Euler equations, which are time reversible.

Basic questions

- **Local wellposedness:** one has local regularity for sufficiently smooth initial data in Sobolev spaces. There are also continuation criteria that guarantee regularity as long as certain quantities are controlled.

- **Long term regularity:** in certain cases one can prove long term (sometimes global) regularity: the solutions exist globally in time if the initial data is "small" in suitable critical spaces or if $d = 2$. The question of global regularity of solutions of the Navier-Stokes or the Euler equations in dimension $d = 3$ is a fundamental open problem in Fluid Mechanics and one of the Millennium Problems.

Basic questions

- **Stability of certain classes of solutions:** shear flows and vortices are sometimes "stable" both at the linear and nonlinear level in 2D.

- **Formation of singularities:** loss of regularity of solutions starting with smooth initial data (completely open).

- **Derived models:** many important equations, like the water waves system, the KdV equation, and the Schrödinger equation can be derived from the basic Fluid equations.

- **Numerical analysis:** solutions of the Euler equations can be very complicated, and numerics have played a critical role in understanding their dynamics.

Theorem 1 (local well-posedness): (i) Assume $\nu \in [0,1]$ and $\phi \in H^s(\mathbb{T}^d)$, $s > d/2+1$ satisfies $\partial_j \phi_j = 0$. Then there is $T = T(s, \|\phi\|_{H^s}) > 0$ and a unique solution $u \in C([0,T] : H^s(\mathbb{T}^d))$ of the initial value problem

$$\begin{aligned} &\partial_t u + u \cdot \nabla u + \nabla p - \nu \Delta u = 0, \qquad \nabla \cdot u = 0, \\ &u(0) = \phi. \end{aligned} \tag{3}$$

Moreover

$$\sup_{t \in [0,T]} \|u(t)\|_{H^s} \lesssim C(s, \|\phi\|_{H^s}).$$

(ii) For any $R \geq 0$ the mapping $\phi \mapsto u$ is a continuous mapping from $B_R(H^s_0(\mathbb{T}^d))$ to $C([0,T] : H^s(\mathbb{T}^d))$.

Local regularity

Duhamel formula: u is a solution of the Navier-Stokes equations (3) if and only if

$$\begin{aligned} &u(t) = e^{\nu t\Delta}\phi + \int_0^t e^{\nu(t-s)\Delta}\mathcal{N}(u,u)(s)\,ds, \\ &\mathcal{N}(u,u) := -(\nabla p + u\cdot\nabla u), \qquad p = p(u) = R_jR_k(u_ju_k). \end{aligned} \tag{4}$$

When $\nu > 0$ this can be solved using a fixed-point argument in the space $Z := C([0,T_\nu] : H^s)$ provided that T_ν is sufficiently small depending on $\nu > 0$ and $\|\phi\|_{H^s}$.

To prove local well-posedness in the Euler case $\nu = 0$ we need to prove apriori control of high order energy functionals.

Local regularity

Energy dissipation:

$$\|u(t)\|_{L^2}^2 = \|u(0)\|_{L^2}^2 - 2\nu \int_0^t \int_{\mathbb{T}^d} |\nabla u(s)|^2 \, dxds.$$

High order energy inequality:

$$\mathcal{E}_s(t) \leq \mathcal{E}_s(0) + C_s \int_0^t \mathcal{E}_s(t') \|\nabla u(t')\|_{L^\infty} \, dt'$$

where

$$\mathcal{E}_s(t) := \int_{\mathbb{T}^d} |J^s u(t,x)|^2 \, dx,$$

and J^s is defined by the Fourier multiplier $\xi \to (1 + |\xi|^2)^{s/2}$. The proof uses the Kato-Ponce inequality: if $s \geq 0$ then

$$\left\| J^s(u\nabla v) - uJ^s\nabla v \right\|_{L^2} \lesssim_s \|\nabla u\|_{L^\infty} \|v\|_{H^s} + \|\nabla v\|_{L^\infty} \|u\|_{H^s}$$

for any $u, v \in H^s(\mathbb{T}^d)$.

Critical well-posedness theory

Set $\nu = 1$, so the incompressible Navier-Stokes equations are

$$\partial_t u + u \cdot \nabla u + \nabla p - \Delta u = 0, \qquad \nabla \cdot u = 0.$$

Solutions are formally invariant under parabolic scaling

$$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x),$$
$$p_\lambda(t,x) = \lambda^2 p(\lambda^2 t, \lambda x)$$

for $\lambda \in \mathbb{R}$. The well-posedness theory and regularity criteria are best expressed in terms of scaling-invariant (critical) norms like

$$L^\infty_t L^d_x, \qquad L^\infty_t \dot{H}^{d/2-1}, \qquad L^p_t L^q_x,\ 2/p + d/q = 1,$$

Critical well-posedness theory

Theorem 2 (Kato 1984): Assume $d \geq 2$, $\phi \in L^d(\mathbb{T}^d)$ satisfies $\partial_j \phi_j = 0$. Then there is a unique solution $u \in C([0, T] : L^d)$ of the initial value problem

$$\partial_t u + u \cdot \nabla u + \nabla p - \Delta u = 0, \qquad \nabla \cdot u = 0,$$
$$u(0) = \phi,$$

where $T = T(\phi) > 0$. Moreover, $T = \infty$ (meaning global regularity) if $\|\phi\|_{L^d} \leq \varepsilon_d \ll 1$.

To prove this we use a fixed-point argument in the space

$$Z_q := \Big\{\|f \in C([0,T] : L^d) : \\ \|f\|_{Z_q} := \sup_{t\in[0,T]} \big[\|f(t)\|_{L^d} + t^a\|f(t)\|_{L^q}\big] < \infty\Big\}.$$

where $q \in (d,\infty)$ and $a = 1/2 - d/(2q)$.

We use the general inequality

$$\|e^{t\Delta}f\|_{L^q} + \sqrt{t}\|e^{t\Delta}\nabla f\|_{L^q} \lesssim t^{-b}\|f\|_{L^p}, \qquad b := \frac{d}{2}\Big(\frac{1}{p} - \frac{1}{q}\Big),$$

which holds for all exponents $1 < p \le q < \infty$ and numbers $t > 0$, provided that $\widehat{f}(0) = 0$.

$2D$ global regularity

The vorticity equation

$$\partial_t \omega_{ij} + u_k \partial_k \omega_{ij} + \omega_{jk} \partial_i u_k - \omega_{ik} \partial_j u_k - \nu \Delta \omega_{ij} = 0$$

In dimension $d = 2$ the equation becomes

$$\partial_t \omega + u \cdot \nabla \omega - \nu \Delta \omega = 0.$$

L^p conservation laws:

$$\|\omega(t)\|_{L^p} \leq \|\omega(0)\|_{L^p} \qquad \text{for any } t \in [0, T] \text{ and } p \in [1, \infty].$$

$2D$ global regularity

Theorem 3 (global well-posedness in 2D): (i) Assume $d = 2$, $\nu \in [0, 1]$ and $\phi \in H^s(\mathbb{T}^d)$, $s > 2$ satisfies $\partial_j \phi_j = 0$. Then there is a unique global solution $u \in C([0, \infty : H^s(\mathbb{T}^2))$ of the initial value problem

$$\begin{aligned} &\partial_t u + u \cdot \nabla u + \nabla p - \nu \Delta u = 0, \qquad \nabla \cdot u = 0, \\ &u(0) = \phi. \end{aligned}$$

Moreover

$$\begin{aligned} &\|u(t)\|_{H^s} \lesssim C(s, t, \|\phi\|_{H^s}) \qquad \text{for any } t \in [0, \infty), \\ &\|\omega(t)\|_{L^\infty} \leq \|\omega_0\|_{L^\infty}. \end{aligned}$$

Theorem 4: Assume $\nu \in [0,1]$, $s > d/2+1$, and $u \in C([0,T] : H^s(\mathbb{T}^d))$, is a solution of the Navier-Stokes equation

$$\partial_t u + u \cdot \nabla u + \nabla p - \nu \Delta u = 0, \qquad \nabla \cdot u = 0.$$

on some interval $[0,T]$.
(i) Then

$$\sup_{t\in[0,T]} \|u(t)\|_{H^s} \leq C\Big(s, \|u(0)\|_{H^s}, \int_0^T \|\nabla_x u(s)\|_{L^\infty}\, ds\Big)$$

(ii) (Beale-Kato-Majda) Moreover

$$\sup_{t\in[0,T]} \|u(t)\|_{H^s} \leq C\Big(s, \|u(0)\|_{H^s}, \int_0^T \|\omega(s)\|_{L^\infty}\, ds\Big),$$

where ω is the associated vorticity of the velocity field

$$\omega_{ij} := \partial_i u_j - \partial_j u_i.$$

Theorem 5: (Prodi-Serrin regularity criterion): Assume $s > d/2 + 1$, and $\phi \in C([0, T] : H^s(\mathbb{T}^d))$, is a solution of the Navier-Stokes equation

$$\partial_t u + u \cdot \nabla u + \nabla p - \Delta u = 0, \qquad \nabla \cdot u = 0.$$

on some interval $[0, T]$. If $2/p + d/q = 1$ then

$$\sup_{t \in [0,T]} \|u(t)\|_{H^s} \le C\Big(s, \|u(0)\|_{H^s}, \|u\|_{L^p_t L^q_x}\Big).$$

Fourier analysis and harmonic analysis

The Fourier transform on $\mathbb{R}^d$:

$$\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}\,dx.$$

The Fourier inversion formula:

$$f(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \widehat{f}(\xi)e^{ix\cdot\xi}\,d\xi.$$

The Plancherel theorem:

$$\|f\|_{L^2} = C_d\|\widehat{f}\|_{L^2}.$$

The heat flow $e^{t\Delta}$ is defined by the Fourier multiplier $\xi \to e^{-t|\xi|^2}$:

$$e^{t\Delta}f := \mathcal{F}^{-1}(\widehat{f}(\xi)e^{-t|\xi|^2}),$$
$$e^{t\Delta}f(x) = f * K_t(x)$$

where K_t is the heat kernel

$$K_t(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t|\xi|^2} e^{ix\cdot\xi}\, d\xi = \frac{C_d}{t^{d/2}} e^{-|x|^2/4t}.$$

- It is easy to see that for $p \in [1, \infty]$

$$\|K_t\|_{L^p} \approx t^{-d/2} t^{d/(2p)}.$$

- Young's inequality

$$\|f * K\|_{L^q} \leq \|f\|_{L^p} \|K\|_{L^r}, \qquad \frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$$

then gives the heat flow estimates

$$\|e^{t\Delta} f\|_{L^q} + \sqrt{t}\|e^{t\Delta}\nabla f\|_{L^q} \lesssim t^{-b}\|f\|_{L^p}, \qquad b := \frac{d}{2}\Big(\frac{1}{p} - \frac{1}{q}\Big),$$

holds for all exponents $1 \leq p \leq q \leq \infty$ and numbers $t > 0$.

- Littlewood-Paley theory: Assume $\varphi : \mathbb{R} \to [0,1]$ is a smooth even function supported in the interval $[-2,2]$ and equal to 1 in the interval $[-2,2]$. For any $k \in \mathbb{Z}$ we define

$$\varphi_k(\xi) := \varphi(|\xi|/2^k) - \varphi(|\xi|/2^{k-1})$$

so φ_k is supported in the annulus $\{|\xi| \in [2^{k-1}, 2^{k+1}]$. We define the Littlewood-Paley projections

$$P_k f := \mathcal{F}^{-1}[\varphi_k(\xi)\widehat{f}(\xi)] = f * L_k,$$
$$L_k(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \varphi_k(\xi) e^{ix\cdot\xi}\,d\xi = 2^{kd} L_0(2^k x).$$

Many properties:

$$f = \sum_{k\in\mathbb{Z}} P_k f, \qquad \|f\|_{H^s}^2 \approx \sum_{k\in\mathbb{Z}} (2^{2ks}+1)\|P_k f\|_{L^2}^2,$$
$$\|P_k f\|_{L^q} \lesssim \|f\|_{L^p} 2^{dk(1/p-1/q)}, \qquad 1 \le p \le q \le \infty.$$

- Calderón-Zygmund theory: if T is an operator defined by a Hörmander-Michlin multiplier m satisfying

$$|D^{\alpha}_{\xi} m(\xi)| \lesssim_{|\alpha|} |\xi|^{-\alpha}$$

then T is a bounded operator on $L^p(\mathbb{R}^d)$, $p \in (1, \infty)$ and

$$\|Tf\|_{L^\infty} \lesssim_s \|f\|_{L^\infty} \big[1 + \log\big(2 + \|f\|_{H^s}/\|f\|_{L^\infty}\big)\big].$$

if $s > d/2$.

- The Kato-Ponce inequality: if $s \geq 0$ then

$$\left\| J^s(u\nabla v) - uJ^s\nabla v \right\|_{L^2} \lesssim_s \|\nabla u\|_{L^\infty}\|v\|_{H^s} + \|\nabla u\|_{L^\infty}\|v\|_{H^s}$$

for any $u, v \in H^s(\mathbb{R}^d)$, where J^s is defined by the Fourier multiplier $\xi \to (1+|\xi|^2)^{s/2}$.

The Euler equations in 2D: Arnold stability

- In vorticity formulation

$$\partial_t\omega + u_1\partial_1\omega + u_2\partial_2\omega = 0,$$
$$u_1 = -\partial_2\psi, \qquad u_2 = \partial_1\psi, \qquad \Delta\psi = \omega.$$

- Shear flows:

$$(u_1, u_2)(x, y) = (U(y), 0), \qquad \omega(x, y) = -U'(y).$$

These are stationary solutions of the 2D Euler equations.

- Vortices:

$$\omega(x, y) = \Omega(\sqrt{x^2 + y^2}), \qquad \psi(x, y) = \Psi(\sqrt{x^2 + y^2}),$$
$$u_1(x, y) = -\Psi'(\sqrt{x^2 + y^2})\frac{y}{\sqrt{x^2 + y^2}}$$
$$u_2(x, y) = \Psi'(\sqrt{x^2 + y^2})\frac{x}{\sqrt{x^2 + y^2}}.$$

These are also stationary solutions of the 2D Euler equations.

Theorem (Arnold stability): (i) Assume $D = \mathbb{T} \times [0,1]$. Then shear flows defined by strictly convex functions $U : [0,1] \to \mathbb{R}$ are globally stable under the norm

$$\int_D |u - \overline{u}|^2 \, dx + \int_D |\omega - \overline{\omega}|^2 \, dx.$$

(ii) Assume $D = B_{R_0}$. Then vortices defined by strictly monotonic functions $\Omega : [0,\infty) \to \mathbb{R}$ are globally stable under the same norm as above.