Self-similar blowup for the Nonlinear Schrödinger Equation and the Complex Ginzburg-Landau Equation

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Last time - Complex Ginzburg-Landau Equation

$$i\frac{\partial u}{\partial t} + (1 - i\epsilon)\Delta u + (1 + i\delta)|u|^{2\sigma} u = 0$$

$$u : \mathbb{R}^d \times (0, T) \to \mathbb{C}$$

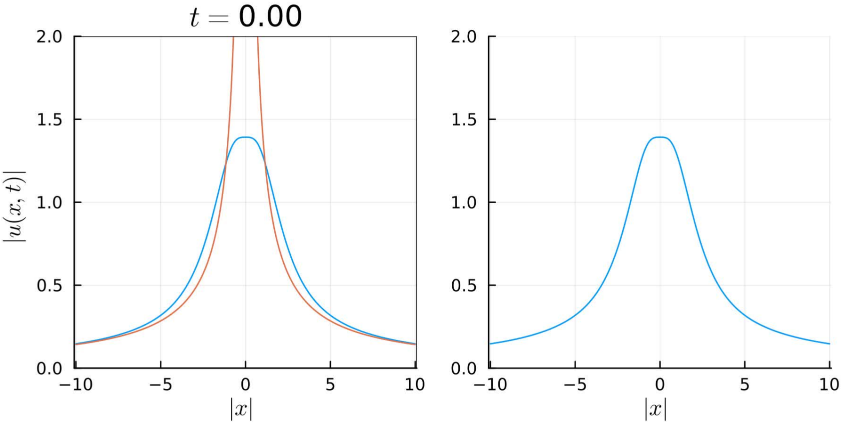
$$u(x, 0) = u_0(x)$$

Parameters $d \in \mathbb{Z}_{\geq 1},\ \sigma, \epsilon, \delta \geq 0$

Nonlinear Schrödinger Equation $\epsilon = \delta = 0$

Last time - Self-similar solutions

$$u(x,t) = \frac{1}{(2\kappa(T-t))^{\frac{1}{2}\left(\frac{1}{\sigma}+i\frac{\omega}{\kappa}\right)}} Q\left(\frac{|x|}{(2\kappa(T-t))^{\frac{1}{2}}}\right)$$



Last time - Self-similar solutions

$$u(x,t) = \frac{1}{(2\kappa(T-t))^{\frac{1}{2}\left(\frac{1}{\sigma}+i\frac{\omega}{\kappa}\right)}} Q\left(\frac{|x|}{(2\kappa(T-t))^{\frac{1}{2}}}\right)$$

$$\begin{cases} (1-i\epsilon)\left(Q'' + \frac{d-1}{\xi}Q'\right) + i\kappa\xi Q' + i\frac{\kappa}{\sigma}Q - \omega Q + (1+i\delta)|Q|^{2\sigma}Q = 0 \\ Q'(0) = 0 \\ Q(\xi) \sim \xi^{-\frac{1}{\sigma}-i\frac{\omega}{\kappa}} \text{ as } \xi \to \infty \end{cases}.$$

Parameters $\omega, \kappa > 0$

Last time - Results for NLS

Theorem (D, Figueras, 2024)

Consider the equation

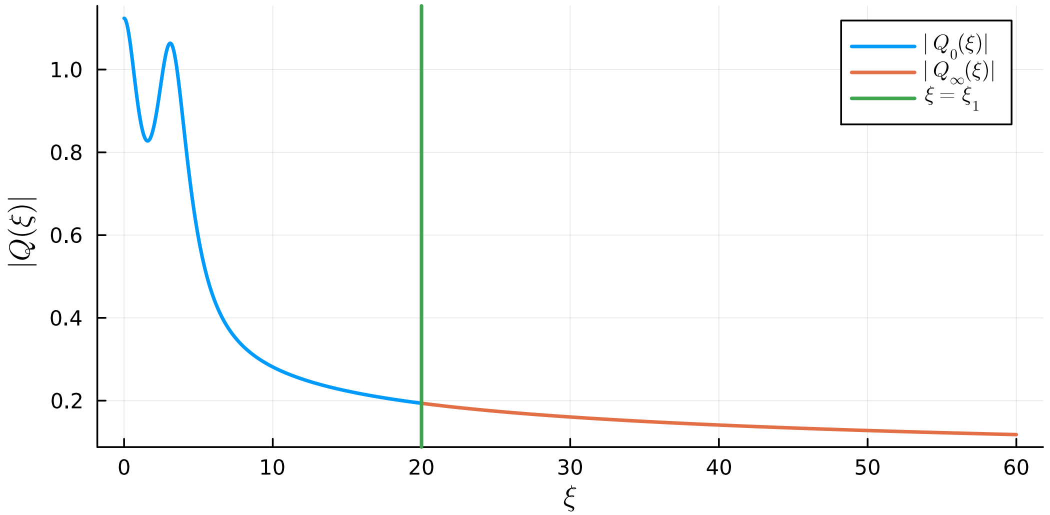
$$\begin{cases} Q'' + \frac{d-1}{\xi}Q' + i\kappa\xi Q' + i\frac{\kappa}{\sigma}Q - \omega Q + |Q|^{2\sigma}Q = 0 \\ Q'(0) = 0, \, Q(\xi) \sim \xi^{-\frac{1}{\sigma} - i\frac{\omega}{\kappa}} \end{cases}.$$

Case I *There exist solutions for at least 8 values of κ.*

Case II *There exist solutions for at least 2 values of κ.*

The number of intervals of monotonicity is given by the index j of the solution.

Last time - Proof idea



Last time - Proving matching condition

$$G(\mu, \gamma, \kappa) = (Q_0(\mu, \kappa; \xi_1) - Q_\infty(\gamma, \kappa; \xi_1), Q_0'(\mu, \kappa; \xi_1) - Q_\infty'(\gamma, \kappa; \xi_1))$$

$$G(\mu, \operatorname{Re}\gamma, \operatorname{Im}\gamma, \kappa) : \mathbb{R}^4 \to \mathbb{R}^4$$

1. Find a numerical approximation
2. Rigorously verify approximation using the interval Newton method
    - Requires computing rigorous interval enclosures for
        - $Q_0(\mu, \kappa; \xi_1), Q_0'(\mu, \kappa; \xi_1), Q_\infty(\gamma, \kappa; \xi_1), Q_\infty'(\gamma, \kappa; \xi_1)$
        - Derivatives w.r.t. μ, γ, κ

j	μ_j	γ_j	κ_j	ξ_1
1	1.88565^{73}_{67}	$1.71360^{13}_{05} - 1.49179^{42}_{35}i$	0.91735^{63}_{59}	60
2	0.8399^{62}_{57}	$13.852^{78}_{46} + 6.034^{59}_{44}i$	0.3212^{41}_{39}	140

A simplified example

$$Q'' + \frac{1}{\xi} Q' - Q + Q^3 = 0$$

$$G(\mu, \gamma) = (Q_0(\mu; \xi_1) - Q_\infty(\gamma; \xi_1), Q_0'(\mu; \xi_1) - Q_\infty'(\gamma; \xi_1))$$

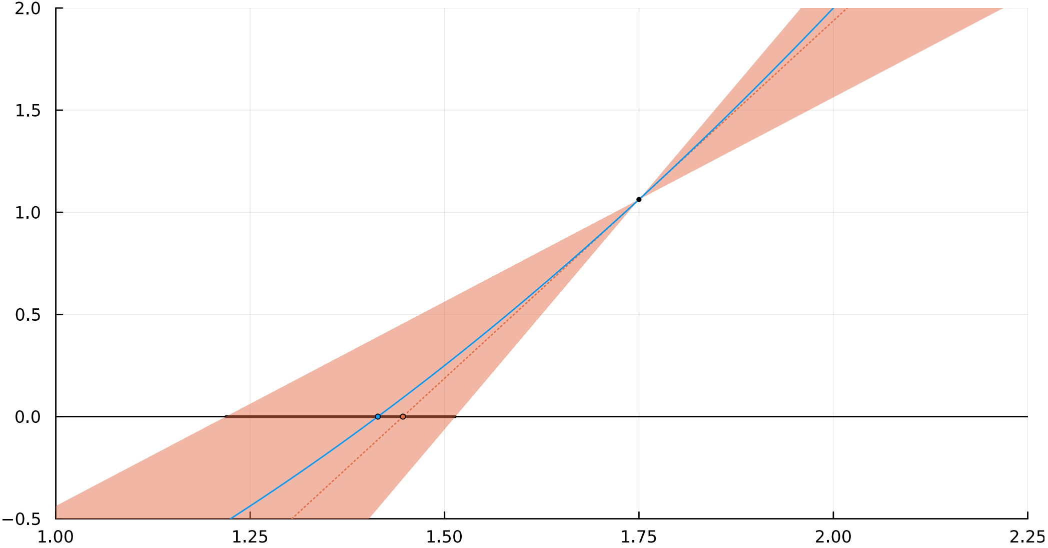
Last time

- How to compute Q_0 approximately (numerical ODE solver)
- How to compute Q_∞ approximately ($\approx \gamma K_0(\xi)$)
- How to find approximate zero of $G(\mu, \gamma)$

This time

- How to compute Q_0 rigorously
- How to compute Q_∞ rigorously
- How to rigorously verify zero of $G(\mu, \gamma)$

Interval Newton method - Idea



Interval Newton method - One variable

Classical Newton

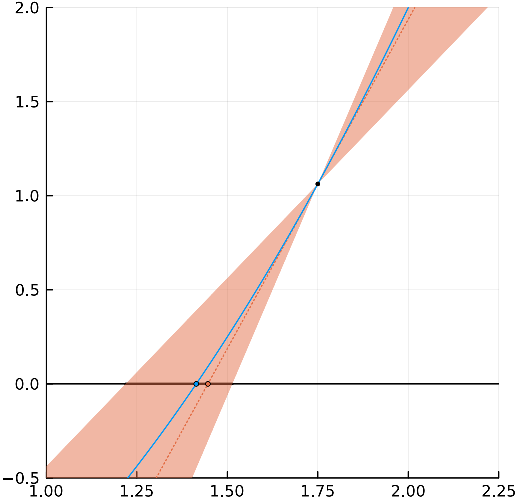
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Interval Newton

$$\boldsymbol{x_0} = [\underline{x}_0, \overline{x}_0]$$

$$\boldsymbol{x_1} = x_0 - \frac{f(x_0)}{f'(\boldsymbol{x_0})} = \left\{ x_0 - \frac{f(x_0)}{f'(x)} : x \in \boldsymbol{x_0} \right\}$$

Success if $\boldsymbol{x_1} \subsetneq \boldsymbol{x_0}$



Interval Newton method - Multiple variables

$$f : \mathbb{R}^n \to \mathbb{R}^n, \;\; J_f : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$$

$$x_0 = (x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(n)}) \in \mathbb{R}^n$$

$$\boldsymbol{x_0} = [\underline{x_0}\overline{x_0}] = [\underline{x}_0^{(1)}\overline{x}_0^{(1)}] \times [\underline{x}_0^{(2)}\overline{x}_0^{(2)}] \times \cdots \times [\underline{x}_0^{(n)}\overline{x}_0^{(n)}] \subset \mathbb{R}^n$$

$$\boldsymbol{x_1} = x_0 - J_f(\boldsymbol{x_0})^{-1} f(x_0)$$

Success if $\boldsymbol{x_1} \subsetneq \boldsymbol{x_0}$

Interval Newton method - Applying it

$$\xi_1 = 15, \quad x_0 = (\mu_0, \gamma_0) = (4.150094036246, 1540.55)$$

$$G(x_0; \xi_1) \approx (2.5204639319664074 \cdot 10^{-10}, -2.942687336480402 \cdot 10^{-10})$$

$$\boldsymbol{x_0} = \boldsymbol{\mu_0} \times \boldsymbol{\gamma_0} = [\mu_0 \pm 2 \cdot 10^{-11}] \times [\gamma_0 \pm 2 \cdot 10^{-2}]$$

Need to verify:

$$\boldsymbol{x_1} = x_0 - J_G^{-1}(\boldsymbol{x_0}; \xi_1) G(x_0; \xi_1) \subsetneq \boldsymbol{x_0}$$

Need to compute:

$$J_G(\boldsymbol{x_0}; \xi_1) \text{ and } G(x_0; \xi_1)$$

Interval Newton method - Applying it

$$G(\mu, \gamma) = (Q_0(\mu; \xi_1) - Q_\infty(\gamma; \xi_1), Q_0'(\mu; \xi_1) - Q_\infty'(\gamma; \xi_1))$$

For $G(x_0; \xi_1)$:

- $Q_0(\mu_0; \xi_1)$, $Q_0'(\mu_0; \xi_1)$
- $Q_\infty(\gamma_0; \xi_1)$, $\mathrm{Q}_\infty{}'(\gamma_0;\ \xi_1)$

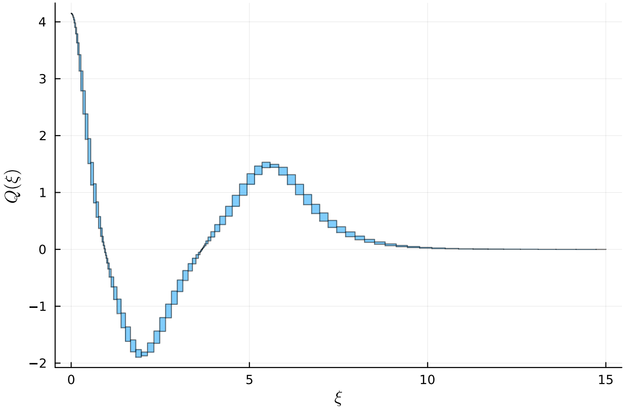
For $J_G(\boldsymbol{x_0}; \xi_1)$:

- $Q_{0,\mu}(\boldsymbol{\mu_0}; \xi_1)$, $Q_{0,\mu}'(\boldsymbol{\mu_0}; \xi_1)$
- $Q_{\infty,\gamma}(\boldsymbol{\gamma_0}; \xi_1)$, $Q_{\infty,\gamma}'(\boldsymbol{\gamma_0}; \xi_1)$

Computing Q_0 - Rigorously

$$\begin{cases} Q_0'' + \frac{1}{\xi} Q_0' - Q_0 + Q_0^3 = 0, \\ Q_0(0) = \mu, \;\; Q_0'(0) = 0. \end{cases}$$

1. Taylor expand at $\xi = 0$
2. Use the rigorous numerical ODE solver from the `CAPD` library.



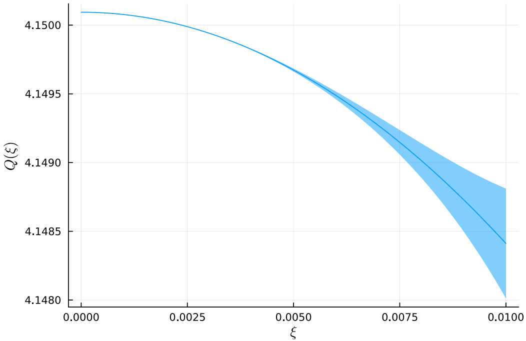
Computing Q_0 - Taylor expansion

$$Q(\xi) = \sum_{n=0}^{\infty} a_n \xi^n \quad \text{with} \quad a_0 = \mu,\ a_1 = 0,\ a_{n+2} = \frac{a_n - u_n}{(n+2)^2},\ u_n = (Q^3)_n$$

$$Q(\xi) = \sum_{n=0}^{N} a_n \xi^n + \sum_{n=N+1}^{\infty} a_n \xi^n$$

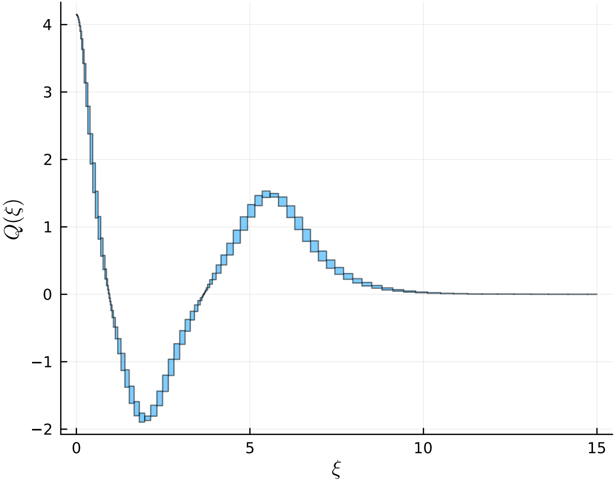
$$|a_n| < r^n \text{ for } n > N$$

$$\left| \sum_{n=N+1}^{\infty} a_n \xi^n \right| \leq \frac{(r\xi)^{N+1}}{1 - r\xi}$$



Computing Q_0 - CAPD

- C++ library for rigorous ODE solvers
- Developed by group in Krakow
  - Kapela, Mrozek, Wilczak, Zgliczyński (2021)
- High order Taylor methods
  - Automatic differentiation
- Clever representations of enclosures



Last time - Computing Q_∞ - Approximately

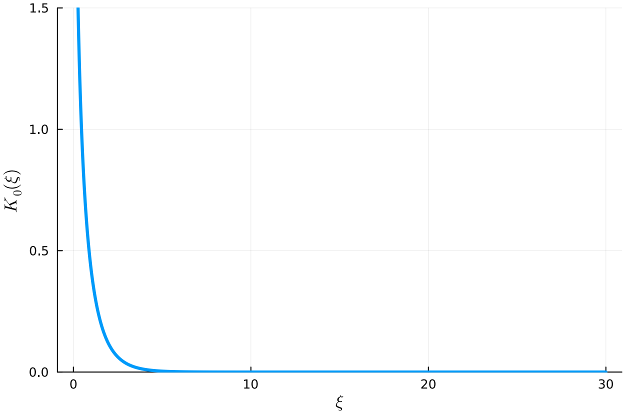
$$\begin{cases} Q''_\infty + \frac{1}{\xi} Q'_\infty - Q_\infty + Q^3_\infty = 0, \\ \lim_{\xi \to \infty} Q_\infty(\xi) = 0. \end{cases}$$

Q_∞ small $\to$ ignore Q^3_∞

$$\begin{cases} Q''_\infty + \frac{1}{\xi} Q'_\infty - Q_\infty = 0, \\ \lim_{\xi \to \infty} Q_\infty(\xi) = 0. \end{cases}$$

- $K_0(\xi) \sim \xi^{-\frac{1}{2}} e^{-\xi}$
- $I_0(\xi) \sim \xi^{-\frac{1}{2}} e^{\xi}$

$$Q_\infty(\xi) \approx \gamma K_0(\xi)$$



Computing Q_∞ - Rigorously

$$Q_\infty'' + \frac{1}{\xi} Q_\infty' - Q_\infty = -Q_\infty^3$$

- Method of variation of parameters

$$Q_\infty(\xi) = c_1(\xi) K_0(\xi) + c_2(\xi) I_0(\xi)$$

$$\begin{aligned} Q_\infty(\xi) = \gamma K_0(\xi) + K_0(\xi) \int_{\xi_1}^{\xi} I_0(\eta) W(\eta)^{-1} Q_\infty(\eta)^3 \, d\eta \\ + I_0(\xi) \int_{\xi}^{\infty} K_0(\eta) W(\eta)^{-1} Q_\infty(\eta)^3 \, d\eta \end{aligned}$$

Computing Q_∞ - Rigorously

$$T[Q](\xi) = \gamma K_0(\xi) + K_0(\xi)\int_{\xi_1}^{\xi} I_0(\eta)W(\eta)^{-1}Q(\eta)^3\ d\eta$$
$$+ I_0(\xi)\int_{\xi}^{\infty} K_0(\eta)W(\eta)^{-1}Q(\eta)^3\ d\eta$$

1. Look for fixed point of T: $Q_\infty = T[Q_\infty](\xi)$
   - $\|Q\|_v = \sup_{\xi\geq\xi_1} \xi^{\frac{1}{2}-v}e^{\xi}|Q(\xi)|$
   - $\|T[Q]\|_v \leq C_{K_0}|\gamma|\xi_1^{-v} + C_T\xi_1^{2v-1}e^{-2\xi_1}\|Q\|_v^3$
   - $\|T[Q_1] - T[Q_2]\|_v \leq$ $C_T\xi_1^{2v-1}e^{-2\xi_1}\|Q_1 - Q_2\|_v(\|Q_1\|_v^2 + \|Q_1\|_v\|Q_2\|_v + \|Q_2\|_v^2)$
2. Improve bounds using bootstrapping

Rigorously verifying zero

1. Enclose $G(x_0;\xi_1)$ and $J_G(\boldsymbol{x_0};\xi_1)$

$$G(x_0;\xi_1) \subseteq ([2.0531\cdot 10^{-10}, 2.0532\cdot 10^{-10}], [-3.3942\cdot 10^{-10}, -3.3941\cdot 10^{-10}])$$

$$J_G(\boldsymbol{x_0};\xi_1) \subseteq \begin{pmatrix} [-400.69 \pm 3.62 \cdot 10^{-3}] & [9.819536 \cdot 10^{-8} \pm 5.85 \cdot 10^{-15}] \\ [-387.09 \pm 1.82 \cdot 10^{-3}] & [-1.0141729 \cdot 10^{-7} \pm 4.69 \cdot 10^{-15}] \end{pmatrix}$$

2. Compute interval Newton iteration

$$\boldsymbol{x_1} = x_0 - J_G(\boldsymbol{x_0})^{-1} G(x_0) \subseteq ([4.150094036246 \pm 8.19\cdot 10^{-13}], [1540.55 \pm 4.04\cdot 10^{-3}])$$

3. Verify $\boldsymbol{x_1} \subsetneq \boldsymbol{x_0}$

$$\begin{aligned} &([4.150094036246 \pm 8.19 \cdot 10^{-13}], [1540.55 \pm 4.04 \cdot 10^{-3}]) \\ &\qquad \subsetneq ([4.150094036246 \pm 2 \cdot 10^{-11}], [1540.55 \pm 2 \cdot 10^{-2}]) \end{aligned}$$

Success!

Back to CGL

- Simplified example

$$Q'' + \frac{1}{\xi}Q' - Q + Q^3 = 0$$

  - $G(\mu, \gamma) : \mathbb{R}^2 \to \mathbb{R}^2$

- Full CGL

$$(1 - i\epsilon)\left(Q'' + \frac{d-1}{\xi}Q'\right) + i\kappa\xi Q' + i\frac{\kappa}{\sigma}Q - \omega Q + (1 + i\delta)|Q|^{2\sigma}Q = 0$$

  - $G(\mu, \gamma, \kappa) : \mathbb{R}^4 \to \mathbb{R}^4$

Results - NLS Case I

$$d = 1,\ \delta = 0,\ \sigma = 2.3,\ \omega = 1$$

j	μ_j	γ_j	κ_j	ξ_1
1	1.2320375^{49}_{02}	$0.758150^{86}_{46} - 0.43437^{14}_{09} i$	0.853108^{97}_{70}	10
2	0.78307^{77}_{65}	$1.7916^{57}_{27} - 1.1049^{210}_{05} i$	0.49322^{33}_{24}	15
3	1.12384^{44}_{11}	$4.331^{34}_{23} - 1.5341^{83}_{05} i$	0.34675^{44}_{29}	20
4	0.8838^{82}_{73}	$9.73^{82}_{77} + 0.046^{66}_{26} i$	0.2667^{61}_{58}	25
5	1.07955^{96}_{88}	$18.156^{91}_{79} + 9.263^{23}_{15} i$	0.216402^{210}_{13}	40
6	0.92718^{58}_{29}	$21.94^{15}_{06} + 36.28^{31}_{20} i$	0.18183^{82}_{76}	45
7	1.054410^{86}_{05}	$-8.221^{77}_{04} + 87.50^{52}_{42} i$	0.15667^{81}_{78}	60
8	0.95114^{63}_{55}	$-130.05^{62}_{38} + 126.99^{28}_{18} i$	0.137564^{44}_{36}	75

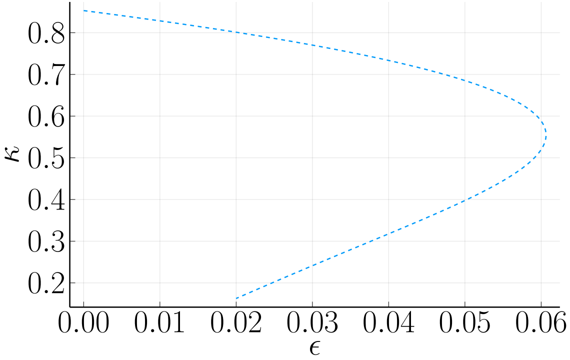
Results - NLS Case II

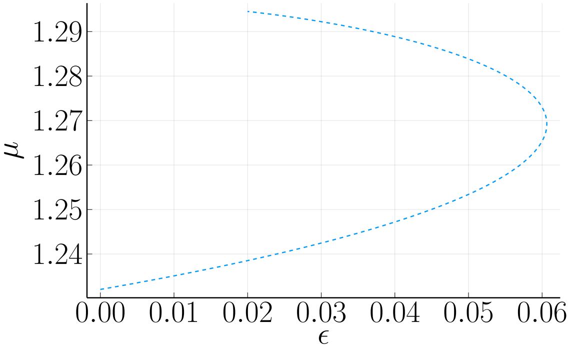
$$d = 3,\ \delta = 0,\ \sigma = 1,\ \omega = 1$$

j	μ_j	γ_j	κ_j	ξ_1
1	1.88565_{67}^{73}	$1.71360_{05}^{13} - 1.49179_{35}^{42} i$	0.91735_{59}^{63}	60
2	0.8399_{57}^{62}	$13.852_{46}^{78} + 6.034_{44}^{59} i$	0.3212_{39}^{41}	140

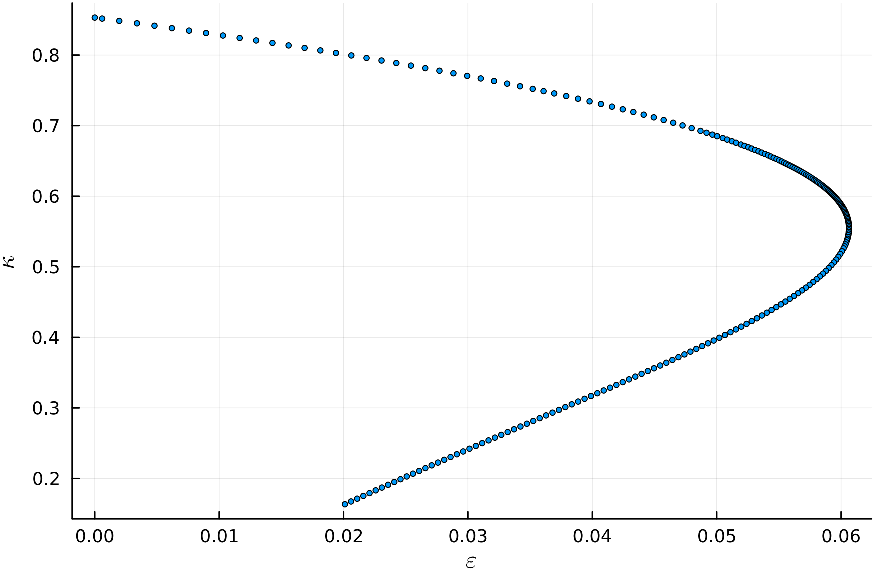
Branches

$$G(\mu(\epsilon), \gamma(\epsilon), \kappa(\epsilon)) = 0$$

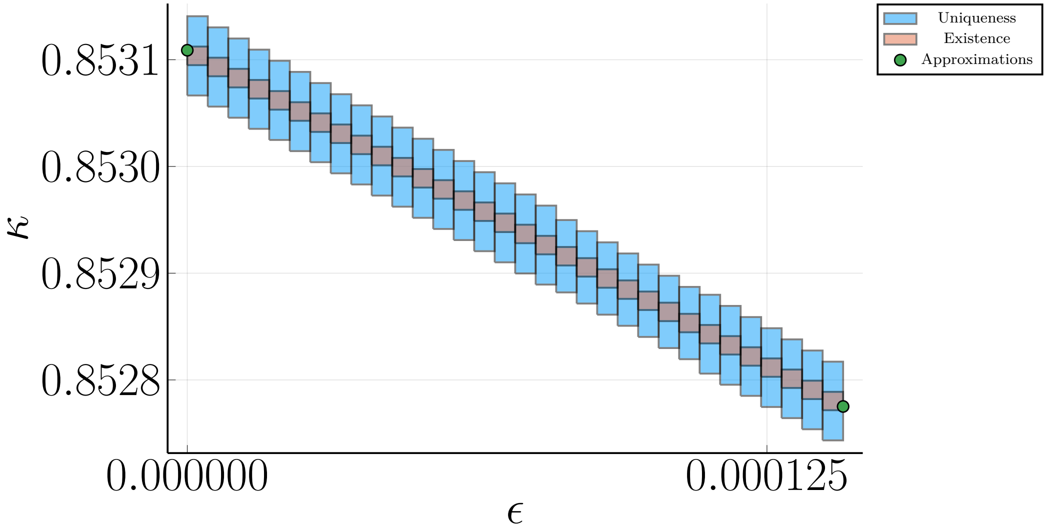




Branches - Pointwise

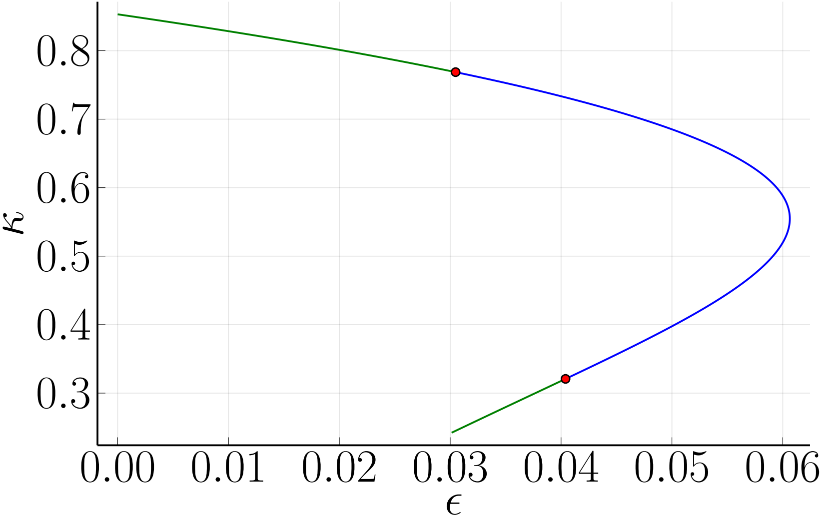


Branches - Enclosure



Branches - Handling turning

$G(\mu(\epsilon), \gamma(\epsilon), \kappa(\epsilon)) = 0$ $G(\mu(\kappa), \gamma(\kappa), \epsilon(\kappa)) = 0$



Results - CGL

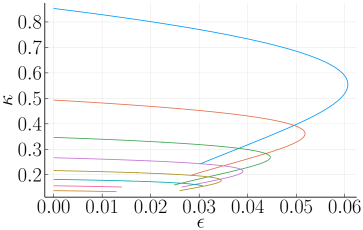
Theorem (D, Figueras, 2024)

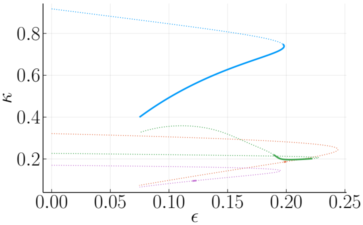
Consider the equation

$$\begin{cases} (1 - i\epsilon)\left(Q'' + \frac{d-1}{\xi}Q'\right) + i\kappa\xi Q' + i\frac{\kappa}{\sigma}Q - \omega Q + (1 + i\delta)|Q|^{2\sigma}Q = 0 \\ Q'(0) = 0,\, Q(\xi) \sim \xi^{-\frac{1}{\sigma} - i\frac{\omega}{\kappa}} \end{cases}.$$

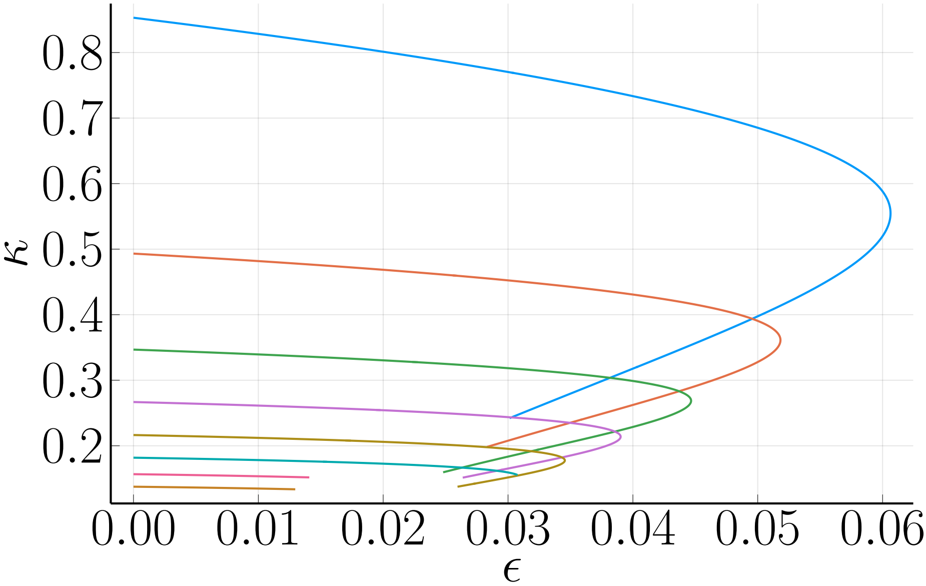
Case I *There exists at least 8 branches of solutions. The number of intervals of monotonicity is constant along these branches.*

Case II *There exists branches of solutions.*





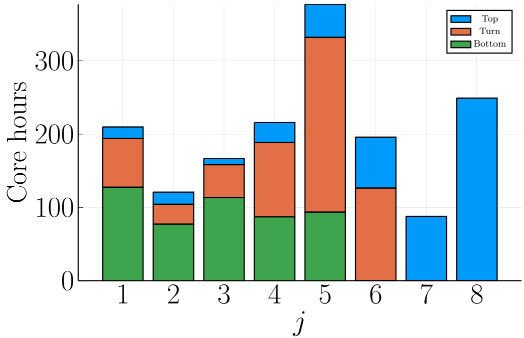
Branches Case I



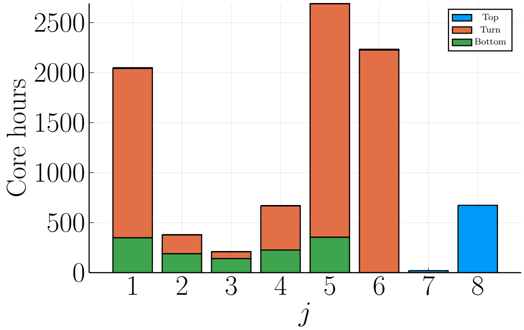
Branches Case I - Runtime

Computations done on Dardel - NAISS computer cluser at KTH

- 256 logical cores per node (AMD EPYC Zen2)

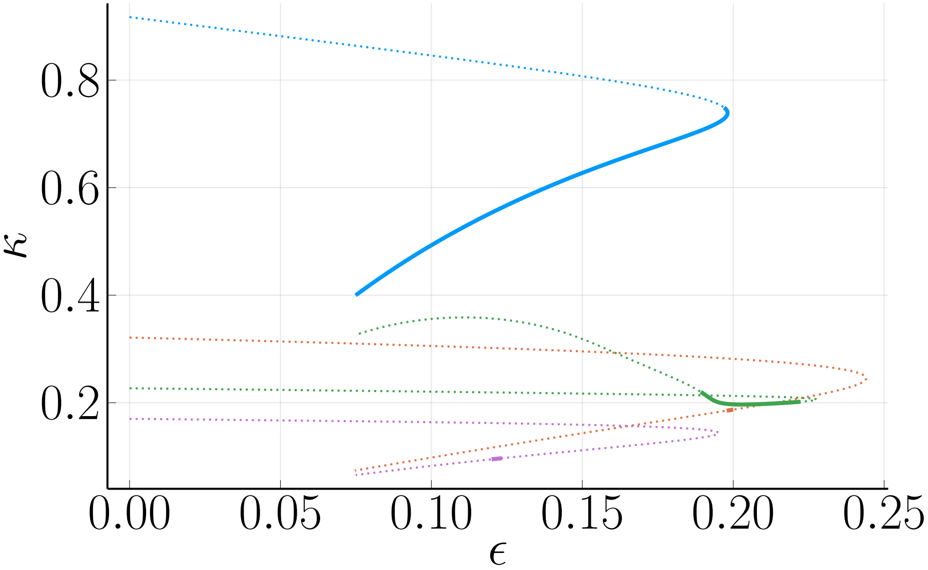


Proving existence



Counting critical points

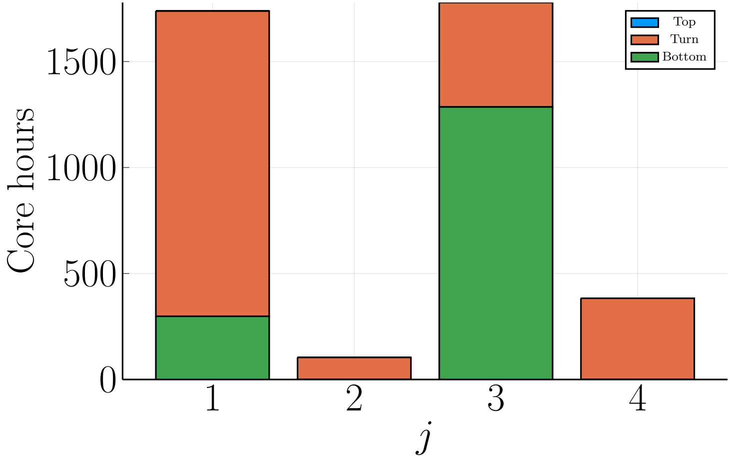
Branches Case II



Branches Case II - Runtime

Computations done on Dardel - NAISS computer cluser at KTH

- 256 logical cores per node (AMD EPYC Zen2)

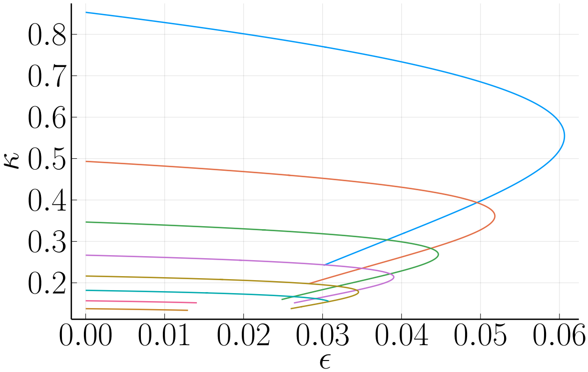


Code

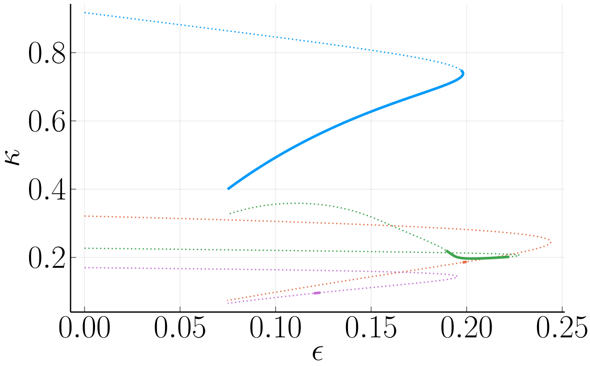
Code available at github.com/Joel-Dahne/CGL.jl

	Simplified example	CGL
Q_0	$\approx$ 800 loc	$\approx$ 2,000 loc
Q_∞	$\approx$ 500 loc	$\approx$ 3,300 loc
Branches	-	$\approx$ 3,300 loc
Total	-	$\approx$ 12,000 loc

Thank you!



κ
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.00
0.01
0.02
0.03
0.04
0.05
0.06
ϵ



κ
0.8
0.6
0.4
0.2
0.00
0.05
0.10
0.15
0.20
0.25
ϵ