Real-Variable Theory and Fourier Integral Operators on Semisimple Lie Groups and Symmetric Spaces of Real Rank One

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Abstract

Let $\mathbb{G}$ be a non-compact connected semisimple Lie group of real rank one with finite center, $\mathbb{K}$ a maximal compact subgroup of $\mathbb{G}$ and $\mathbb{X} = \mathbb{G}/\mathbb{K}$ an associated symmetric space of real rank one. We will prove that $L^{2,1}(\mathbb{G}) * L^{2,1}(\mathbb{G}) \subseteq L^{2,\infty}(\mathbb{G})$, which is a sharp endpoint estimate for the Kunze-Stein phenomenon. We will also show that the noncentered maximal operator

$$\mathcal{M}_2 f(z) = \sup_{z \in B} \frac{1}{|B|} \int_B f(z') dz'$$

is bounded from $L^{2,1}(\mathbb{X})$ to $L^{2,\infty}(\mathbb{X})$ and from $L^p(\mathbb{X})$ to $L^p(\mathbb{X})$ in the sharp range of exponents $p \in (2, \infty]$. The supremum in the definition of $\mathcal{M}_2 f(z)$, is taken over all balls B containing the point z.

In the second part of this thesis we investigate L^p boundedness properties of a certain class of radial Fourier integral operators on the symmetric space $\mathbb{X}$. We will prove that if u_τ is the solution at some fixed time τ of the natural wave equation on $\mathbb{X}$ with initial data f and g and $1 < p < \infty$ then

$$||u_\tau||_{L^p(\mathbb{X})} \leq C_p(\tau) \left(||f||_{L^p_{b_p}(\mathbb{X})} + (1+\tau) ||g||_{L^p_{b_p - 1}(\mathbb{X})} \right).$$

We will obtain both the precise behavior in τ of the norm $C_p(\tau)$ and the sharp regularity assumptions on the functions f and g (i.e. the exponent b_p) that make this inequality possible. Our last theorem is concerned with the analog of Stein's maximal spherical averages introduced in [23] and we prove exponential decay estimates (of a highly non-Euclidean nature) on the L^p norm of $\sup_{T \leq \tau \leq T+1} |f * d\sigma_\tau(z)|$.

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Contents

1 Introduction

In this thesis we study boundedness properties of certain convolution-type operators on semisimple Lie Groups and on associated symmetric spaces. A central result in the theory of such operators is the Kunze-Stein phenomenon, which, in its classical form, states that if $\mathbb{G}$ is a connected semisimple Lie group with finite center and $p \in [1, 2)$ then

$$L^2(\mathbb{G}) * L^p(\mathbb{G}) \subseteq L^2(\mathbb{G}) \tag{1.1}$$

(the usual convention, which will be used throughout this thesis, is that if $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ are Banach spaces of functions on $\mathbb{G}$ then $\mathcal{U} * \mathcal{V} \subseteq \mathcal{W}$ means both the set inclusion and the associated norm inequality). The inclusion (1.1) has been established by Kunze and Stein [14] in the case when the group $\mathbb{G}$ is $\mathbb{SL}(2, \mathbb{R})$ (and, later on, for a number of other particular groups) and by Cowling [7] in the general case stated above. For a more complete account of the development of ideas leading to (1.1) we refer the reader to [7] and [8].

More recently, Cowling, Meda and Setti noticed that if the group $\mathbb{G}$ has real rank one then the inclusion (1.1) can be strengthened. Following earlier work of Lohoué and Rychener [17], the key ingredient in their approach is the use of Lorentz spaces $L^{p,q}(\mathbb{G})$ and it is proved in [8] that if $\mathbb{G}$ is a connected semisimple Lie group of real rank one with finite center, $p \in (1, 2)$ and $(\alpha, \beta, \gamma) \in [1, \infty]^3$ have the property that $1 + 1/\gamma \leq 1/\alpha + 1/\beta$, then

$$L^{p,\alpha}(\mathbb{G}) * L^{p,\beta}(\mathbb{G}) \subseteq L^{p,\gamma}(\mathbb{G}). \tag{1.2}$$

In particular, $L^{p,1}$ convolves L^p into L^p for any $p \in [1, 2)$. Our first theorem is an endpoint estimate for (1.2) that shows what happens when $p = 2$.

Theorem A. *If $\mathbb{G}$ is a non-compact connected semisimple Lie group of real rank one* *with finite center then*

$$L^{2,1}(\mathbb{G}) * L^{2,1}(\mathbb{G}) \subseteq L^{2,\infty}(\mathbb{G}). \tag{1.3}$$

Notice that (1.2) follows from Theorem A and a bilinear interpolation theorem ([8, Theorem 1.2]). However, unlike the classical proof of the Kunze-Stein phenomenon, our proof of Theorem A will be based on real-variable techniques only. Easy examples, involving only $\mathbb{K}$-biinvariant functions, show that the inclusion (1.3) is sharp in the sense that none of the $L^{2,1}$ spaces or the $L^{2,\infty}$ space can be replaced with some $L^{2,\alpha}$ space for some $\alpha \in (1,\infty)$.

Assume from now on that the group $\mathbb{G}$ satisfies the hypothesis stated in Theorem A. Let $\mathfrak{g}$ denote its Lie algebra, θ a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the associated Cartan decomposition. Let $\mathbb{K} = \exp \mathfrak{k}$ be a maximal compact subgroup of $\mathbb{G}$ and let $\mathbb{X} = \mathbb{G}/\mathbb{K}$ be an associated symmetric space with origin $\mathbf{0} = \{\mathbb{K}\}$. The Killing form on $\mathfrak{g}$ induces a $\mathbb{G}$-invariant distance function d on $\mathbb{X}$. Let $B(x,r)$ denote the ball in $\mathbb{X}$ centered at x of radius r (with respect to the distance function d) and let $|A|$ denote the measure of the set $A \subset \mathbb{X}$. For any locally integrable function f on $\mathbb{X}$, let

$$\widetilde{\mathcal{M}_2} f(z) = \sup_{r \geq 1} \frac{1}{|B(z,r)|^{1/2}} \int_{B(z,r)} f(z')dz' \tag{1.4}$$

and

$$\mathcal{M}_2 f(z) = \sup_{z \in B} \frac{1}{|B|} \int_B f(z')dz', \tag{1.5}$$

where the supremum in the definition of $\mathcal{M}_2 f(x)$ is taken over all balls B containing z. We will prove the following two theorems on these maximal operators:

Theorem B. *The operator $\widetilde{\mathcal{M}_2}$ is a bounded operator from $L^{2,1}(\mathbb{X})$ to $L^{2,\infty}(\mathbb{X})$.*

Theorem C. *The operator $\mathcal{M}_2$ is a bounded operator from $L^{2,1}(\mathbb{X})$ to $L^{2,\infty}(\mathbb{X})$ and from $L^p(\mathbb{X})$ to $L^p(\mathbb{X})$ in the sharp range of exponents $p \in (2,\infty]$.*

Notice that the operator $\widetilde{\mathcal{M}_2}$ does not have a suitable Euclidean analogue and its

$L^{2,1} \to L^{2,\infty}$ boundedness is related to Theorem A. On the other hand, unlike in the case of Euclidean spaces, the exponential increase of the volume of large balls shows that the noncentered maximal operator $\mathcal{M}_2$ is not bounded from $L^p(\mathbb{X})$ to $L^p(\mathbb{X})$ if $p \in [1, 2]$. We recall that the more standard centered maximal operator

$$\mathcal{M}_1 f(z) = \sup_{r>0} \frac{1}{|B(z,r)|} \int_{B(z,r)} f(z') dz'$$

is bounded from $L^1(\mathbb{X})$ to $L^{1,\infty}(\mathbb{X})$ and from $L^p(\mathbb{X})$ to $L^p(\mathbb{X})$ for any $p > 1$, as shown in [5] and [26] (without the assumption that $\mathbb{G}$ has real rank one). Our proof of Theorem B follows the same idea as in [26], which is understanding the connection between the Iwasawa decomposition of $\mathbb{G}$ and its Cartan decomposition. Theorem C will turn out to be an easy consequence of Theorem B.

In order to state our last two theorems we need to introduce some more notation. Recall that the Lie algebra $\mathfrak{g}$ of the Lie group $\mathbb{G}$ has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and, since $\mathbb{G}$ has real rank one, $\dim_{\mathbb{R}} \mathfrak{a} = 1$. Let $\mathfrak{a}^*_{\mathbb{R}}$ denote the real dual of $\mathfrak{a}$ and, for $\alpha \in \mathfrak{a}^*_{\mathbb{R}}$, let $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$. Let $\Sigma = \{\alpha \in \mathfrak{a}^*_{\mathbb{R}} \setminus \{0\} : \dim_{\mathbb{R}} \mathfrak{g}_\alpha > 0\}$ be the set of nonzero roots; it is well known that Σ is either of the form $\{-\alpha, \alpha\}$ or of the form $\{-2\alpha, -\alpha, \alpha, 2\alpha\}$. Assume that the Lie algebras $\mathfrak{k}$ and $\mathfrak{a}$ are fixed once and for all. In order to simplify the exposition, we identify from the very beginning the space $\mathfrak{a}^*_{\mathbb{C}}$ (the complex dual of $\mathfrak{a}$) with $\mathbb{C}$ using the map $\lambda \to \lambda\alpha$ for $\lambda \in \mathbb{C}$. We also renormalize the Killing form on $\mathfrak{g}$ such that $|H_0| = 1$ where H_0 is the unique vector in $\mathfrak{a}$ with the property that $\alpha(H_0) = 1$.

One has a Fourier transform on the symmetric space $\mathbb{X}$ that associates to any smooth compactly supported function f on $\mathbb{X}$ a function $\tilde{f} : \mathfrak{a}^*_{\mathbb{C}} \times \mathbb{K}/\mathbb{M} \to \mathbb{C}$, where $\mathbb{M}$ is the centralizer of $\mathfrak{a}$ in $\mathbb{K}$ ([12, Chapter III]). By Plancherel's theorem and the inversion formula, any bounded even function $m : \mathbb{R} \to \mathbb{C}$ defines a bounded operator

T_m on $L^2(\mathbb{X})$ given by $\widetilde{T_m f}(\lambda, b) = m(\lambda)\tilde{f}(\lambda, b)$ (recall that we identified $\mathfrak{a}^*_{\mathbb{C}}$ with $\mathbb{C}$ and this gives an identification of $\mathfrak{a}^*_{\mathbb{R}}$ with $\mathbb{R}$). The question of L^p boundedness of operators defined by analytic multipliers m that satisfy suitable symbol estimates has been subject of extensive research (see [5] for the case of complex groups $\mathbb{G}$, [22] for real rank one groups, [2] when $\mathbb{G}$ is a normal real form and [1] for groups of arbitrary real rank). More details on the development of these ideas can be found in Anker's work [1]. Suitable classes of symbols are defined as follows: for any $a \geq 0$ and $b \in \mathbb{R}$, let S^b_a be the set of continuous functions $m(\lambda)$ defined on the tube $\{\lambda \in \mathbb{C}, |\Im\lambda| \leq a\}$, analytic in the interior of the tube, infinitely differentiable on the two lines $|\Im\lambda| = a$ and which satisfy the symbol inequalities

$$\left|\frac{\partial^\alpha}{\partial\lambda^\alpha} m(\lambda)\right| \leq C(1 + |\Re\lambda|)^{b-\alpha} \text{ for any } \alpha = 0, 1 \ldots N \text{ and } |\Im\lambda| \leq a, \tag{1.6}$$

where $N = \lfloor n/2 \rfloor + 1$ is a "large" integer (n is the dimension of $\mathbb{X}$).

The main objects of study in the second part of this thesis are operators defined by Fourier multipliers of the form $m_\tau(\lambda)\cos(\lambda\tau)$ or $m_\tau(\lambda)\lambda^{-1}\sin(\lambda\tau)$ where the symbol m_τ belongs to a suitable class S^b_a. Our next theorem is the following L^p estimate:

Theorem D. *If $1 < p < \infty$, $\tau \geq 0$, $a = |\rho|\alpha_p$, $b = -d\alpha_p$ and $m \in S^b_a$ (the notation is explained in* (1.11)*) then the operators $T_{1,\tau}$ and $T_{2,\tau}$ defined by the Fourier multipliers $[m(\lambda)\cos(\lambda\tau)]$, respectively $\left[m(\lambda)\left(\lambda^2 + |\rho|^2\right)^{1/2}\lambda^{-1}\sin(\lambda\tau)\right]$ are bounded from $L^p(\mathbb{X})$ to $L^p(\mathbb{X})$ and*

$$\begin{cases} ||T_{1,\tau}||_{L^p \to L^p} \leq C_p e^{|\rho|\alpha_p\tau}; \\ ||T_{2,\tau}||_{L^p \to L^p} \leq C_p e^{|\rho|\alpha_p\tau}(1+\tau). \end{cases} \tag{1.7}$$

Let u_τ be the solution to the Cauchy problem

$$\begin{cases} \frac{\partial^2}{\partial\tau^2} u_\tau = (\Delta + |\rho|^2)u_\tau, \\ u_0 = f; \frac{\partial}{\partial\tau} u_\tau|_{\tau=0} = g. \end{cases} \tag{1.8}$$

Recall that the spectrum of the Laplace-Beltrami operator Δ is $(-\infty, -|\rho|^2]$, therefore the Fourier transform of u_τ is given (formally) by $\widetilde{u_\tau}(\lambda, b) = \cos(\lambda\tau)\tilde{f}(\lambda, b) + \lambda^{-1}\sin(\lambda\tau)\tilde{g}(\lambda, b)$. Theorem D gives the following:

Corollary 1.1. *If $p \in (1, \infty)$ and $\tau \geq 0$ then*

$$||u_\tau||_{L^p} \leq C_p e^{|\rho|\alpha_p\tau} \left(||f||_{L^p_{d\alpha_p}} + (1+\tau)||g||_{L^p_{d\alpha_p - 1}} \right). \tag{1.9}$$

The Euclidean counterpart of Corollary 1.1 has been proven by Peral [18] and a local variable coefficient version has been considered in [19]. As pointed out in these two papers, the exponents $d\alpha_p$, respectively $d\alpha_p - 1$ that appear in the Sobolev spaces on the right hand side of (1.9) are sharp. In addition, the exponential part of the bound depending on τ in (1.9) (i.e. $e^{|\rho|\alpha_p\tau}$) is best possible (this can be checked easily if the dimension of $\mathbb{X}$ is odd using an explicit formula for u_τ ([12, Chapter 5])). Strichartz-type estimates (i.e. $L^p \to L^{p'}$, $p \leq 2$) on the solution of the wave equation on hyperbolic spaces have been obtained by Tataru [27]. The problem of finding $L^p \to L^q$ bounds on the solution of the heat equation on symmetric spaces of arbitrary real rank has been considered in [9].

The question of L^p boundedness of "pseudo-differential" operators on non-compact symmetric spaces (defined by multipliers m that satisfy suitable symbol-type estimates) seems completely settled by the results in [1] (except possibly for the precise hypothesis one needs to make on the behavior of the symbol m on the boundary of the tube in which it is analytic). On the other hand, much less is known about the L^p boundedness of "Fourier integral" operators on symmetric spaces defined by kernels with large singular supports (as it is the case with the solution of the wave equation at large time). A slightly weaker result than Theorem D (without the sharp regularity assumption) has been obtained in [11] at time $\tau = 1$. Some L^p estimates on the solution of the wave equation on manifolds satisfying very general conditions have been obtained by Lohoué [16].

In our last theorem we deal with the analog of Stein's maximal spherical operator on symmetric spaces. Let $d\sigma_\tau$ be the normalized spherical measure such that

$$\int_{\mathbb{X}} f d\sigma_\tau = \int_{\mathbb{K}} f(ka(\tau)\cdot \mathbf{0})dk$$

for any continuous function $f : \mathbb{X} \to \mathbb{C}$. For any continuous compactly supported function f and for any $T \geq 0$ we define the maximal operator

$$\mathcal{M}_T f(z) = \sup_{\tau\in[T,T+1]} |f * d\sigma_\tau(z)|.$$

With the notation in (1.11), we have the following:

Theorem E. *If* $\frac{n}{n-1} < p < \infty$ *and* $T \geq 0$ *then:*

$$||\mathcal{M}_T f||_{L^p(\mathbb{X})} \leq C_p e^{-|\rho|(1-\alpha_p)T}(T+1)^\beta ||f||_{L^p(\mathbb{X})}. \tag{1.10}$$

The constant β *may be taken* $\beta = 1$ *if* $n \geq 3$ *and* $\beta = 2$ *if* $n = 2$.

Corollary 1.2. *If* $\frac{n}{n-1} < p \leq \infty$ *then*

$$\left|\left| \sup_{0\leq\tau<\infty} |f * d\sigma_\tau(z)| \right|\right|_p \leq C_p ||f||_p.$$

The Euclidean counterpart of Corollary 1.2 has been first proven by Stein [23] in the case $n \geq 3$ and by Bourgain [3] in the case $n = 2$; the corollary (which clearly follows from Theorem B by summation over integers $T \geq 0$ for any $p < \infty$) has been proven in the case of hyperbolic spaces of dimension $n \geq 3$ by Kohen [13]. As in Euclidean spaces, the proof of Theorem B is harder when $n = 2$, in which case an extra argument, based on the proof of the main theorem in [20], is needed. Moreover the exponential part of the decay of the norm in (1.10) (i.e. $e^{-|\rho|(1-\alpha_p)T}$) is sharp.

The thesis is organized as follows: in Section 2 we prove theorems A, B and C together with two corollaries: a covering lemma on $\mathbb{X}$ and a general rearrangement inequality. In Section 3 we prove theorems D and E and in Appendix A we prove certain estimates (somewhat sharper than the ones I found in the literature) on the Harish-Chandra function and on the elementary spherical functions on $\mathbb{X}$. These estimates are used in the proofs of theorems D and E.

1.1 Notation

The following table summarizes most of our notation and we will often refer to it:

$$
\begin{aligned}
&\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \text{ – a Cartan decomposition of the semisimple Lie algebra } \mathfrak{g};\\
&\mathfrak{a} \text{ – a maximal abelian subspace of } \mathfrak{p};\ \mathbb{A} = \exp \mathfrak{a};\ \mathbb{K} = \exp \mathfrak{k};\\
&\Sigma = \{-2\alpha, -\alpha, \alpha, 2\alpha\} \text{ (or } \Sigma = \{-\alpha, \alpha\}) \text{ – the set of nonzero roots;}\\
&m_1, m_2 \text{ – the dimensions of the root spaces } \mathfrak{g}_{-\alpha}, \text{ respectively } \mathfrak{g}_{-2\alpha};\\
&\overline{\mathfrak{n}} = \mathfrak{g}_{-\alpha} + \mathfrak{g}_{-2\alpha};\ \overline{\mathbb{N}} = \exp \overline{\mathfrak{n}};\\
&n = m_1 + m_2 + 1 \text{ – the dimension of the symmetric space } \mathbb{X};\\
&d = (n-1)/2 = (m_1 + m_2)/2;\ |\rho| = (m_1 + 2m_2)/2;\\
&\alpha_p = |1 - 2/p|,\ 1 \le p \le \infty;\\
&S_a^b \text{ – the set of analytic symbols inside the tube } |\Im\lambda| \le a \text{ satisfying (1.6).}
\end{aligned}
\tag{1.11}
$$

2 Real-Variable Theory on $\mathbb{G}$ and $\mathbb{X}$

It is well known that the group $\mathbb{G}$ possesses an Iwasawa decomposition $\mathbb{G} = \overline{\mathbb{N}}\mathbb{A}\mathbb{K}$ and a Cartan decomposition $\mathbb{K}\overline{\mathbb{A}_+}\mathbb{K}$ where $\mathbb{A}_+ = \exp \mathfrak{a}_+$ and $\mathfrak{a}_+ = \{H \in \mathfrak{a} : \alpha(H) > 0\}$. Our proofs of theorems A, B and C in this section are based on relating these two decompositions and, fortunately, for real rank one groups, one has a very precise formula ([12, Ch.2, Theorem 6.1]) in this sense. A similar idea has been used by

Strömberg [26] for groups of arbitrary real rank. Let $H_0 \in \mathfrak{a}_+$ be the unique element of $\mathfrak{a}$ for which $\alpha(H_0) = 1$ and let $a(s) = \exp(sH_0)$ for $s \in \mathbb{R}$ be a parametrization of the subgroup $\mathbb{A}$. Clearly $\mathbb{A}_+ = \{a(s) : s > 0\}$. Using [12, Ch.2, Theorem 6.1] one can identify the Lie algebra $\overline{\mathfrak{n}}$ with $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ and find suitable constants c_1 and c_2 such that the diffeomorphism $\overline{n} : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \overline{\mathbb{N}}$, $\overline{n}(X,Y) = \exp(c_1 X + c_2 Y)$ has the property that $\overline{n}(X,Y)a(s) \in \mathbb{K}a(l)\mathbb{K}$ if and only if $l \geq 0$ and

$$(\cosh l)^2 = \left[\cosh s + e^s |X|^2\right]^2 + e^{2s}|Y|^2. \tag{2.1}$$

In addition

$$a(t)\overline{n}(X,Y)a(-t) = \overline{n}(e^{-t}X, e^{-2t}Y). \tag{2.2}$$

Let $\rho = \frac{1}{2}(m_1 \cdot \alpha + m_2 \cdot 2\alpha)$ (such that $\rho(\log[a(s)]) = |\rho| s$ for all $s \in \mathbb{R}$) and let dg, $d\overline{n}$ and dk denote Haar measures on $\mathbb{G}$, $\overline{\mathbb{N}}$ and $\mathbb{K}$, the last one normalized such that $\int_{\mathbb{K}} 1 dk = 1$. Then the following integral formulae hold for any continuous function f with compact support:

$$\int_{\mathbb{G}} f(g)dg = C_1 \int_{\mathbb{K}} \int_{\mathbb{R}_+} \int_{\mathbb{K}} f(k_1 a(l) k_2)(\sinh l)^{m_1} (\sinh 2l)^{m_2} dk_2 dl dk_1 \tag{2.3}$$

and

$$\begin{aligned} \int_{\mathbb{G}} f(g)dg &= C_2 \int_{\mathbb{K}} \int_{\mathbb{R}} \int_{\overline{\mathbb{N}}} f(\overline{n} a(s) k) e^{2|\rho| s} d\overline{n} ds dk \\ &= C_2' \int_{\mathbb{K}} \int_{\mathbb{R}} \int_{\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}} f(\overline{n}(X,Y) a(s) k) e^{2|\rho| s} dX dY ds dk. \end{aligned} \tag{2.4}$$

The following simple proposition explains the role of the $L^{2,1}$ spaces:

Proposition 2.1. *If f is a $\mathbb{K}$-biinvariant function (i.e. $f(kgk') = f(g)$ for any*

$k, k' \in \mathbb{K}$, $g \in \mathbb{G}$) then:

$$\int_{\mathbb{N}} f(\overline{n}a)d\overline{n} \le Ce^{-\rho(\log a)}||f||_{L^{2,1}(\mathbb{G})}.$$

In other words, the Abel transform $\mathcal{A}f(a) = e^{\rho(\log a)}\int_{\mathbb{N}} f(\overline{n}a)d\overline{n}$ is bounded from $L^{2,1}(\mathbb{G}//\mathbb{K})$ to $L^{\infty}(\mathbb{A})$.

Proof of Proposition 2.1. The usual theory of Lorentz spaces (see, for example, [25, Chapter V]) shows that it suffices to prove the proposition under the additional assumption that f is the characteristic function of an open $\mathbb{K}$-biinvariant set of finite measure. For any $l \ge 0$, let $F(l) = f(ka(l)k')$ such that

$$||f||_{L^{2,1}(\mathbb{G})} = C\left[\int_{\mathbb{R}_+} F(l)(\sinh l)^{m_1}(\sinh 2l)^{m_2}dl\right]^{1/2}. \tag{2.5}$$

Let $a = a(s)$ and for any $l \ge |s|$ let

$$T_{l,s} = \{(X,Y) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} : (\cosh l)^2 = \left[\cosh s + e^s|X|^2\right]^2 + e^{2s}|Y|^2\} \tag{2.6}$$

be the set of points $P \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ with the property that $\overline{n}(P)a(s) \in \mathbb{K}a(l)\mathbb{K}$ (these surfaces will play a key role in the proof of Theorem A). Let $d\omega_{l,s}$ be the induced measure on $T_{l,s}$ such that $\int_{\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}} \phi(X,Y)dXdY = \int_{l\ge|s|}\left[\int_{T_{l,s}} \phi(P)d\omega_{l,s}(P)\right]dl$. Then, since the function f is $\mathbb{K}$-biinvariant

$$\int_{\mathbb{N}} f(\overline{n}a)d\overline{n} = C\int_{\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}} f(\overline{n}(X,Y)a(s))dXdY = C\int_{l\ge|s|} F(l)\left[\int_{T_{l,s}} 1d\omega_{l,s}\right]dl. \tag{2.7}$$

Let $\psi(l,s) = e^{|\rho|s}\int_{T_{l,s}} 1d\omega_{l,s}$. The substitutions $X = \left[e^{-s}(\alpha\cosh l - \cosh s)\right]^{1/2}\omega_X$ and $Y = e^{-s}\cosh l(1-\alpha^2)^{1/2}\omega_Y$ for $\omega_X \in \mathbb{S}^{m_1-1}$, $\omega_Y \in \mathbb{S}^{m_2-1}$ (assume that $m_2 \ge 1$) and

$\alpha \in [\frac{\cosh s}{\cosh l}, 1]$ show that

$$\psi(l,s) = C \sinh l (\cosh l)^{m_2} \int_{\frac{\cosh s}{\cosh l}}^{1} (\alpha \cosh l - \cosh s)^{(m_1-2)/2} (1-\alpha^2)^{(m_2-2)/2} d\alpha$$

which shows that

$$\psi(l,s) \approx \sinh l (\cosh l)^{m_2/2} (\cosh l - \cosh s)^{(m_1+m_2-2)/2}. \tag{2.8}$$

The computation of the function ψ is slightly easier if $m_2 = 0$ and the result is also given by (2.8). In view of (2.5) and (2.7), it suffices to prove that

$$\int_{l \geq |s|} F(l)\psi(l,s) dl \leq C \left[\int_{\mathbb{R}_+} F(l) (\sinh l)^{m_1} (\sinh 2l)^{m_2} dl \right]^{1/2} \tag{2.9}$$

for any measurable function $F : \mathbb{R}_+ \to \{0,1\}$. Notice that if $l \geq 1 + |s|$ then $\psi(l,s) \approx e^{|\rho| l}$, $(\sinh l)^{m_1} (\sinh 2l)^{m_2} \approx e^{2|\rho| l}$ and it follows from Lemma 2.2 below that

$$\int_{l \geq |s|+1} F(l)\psi(l,s) dl \leq C \left[\int_{l \geq |s|+1} F(l) (\sinh l)^{m_1} (\sinh 2l)^{m_2} dl \right]^{1/2}. \tag{2.10}$$

In order to deal with the integral over $l \in [|s|, |s|+1]$, we consider two cases: $|s| \geq 1$ and $|s| \leq 1$. If $|s| \geq 1$ and $l \in [|s|, |s|+1]$, then $\psi(l,s) \approx e^{|\rho||s|} (l - |s|)^{(m_1+m_2-2)/2}$, $(\sinh l)^{m_1} (\sinh 2l)^{m_2} \approx e^{2|\rho||s|}$ and, since $(m_1 + m_2 - 2)/2 \geq -1/2$, it follows that

$$\int_{|s|}^{|s|+1} F(l)\psi(l,s) dl \leq C e^{|\rho||s|} \int_{|s|}^{|s|+1} F(l)(l-|s|)^{-1/2} dl = C e^{|\rho||s|} \int_0^1 F(|s| + u^2) du$$

$$\leq C \left[e^{2|\rho||s|} \int_0^1 F(|s| + u^2) u du \right]^{1/2} \leq C \left[\int_{|s|}^{|s|+1} F(l) (\sinh l)^{m_1} (\sinh 2l)^{m_2} dl \right]^{1/2}$$

(one of the inequalities follows from (2.11) below) which, together with (2.10), completes the proof of the lemma in the case $|s| \geq 1$. The estimation of the integrals over the interval $[|s|, |s|+1]$ is similar in the case $|s| \leq 1$. $\square$

Lemma 2.2. *If* $\beta \neq 0$ *and* $d\mu_1(t) = e^{\beta t}dt$, $d\mu_2(t) = e^{2\beta t}dt$ *are two measures on* $\mathbb{R}$ *then*

$$||f||_{L^1(\mathbb{R},d\mu_1)} \leq C_\beta ||f||_{L^{2,1}(\mathbb{R},d\mu_2)}.$$

Proof of Lemma 2.2. One can assume that f is the characteristic function of some measurable set. The change of variable $t = (\log s)/\beta$ and the substitution $g(s) = f((\log s)/\beta))$ show that it suffices to prove that

$$\frac{1}{|\beta|}\int_{\mathbb{R}_+} g(s)ds \leq C_\beta \left[\frac{1}{|\beta|}\int_{\mathbb{R}_+} g(s)sds\right]^{1/2} \tag{2.11}$$

for any measurable function $g : \mathbb{R}_+ \to \{0,1\}$, which follows by a rearrangement argument. □

2.1 Proof of the Maximal Theorems B and C

We will first prove the maximal theorems B and C since they are easier than Theorem A and still capture the main idea of our approach. Let $\mathbf{0}$ be the origin of the space $\mathbb{X} = \mathbb{G}/\mathbb{K}$ and let χ_r be the characteristic function of the $\mathbb{K}$-biinvariant set $\{g \in \mathbb{G} : d(g \cdot \mathbf{0}, \mathbf{0}) \leq r\}$. Since the measure of a ball of radius r in $\mathbb{X}$ is proportional to $e^{2|\rho|r}$ if $r \geq 1$, it follows that

$$\widetilde{\mathcal{M}_2}f(g\cdot\mathbf{0}) \approx \sup_{r\geq 1}\left[e^{-|\rho|r}\int_{\mathbb{G}} f(g'\cdot\mathbf{0})\chi_r({g'}^{-1}g)dg'\right].$$

The changes of variables $g = \overline{n}a(t)k$, $g' = \overline{n}'a(t')k'$ and the formula (2.4) show that

$$\begin{aligned}&\widetilde{\mathcal{M}_2}f(\overline{n}a(t)\cdot\mathbf{0})\\ &\quad\leq C\sup_{r\geq 1}\left[e^{-|\rho|r}\int_{\mathbb{R}}\left(\int_{\overline{\mathbb{N}}} f(\overline{n}'a(t')\cdot\mathbf{0})\chi_r(a(-t')\overline{n}'^{-1}\overline{n}a(t))d\overline{n}'\right)e^{2|\rho|t'}dt'\right].\end{aligned} \tag{2.12}$$

We first deal with the integral over the space $\overline{\mathbb{N}}$ and dominate the right hand side of (2.12) using a standard maximal operator along the nilpotent group $\overline{\mathbb{N}}$. For any $u > 0$ let $\mathcal{B}_u$ be the ball in $\overline{\mathbb{N}}$ defined as the set $\{\overline{n}(X,Y) : |X| \leq u \text{ and } |Y| \leq u^2\}$. Clearly $\int_{\mathcal{B}_u} 1 d\overline{n} = Cu^{2|\rho|}$. The group $\overline{\mathbb{N}}$ is equipped with nonisotropic dilations $\delta_u(\overline{n}(X,Y)) = \overline{n}(uX, u^2Y)$, which are group automorphisms, therefore the maximal operator

$$Mg(\overline{n}) = \sup_{u>0} \left[\frac{1}{u^{2|\rho|}} \int_{\mathcal{B}_u} |g(\overline{n}m^{-1})| dm \right]$$

is bounded from $L^p(\overline{\mathbb{N}})$ to $L^p(\overline{\mathbb{N}})$ for any $p > 1$ ([29, Lemma 2.2]). For any measurable function $f : \mathbb{X} \to \mathbb{R}_+$ let

$$Mf(\overline{n}a \cdot \mathbf{0}) = \sup_{u>0} \left[\frac{1}{u^{2|\rho|}} \int_{\mathcal{B}_u} |f(\overline{n}m^{-1}a \cdot \mathbf{0})| dm \right]$$

and it follows that $||Mf||_{L^p(\mathbb{X})} \leq C_p ||f||_{L^p(\mathbb{X})}$ for any $p > 1$. We will now use the function Mf to control the $\overline{\mathbb{N}}$ integral in (2.12). Notice that (2.1) and (2.2), together with the fact that $d(ka(l) \cdot \mathbf{0}, \mathbf{0}) = l$ for any $l \geq 0$, $k \in \mathbb{K}$, show that if $\chi_r(a(-t')ma(t)) = 1$ for some $m \in \overline{\mathbb{N}}$ then m has to belong to the ball $\mathcal{B}_{e^{(r-t-t')/2}}$, therefore

$$\begin{aligned}
\int_{\overline{\mathbb{N}}} f(\overline{n}'a(t') \cdot \mathbf{0}) \chi_r(a(-t')\overline{n}'^{-1}\overline{n}a(t)) d\overline{n}' &= \int_{\overline{\mathbb{N}}} f(\overline{n}m^{-1}a(t') \cdot \mathbf{0}) \chi_r(a(-t')ma(t)) dm \\
&\leq \int_{\mathcal{B}_{e^{(r-t-t')/2}}} f(\overline{n}m^{-1}a(t') \cdot \mathbf{0}) dm \\
&\leq Ce^{|\rho|(r-t-t')} Mf(\overline{n}a(t') \cdot \mathbf{0}).
\end{aligned}$$

If we substitute this inequality in (2.12) we conclude that

$$\widetilde{\mathcal{M}_2} f(\overline{n}a(t) \cdot \mathbf{0}) \leq Ce^{-|\rho|t} \int_{\mathbb{R}} Mf(\overline{n}a(t') \cdot \mathbf{0}) e^{|\rho|t'} dt'. \tag{2.13}$$

We can now estimate the $L^{2,\infty}$ norm of $\widetilde{\mathcal{M}_2} f$: for some $\lambda > 0$, the set $E_\lambda = \{z \in \mathbb{X} : \widetilde{\mathcal{M}_2} f(z) > \lambda\}$ is included in the set $\{\overline{n}a(t) \cdot \mathbf{0} : e^{-|\rho|t} \int_{\mathbb{R}} Mf(\overline{n}a(t') \cdot \mathbf{0}) e^{|\rho|t'} dt' > \lambda/C\}$. Since the density measure dz in $\mathbb{X}$ is proportional to the density measure

$e^{2|\rho|t}d\overline{n}dt$ in $\overline{\mathbb{N}} \times \mathbb{R}$ under the identification $z = \overline{n}a(t)\cdot\mathbf{0}$, the measure of this last set is equal to

$$\frac{C\int_{\overline{\mathbb{N}}}\left[\int_{\mathbb{R}} Mf(\overline{n}a(t')\cdot\mathbf{0})e^{|\rho|t'}dt'\right]^2 d\overline{n}}{\lambda^2}$$

therefore

$$||\widetilde{\mathcal{M}_2}f||^2_{L^{2,\infty}} \leq C\int_{\overline{\mathbb{N}}}\left[\int_{\mathbb{R}} Mf(\overline{n}a(t')\cdot\mathbf{0})e^{|\rho|t'}dt'\right]^2 d\overline{n}. \tag{2.14}$$

One can now use the following simple lemma to dominate the right hand side of (2.14):

Lemma 2.3. *If A and B are two measure spaces with measures da, respectively db, and H is a measurable function $H : A \times B \to \mathbb{R}_+$ then*

$$\left[\int_A ||H(a,.)||^2_{L^{2,1}(B,db)}da\right]^{1/2} \leq C||H||_{L^{2,1}(A\times B,dadb)}.$$

The proof of this lemma is straightforward. Combining Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned}\int_{\overline{\mathbb{N}}}\left[\int_{\mathbb{R}} Mf(\overline{n}a(t')\cdot\mathbf{0})e^{|\rho|t'}dt'\right]^2 d\overline{n} &\leq C\int_{\overline{\mathbb{N}}}||Mf(\overline{n}a(.)\cdot\mathbf{0})||^2_{L^{2,1}(\mathbb{R},e^{2|\rho|t'}dt')}d\overline{n} \\ &\leq C||Mf(\overline{n}a(t')\cdot\mathbf{0})||^2_{L^{2,1}(\overline{\mathbb{N}}\times\mathbb{R},e^{2|\rho|t'}d\overline{n}dt')} \\ &\leq C||Mf||^2_{L^{2,1}(\mathbb{G}/\mathbb{K})}.\end{aligned} \tag{2.15}$$

Finally, since $||Mf||_{L^p(\mathbb{X})} \leq C_p||f||_{L^p(\mathbb{X})}$ for any $p > 1$ one also has $||Mf||_{L^{2,1}(\mathbb{X})} \leq C||f||_{L^{2,1}(\mathbb{X})}$ (by the general form of Marcinkiewicz interpolation theorem) and Theorem B follows from (2.14) and (2.15).

Theorem C is an easy consequence of Theorem B: let

$$\begin{cases} \mathcal{M}^0_2 f(z) = \sup\limits_{z\in B, r(B)\leq 1} \frac{1}{|B|}\int_B f(z')dz', \\ \mathcal{M}^1_2 f(z) = \sup\limits_{z\in B, r(B)\geq 1} \frac{1}{|B|}\int_B f(z')dz', \end{cases}$$

where $r(B)$ is the radius of the ball B. The operator $\mathcal{M}_2^0$, the local part of $\mathcal{M}_2$ is clearly bounded from $L^p(\mathbb{X}) \to L^p(\mathbb{X})$ for any $p > 1$. On the other hand, if z belongs to a ball B of radius $r \geq 1$, then $B(z, 2r)$ contains the ball B and $|B(z, 2r)| \approx e^{2|\rho|\cdot 2r} \approx |B|^2$. Therefore

$$\frac{1}{|B|}\int_B f(z')dz' \leq \frac{C}{|B(z,2r)|^{1/2}}\int_{B(z,2r)} f(z')dz'$$

which shows that $\mathcal{M}_2^1 f(z) \leq C\widetilde{\mathcal{M}_2} f(z)$ and the conclusion of Theorem C follows by interpolation with the trivial L^∞ estimate.

2.2 A Covering Lemma

A simple connection between covering lemmata and boundedness of maximal operators is explained in [6]. In our setting we have:

Corollary 2.4. *If a collection of balls $B_i \subset \mathbb{X}$, $i \in I$ has the property that $|\cup B_i| < \infty$ then one can select a finite subset $J \subset I$ such that*

$$\begin{aligned} &\textit{(i)}\ c\left|\underset{i\in I}{\cup} B_i\right| \leq \left|\underset{j\in J}{\cup} B_j\right|; \\ &\textit{(ii)}\ \left|\left|\sum_{j\in J}\chi_{B_j}\right|\right|_{L^{2,\infty}(\mathbb{X})} \leq C\left|\underset{i\in I}{\cup} B_i\right|^{1/2}. \end{aligned} \tag{2.16}$$

The corollary is an immediate consequence of Theorem C and the proof of Proposition 1 in [6] (in fact Corollary 2.4 and Theorem C are equivalent statements). The inequality (ii) in (2.16) is the natural analog of the requirement that the selected balls are disjoint: if B_i, $i \in I$ are standard balls in some Euclidean space, then one can select *disjoint* balls B_j, $j \in J$ that satisfy inequality (i) in (2.16). Notice that the disjointness property of the balls B_j is equivalent to $\left|\left|\sum_{j\in J}\chi_{B_j}\right|\right|_{L^\infty} \leq ||\chi_{\cup B_i}||_{L^\infty}$. Since balls on symmetric spaces do not have the basic doubling property (i.e. $|B(z,r)|$ is not proportional to $|B(z,2r)|$) the disjointness property of the selected balls has to be replaced by (2.16)(ii).

2.3 Proof of Theorem A

In this subsection we will prove Theorem A. In view of the general theory of Lorentz spaces, it suffices to prove that

$$\iint_{\mathbb{G}\times\mathbb{G}} f(z)g(z^{-1}z')h(z')dz'dz \leq C||f||_{L^{2,1}}||g||_{L^{2,1}}||h||_{L^{2,1}} \tag{2.17}$$

whenever $f, g, h : \mathbb{G} \to \{0,1\}$ are characteristic functions of open sets of finite measure. We can also assume that g is supported far from the origin of the group, for example in the set $\underset{l>1}{\cup}\mathbb{K}a(l)\mathbb{K}$. The main part of our argument is devoted to proving that the left hand side of (2.17) is controlled by an integral involving suitable rearrangements of the functions f, g and h, as in (2.34). Let $z = \overline{n}a(t)k$, $z' = \overline{n}'a(t')k'$ and the left hand side of (2.17) becomes

$$\begin{aligned}&\int_{\mathbb{K}}\int_{\mathbb{K}}\int_{\mathbb{R}}\int_{\mathbb{R}} dt'dtdk'dk\ e^{2|\rho|(t+t')}\cdot \\ &\qquad \left[\iint_{\overline{\mathbb{N}}\times\overline{\mathbb{N}}} f(\overline{n}a(t)k)g(k^{-1}a(-t)\overline{n}^{-1}\overline{n}'a(t')k')h(\overline{n}'a(t')k')d\overline{n}'d\overline{n}\right].\end{aligned} \tag{2.18}$$

We will show how to dominate the expression in (2.18) in four steps.

Step1. Integration along the subgroup $\overline{\mathbb{N}}$. As in the proof of the maximal theorems, we start by integrating along $\overline{\mathbb{N}}$. Define $F_1, H_1 : \mathbb{K}\times\mathbb{R} \to \mathbb{R}_+$ as $F_1(k,t) = \int_{\overline{\mathbb{N}}} f(\overline{n}a(t)k)d\overline{n}$ and $H_1(k',t') = \int_{\overline{\mathbb{N}}} h(\overline{n}'a(t')k')d\overline{n}'$. Using the simple inequality

$$\begin{aligned}&\iint_{\overline{\mathbb{N}}\times\overline{\mathbb{N}}} a(\overline{n})b(\overline{n}^{-1}\overline{n}')c(\overline{n}')d\overline{n}'d\overline{n} \\ &\qquad\qquad \leq \left(\int_{\overline{N}} b(\overline{n})d\overline{n}\right)\left[\min\left(\left(\int_{\overline{\mathbb{N}}} a(\overline{n})d\overline{n}\right),\left(\int_{\overline{\mathbb{N}}} c(\overline{n})d\overline{n}\right)\right)\right]\end{aligned}$$

for any measurable functions $a, b, c : \overline{\mathbb{N}} \to [0,1]$, it follows that the inner integral in

(2.18) is dominated by

$$\min\left[F_1(k,t), H_1(k',t')\right]\left[\int_{\overline{\mathbb{N}}} g(k^{-1}a(-t)\overline{n}_1 a(t')k')d\overline{n}_1\right]. \tag{2.19}$$

Using (2.2), $\overline{n}_1 \to a(-t)\overline{n}_1 a(t) = \overline{n}_2$ is a dilation of $\overline{\mathbb{N}}$ with $d\overline{n}_1 = e^{-2|\rho|t}d\overline{n}_2$, therefore

$$\begin{aligned}\int_{\overline{\mathbb{N}}} g(k^{-1}a(-t)\overline{n}_1 a(t')k')d\overline{n}_1 &= e^{-2|\rho|t}\int_{\overline{\mathbb{N}}} g(k^{-1}\overline{n}_2 a(t'-t)k')d\overline{n}_2 \\ &= Ce^{-2|\rho|t}\int_{\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}} g(k^{-1}\overline{n}(X,Y)a(t'-t)k')dXdY \\ &= Ce^{-2|\rho|t}\int_{l\geq|t'-t|}\int_{T_{l,t'-t}} g(k^{-1}\overline{n}(P)a(t'-t)k')d\omega_{l,t'-t}(P)dl.\end{aligned} \tag{2.20}$$

The surfaces $T_{l,s}$ defined in (2.6) for $\{(l,s)\in\mathbb{R}_+\times\mathbb{R} : l\geq|s|\}$ and the associated measures $d\omega_{l,s}$ have the same meaning as in the proof of Proposition 2.1. Let

$$G_1(k,k',l,s) = \left(\int_{T_{l,s}} 1 d\omega_{l,s}\right)^{-1}\left[\int_{T_{l,s}} g(k^{-1}\overline{n}(P)a(s)k')d\omega_{l,s}(P)\right] \tag{2.21}$$

be a suitable average of the function $P \to g(k^{-1}\overline{n}(P)a(s)k')$ on the surface $T_{l,s}$ (clearly, the domain of definition of G_1 is $\{(k,k',l,s)\in\mathbb{K}\times\mathbb{K}\times\mathbb{R}_+\times\mathbb{R} : l\geq|s|\}$ and $G_1(k,k',l,s)\in[0,1]$). If we substitute this definition in (2.20), we conclude that

$$\int_{\overline{\mathbb{N}}} g(k^{-1}a(-t)\overline{n}_1 a(t')k')d\overline{n}_1 = Ce^{-|\rho|(t+t')}\int_{l\geq|t'-t|} G_1(k,k',l,t'-t)\psi(l,t'-t)dl.$$

The function $\psi(l,s)$ has been defined in the proof of Proposition 2.1 and is given by (2.8). Finally, if we substitute this last formula in (2.19), we find that the inner integral in (2.18) is dominated by

$$Ce^{-|\rho|(t+t')}\min\left[F_1(k,t), H_1(k',t')\right]\int_{l\geq|t'-t|} G_1(k,k',l,t'-t)\psi(l,t'-t)dl$$

which shows that the left hand side of (2.17) is dominated by

$$C\int_{\mathbb{K}}\int_{\mathbb{K}} dk'dk \left[\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{l\geq|t'-t|} \min\left[F_1(k,t),H_1(k',t')\right] G_1(k,k',l,t'-t)\psi(l,t'-t)e^{|\rho|(t+t')}dldt'dt\right]. \tag{2.22}$$

For later use, we record the following properties of the functions F_1 and H_1:

$$\begin{cases} ||f||_{L^{2,1}(\mathbb{G})} = \left[C_2\int_{\mathbb{K}}\int_{\mathbb{R}} F_1(k,t)e^{2|\rho|t}dtdk\right]^{1/2}, \\ ||h||_{L^{2,1}(\mathbb{G})} = \left[C_2\int_{\mathbb{K}}\int_{\mathbb{R}} H_1(k',t')e^{2|\rho|t'}dt'dk'\right]^{1/2}. \end{cases} \tag{2.23}$$

Step 2. Integration along the subgroup $\mathbb{A}$. Let χ_1, respectively χ_2, be the characteristic function of the set $\{(k,k',t,t') : F_1(k,t)\leq H_1(k',t')\}$, respectively $\{(k,k',t,t') : H_1(k',t')\leq F_1(k,t)\}$ so, for any k,k',t,t' one has

$$\begin{cases} F_1(k,t)\chi_1(k,k',t,t')\leq H_1(k',t'), \\ H_1(k',t')\chi_2(k,k',t,t')\leq F_1(k,t) \end{cases} \tag{2.24}$$

and the expression (2.22) is the sum of two similar expressions of the form

$$C\int_{\mathbb{K}}\int_{\mathbb{K}} dk'dk \left[\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{l\geq|t'-t|} F_1(k,t)\chi_1(k,k',t,t')G_1(k,k',l,t'-t)\psi(l,t'-t)e^{|\rho|(t+t')}dldt'dt\right].$$

The change of variable $t'=t+s$ in the expression above shows that it is equal to

$$C\int_{\mathbb{K}}\int_{\mathbb{K}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{l\geq|s|} F_1(k,t)\chi_1(k,k',t,t+s)G_1(k,k',l,s)\psi(l,s)e^{2|\rho|t}e^{|\rho|s}dldtdsdk'dk. \tag{2.25}$$

and the first of the inequalities in (2.24) becomes

$$F_1(k,t)\chi_1(k,k',t,t+s)e^{2|\rho|t} \leq H_1(k',t+s)e^{2|\rho|t}. \tag{2.26}$$

Let $F(k) = \left[\int_{\mathbb{R}} F_1(k,t)e^{2|\rho|t}dt\right]^{1/2}$, $H(k') = \left[\int_{\mathbb{R}} H_1(k',t')e^{2|\rho|t'}dt'\right]^{1/2}$ and let

$$A(k,k',s) = \int_{\mathbb{R}} F_1(k,t)\chi_1(k,k',t,t+s)e^{2|\rho|t}dt.$$

The expression (2.25) becomes

$$C\int_{\mathbb{K}}\int_{\mathbb{K}}\int_{\mathbb{R}}\int_{l\geq|s|} A(k,k',s)G_1(k,k',l,s)\psi(l,s)e^{|\rho|s}dldsdk'dk. \tag{2.27}$$

Clearly, $A(k,k',s) \leq F(k)^2$ (since $\chi_1 \leq 1$) and $A(k,k',s) \leq e^{-2|\rho|s}H(k')^2$ by (2.26), therefore

$$e^{|\rho|s}A(k,k',s) \leq \begin{cases} e^{|\rho|s}F(k)^2 & \text{if } e^{|\rho|s} \leq H(k')/F(k), \\ e^{-|\rho|s}H(k')^2 & \text{if } e^{|\rho|s} \geq H(k')/F(k). \end{cases}$$

If we substitute this inequality in (2.27) we find that the left hand side of (2.17) is dominated by

$$\begin{aligned} &C\int_{\mathbb{K}}\int_{\mathbb{K}}\int_{e^{|\rho|s}\leq H(k')/F(k)}\int_{l\geq|s|} F(k)^2G_1(k,k',l,s)\psi(l,s)e^{|\rho|s}dldsdk'dk+ \\ &C\int_{\mathbb{K}}\int_{\mathbb{K}}\int_{e^{|\rho|s}\geq H(k')/F(k)}\int_{l\geq|s|} H(k')^2G_1(k,k',l,s)\psi(l,s)e^{-|\rho|s}dldsdk'dk. \end{aligned} \tag{2.28}$$

We pause for a moment to notice that our estimations so far, together with the proof of Proposition 2.1, would suffice to prove that $L^{2,1}(\mathbb{G}) * L^{2,1}(\mathbb{G}//\mathbb{K}) \subseteq L^{2,\infty}(\mathbb{G})$: if g is a $\mathbb{K}$-biinvariant function, then $G_1(k,k',l,s)$ depends only on l and (2.9) shows that $\int_{l\geq|s|} G_1(k,k',l,s)\psi(l,s)dl \leq C||g||_{L^{2,1}}$. As a consequence, both terms in (2.28) are dominated by $C||g||_{L^{2,1}} \int_{\mathbb{K}} \int_{\mathbb{K}} F(k)H(k')dk'dk$, therefore

$$\iint_{\mathbb{G}\times\mathbb{G}} f(z)g(z^{-1}z')h(z')dz'dz \leq C||g||_{L^{2,1}} \int_{\mathbb{K}} \int_{\mathbb{K}} F(k)H(k')dk'dk$$
$$\leq C||g||_{L^{2,1}} \left[\int_{\mathbb{K}} \int_{\mathbb{K}} F(k)^2 H(k')^2 dk'dk\right]^{1/2} = C||f||_{L^{2,1}}||g||_{L^{2,1}}||h||_{L^{2,1}}.$$

Here we used the fact that, as a consequence of (2.23)

$$\begin{cases} ||f||_{L^{2,1}(\mathbb{G})} = \left[C_2 \int_{\mathbb{K}} F(k)^2 dk\right]^{1/2}, \\ ||h||_{L^{2,1}(\mathbb{G})} = \left[C_2 \int_{\mathbb{K}} H(k')^2 dk'\right]^{1/2}. \end{cases} \tag{2.29}$$

Step 3. A rearrangement inequality. In the general case (if g is not $\mathbb{K}$-biinvariant) we will show that both terms in (2.28) are dominated by some expression of the form

$$C \int_0^1 \int_0^1 \int_{\mathbb{R}_+} F^*(x)H^*(y)G^{**}(x,y,l)e^{|\rho|l} dl dy dx$$

where $F^*, H^* : (0,1] \to \mathbb{R}_+$ are the usual nonincreasing rearrangements of the functions F and H (recall that the measure of $\mathbb{K}$ is equal to 1) and $G^{**} : (0,1] \times (0,1] \times \mathbb{R}_+ \to \{0,1\}$ is a suitable "double" rearrangement of g. The precise definitions are the following: if $a : \mathbb{K} \to \mathbb{R}_+$ is a measurable function then the nonincreasing rearrangement $a^* : (0,1] \to \mathbb{R}_+$ is the right semicontinuous nonincreasing function with the property that

$$|\{k \in \mathbb{K} : a(k) > \lambda\}| = \sup\left(\{x \in (0,1] : a^*(x) > \lambda\}\right) \text{ for any } \lambda \in [0,\infty).$$

Next, assume that $a : \mathbb{K} \times \mathbb{K} \to \mathbb{R}_+$ is a measurable function. For a.e. $k \in \mathbb{K}$ let $a^*(k,y)$, $y \in (0,1]$, be the nonincreasing rearrangement of the function $k' \to a(k,k')$ and let $a^{**}(x,y)$ be the nonincreasing rearrangement of the function $k \to a^*(k,y)$ (clearly $a^{**} : (0,1] \times (0,1] \to \mathbb{R}_+$). The following simple lemma summarizes some of the properties of the functions a^* and a^{**}:

Lemma 2.5. *(a) If $a : \mathbb{K} \to \mathbb{R}_+$ is a measurable function then*

$$\left[\int_{\mathbb{K}} a(k)^2 dk\right]^{1/2} = \left[\int_{(0,1]} a^*(x)^2 dx\right]^{1/2}.$$

(b) If $a : \mathbb{K} \times \mathbb{K} \to \mathbb{R}_+$ is a measurable function then

(i)

$$\int_{\mathbb{K}} \int_{\mathbb{K}} a(k,k')dk'dk = \int_0^1 \int_0^1 a^{**}(x,y)dydx.$$

(ii) The function a^{**} is nonincreasing: $a^{**}(x,y) \leq a^{**}(x',y')$ whenever $x \geq x'$ and $y \geq y'$.

(iii) For any measurable sets $D, E \subset \mathbb{K}$ with measures $|D|$ and $|E|$

$$\int_D \int_E a(k,k')dk'dk \leq \int_0^{|D|} \int_0^{|E|} a^{**}(x,y)dydx.$$

The proofs of the statements of the lemma are straightforward. Returning to our setting, let F^* and H^* be the nonincreasing rearrangements of F and H, let $\tilde{g} : \mathbb{K}\times\mathbb{K}\times\mathbb{R}_+ \to \{0,1\}$ be given by $\tilde{g}(k,k',l) = g(k^{-1}a(l)k')$ and let $G^{**} : (0,1]\times(0,1]\times\mathbb{R}_+ \to \{0,1\}$ be the double rearrangement of the function $\tilde{g}$ (such that $G^{**}(.,.,l)$ is the double rearrangement of $\tilde{g}(.,.,l)$ for all $l \geq 0$). Recall that we assumed that the function g is supported in the set $\underset{l>1}{\cup} \mathbb{K}a(l)\mathbb{K}$, therefore

$$||g||_{L^{2,1}(\mathbb{G})} \approx \left[\int_{\mathbb{R}_+} \int_0^1 \int_0^1 G^{**}(x,y,l)e^{2|\rho|l} dydxdl\right]^{1/2}. \tag{2.30}$$

We will now show how to use these rearrangements to dominate the two expressions in (2.28). For any integers m, n let $D_m = \{k \in \mathbb{K} : F(k) \in [e^{|\rho|m}, e^{|\rho|(m+1)}]\}$, $E_n = \{k' \in \mathbb{K} : H(k') \in [e^{|\rho|n}, e^{|\rho|(n+1)}]\}$ and let $D_{-\infty} = \{k \in \mathbb{K} : F(k) = 0\}$, $E_{-\infty} = \{k' \in \mathbb{K} : H(k') = 0\}$ such that $\mathbb{K} = \underset{m}{\cup} D_m = \underset{n}{\cup} E_n$. Let δ_m, respectively ε_n, be the measures of the sets D_m, respectively E_n, as subsets of $\mathbb{K}$. The first of the two expressions in (2.28) is dominated by

$$C\sum_{m,n}\int_{D_m}\int_{E_n}\int_{s\leq(n-m+1)}\int_{l\geq|s|} e^{2|\rho|(m+1)}G_1(k,k',l,s)\psi(l,s)e^{|\rho|s}dldsdk'dk. \quad (2.31)$$

Combining the definition (2.21) of the function G_1 (recall that the surfaces $T_{l,s}$ are defined as the set of points $P\in\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}$ with the property that $\overline{n}(P)a(s)\in\mathbb{K}a(l)\mathbb{K}$), the fact that dk is a Haar measure on $\mathbb{K}$ and the last statement of Lemma 2.5 we can conclude that

$$\int_{D_m}\int_{E_n} G_1(k,k',l,s)dk'dk \leq \int_0^{\delta_m}\int_0^{\varepsilon_n} G^{**}(x,y,l)dydx$$

for any s with the property that $|s|\leq l$. Substituting this inequality in (2.31) we find that the expression in (2.31) is dominated by

$$C\sum_{m,n}\int_{\mathbb{R}_+} e^{2|\rho|m}\left[\int_0^{\delta_m}\int_0^{\varepsilon_n} G^{**}(x,y,l)dydx\right]\left[\int_{s\leq(n-m+1),|s|\leq l}\psi(l,s)e^{|\rho|s}ds\right]dl. \quad (2.32)$$

The formula (2.8) shows that the last of the integrals in the expression above is dominated by $Ce^{|\rho|l}e^{|\rho|(n-m)}$, therefore the first of the two expressions in (2.28) is dominated by

$$C\int_{\mathbb{R}_+}\sum_{m,n}\left[e^{|\rho|(m+n)}\int_0^{\delta_m}\int_0^{\varepsilon_n} G^{**}(x,y,l)dydx\right]e^{|\rho|l}dl. \quad (2.33)$$

Let

$$S(x,y)=\sum_{m,n}\left[e^{|\rho|(m+n)}\chi_{\delta_m}(x)\chi_{\varepsilon_n}(y)\right]$$

where χ_{δ_m}, χ_{ε_n} are the characteristic functions of sets $(0,\delta_m)$, respectively $(0,\varepsilon_n)$. If $m_x=\max\{m:\delta_m>x\}$ and $n_y=\max\{n:\varepsilon_n>y\}$ then $S(x,y)\leq Ce^{|\rho|(m_x+n_y)}$.

Clearly $F^*(x) \geq e^{|\rho|m_x}$, $H^*(y) \geq e^{|\rho|n_y}$ so the expression (2.33) is dominated by

$$C \int_{\mathbb{R}_+} \int_0^1 \int_0^1 F^*(x)H^*(y)G^{**}(x,y,l)e^{|\rho|l} dydxdl.$$

One can deal with the second of the two expressions in (2.28) in a similar way, therefore

$$\iint_{\mathbb{G}\times\mathbb{G}} f(z)g(z^{-1}z')h(z')dz'dz \leq C \int_{\mathbb{R}_+} \int_0^1 \int_0^1 F^*(x)H^*(y)G^{**}(x,y,l)e^{|\rho|l} dydxdl. \tag{2.34}$$

Step 4. Final estimates. Let $\mathcal{K}$ be a suitable constant (to be chosen later) and let $\mathcal{U} = \{(x,y,l) : F^*(x)H^*(y) \leq \mathcal{K}e^{|\rho|l}\}$ and $\mathcal{V} = \{(x,y,l) : F^*(x)H^*(y) \geq \mathcal{K}e^{|\rho|l}\}$. Using (2.30)

$$\begin{aligned}\int_{\mathcal{U}} F^*(x)H^*(y)G^{**}(x,y,l)e^{|\rho|l} dydxdl &\leq \int_{\mathbb{R}_+} \int_0^1 \int_0^1 \mathcal{K}G^{**}(x,y,l)e^{2|\rho|l} dydxdl \\ &\leq C\mathcal{K}||g||^2_{L^{2,1}}.\end{aligned}$$

Using Lemma 2.5(a), (2.29) and the fact that $G^{**}(x,y,l) \leq 1$

$$\begin{aligned}\int_{\mathcal{V}} F^*(x)H^*(y)G^{**}(x,y,l)e^{|\rho|l} dydxdl &\leq C \int_0^1 \int_0^1 \frac{[F^*(x)H^*(y)]^2}{\mathcal{K}} dydx \\ &\leq C\frac{||f||^2_{L^{2,1}}||h||^2_{L^{2,1}}}{\mathcal{K}}.\end{aligned}$$

Finally, if one lets $\mathcal{K} = (||g||_{L^{2,1}})^{-1}(||f||_{L^{2,1}}||h||_{L^{2,1}})$, the theorem follows from (2.34).

2.4 A General Rearrangement Inequality

We will now extend the rearrangement inequality (2.34) to the case when f, g, h are arbitrary measurable functions (not just characteristic functions of sets). For any measurable function $f : \mathbb{G} \to \mathbb{R}_+$ we define the function $F^* : (0,1] \to \mathbb{R}_+$ by the following procedure: first, let $\tilde{f} : \mathbb{K} \times (0,\infty) \to \mathbb{R}_+$ be defined, for a.e.

$k \in \mathbb{K}$, as the usual nonincreasing rearrangement of the function $\overline{n}a \to f(\overline{n}ak)$ with respect to the measure $e^{2\rho(\log a)}d\overline{n}da$. Using the function $\tilde{f}$ we define the function $\tilde{F} : (0,1] \times (0,\infty) \to \mathbb{R}_+$: for each $u > 0$ fixed, the function $\tilde{F}(.,u)$ is the usual the nonincreasing rearrangement of the function $k \to \tilde{f}(k,u)$. Finally, let

$$F^*(x) = \frac{1}{2}\int_0^\infty \tilde{F}(x,u)u^{-1/2}du \tag{2.35}$$

be the $L^{2,1}$ norm of the function $u \to \tilde{F}(x,u)$. Notice that this definition of the function F^* agrees with our earlier definition if f is a characteristic function.

Corollary 2.6. *If $f, g, h : \mathbb{G} \to \mathbb{R}_+$ are measurable functions then*

$$\iint_{\mathbb{G}\times\mathbb{G}} f(z)g(z^{-1}z')h(z')dz'dz \le C\int_{\mathbb{R}_+}\int_0^1\int_0^1 F^*(x)H^*(y)G^{**}(x,y,l)\phi(l)dydxdl \tag{2.36}$$

where $G^{**} : (0,1] \times (0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is the double rearrangement of the function $(k,k',l) \to g(k^{-1}a(l)k')$ (the same meaning as before), F^* and H^* are defined in the previous paragraph and $\phi(l) = \sup_{d\in[-l,l]} e^{-|\rho|d}\int_{s\le d,|s|\le l}\psi(l,s)e^{|\rho|s}ds$ (clearly, $\phi(l) \approx l^{m_1+m_2}$ if $l \le 1$ and $\phi(l) \approx e^{|\rho|l}$ if $l \ge 1$).

Notice that if f and h are characteristic functions of sets then (2.36) is equivalent to (2.34). If f, h are simple positive functions, one can write (uniquely up to sets of measure zero) $f = \sum_1^M c_i f_i$, $h = \sum_1^N d_j h_j$ where $c_i, d_j > 0$ and f_i, respectively h_j, are characteristic functions of sets U_i, respectively V_j with the property that for all i, respectively j, $U_{i+1} \subset U_i$, respectively $V_{j+1} \subset V_j$. Simple manipulations involving rearrangements show that $F^* = \sum_1^M c_i F_i^*$ and $H^* = \sum_1^N d_j H_j^*$ (this explains the reason why we chose the apparently complicated definition of the function F^* in (2.35)) and (2.36) follows by summation. Finally, a standard limiting argument shows that (2.36) holds for arbitrary measurable functions f, g and h for which the right hand side integral in (2.36) converges.

3 Radial Fourier Integral Operators on $\mathbb{X}$

In this section we will prove theorems D and E. We start however with a rather surprising BMO theory on symmetric spaces, well adapted to the geometry of these spaces. The motivation for this BMO theory is the following: in Theorem D we prove $L^p \to L^p$ boundedness properties of a Fourier integral operator under sharp regularity assumptions. As in Euclidean spaces, this does not seem to be possible by interpolating with a suitable $L^1 \to L^1$ (or $L^\infty \to L^\infty$) estimate. The standard way to deal with this difficulty is to prove an $H^1_{\text{loc}} \to L^1_{\text{comp}}$ estimate ([18], [19]). In our case, however, the $\mathbb{K}$-invariant kernels of the operators $T_{1,\tau}$ and $T_{2,\tau}$ are singular on the sphere of radius τ (which is large if τ large). Thus, it appears that the best approach to keep both the regularity assumption and the exponential behavior of the norm (1.7) sharp is to work with a genuine H^1 or BMO space that may substitute for interpolation purposes the space L^1, respectively L^∞ (a slightly different line of approach has been pointed out by A. Seeger). It is more convenient to define the space BMO($\mathbb{X}$) and prove a suitable $L^\infty \to$ BMO estimate.

Recall that we identified the subgroup $\mathbb{A}$ of $\mathbb{G}$ with the real line $\mathbb{R}$ (as explained at the beginning of the second section) and we also identified $\mathfrak{a}^*_{\mathbb{C}}$, the complex dual of the Lie algebra $\mathfrak{a}$, with the complex plane $\mathbb{C}$ (as explained in the introduction). It is well known that

$$d(a(s) \cdot \mathbf{0}, a(s') \cdot \mathbf{0}) = |s - s'| \text{ for all } s, s' \in \mathbb{R}. \tag{3.1}$$

In view of the Cartan decomposition $\mathbb{G} = \mathbb{K}\overline{\mathbb{A}_+}\mathbb{K}$, we can identify any $\mathbb{K}$-invariant function $K : \mathbb{X} \to \mathbb{C}$ (i.e. $K(k \cdot z) = K(z)$ for all $z \in \mathbb{X}$, $k \in \mathbb{K}$) with the function $K : \mathbb{R}_+ \to \mathbb{C}$ given by $K(s) = K(a(s) \cdot \mathbf{0})$; in this section we will always use the same letter to denote a $\mathbb{K}$-invariant function on $\mathbb{X}$ and the associated function defined on $\mathbb{R}_+$. Using this convention, the convolution between a smooth compactly supported function $f : \mathbb{X} \to \mathbb{C}$ and a $\mathbb{K}$-invariant locally integrable kernel K is

$$f * K(z) = \int_{\mathbb{G}} f(g \cdot \mathbf{0}) K(g^{-1} \cdot z) dg = \int_{\mathbb{X}} f(z') K(d(z, z')) dz'. \tag{3.2}$$

3.1 BMO Theory on Symmetric Spaces

For any locally integrable function f on $\mathbb{X}$ let

$$f^{\sharp}(z) = \sup_{z \in B,\, r(B) \leq 1} \frac{1}{|B|} \int_B |f(z') - f_B| dz',$$

and, if $d \geq 0$,

$$\mathcal{M}_d f(z) = \sup_{z \in B,\, r(B) \leq d} \frac{1}{|B|} \int_B |f(z')| dz'.$$

The supremum in the two definitions is taken over all the balls B containing z of radius ≤ 1, respectively of radius $\leq d$, and, for any measurable set Q, $f_Q = \frac{1}{|Q|} \int_Q f(z) dz$. Let $B(z, r)$ denote the open ball centered at $z \in \mathbb{X}$ of radius r. Let C_0 be a fixed constant such that

$$|B(z, 2r)| \leq C_0 |B(z, r)|$$

for any point $z \in \mathbb{X}$ and any $r \in [0, 1]$. We define

$$||f||_{\mathrm{BMO}(\mathbb{X})} = ||f^{\sharp}||_{L^{\infty}(\mathbb{X})}.$$

One clearly has

$$||f^{\sharp}||_p \leq C_p ||f||_p, \tag{3.3}$$

for any $p > 1$ since $f^{\sharp}(z) \leq M_1(z)$ and M_1 is a bounded L^p operator if $p > 1$. Notice, however, that this inequality would not hold for any $p \leq 2$ if the supremum in the definition of $f^{\sharp}(z)$ was taken over all balls containing z (as it is done in the setting of Euclidean spaces). The main step in proving an interpolation theorem is the following proposition that shows that inequality (3.3) can be reversed.

Proposition 3.1. Converse Inequality. *If $1 \leq p < \infty$ and $f \in L^p$, then*

$$||f||_p \leq A_p ||f^\sharp||_p. \tag{3.4}$$

The bound A_p depends only on p and n.

Easy examples (characteristic functions of large balls) show that the converse inequality (3.4) fails to hold in the setting of Euclidean spaces if in the definition of the sharp function $f^\sharp$ the supremum is taken only over balls of radius ≤ 1. The relevant difference is based on the observation that a positive fraction of the volume of any set $\subset \mathbb{X}$ lies close to the boundary of the set. More precisely:

Lemma 3.2. *For any $\delta > 0$, there exists $\varepsilon(\delta) > 0$ such that for any measurable set of finite measure $A \subset \mathbb{X}$, the measure of the set $A(\delta) = \{z \in A : B(z, \delta) \subset A\}$ satisfies the inequality*

$$|A(\delta)| \leq (1 - \varepsilon(\delta))\, |A|. \tag{3.5}$$

Proof of Lemma 3.2. One can clearly assume that the set A is bounded, open, $\mathbf{0} \in A$ and, for any $k \in \mathbb{K}$ let $A_k = \{a \in \mathbb{A}_+ : ka \cdot \mathbf{0} \in A\}$. As explained before, the set A_k is identified with $\{s \in \mathbb{R}_+ : ka(s) \cdot \mathbf{0} \in A\}$; the relevant measure on $\mathbb{R}_+$ is $d\mu = (\sinh s)^{m_1} (\sinh 2s)^{m_2} ds$. For any bounded, nonempty open set $O \subset \mathbb{R}_+$ let $O(\delta) = \{s \in O : [(s - \delta, s + \delta) \cap \mathbb{R}_+] \subset O\}$. Since the set O is assumed to be bounded and nonempty, the set $O(\delta)$ is bounded as well and let $u = \sup O(\delta)$. It follows that

$$\frac{\mu\left(O(\delta)\right)}{\mu(O)} \leq \frac{\mu\left(O(\delta)\right)}{\mu\left(O(\delta)\right) + \mu([u, u + \delta))} \leq \frac{\mu([0, u))}{\mu([0, u + \delta))}.$$

An elementary calculation shows that

$$\frac{\mu([0, u))}{\mu([0, u + \delta))} \leq (1 - \varepsilon(\delta))$$

for any $u \in \mathbb{R}_+$ (it is here that the exponential increase of the measure is important), therefore

$$\frac{\mu\left(O(\delta)\right)}{\mu(O)} \leq (1 - \varepsilon(\delta))$$

for all bounded, nonempty open sets O. This inequality can be applied to the sets A_k (clearly $(A(\delta))_k \subset (A_k)(\delta)$ by (3.1)) and one integrates over $k \in \mathbb{K}$ to prove (3.5). □

A consequence of Lemma 3.2 is the following covering lemma:

Lemma 3.3. *If $O \subset \mathbb{X}$ is an open set, $|O| < \infty$ and $O \subset \bigcup_{i \in I} B_i$, where B_i are open balls of radius ≤ 1, then one can select a finite subset of disjoint balls $B_1,\ B_2 \ldots B_k$ such that*

(i) $|B_1| + |B_2| + \ldots |B_k| \geq c_0 |O|$;

(ii) the balls $B_1,\ B_2 \ldots B_k$ are close to the boundary of O in the sense that $d(B_j, {}^cO) \leq 1/10$ for any $j = 1, 2 \ldots k$.

Proof of Lemma 3.3. Let $\widetilde{O} = \{z \in O : d(z, {}^cO) < 1/10\}$ and $J = \{i \in I : B_i \cap \widetilde{O} \neq \varnothing\}$. By Lemma 1, $|\widetilde{O}| \geq \varepsilon(.1)|O|$ and clearly $\widetilde{O} \subset \bigcup_{i \in J} B_i$. One may now use standard arguments (as in [24, Chapter 1]) to select a finite set of disjoint balls $B_{i_1},\ B_{i_2} \ldots B_{i_k}$ such that $i_1, i_2 \ldots i_k \in J$ with the property that $|B_{i_1}| + |B_{i_2}| + \ldots |B_{i_k}| \geq |\widetilde{O}|/(2C_0)$. These balls satisfy the two conditions (i) and (ii) in the lemma and one may take $c_0 = \varepsilon(.1)/(2C_0)$. □

Proof of Proposition 3.1. The proposition is an easy consequence of the following distributional inequality relating $f^\sharp$ and the maximal function $\mathcal{M}_{1/4}$:

$$|\{z : \mathcal{M}_{1/4} f(z) > \alpha, f^\sharp(z) \leq \varepsilon\alpha\}| \leq a|\{z : \mathcal{M}_{1/4} f(z) > b\alpha\}|, \tag{3.6}$$

for any $\alpha > 0$, for some constants b close to 1, ε close to 0 and $a = 1 - \delta(b, \varepsilon)$ (the precise conditions on b and ε and the formula for a will become clear during the proof). To prove (3.6), let $A = \{z \in \mathbb{X} : \mathcal{M}_{1/4} f(z) > b\alpha\}$ and notice that for any

$z \in A$ one can find a ball B_z containing z such that $|f|_{B_z} > b\alpha$ and with the following maximality property: *either* $1/8 \leq r(B_z) \leq 1/4$ *or, for any ball* B'_z *containing* z *of radius* $r(B'_z) \geq 2r(B_z)$, *one has* $|f|_{B'_z} \leq b\alpha$. Clearly, $A = \bigcup_{z\in A} B_z$, $|A| < \infty$, so one can apply Lemma 3.3 to select a finite number of disjoint balls B_{z_i} close to the boundary of A such that

$$|B_{z_1}| + |B_{z_2}| + \ldots |B_{z_k}| \geq c_0|A|. \tag{3.7}$$

We will first prove that for any of the maximal balls B_{z_i} selected above, which will be denoted by B in the next paragraphs, one has

$$|\{z \in B : \mathcal{M}_{1/4}f(z) > \alpha, f^\sharp(z) \leq \varepsilon\alpha\} \leq a'|B| \tag{3.8}$$

where $a' = \frac{C\cdot\varepsilon}{1-b}$, C is a large constant depending only on n, and the numbers b and ε are such that $b + C\cdot\varepsilon < 1$. To prove (3.8), we have to analyze two different cases.

Case 1: $r(B) \geq 1/10$. Let $Q = \{z \in B : \mathcal{M}_{1/4}f(z) > \alpha, f^\sharp(z) \leq \varepsilon\alpha\}$. One can cover the set Q with a reunion of balls of radius $\leq 1/4$ such that $|f|_{B'} > \alpha$ for any of these balls B' and then, using Lemma 3.3, one can select a set of disjoint balls $B'_1, B'_2 \ldots B'_{k'}$, all of them intersecting the ball B, with the properties that

$$|B'_1| + |B'_2| + \ldots |B'_{k'}| \geq c_0|Q| \tag{3.9}$$

and $|f|_{B'_i} > \alpha$ for $i = 1, 2 \ldots k'$. Since the ball B is close to the boundary of the region A, there exists a ball $\widetilde{B}$, say of radius $1/10$, such that $|f|_{\widetilde{B}} \leq b\alpha$ and $d(B, \widetilde{B}) \leq 1/10$. Clearly, one can now find a larger ball B^* of radius 1 containing all the balls B, $\widetilde{B}$, $B'_1, B'_2 \ldots B'_{k'}$. If Q is not empty then

$$\frac{1}{|B^*|}\int_{B^*} |f(z') - f_{B^*}|dz' \leq \varepsilon\alpha$$

therefore

$$\begin{cases} \int_{\widetilde{B}} |f(z') - f_{B^*}|dz' \leq \varepsilon\alpha|B^*| \\ \sum \int_{B'_i} |f(z') - f_{B^*}|dz' \leq \varepsilon\alpha|B^*| \end{cases}$$

which shows that

$$\begin{cases} |\widetilde{B}|\,(|f_{B^*}| - b\alpha) \leq \varepsilon\alpha|B^*| \\ (\alpha - |f_{B^*}|)\,(|B'_1| + |B'_2| + \dots |B'_{k'}|) \leq \varepsilon\alpha|B^*|. \end{cases}$$

Clearly, (3.8) follows in this case by using inequality (3.9) and eliminating $|f_{B^*}|$ in the inequalities above.

Case 2: $r(B) < 1/10$. We start by defining the sets Q and the balls B'_i as in the first case. Let $r' = \max(r(B), r(B'_1) \dots r(B'_{k'}))$ and let B^* be a ball of radius $2r'$ containing all the balls $B, B'_1 \dots B'_{k'}$. By the maximality assumption on the ball B, one either has $|f|_{B^*} \leq b\alpha$ or $r' \geq 1/8$. If $r' \geq 1/8$, it follows by the same argument as in the first case that the set Q is empty provided that one takes $C\varepsilon < 1 - b$ for a large enough constant C. If $|f|_{B^*} \leq b\alpha$ and Q is not empty, then

$$\frac{1}{|B^*|}\int_{B^*} |f(z') - f_{B^*}|dz' \leq \varepsilon\alpha$$

therefore

$$\sum \int_{B'_i} |f(z') - f_{B^*}|dz' \leq \varepsilon\alpha|B^*|$$

which shows that

$$(1 - b)\,(|B'_1| + |B'_2| + \dots |B'_{k'}|) \leq \varepsilon|B^*| \tag{3.10}$$

This last equation shows in particular that the only nontrivial case is when $r' = r$; otherwise, $r(B^*) = 2r(B'_i)$ for some i, so inequality (3.10) could not hold if one lets $\varepsilon \ll (1-b)$. Moreover, if $r = r'$, it follows that $|B^*| \leq C_0|B|$, so one can combine (3.9) and (3.10) to complete the proof of the inequality (3.8).

Let $\widetilde{A} = \underset{i}{\cup} B_{z_i}$. The inequality (3.8) clearly shows that

$$|\{z \in \widetilde{A} : \mathcal{M}_{1/4} f(z) > \alpha, f^{\sharp}(z) \leq \varepsilon\alpha\}| \leq \frac{C\varepsilon}{1-b}|\widetilde{A}| \leq \frac{C\varepsilon}{1-b}|A|.$$

The main distributional inequality (3.6) follows by using (3.7). One has to assume that $\varepsilon \ll 1-b$ and the bound a in (3.6) is $a = 1 - \left(c_0 - \frac{C\cdot\varepsilon}{1-b}\right)$.

We are now in the position to use the general lemma in [24, page 152] to conclude that $||\mathcal{M}_{1/4} f||_p \leq A_p ||f^{\sharp}||_p$ for $p < \infty$. We only need to choose suitable constants b and ε such that $a < b^p$. For given p, we first choose b such that $b^p = 1 - \frac{c_0}{4}$ and then we choose ε small enough such that $a \leq 1 - \frac{c_0}{2}$. The conclusion of the proposition follows with the constant A_p in (3.4) satisfying $A_p \leq C \cdot p$. □

We conclude this discussion with an interpolation theorem.

Theorem 3.4. Analytic interpolation. *Let $\mathcal{S}$ denote the closed strip $0 \leq \Re\sigma \leq 1$ and assume that for any $\sigma \in \mathcal{S}$ one has a bounded linear operator $T_\sigma : L^2(\mathbb{X}) \to L^2(\mathbb{X})$ with the following properties:*
(i) There exists $K \geq 0$ such that $||T_\sigma(f)||_2 \leq K||f||_2$ for all $\sigma \in \mathcal{S}$ and any simple function f. The uniform bound K will not enter in the quantitative conclusion below.
(ii) For any simple functions f, g, the function $\sigma \to \int_{\mathbb{X}} T_\sigma(f)(z)g(z)dz$ is continuous in $\mathcal{S}$ and analytic in the interior of $\mathcal{S}$.
(iii) There exist bounds A_0 and A_1 such that for any simple function f

$$||T_\sigma(f)||_2 \leq A_0||f||_2 \text{ if } \Re\sigma = 0,$$

$$||T_\sigma(f)||_{\mathrm{BMO}} \leq A_1||f||_\infty \text{ if } \Re\sigma = 1.$$

Then for any $p \in [2, \infty)$ and any simple function f

$$||T_\sigma(f)||_p \leq A_p||f||_p$$

if $\Re\sigma = (p-2)/p$. *Moreover, the bound* A_p *satisfies the inequality*

$$A_p \leq C_p \cdot A_0^{2/p} A_1^{(p-2)/p}, \tag{3.11}$$

where C_p *is a constant depending only on* p.

Since both inequalities (3.3) and (3.4) hold, the proof of the corresponding Euclidean interpolation theorem ([10, page 156]) goes through with only straightforward modifications.

3.2 Proof of Theorem D

All of our L^p estimates in this section will be proved in *a priori* forms. This means that, in order to insure the convergence of the integrals throughout, we will always assume that all the symbols $m(\lambda)$ that appear at different places are premultiplied with symbols of the form $e^{-\delta^2\lambda^2}$. This approach is based on the observation that if $m \in S_a^b$, then the symbols $m_\delta(\lambda) = m(\lambda)e^{-\delta^2\lambda^2}$ belong to S_a^b uniformly in $\delta \in [0,1]$. Of course, our estimates will be independent of $\delta \in (0,1]$ and they will depend only on the constant C that appears in the definition of the symbol m. We will also assume that all the functions f on which various operators are tested are complex-valued smooth compactly supported functions on $\mathbb{X}$. Once one proves suitable estimates uniform in $\delta \in (0,1]$, standard limiting arguments allow one to pass to the general theorems. These assumptions will be implicit in all the computations we make and the subscripts δ will be omitted.

The following proposition is the real rank one version of the main theorem in [1]:

Proposition 3.5. *If* $1 < p < \infty$ *and* $m \in S^0_{|\rho|\alpha_p}$ *is an even symbol then the operator defined by the Fourier multiplier* m *is bounded from* $L^p(\mathbb{X})$ *to itself.*

An application of Proposition 3.5 shows that one can assume that the symbol m in Theorem D is of the form $(\lambda^2 + \rho'^2)^{-d\alpha_p/2}$ where $\rho' = |\rho| + 1/10 > |\rho|$. This allows one to expand the region in which m is analytic and satisfies suitable symbol estimates. Notice also that it suffices to prove Theorem D for $p \in [2, \infty)$, since the operators $T_{1,\tau}$ and $T_{2,\tau}$ are essentially selfadjoint. The theorem follows by analytic interpolation (Theorem 3.4 in the previous subsection) once one proves the following $L^\infty \to \mathrm{BMO}$ estimate:

Proposition 3.6. *If $m \in S^{-d}_{|\rho|}$ is an even symbol and the operators $T_{1,\tau}$, $T_{2,\tau}$ are defined by the multipliers $[m(\lambda)\cos(\lambda\tau)]$, respectively $\left[m(\lambda)\left(\lambda^2 + \rho'^2\right)^{1/2}\lambda^{-1}\sin(\lambda\tau)\right]$ then*

$$\begin{cases} ||T_{1,\tau} f||_{\mathrm{BMO}(\mathbb{X})} \leq C e^{|\rho|\tau} ||f||_{L^\infty(\mathbb{X})}; \\ ||T_{2,\tau} f||_{\mathrm{BMO}(\mathbb{X})} \leq C e^{|\rho|\tau}(1+\tau) ||f||_{L^\infty(\mathbb{X})}. \end{cases} \tag{3.12}$$

The notation is explained in (1.11). We will need the following easy lemma:

Lemma 3.7. *If $2 \leq q < \infty$, $b = \frac{n}{q} - \frac{n}{2}$, and $m \in S^b_0$ is an even symbol then the operator U defined by the Fourier multiplier m satisfies the inequality*

$$||Uf||_q \leq C_q ||f||_2.$$

This Sobolev-type lemma is a particular instance of a general situation covered in [15]. As it stands, the lemma follows also from [22, Theorem 6.1(c)(ii)]. One starts by writing down explicitly an integral formula of the $\mathbb{K}$-invariant kernel K of the operator U (using the inversion formula of the Fourier transform); next, one uses estimates on the spherical functions and the Harish-Chandra function (Propositions A1 and A2 in the appendix) to show that if $s = d(\mathbf{0}, z)$ then $|K(z)| \leq C s^{-(b+n)}$ if $s \leq 1$ and $|K(z)| \leq C e^{-|\rho| s} s^{-3}$ if $s \geq 1/2$. Finally, one uses a local version of the Hardy-Littlewood-Sobolev inequality to deal with the local part of the operator U (if $b < 0$) and a variant of the Kunze-Stein phenomenon to deal with its non-local part. We would like to point out the following endpoint estimate related to this lemma: if $b = -n/2$ and $m \in S^b_0$ is an even symbol then the operator U defined by the Fourier multiplier m is bounded from $L^2(\mathbb{X})$ to $\mathrm{BMO}(\mathbb{X})$. The proof of this endpoint estimate is similar, but easier than the proof of Proposition 3.6 below.

Proof of Proposition 3.6. We will only prove the estimate (3.12) for the operator $T_{1,\tau}$ since the estimate for $T_{2,\tau}$ is similar. Notice first that Plancherel's theorem and Lemma 3.7 show that

$$||T_{1,\tau}f||_{2n} \leq C||f||_2 \tag{3.13}$$

(since $\left|(\lambda^2+\rho'^2)^{1/2}\lambda^{-1}\sin(\lambda\tau)\right| \leq C(1+\tau)$, the estimate (3.13) for $T_{2,\tau}f$ becomes $||T_{2,\tau}f||_{2n} \leq C(1+\tau)||f||_2$). Assume first that $\tau \geq 1/2$. Let $B = B(z_0, r)$ be any ball in $\mathbb{X}$ with radius $r \leq 1$ and let $B^* = \{z \in \mathbb{X} : d(z, z_0) \in [\tau - 10r, \tau + 10r]\}$ be the main "region of influence" of B. Clearly $|B| \approx r^n$, $|B^*| \approx re^{2|\rho|\tau}$ and it suffices to prove that

$$\frac{1}{|B|}\int_B |T_{1,\tau}f(z) - (T_{1,\tau}f)_B|dz \leq Ce^{|\rho|\tau}||f||_\infty \tag{3.14}$$

with a constant C independent of the function f and the radius of the ball B. Let $f = f_1 + f_2$ where $f_1 = f\cdot(1-\chi_{B^*})$ and $f_2 = f\cdot\chi_{B^*}$. To deal with the function f_2, we use (3.13):

$$\begin{aligned}\frac{1}{|B|}\int_B |T_{1,\tau}f_2(z) - (T_{1,\tau}f_2)_B|dz &\leq \frac{2}{|B|}\int_B |T_{1,\tau}f_2(z)|dz \leq 2\left(\frac{1}{|B|}\right)^{1/2n}||T_{1,\tau}f_2||_{2n}\\ &\leq Cr^{-1/2}||f_2||_2 \leq Cr^{-1/2}||f||_\infty \cdot |B^*|^{1/2}\\ &\leq Ce^{|\rho|\tau}||f||_\infty.\end{aligned} \tag{3.15}$$

Let $K_{1,\tau}$ be the kernel of the operator $T_{1,\tau}$, which is a smooth function on $\mathbb{X}$ in view of the a priori assumption on the symbol m. The inversion formula of the spherical Fourier transform shows that

$$K_{1,\tau}(z) = c_1\int_{\mathbb{R}}(m(\lambda)\cos(\lambda\tau))\Phi_\lambda(z)|\mathbf{c}(\lambda)|^{-2}d\lambda \tag{3.16}$$

where $\Phi_\lambda(z)$ are the elementary spherical functions and $\mathbf{c}$ is the Harish-Chandra function. We will use from now on the convention explained in the paragraph preceding (3.2). To deal with the function f_1 we evaluate the left hand side of (3.14):

$$\begin{aligned}
&\frac{1}{|B|}\int_B |T_{1,\tau}f_1(z)-(T_{1,\tau}f_1)_B|dz = \\
&= \frac{1}{|B|}\int_B \left|\int_{\mathbb{X}} f_1(z')K_{1,\tau}(d(z,z'))dz' - \frac{1}{|B|}\int_B\int_{\mathbb{X}} f_1(z')K_{1,\tau}(d(z'',z'))dz'dz''\right| dz \\
&= \frac{1}{|B|^2}\int_B \left|\int_{\mathbb{X}} f_1(z')\left(\int_B K_{1,\tau}(d(z,z'))-K_{1,\tau}(d(z'',z'))dz''\right)dz'\right| dz \\
&\leq ||f||_\infty \cdot \frac{1}{|B|^2}\iint\limits_{B\times B}\left(\int_{^cB^*}|K_{1,\tau}(d(z,z'))-K_{1,\tau}(d(z'',z'))|dz'\right)dzdz''.
\end{aligned} \tag{3.17}$$

It would therefore suffice to prove that

$$\int_{^cB^*}|K_{1,\tau}(d(z,z'))-K_{1,\tau}(d(z'',z'))|dz' \leq Ce^{|\rho|\tau} \tag{3.18}$$

for any $z,z''\in B$. By the inversion formula (3.16)

$$K_{1,\tau}(s) = c_1\int_{\mathbb{R}}(m(\lambda)\cos(\lambda\tau))\,\Phi_\lambda(s)|\mathbf{c}(\lambda)|^{-2}d\lambda \tag{3.19}$$

Let $A_{1,\tau}(s)=\phi_\tau(s)K_{1,\tau}(s)$ and $B_{1,\tau}(s)=(1-\phi_\tau(s))K_{1,\tau}(s)$ where the function ϕ_τ is a C^∞ cutoff function with the properties that $\phi_\tau(s)=1$ if $|s-\tau|\leq 1/10$ and $\phi_\tau(s)=0$ if $|s-\tau|\geq 2/10$ such that $K_{1,\tau}=A_{1,\tau}+B_{1,\tau}$. The main estimate on the kernel $B_{1,\tau}$ is

$$|B_{1,\tau}(s)| \leq \begin{cases} Ce^{|\rho|\tau}e^{-2|\rho|s}(1+|\tau-s|)^{-2} & \text{if } s\geq\tau; \\ Ce^{-|\rho|s}(1+|\tau-s|)^{-2} & \text{if } 1/10\leq s\leq\tau; \\ Cs^{-(d+1)} & \text{if } s\leq 1/10. \end{cases} \tag{3.20}$$

To prove this estimate when $s \geq 1/10$, one starts from Proposition A2(c) and writes:

$$B_{1,\tau}(s) = 2c_1\,(1-\phi_\tau(s))\,e^{-|\rho|s}\int_{\mathbb{R}}(m(\lambda)\cos(\lambda\tau))\,e^{i\lambda s}a_2(\lambda,s)\mathbf{c}(-\lambda)^{-1}d\lambda.$$

Notice that the function under the integral above is analytic in the region $0 \leq \Im\lambda \leq |\rho|$. If $s \geq \tau$, we first move the contour of integration to the line $i|\rho| + \mathbb{R}$ in order to get the essential decreasing factor $e^{-|\rho|s}e^{-|\rho|(s-\tau)}$. Next, we use (A.6) and (A.2) together with classical estimates on Fourier transforms of symbols ([24, page 241]) to prove (3.20) in this case. A similar argument (without changing the contour of integration) shows that (3.20) holds if $1/10 \leq s \leq \tau$. To prove the estimate for small s let η_0 be an even, smooth cutoff function on $\mathbb{R}$ such that $\eta_0(\mu) = 1$ if $|\mu| \leq 1$ and $\eta_0(\mu) = 0$ if $|\mu| \geq 2$ and notice that, using proposition A2(b), the kernel $B_{1,\tau}$ can be written in the form

$$\begin{aligned}B_{1,\tau}(s) &= 2c_1(1-\phi_\tau(s))\int_{\mathbb{R}}(1-\eta_0(\lambda s))m(\lambda)\cos(\lambda\tau)e^{i\lambda s}a_1(\lambda,s)|\mathbf{c}(\lambda)|^{-2}d\lambda\\ &\quad + c_1(1-\phi_\tau(s))\int_{\mathbb{R}}\left[\eta_0(\lambda s)\Phi_\lambda(s) + (1-\eta_0(\lambda s))O(\lambda,s)\right]m(\lambda)\cos(\lambda\tau)|\mathbf{c}(\lambda)|^{-2}d\lambda.\end{aligned}$$

By (A.2), (A.4) and (A.5), the second of the integrals in the expression above is dominated by $Cs^{-(d+1)}$. Also, one integrates by parts twice in λ and uses (A.5) and (A.2) to prove that the first integral is dominated by Cs^{-d}, which completes the proof of (3.20). An immediate consequence of (3.20) and (2.3) is that $||B_\tau||_{L^1(\mathbb{X})} \leq Ce^{|\rho|\tau}$, therefore

$$\int_{^cB^*}|B_{1,\tau}(d(z,z')) - B_{1,\tau}(d(z'',z'))|dz' \leq 2||B_{1,\tau}||_{L^1(\mathbb{X})} \leq Ce^{|\rho|\tau}$$

for any $z, z'' \in B$. It remains to prove a similar inequality for the kernel $A_{1,\tau}$ which, since $\tau \geq 1/2$, is given by the formula

$$A_{1,\tau}(s) = 2c_1\phi_\tau(s)e^{-|\rho|s}\int_{\mathbb{R}}(m(\lambda)\cos(\lambda\tau))\,e^{i\lambda s}a_2(\lambda,s)\mathbf{c}(-\lambda)^{-1}d\lambda. \tag{3.21}$$

Since the function $\lambda \to m(\lambda)a_2(\lambda,s)\mathbf{c}(-\lambda)^{-1}$ is a symbol on the real line of order 0 one has

$$\left|\frac{\partial}{\partial s}A_{1,\tau}(s)\right| \leq Ce^{-|\rho|\tau}\frac{1}{|\tau-s|^2}$$

if $|\tau - s| \leq 2/10$ which shows that

$$\begin{aligned}\int_{^cB^*}|A_{1,\tau}(d(z,z')) - A_{1,\tau}(d(z'',z'))|dz' &\leq C\int_{^cB^*} r\cdot\sup_{z\in B}\left|\frac{\partial}{\partial s}A_{1,\tau}(d(z,z'))\right|dz' \\ &\leq C{\cdot}re^{|\rho|\tau}\int_{5r\leq|s-\tau|\leq 2/10}\frac{1}{|\tau-s|^2}ds \leq Ce^{|\rho|\tau}.\end{aligned} \tag{3.22}$$

This finishes the proof of the proposition in the case $\tau \geq 1/2$. The proof if $\tau \leq 1/2$ proceeds along the same line. Let $B = B(z_0, r)$ be any ball in $\mathbb{X}$ and it suffices again to prove inequality (3.14). Let $B^* = \{z \in \mathbb{X} : d(z,z_0) \in [\tau - 10r, \tau + 10r] \cup [0, 10r]\}$ such that $|B| \approx r^n$, $|B^*| \leq C \cdot r$. Let $f_1 = f(1 - \chi_{B^*})$, $f_2 = f\chi_{B^*}$; the inequalities in (3.15) and (3.17) do not change, so it suffices again to prove (3.18). We define the kernels $A_{1,\tau}(s) = \phi_0(s)K_{1,\tau}(s)$ and $B_{1,\tau}(s) = (1 - \phi_0(s))\,K_{1,\tau}(s)$ using a smooth function $\phi_0 : \mathbb{R}_+ \to [0,1]$ such that $\phi_0(s) = 1$ if $s \leq 3/4$ and $\phi_0(s) = 0$ if $s \geq 1$. The estimate (3.20) becomes

$$|B_{1,\tau}(s)| \leq Ce^{-2|\rho|s}(1 + |\tau - s|)^{-2}$$

which shows that $||B_{1,\tau}||_{L^1(\mathbb{X})} \leq C$. To deal with the kernel $A_{1,\tau}$ one uses again the cutoff function η_0 defined in the proof of (3.20) and Proposition A2(b):

$$\begin{aligned} A_{1,\tau}(s) &= c_1\phi_0(s)\int_{\mathbb{R}}\left[\eta_0(\lambda s)\Phi_\lambda(s)+(1-\eta_0(\lambda s))O(\lambda,s)\right]m(\lambda)\cos(\lambda\tau)|\mathbf{c}(\lambda)|^{-2}d\lambda \\ &\quad + 2c_1\phi_0(s)\int_{\mathbb{R}}(1-\eta_0(\lambda s))m(\lambda)\cos(\lambda\tau)e^{i\lambda s}a_1(\lambda,s)|\mathbf{c}(\lambda)|^{-2}d\lambda \\ &\quad = I_{1,\tau}(s)+J_{1,\tau}(s). \end{aligned}$$

By (A.2), (A.4) and (A.5) $I_{1,\tau}(s)$, the first of the two integrals above, is bounded by $Cs^{-(d+1)}$ i.e. it is an L^1 function. In addition

$$\left|\frac{\partial}{\partial s}J_{1,\tau}(s)\right| \le C\frac{1}{s^d}\left(\frac{1}{|\tau-s|^2}+\frac{1}{s|\tau-s|}\right)$$

by (A.5) and standard estimates on Fourier transforms of symbols on the real line. An estimate similar to (3.22) completes the proof of the proposition. □

3.3 Proof of Theorem E. Part I

As in the previous subsection, we make the a priori assumption that all the functions f on which various operators are tested are smooth, compactly supported functions on $\mathbb{X}$. Notice first that the "local" part of Theorem E, that is if, for example, $T \le 10$, follows from the more general maximal operators studied in [21] if $n \ge 3$ and [20] if $n = 2$ (see the remark following Corollary 2.2 in [20]). Assume therefore that $T \ge 10$. If $\tau \ge 10$ then the Fourier transform of $d\sigma_\tau$ is

$$\widetilde{d\sigma_\tau}(\lambda) = \Phi_\lambda(\tau) = e^{-|\rho|\tau}\left(e^{i\lambda\tau}\mathbf{c}(\lambda)a_2(\lambda,\tau)+e^{-i\lambda\tau}\mathbf{c}(-\lambda)a_2(-\lambda,\tau)\right).$$

Let $\phi_T : \mathbb{R}_+ \to [0,1]$ be a C^∞ cutoff function $\phi_T : \mathbb{R}_+ \to [0,1]$ such that $\phi_T(s) = 1$ if $s \in [T-1/2, T+3/2]$ and $\phi_T(s) = 0$ if $s \notin [T-1, T+2]$. Let η_0 be the even cutoff function defined in the proof of (3.20) and, for $j = 1, 2, \ldots$, let $\eta_j(\mu) = \eta_0(2^{-j}\mu) - \eta_0(2^{-j+1}\mu)$. Clearly, supp $\eta_j \subset \{\mu \in \mathbb{R} : |\mu| \in [2^{j-1}, 2^{j+1}]\}$ for any $j \ge 1$.

If $\tau \in [T, T+1]$, the Littlewood-Paley decomposition of the singular kernel $d\sigma_\tau$ is $d\sigma_\tau = \sum_{j=0}^{\infty} A_\tau^j$ (in the sense of distributions) where

$$\begin{aligned} A_\tau^j(s) &= c_1 \phi_T(s) \int_{\mathbb{R}} \eta_j(\lambda) \Phi_\lambda(\tau) \Phi_\lambda(s) |\mathbf{c}(\lambda)|^{-2} d\lambda \\ &= 2c_1 \phi_T(s) e^{-|\rho|(s+\tau)} \int_{\mathbb{R}} \eta_j(\lambda) a_2(\lambda, \tau) e^{i\lambda\tau} \left(\frac{\mathbf{c}(\lambda)}{\mathbf{c}(-\lambda)} a_2(\lambda, s) e^{i\lambda s} + a_2(-\lambda, s) e^{-i\lambda s} \right) d\lambda \end{aligned} \tag{3.23}$$

and, as before, $A_\tau^j(z) = A_\tau^j(d(\mathbf{0}, z))$ for $z \in \mathbb{X}$. The estimate (A.6) and integration by parts show that for any $\tau \in [T, T+1]$

$$|A_\tau^j(s)| \leq C \cdot 2^j e^{-2|\rho|T} (1 + 2^j |\tau - s|)^{-N} \tag{3.24}$$

if $s \in [T-1, T+2]$ and $A_\tau^j(s) = 0$ otherwise. Let

$$\mathcal{M}_T^j f(z) = \sup_{\tau \in [T, T+1]} |f * A_\tau^j(z)|.$$

The estimate (3.24) and the integral formula (2.3) show that

$$\int_{\mathbb{X}} |A_\tau^j(z)| dz \leq C$$

uniformly in $\tau \in [T, T+1]$ and $j \geq 0$, therefore

$$||\mathcal{M}_T^j f||_\infty \leq C ||f||_\infty \tag{3.25}$$

with a universal constant C. There is also a very crude L^1 estimate: notice that

$$\sup_{\tau \in [T, T+1]} |A_\tau^j(s)| \leq \begin{cases} C \cdot 2^j e^{-2|\rho|T} & \text{if } s \in [T-1, T+2], \\ 0 & \text{otherwise,} \end{cases}$$

which shows that

$$||\mathcal{M}_T^j f||_1 \leq \left|\left| |f| * \sup_{\tau \in [T,T+1]} |A_\tau^j| \right|\right|_1 \leq C \cdot 2^j ||f||_1. \tag{3.26}$$

Our next task, which will suffice if $n \geq 3$, is to prove the L^2 estimate

$$||\mathcal{M}_T^j f||_2 \leq C \cdot 2^{-j(n-2)/2} \cdot e^{-|\rho|T}(T+1)||f||_2. \tag{3.27}$$

This would be a standard consequence of the following two estimates:

$$\begin{cases} ||f * A_\tau^j||_2 \leq C \cdot 2^{-jd} \cdot e^{-|\rho|T}(T+1)||f||_2, \\ \left|\left|\frac{\partial}{\partial \tau}(f * A_\tau^j)\right|\right|_2 \leq C \cdot 2^{-j(d-1)} \cdot e^{-|\rho|T}(T+1)||f||_2 \end{cases} \tag{3.28}$$

for any $\tau \in [T, T+1]$. To prove (3.28) let

$$\begin{aligned} B_\tau^j(s) &= c_1(1-\phi_T(s)) \int_{\mathbb{R}} \eta_j(\lambda)\Phi_\lambda(\tau)\Phi_\lambda(s)|\mathbf{c}(\lambda)|^{-2} d\lambda \\ &= 2c_1(1-\phi_T(s))e^{-|\rho|(s+\tau)} \int_{\mathbb{R}} \eta_j(\lambda)a_2(\lambda,\tau)\mathbf{c}(\lambda)\mathbf{c}(-\lambda)^{-1}a_2(\lambda,s)e^{i\lambda(\tau+s)}d\lambda \\ &+ 2c_1(1-\phi_T(s))e^{-|\rho|(s+\tau)} \int_{\mathbb{R}} \eta_j(\lambda)a_2(\lambda,\tau)a_2(-\lambda,s)e^{-i\lambda(\tau-s)}d\lambda \end{aligned}$$

be the complementary kernel of A_τ^j such that $(\widetilde{A_\tau^j + B_\tau^j})(\lambda) = \eta_j(\lambda)\Phi_\lambda(\tau)$. Using Plancherel's theorem, the estimates in Proposition A2(c) and (A.2) one has

$$||f * (A_\tau^j + B_\tau^j)||_2 \leq C \cdot 2^{-jd} \cdot e^{-|\rho|T}(T+1)||f||_2.$$

(the factor $(T+1)$ appears only if $j=0$). Easy estimates on $|B_\tau^j|$ (similar to the ones in Proposition 3.6) show that

$$|B_\tau^j(s)| \leq Ce^{-|\rho|(s+\tau)}2^{-Nj}(1+|\tau-s|)^{-N}$$

therefore, by Plancherel's theorem and (A.4)

$$
\begin{aligned}
||f * B^j_\tau||_2 &\leq C||f||_2 \int_0^\infty |B^j_\tau(s)| e^{-|\rho| s}(s+1)(\sinh s)^{m_1}(\sinh 2s)^{m_2} ds \\
&\leq C 2^{-Nj} e^{-|\rho| T}(T+1)||f||_2
\end{aligned}
$$

and the first of the estimates in (3.28) follows. The proof of the second estimate in (3.28) is similar, the only difference being that differentiation with respect to τ may bring down an extra factor of $\lambda \approx 2^j$. One can now apply the general lemma in [24, page 499] to complete the proof of the main L^2 estimate (3.27).

If $n \geq 3$, we interpolate between the estimates (3.27) and (3.25) or between (3.27) and (3.26) and conclude that for any $p \in (n/(n-1), \infty)$ there exists $\varepsilon(p) > 0$ ($\varepsilon(p) = (n-2)/p$ if $p \geq 2$ and $\varepsilon(p) = n - 1 - n/p$ if $p \leq 2$) such that

$$
||\mathcal{M}^j_T f||_p \leq C \cdot 2^{-\varepsilon(p) j} \cdot e^{-(1-\alpha_p)|\rho| T}(T+1)||f||_p.
$$

A final summation over positive integers j finishes the proof of Theorem B when $n \geq 3$.

3.4 Proof of Theorem E. Part II ($n = 2$)

If $n = 2$ then the only possibility is that $\mathbb{X} = \mathbb{H}^2$ (the hyperbolic space of dimension 2) thus $m_1 = 1$, $m_2 = 0$ and $|\rho| = 1/2$. The estimate (3.27) becomes

$$
||\mathcal{M}^j_T f||_2 \leq C \cdot e^{-T/2}(T+1)||f||_2. \tag{3.29}
$$

This is not sufficient since one has to sum over j. The essential step in proving the theorem in this case is the following:

Lemma 3.8. *There exist universal constants $\varepsilon_0 > 0$ and N_0 such that*

$$||\mathcal{M}_T^j f||_4 \leq C \cdot 2^{-\varepsilon_0 j} e^{N_0 T} ||f||_4. \tag{3.30}$$

Let us first see how one can use Lemma 3.8 to complete the proof of the theorem when $n = 2$. If one interpolates between (3.30) and (3.25) or between (3.30) and (3.29), one finds that for any $p \in (2, \infty)$ there exists $\varepsilon_0(p) > 0$ such that

$$||\mathcal{M}_T^j f||_p \leq C \cdot 2^{-\varepsilon_0(p) j} e^{N_0 T} ||f||_p. \tag{3.31}$$

One can also interpolate between (3.25) and (3.29) to conclude that

$$||\mathcal{M}_T^j f||_p \leq C \cdot e^{-(1-\alpha_p)T/2}(T+1)||f||_p, p \in [2, \infty]. \tag{3.32}$$

In order to sum over j, one uses (3.32) for $j \leq C_0(p)T$ and (3.31) for $j \geq C_0(p)T$ where $C_0(p) = \frac{N_0 + (1-\alpha_p)/2}{\log 2 \cdot \varepsilon_0(p)}$ is such that the two norms in (3.31) and (3.32) are essentially equal. The result is:

$$||\mathcal{M}_T f||_p \leq \sum_{j=0}^{j \leq C_0(p)T} ||\mathcal{M}_T^j f||_p + \sum_{j \geq C_0(p)T} ||\mathcal{M}_T^j f||_p \leq C_p \cdot e^{-(1-\alpha_p)T/2}(T+1)^2 ||f||_p.$$

which proves Theorem B in the case $n = 2$.

Proof of Lemma 3.8. Roughly speaking, the favorable factor $2^{-\varepsilon_0 j}$ in (3.30) comes from the proof of the main theorem in [20] while the unfavorable but (fortunately) not very important factor $e^{N_0 T}$ is due to several localizations we have to make and to quantitative estimates on the rotational curvature of defining functions of circles of radius $\approx T$. We start by localizing the operator $\mathcal{M}_T^j$. Notice that it suffices to prove that for any smooth cutoff functions ψ_0, $\psi_1 : \mathbb{H}^2 \to [0, 1]$ with small supports (say of diameter at most c_0, where c_0 is a small constant to be fixed later), one has

$$\left\|\sup_{\tau\in[T,T+1]}\left|\psi_0(z)\int_{\mathbb{H}^2} f(z')\psi_1(z')A_\tau^j(d(z,z'))dz'\right|\right\|_4 \leq C\cdot 2^{-\varepsilon_0 j}e^{N_0'T}||f\psi_1||_4. \quad (3.33)$$

To show that (3.33) suffices we define a suitable family of smooth cutoff functions with small supports ψ_i, indexed over a countable sets I, with the properties that $\sum_{i\in I}\psi_i = 1$ and any ball $B\subset\mathbb{H}^2$ of radius 1 intersects at most a constant number C of the supports of the functions ψ_i (C depends only on c_0, the size of the supports of the functions ψ_i) . For any $i\in I$, let $L_i = \{i'\in I : \exists z\in \text{supp}(\psi_i), z'\in \text{supp}(\psi_{i'})$ such that $d(z,z')\in[T-1,T+2]\}$. Clearly, each set L_i has at most Ce^T elements. Recall also that the kernels $A_\tau^j(d(z,z'))$ vanish unless $d(z,z')\in[T-1,T+2]$, therefore

$$\begin{aligned}
\int_{\mathbb{H}^2}\left|\mathcal{M}_T^j f(z)\right|^4 dz &\leq C\sum_{i\in I}\int_{\mathbb{H}^2}\left|\psi_i(z)\mathcal{M}_T^j f(z)\right|^4 dz\\
&\leq C\sum_{i\in I}\int_{\mathbb{H}^2}|\psi_i(z)\sum_{i'\in L_i}\mathcal{M}_T^j(\psi_{i'}f)(z)|^4 dz\\
&\leq Ce^{3T}\sum_{i\in I}\sum_{i'\in L_i}\int_{\mathbb{H}^2}|\psi_i(z)\mathcal{M}_T^j(\psi_{i'}f)(z)|^4 dz\\
&\leq Ce^{3T}\cdot\left(2^{-\varepsilon_0 j}e^{N_0'T}\right)^4\sum_{i\in I}\sum_{i'\in L_i}\int_{\mathbb{H}^2}|(\psi_{i'}f)(z')|^4 dz'\\
&\leq Ce^{3T}\cdot\left(2^{-\varepsilon_0 j}e^{N_0'T}\right)^4\cdot e^T\int_{\mathbb{H}^2}|f(z')|^4 dz'
\end{aligned}$$

which proves (3.30) with $N_0 = N_0'+1$. It remains to prove (3.33). By the $\mathbb{G}$-invariance of the measure on $\mathbb{X}$, we may assume that the cutoff function ψ_1 in (3.33) has small support around the point $\mathbf{0}\in\mathbb{H}^2$ and ψ_0 has small support around the point $a(T_0)\cdot\mathbf{0}\in\mathbb{H}^2$ (clearly, the only nontrivial case is when $T_0\in[T-1,T+2]$). The formula (3.23) shows that we may also replace the kernel $\psi_0(z)\psi_1(z')A_\tau^j(d(z,z'))$

with a kernel $K^j_\tau(z,z')$ of the form

$$K^j_\tau(z,z') = 2c_1 e^{-\tau}\phi_T(\tau)\psi_0(z)\psi_1(z')\int_{\mathbb{R}} \eta_j(\lambda) e^{i\lambda(\tau-d(z,z'))} b(\lambda,\tau)d\lambda \tag{3.34}$$

where b is a symbol of order 0 (uniformly in $\tau \in [T-1, T+2]$) and it remains to prove that

$$\left|\left| \sup_{\tau\in[T,T+1]} \left| \int_{\mathbb{H}^2} f(z')K^j_\tau(z,z')dz' \right| \right|\right|_4 \leq C\cdot 2^{-\varepsilon_0 j} e^{N_0' T}||f||_4 \tag{3.35}$$

(the error made in replacing $\psi_0(z)\psi_1(z')A^j_\tau(d(z,z'))$ by $K^j_\tau(z,z')$ is controlled by $C\psi_0(z)\psi_1(z')e^{-T}(1+2^j|\tau - d(z,z')|)^{-N}$ and it is easily seen that the $L^4 \to L^4$ norm of the corresponding maximal operator is dominated by $Ce^{-T}2^{-3j/4}$).

The estimate (3.35) will follow from the following simplified version of Sogge's main theorem in [20]. Let X and Y be two Riemannian manifolds of dimension 2 and let $\widetilde{\psi_0}(x)$ and $\widetilde{\psi_1}(y)$ be two cutoff functions with small compact supports included in small open sets $K \subset X$, respectively $L \subset Y$. Let $\Psi : K \times L \to [T-1, T+2]$ be a smooth function with the following properties

$$\left| \det \begin{bmatrix} 0 & \partial\Psi/\partial x \\ \frac{\partial \Psi}{\partial y} & \frac{\partial^2\Psi}{\partial x \partial y} \end{bmatrix} \right| \geq c > 0 \text{ for all } x \in K,\ y \in L$$

and

$$||\Psi'_x(x,y)|| \equiv 1 \text{ for all } x \in K,\ y \in L.$$

The first property is usually referred to as rotational curvature while the second property is a simplified version of Sogge's cinematic curvature hypothesis (the norm of the vector $\Psi'_x(x,y)$ is related to the Riemannian metric on X). Using the functions

b and η_j from (3.34) let

$$\widetilde{K^j_\tau}(x,y) = \widetilde{\psi_0}(x)\widetilde{\psi_1}(y)\int_{\mathbb{R}} \eta_j(\lambda)e^{i\lambda(\tau-\Psi(x,y))}b(\lambda,\tau)d\lambda.$$

Theorem. (C.D. Sogge [20]). With this notation, there exists $\varepsilon_0 > 0$ such that for any $j \geq 0$

$$\left|\left|\sup_{\tau\in[T,T+1]}\left|\int_Y f(y)\widetilde{K^j_\tau}(x,y)dy\right|\right|\right|_{L^4(X)} \leq C\cdot 2^{-\varepsilon_0 j}||f||_{L^4(Y)}.$$

Remark. Most calculations in [20] are done using an apparently different form of the kernels $\widetilde{K^j_\tau}$ (see equation (3.18) in [20]). However, as explained at various places in [20] the two forms are equivalent modulo $O(2^{-Nj})$ errors.

In order to apply Sogge's theorem and prove (3.35) one has to rescale the problem (our situation is somewhat degenerate in the sense that the Monge-Ampere determinant associated to $d(.,.)$ is $\approx e^{-T}$). We will use natural coordinates on $\mathbb{H}^2$ induced by the Iwasawa decomposition of the group $\mathbb{G} = \mathbb{SO}_e(2,1)$. Using the notation in [4], one has the Iwasawa decomposition $\mathbb{SO}_e(2,1) = \mathbb{NAK}$ and there exists a diffeomorphism $n : \mathbb{R} \to \mathbb{N}$ such that

$$a(u)n(v) = n(e^u v)a(u) \text{ for all } u,v\in\mathbb{R} \tag{3.36}$$

and

$$\cosh[d(n(v)a(u)\cdot\mathbf{0},\mathbf{0})] = \cosh u + e^{-u}v^2/2 \text{ for all } u,v\in\mathbb{R}. \tag{3.37}$$

Furthermore, one can identify $\mathbb{H}^2$ with $\mathbb{R}\times\mathbb{R}$ using the map $(u,v)\to n(v)a(u)\cdot\mathbf{0}$ and the change of measure is $dz = C_2e^{-u}dudv$. The functions ψ_0 and ψ_1 in the formula (3.34) have small supports around the points $a(T_0)\cdot\mathbf{0}$ and $\mathbf{0}$; if one lets $z = a(T_0)n(v)a(u)\cdot\mathbf{0}$ and $z' = n(v')a(u')\cdot\mathbf{0}$, a simple calculation using (3.36), (3.37) and the $\mathbb{G}$-invariance of the distance function shows that $\cosh[d(z,z')] = \cosh(T_0 +$

$u - u') + e^{T_0 - u - u'}(v - e^{-T_0}v')^2/2$. This suggest to rescale v'.

Let therefore $z(u,v) = a(T_0)n(v)a(u) \cdot \mathbf{0}$ and $z'(u^*,v^*) = n(e^{T_0}v^*)a(u^*) \cdot \mathbf{0}$ for $|u|, |v|, |u^*| \leq c_0$ and $|v^*| \leq c_0 e^{-T_0}$. One has

$$d(z,z') = \Psi_{T_0}((u,v),(u^*,v^*)) = \operatorname{arccosh}\left[\cosh(T_0 + u - u^*) + e^{T_0 - u - u^*}(v - v^*)^2/2\right]. \tag{3.38}$$

Notice that the problem in $[(u,v),(u^*,v^*)]$-coordinates is *not* degenerate any longer. Indeed, one can easily check that $[(\partial\Psi_{T_0}/\partial u)^2 + e^{2u}(\partial\Psi_{T_0}/\partial v)^2]^{1/2} \equiv 1$, which is the simplified version of the cinematic curvature condition. Also, the function Ψ_{T_0} can be written in the form

$$\Psi_{T_0}((u,v),(u^*,v^*)) = T_0 + u - u^* + (v - v^*)^2 \cdot C_{T_0}(u,v,u^*,v^*)$$

where $C^{-1} \leq C_{T_0}(0,0,0,0) \leq C$ (uniformly if $T_0 \geq 1$) and all the first and second order derivatives of the function C around the point $(0,0,0,0)$ are bounded by an absolute constant (independent of T_0). Thus the rotational curvature hypothesis is satisfied if one chooses c_0 small enough (depending only on this absolute constant). Sogge's theorem applies to the maximal operator with kernels $K^j_\tau((u,v),(u^*,v^*))$ defined as in (3.34) (replacing of course $d(z,z')$ by $\Psi_{T_0}((u,v),(u^*,v^*))$ and $\psi_1(z')$ by a suitable non-degenerate cutoff function $\widetilde{\psi_1}(u^*,v^*)$). One can finally trace back the e^T factors and conclude that (3.35) holds with a small $\varepsilon_0 > 0$ and $N_0' = -1 + 3/4 = -1/4$ (the term -1 comes from the factor $e^{-\tau} \approx e^{-T}$ in front of the integral in (3.34)) and the lemma follows with $N_0 = 3/4$. $\square$

A Estimates on the Harish-Chandra Function and the Spherical Functions

Throughout this section we will use the notation summarized in (1.11) and the identifications described in the paragraph preceding (3.2). In particular, the Harish-Chandra function $\mathbf{c}(\lambda)$ is defined for $\lambda \in \mathbb{C}$ and the elementary spherical functions $\Phi_\lambda(s)$ are defined for $\lambda \in \mathbb{C}$ and $s \in \mathbb{R}_+$. Also, let $\rho' = |\rho| + 1/10$ be a fixed number slightly greater than $|\rho|$. We will prove the following two propositions:

Proposition A1. *Let* $\mathbf{c}$ *be the Harish-Chandra function on* $\mathbb{X}$*.*

(a) For all $\lambda \in \mathbb{R}$

$$|\mathbf{c}(\lambda)|^{-2} = \mathbf{c}(\lambda)^{-1}\mathbf{c}(-\lambda)^{-1}. \tag{A.1}$$

(b) The function $\lambda \to \lambda^{-1}\mathbf{c}(-\lambda)^{-1}$ *is analytic inside the region* $\Im\lambda \geq 0$ *and*

$$\left|\frac{\partial^\alpha}{\partial\lambda^\alpha}(\lambda^{-1}\mathbf{c}(-\lambda)^{-1})\right| \leq C(1+|\Re\lambda|)^{d-1-\alpha} \tag{A.2}$$

for all integers $\alpha \in [0, N]$ *and for all* λ *with the property* $0 \leq \Im\lambda \leq \rho'$*.*

(c) The function $\lambda \to \lambda\mathbf{c}(\lambda)$ *is analytic in a neighborhood of the real axis and*

$$\left|\frac{\partial^\alpha}{\partial\lambda^\alpha}(\lambda\mathbf{c}(\lambda))\right| \leq C(1+|\Re\lambda|)^{1-d-\alpha} \tag{A.3}$$

for all integers $\alpha \in [0, N]$ *and for all* $\lambda \in \mathbb{R}$*.*

Proposition A2. *(a) If* $\lambda \in \mathbb{R}$ *then*

$$|\Phi_\lambda(s)| \leq Ce^{-|\rho|s}(s+1). \tag{A.4}$$

(b) If $s \leq 1$*,* $\lambda \in \mathbb{R}$ *and* $s|\lambda| \geq 1$ *then* $\Phi_\lambda(s)$ *can be written in the form*

$$\Phi_\lambda(s) = e^{i\lambda s}a_1(\lambda, s) + e^{-i\lambda s}a_1(-\lambda, s) + O(\lambda, s)$$

where the functions $a_1, O : \{(\lambda, s) \in \mathbb{R} \times [0,1] : s|\lambda| \geq 1\} \to \mathbb{C}$ *satisfy*

$$\begin{cases} \left| \frac{\partial^\alpha}{\partial \lambda^\alpha} \frac{\partial^l}{\partial s^l} a_1(\lambda, s) \right| \leq C[s(1+|\lambda|)]^{-d} s^{-l} (1+|\lambda|)^{-\alpha}, \\ |O(\lambda, s)| \leq C[s(1+|\lambda|)]^{-d-N-1}, \end{cases} \tag{A.5}$$

for all integers $\alpha \in [0, N]$, $l \in \{0, 1\}$ *and* s, λ *in the suitable ranges stated above.*

(c) If $s \geq 1/10$ *then* $\Phi_\lambda(s)$ *can be written in the form*

$$\Phi_\lambda(s) = e^{-|\rho|s}(e^{i\lambda s}\mathbf{c}(\lambda)a_2(\lambda, s) + e^{-i\lambda s}\mathbf{c}(-\lambda)a_2(-\lambda, s))$$

where the function a_2 *satisfies the inequalities*

$$\left| \frac{\partial^\alpha}{\partial \lambda^\alpha} \frac{\partial^l}{\partial s^l} a_2(\lambda, s) \right| \leq C[(1 + |\Re\lambda|)]^{-\alpha} \tag{A.6}$$

for all integers $\alpha \in [0, N]$, $l \in \{0, 1\}$ *and for all* $s \geq 1/10$ *and* λ *in the region* $0 \leq \Im\lambda \leq \rho'$. *Also, the function* $\lambda \to a_2(\lambda, s)$ *is analytic inside the region* $\Im\lambda \geq 0$.

As usual, C denotes an absolute constant independent of s and λ. Proposition A1 follows easily from the formula

$$\mathbf{c}(\lambda) = c \frac{\Gamma(i\lambda)\Gamma\left(\frac{1}{2}\left(i\lambda + \frac{m_1}{2}\right)\right)}{\Gamma\left(i\lambda + \frac{m_1}{2}\right)\Gamma\left(\frac{1}{2}(i\lambda + |\rho|)\right)},$$

which can be found in [22, Section 3]. To prove (A.1) one only uses the fact that $\Gamma(\overline{z}) = \overline{\Gamma(z)}$ for all complex numbers z. Also, (A.2) and (A.3) are easy consequences of Stirling's formula ([28, Chapter 4]).

Proof of Proposition A2. The function $\Phi_\lambda(s)$ has the integral formula

$$\Phi_\lambda(s) = c(\cosh s)^{-m_2/2}(\sinh s)^{1-2d} \int_{-s}^{s} e^{i\lambda\mu}(\cosh s - \cosh \mu)^{d-1} F\left(\frac{m_2}{2}, 1 - \frac{m_2}{2}; 2d; z(s, \mu)\right) d\mu$$

([22, Lemma 2.2]) where $z(s,\mu) = (\cosh s - \cosh \mu)/(2\cosh s)$ and F is the hypergeometric function. Part (a) of the proposition follows easily once one notices that the expression involving the hypergeometric function is bounded by an absolute constant.

For part (b), we use Theorem 2.1 in [22]. If $s \le 1$, $\Phi_\lambda(s)$ can be written as

$$\Phi_\lambda(s) = c\left[\frac{s^{n-1}}{D(s)}\right]^{1/2} \sum_{j=0}^{N} s^{2j} a_j(s) \mathfrak{J}_{(n-2)/2+j}(\lambda s) + E(\lambda, s)$$

where $D(s) = (\sinh s)^{m_1}(\sinh 2s)^{m_2}$, $E(\lambda,s) \le C(\lambda s)^{-d-N-1}$, $|a_j(s)| \le C$, $|a_j'(s)| \le C$ and

$$\mathfrak{J}_m(\mu) = \int_{-1}^{1} e^{i\mu r}(1-r^2)^{m-1/2}.$$

The estimate $|a_j'(s)| \le C$ is not stated as part of the theorem but follows easily, at least if $s \le 1$. Also, it is well known that if $|\mu| \ge 1$ and $m > -1/2$ then $\mathfrak{J}_m(\mu)$ can be written as

$$\mathfrak{J}_m(\mu) = e^{i\mu}\psi_m(\mu) + e^{-i\mu}\psi_m(-\mu) + O_m(\mu)$$

where for all integers $\alpha \in [0, N]$ and real numbers μ, $|\mu| \ge 1$

$$\left|\frac{\partial^\alpha}{\partial \mu^\alpha}\psi_m(\mu)\right| \le C_m|\mu|^{-m-1/2-\alpha}. \tag{A.7}$$

Also $|O_m(\mu)| \le C_m|\mu|^{-N-d-1}$. Let therefore

$$\begin{cases} a_1(\lambda, s) = c\left[\frac{s^{n-1}}{D(s)}\right]^{1/2} \sum_{j=0}^{N} s^{2j} a_j(s)\psi_{(n-2)/2+j}(\lambda s), \\ O(\lambda, s) = E(\lambda, s) + c\left[\frac{s^{n-1}}{D(s)}\right]^{1/2} \sum_{j=0}^{N} s^{2j} a_j(s) O_{(n-2)/2+j}(\lambda s), \end{cases}$$

and (A.5) follows from (A.7) and the estimates on the error terms.

To prove part (c), we start from the formula

$$\Phi_\lambda(s) = e^{-|\rho|t}(\mathbf{c}(\lambda)e^{i\lambda t}a_2(\lambda, t) + \mathbf{c}(-\lambda)e^{-i\lambda t}a_2(-\lambda, t))$$

where

$$a_2(\lambda, t) = \sum_{k=0}^{\infty} \Gamma_k(\lambda) e^{-2kt}$$

and the functions Γ_k satisfy the recursion

$$\Gamma_k(\lambda) = \sum_{j=0}^{k-1} \alpha_j^k(\lambda)\Gamma_j(\lambda) \text{ for } k \geq 1 \text{ and } \Gamma_0(\lambda) = 1. \tag{A.8}$$

This is shown in [22, Theorem 3.1]. The coefficients $\alpha_j^k(\lambda)$ have the formula

$$\alpha_j^k(\lambda) = \frac{(m_1/2 + \delta_j^k m_2)}{k}\left(1 + \frac{2j + |\rho| - k}{k - i\lambda}\right) \tag{A.9}$$

where $\delta_j^k = 1$ if $j \equiv k(\text{mod } 2)$ and $\delta_j^k = 0$ otherwise. We will prove that for all integers $\alpha \in [0, N]$ there exist constants A and b_α such that

$$\left|\frac{\partial^\alpha}{\partial\lambda^\alpha}\Gamma_k(\lambda)\right| \leq Ak^{b_\alpha}(1 + |\Re\lambda|)^{-\alpha} \tag{A.10}$$

for all integers $k \geq 1$ and all complex numbers λ, $0 \leq \Im\lambda \leq \rho'$. This would clearly suffice to prove the estimates (A.6). In [22, Theorem 3.2], the authors prove weaker estimates on the functions Γ_k (involving an exponential increase in k); their estimates would only suffice to prove (A.6) for $s \geq R_0 > 1$. Notice that for all integers $k \geq 2$ and real numbers $b \geq 4$

$$1 + \sum_{j=1}^{k-1} j^b \leq \frac{k^{b+1}}{b}. \tag{A.11}$$

Also, the formula (A.9) shows that there exists an absolute constant $A \geq 4$ such that

$$\left|\frac{\partial^\alpha}{\partial\lambda^\alpha}\alpha_j^k(\lambda)\right| \leq \frac{A}{k(1 + |\Re\lambda|)^\alpha} \tag{A.12}$$

for all integers $k \geq 1$, $j \leq k - 1$, $\alpha \in [0, N]$ and all complex numbers λ, $0 \leq \Im\lambda \leq \rho'$ (this is a simple consequence of the fact that $|k + 1 - i\lambda| \geq \max(k, |\Re\lambda|)$). We now prove (A.10) for $\alpha = 0$ by induction over $k \geq 1$. Clearly $|\Gamma_1(\lambda)| \leq A$ by (A.12). Assume that (A.10) holds for all $1 \leq j \leq k-1$ (a suitable power b_0 will be fixed momentarily). Then, by (A.8), (A.11) and (A.12) with $\alpha = 0$

$$|\Gamma_k(\lambda)| \leq \sum_{j=0}^{k-1} \frac{A}{k}|\Gamma_j(\lambda)| \leq \frac{A}{k}\frac{Ak^{b_0+1}}{b_0} = Ak^{b_0}\frac{A}{b_0}$$

The induction works if we set $b_0 = A$. To prove (A.10) for an arbitrary integer $\alpha \leq N$, assume, by induction, that we found suitable powers b_β, such that (A.10) holds for all $\beta \in \{0, 1 \ldots \alpha - 1\}$ and for all k. We can also assume that $b_0 \leq b_1 \leq \ldots \leq b_{\alpha-1}$. Clearly $\left|\frac{\partial^\alpha}{\partial\lambda^\alpha}\Gamma_1(\lambda)\right| \leq A(1 + |\Re\lambda|)^{-\alpha}$ by (A.12) and we only need to find a suitable number $b_\alpha \geq b_{\alpha-1}$ that would allow us to prove (A.10) by induction over k. Assume that (A.10) holds for α and for all $j \in \{1, 2 \ldots k-1\}$. Then, by (A.8), (A.11), (A.12) and the induction hypothesis

$$\begin{aligned}
\left|\frac{\partial^\alpha}{\partial\lambda^\alpha}\Gamma_k(\lambda)\right| &\leq 2^\alpha \sum_{\beta=0}^{\alpha}\sum_{j=0}^{k-1}\left|\frac{\partial^{(\alpha-\beta)}}{\partial\lambda^{(\alpha-\beta)}}\alpha_j^k(\lambda)\right|\left|\frac{\partial^\beta}{\partial\lambda^\beta}\Gamma_j(\lambda)\right| \\
&\leq 2^\alpha \sum_{\beta=0}^{\alpha}\sum_{j=0}^{k-1}\frac{A}{k(1+|\Re\lambda|)^{\alpha-\beta}}\frac{A\max(j,1)^{b_\beta}}{(1+|\Re\lambda|)^\beta} \\
&\leq 2^\alpha(1+|\Re\lambda|)^{-\alpha}\sum_{\beta=0}^{\alpha}\frac{A^2k^{b_\beta}}{b_\beta} \leq Ak^{b_\alpha}(1+|\Re\lambda|)^{-\alpha}\frac{A2^\alpha(\alpha+1)}{b_\alpha}.
\end{aligned}$$

Clearly, the induction works as long as $b_\alpha \geq \max(b_{\alpha-1}, A2^\alpha(\alpha+1))$. Notice that in fact one can set $b_\alpha = A2^\alpha(\alpha+1)$ for all integers $\alpha \in [0, N]$. □

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