Self-similar blowup for the Nonlinear Schrödinger Equation and the Complex Ginzburg-Landau Equation

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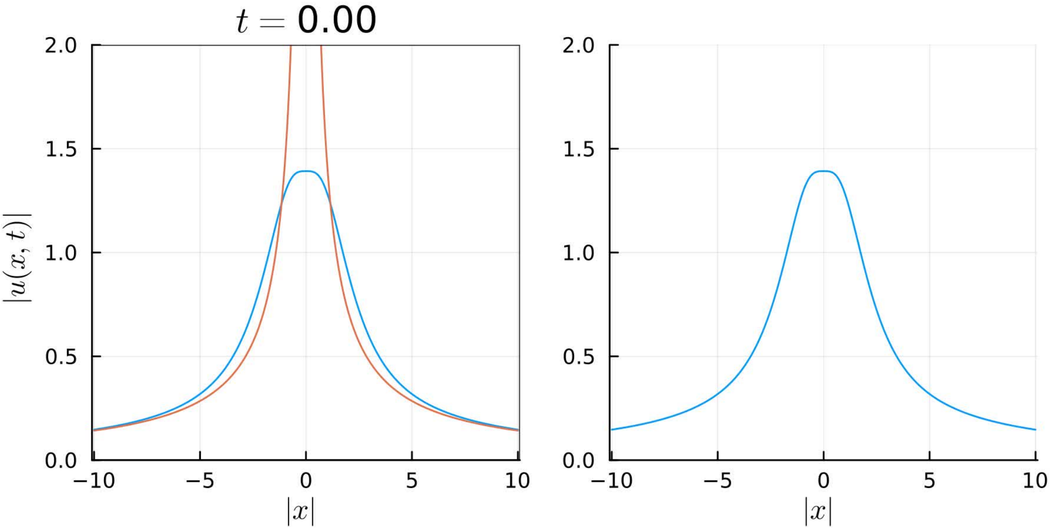
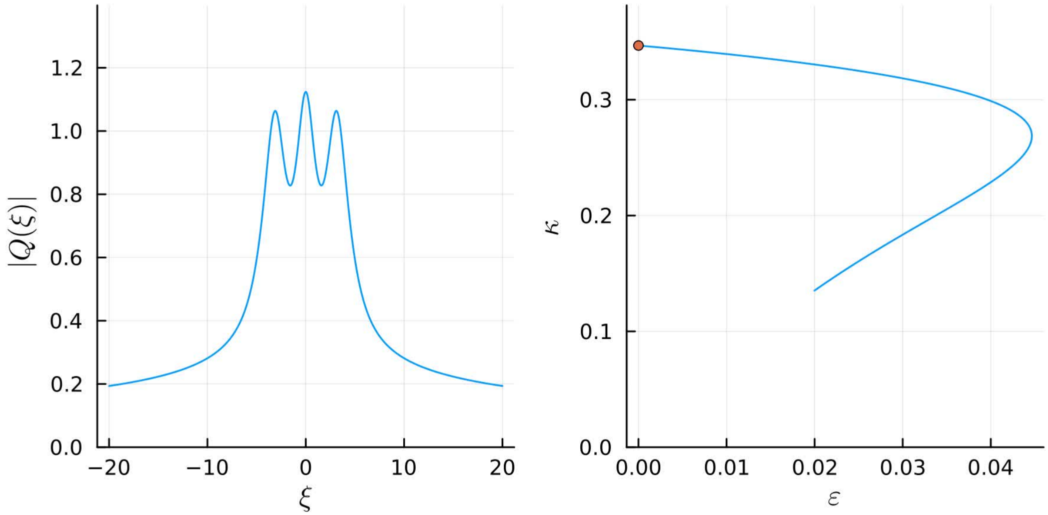
Whether three-dimensional incompressible flows develop singularities in finite time and whether (weak) solutions of Navier-Stokes equations are unique, are two of the most important problems in mathematical fluid dynamics. Any progress towards resolving these problems would have significant implications for the entire field.

Self-similar blowup



Branches with self-similar blowup



The Nonlinear Schrödinger Equation (NLS)

$$i\frac{\partial u}{\partial t} + \Delta u + |u|^{2\sigma} u = 0$$

$$u(x,t) : \mathbb{R}^d \times (0, T) \to \mathbb{C}$$

$$u(x,0) = u_0(x)$$

Parameters $d \in \mathbb{Z}_{\geq 1}$, $\sigma \geq 0$

The Complex Ginzburg-Landau Equation (CGL)

$$i\frac{\partial u}{\partial t} + (1 - i\epsilon)\Delta u + (1 + i\delta)|u|^{2\sigma} u = 0$$

$$u : \mathbb{R}^d \times (0, T) \to \mathbb{C}$$

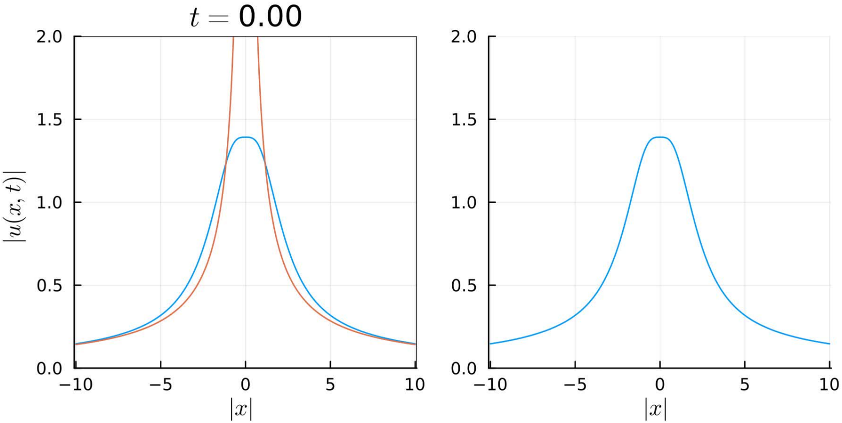
$$u(x, 0) = u_0(x)$$

Parameters $d \in \mathbb{Z}_{\geq 1},\ \sigma, \epsilon, \delta \geq 0$

Nonlinear Schrödinger Equation $\epsilon = \delta = 0$

Self-similar solutions

$$u(x,t) = \frac{1}{(2\kappa(T-t))^{\frac{1}{2}\left(\frac{1}{\sigma}+i\frac{\omega}{\kappa}\right)}} Q\left(\frac{|x|}{(2\kappa(T-t))^{\frac{1}{2}}}\right)$$



Self-similar solutions

$$u(x,t) = \frac{1}{(2\kappa(T-t))^{\frac{1}{2}\left(\frac{1}{\sigma}+i\frac{\omega}{\kappa}\right)}} Q\left(\frac{|x|}{(2\kappa(T-t))^{\frac{1}{2}}}\right)$$

$$\begin{cases} (1-i\epsilon)\left(Q'' + \frac{d-1}{\xi}Q'\right) + i\kappa\xi Q' + i\frac{\kappa}{\sigma}Q - \omega Q + (1+i\delta)|Q|^{2\sigma}Q = 0 \\ Q'(0) = 0 \\ Q(\xi) \sim \xi^{-\frac{1}{\sigma}-i\frac{\omega}{\kappa}} \text{ as } \xi \to \infty \end{cases} .$$

Parameters $\omega, \kappa > 0$

Previous work - NLS

Book Sulem, Sulem (1999)

$$s_c = \frac{d}{2} - \frac{1}{\sigma}$$

$s_c < 0$ Global existence

$s_c = 0$
- Merle, Raphaël (CMP (2004), Inventiones (2004), Annals (2005), JAMS (2006))

$0 < s_c \ll 1$
- Merle, Raphaël, Szeftel (2010)
- Bahri, Martel, Raphaël (2021)

$s_c = \frac{1}{2}$
- Donninger, Schörkhuber (2024) (3D cubic NLS)

Sulem, Sulem (1999) - Chapter 7: Supercritical collapse

> *We are interested in complex solutions Q of (7.1.2) with a monotonically decreasing amplitude $|Q|$ and zero Hamiltonian, which provide the limiting profiles of singular solutions of the NLS equation. We call such solutions "admissible solutions".*

Conjecture (based on numerics by Budd, Chen, Russel (1999))

The supercritical NLS equation has a countable number of nontrivial radial self-similar singular solutions. In certain regimes, these solutions are characterized by the number, j, of monotone intervals of the profile $|Q|$. Except for $j = 1$, these solutions are all unstable.

Previous work - NLS

Theorem (Donninger, Schörkhuber (2024))

There exist a nontrivial, radial function $Q \in L^4(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3)$ *and a* $\kappa > 0$ *such that*

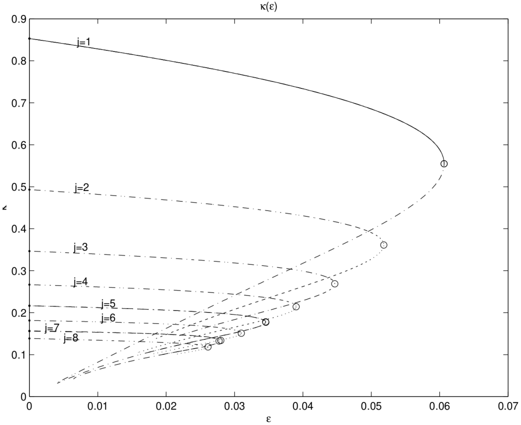
$$u(x,t) = \frac{1}{(2\kappa(T-t))^{\frac{1}{2}(1+\frac{i}{\kappa})}} Q\left(\frac{x}{(2\kappa(T-t))^{\frac{1}{2}}}\right)$$

is a self-similar singular solution to the 3D cubic NLS equation

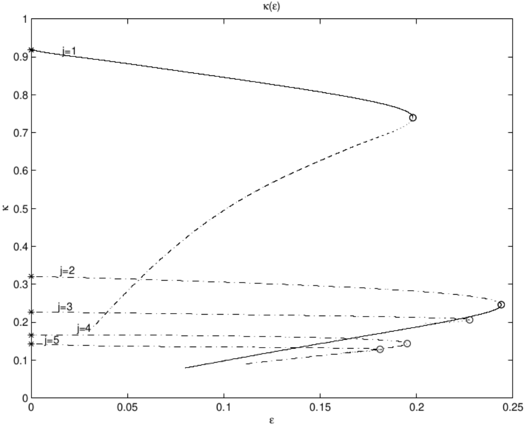
$$i\frac{\partial u}{\partial t} + \Delta u + |u|^2 u = 0.$$

Branches of self-similar singular solutions

Case I: $d = 1$, $\sigma = 2.3$, $\delta = 0$ and $\omega = 1$



Case II: $d = 3$, $\sigma = 1$, $\delta = 0$ and $\omega = 1$



Plecháč and Šverák (2001)

Equation to solve

$$\begin{cases} (1-i\epsilon)\left(Q'' + \frac{d-1}{\xi}Q'\right) + i\kappa\xi Q' + i\frac{\kappa}{\sigma}Q - \omega Q + (1+i\delta)|Q|^{2\sigma}Q = 0 \\ Q'(0) = 0 \\ Q(\xi) \sim \xi^{-\frac{1}{\sigma} - i\frac{\omega}{\kappa}} \text{ as } \xi \to \infty \end{cases}.$$

Fix d, σ, δ, ω

Case I $d = 1$, $\delta = 0$, $\sigma = 2.3$, $\omega = 1$

Case II $d = 3$, $\delta = 0$, $\sigma = 1$, $\omega = 1$

Vary κ, ϵ

Results - NLS

Theorem (D, Figueras, 2024)

Consider the equation

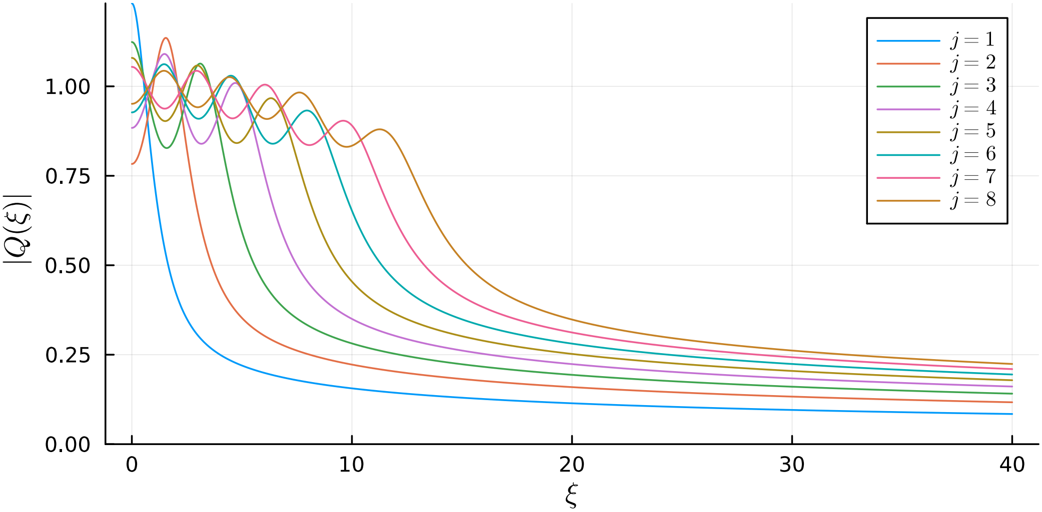
$$\begin{cases} Q'' + \frac{d-1}{\xi} Q' + i\kappa\xi Q' + i\frac{\kappa}{\sigma} Q - \omega Q + |Q|^{2\sigma} Q = 0 \\ Q'(0) = 0, \, Q(\xi) \sim \xi^{-\frac{1}{\sigma} - i\frac{\omega}{\kappa}} \end{cases} .$$

Case I *There exist solutions for at least 8 values of* κ.

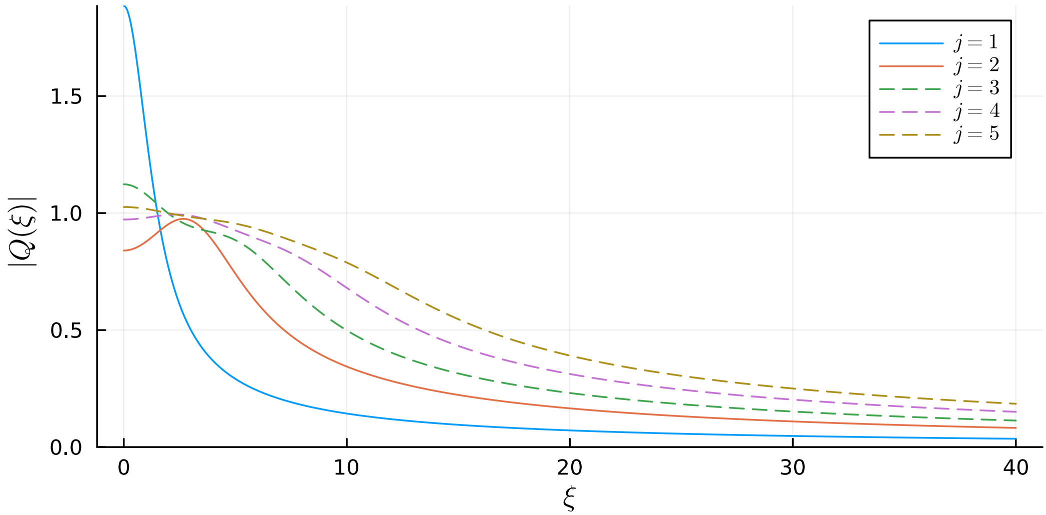
Case II *There exist solutions for at least 2 values of* κ.

The number of intervals of monotonicity is given by the index j *of the solution.*

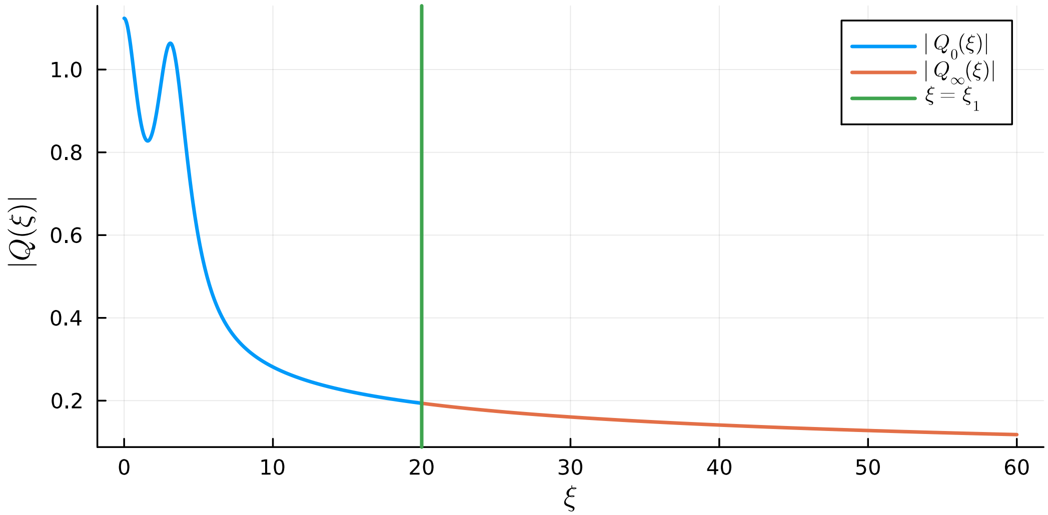
Solutions for Case I



Solutions for Case II



Proof idea - Matching solutions



Q_0 and Q_∞

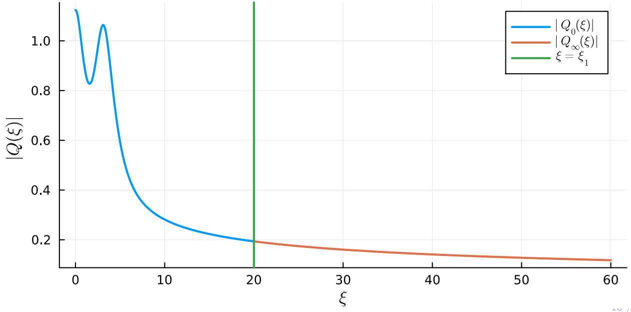
$$\begin{cases} (1-i\epsilon)\left(Q'' + \frac{d-1}{\xi}Q'\right) + i\kappa\xi Q' + i\frac{\kappa}{\sigma}Q - \omega Q + (1+i\delta)|Q|^{2\sigma}Q = 0 \\ Q'(0) = 0,\, Q(0) = \mu > 0,\, \lim_{\xi\to\infty} Q(\xi) = 0 \end{cases}$$

$Q_0(\xi) = Q_0(\mu, \kappa; \xi)$ Solution with $Q_0'(0) = 0, Q_0(0) = \mu$

$Q_\infty(\xi) = Q_\infty(\gamma, \kappa; \xi)$ Solution with $Q_\infty(\xi) \sim \xi^{-\frac{1}{\sigma} - i\frac{\omega}{\kappa}}$ as $\xi \to \infty$

Fix $\xi_1 > 0$. Want to find μ, γ, κ s.t.

- $Q_0(\mu, \kappa; \xi_1) = Q_\infty(\gamma, \kappa; \xi_1)$
- $Q_0'(\mu, \kappa; \xi_1) = Q_\infty'(\gamma, \kappa; \xi_1)$



Proving matching condition

$$G(\mu,\gamma,\kappa) = \left(Q_0(\mu,\kappa;\xi_1) - Q_\infty(\gamma,\kappa;\xi_1),\, Q_0'(\mu,\kappa;\xi_1) - Q_\infty'(\gamma,\kappa;\xi_1)\right)$$

$$G(\mu, \operatorname{Re}\gamma, \operatorname{Im}\gamma, \kappa) : \mathbb{R}^4 \to \mathbb{R}^4$$

1. Find a numerical approximation
2. Rigorously verify approximation using the interval Newton method
   - Requires computing rigorous interval enclosures for
     - $Q_0(\mu,\kappa;\xi_1), Q_0'(\mu,\kappa;\xi_1), Q_\infty(\gamma,\kappa;\xi_1), Q_\infty'(\gamma,\kappa;\xi_1)$
     - Derivatives w.r.t. μ, γ, κ

j	μ_j	γ_j	κ_j	ξ_1
1	1.88565_{67}^{73}	$1.71360_{05}^{13} - 1.49179_{35}^{42} i$	0.91735_{59}^{63}	60
2	0.8399_{57}^{62}	$13.852_{46}^{78} + 6.034_{44}^{59} i$	0.3212_{39}^{41}	140

Things to explain

$$G(\mu,\gamma,\kappa) = (Q_0(\mu,\kappa;\xi_1) - Q_\infty(\gamma,\kappa;\xi_1), Q_0'(\mu,\kappa;\xi_1) - Q_\infty'(\gamma,\kappa;\xi_1))$$

1. How to find an approximate numerical solution
2. What is the interval Newton method?
3. How to compute, approximately and rigorously:
    3.1 $Q_0(\mu,\kappa;\xi_1)$, $Q_0'(\mu,\kappa;\xi_1)$, $Q_{0,\mu}(\mu,\kappa;\xi_1)$, $Q_{0,\mu}'(\mu,\kappa;\xi_1)$, $Q_{0,\kappa}'(\mu,\kappa;\xi_1)$, $Q_{0,\kappa}'(\mu,\kappa;\xi_1)$
    3.2 $Q_\infty(\gamma,\kappa;\xi_1)$, $Q_\infty'(\gamma,\kappa;\xi_1)$, $Q_{\infty,\gamma}(\gamma,\kappa;\xi_1)$, $Q_{\infty,\gamma}'(\gamma,\kappa;\xi_1)$, $Q_{\infty,\kappa}'(\gamma,\kappa;\xi_1)$, $Q_{\infty,\kappa}'(\gamma,\kappa;\xi_1)$

A simplified example

$$(1 - i\epsilon)\left(Q'' + \frac{d-1}{\xi}Q'\right) + i\kappa\xi Q' + i\frac{\kappa}{\sigma}Q - \omega Q + (1 + i\delta)|Q|^{2\sigma}Q = 0$$

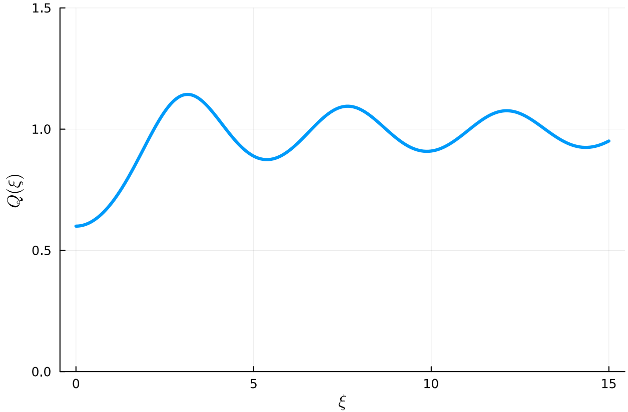
- $\epsilon = \delta = 0$ (NLS)
- $d = 2$, $\sigma = 1$ (2D cubic NLS)
- $\omega = 1$, $\kappa = 0$ ("Excited states")
- Q real valued

$$Q'' + \frac{1}{\xi}Q' - Q + Q^3 = 0$$

Computing Q_0 - Approximately

$$\begin{cases} Q_0'' + \frac{1}{\xi} Q_0' - Q_0 + Q_0^3 = 0, \\ Q_0(0) = \mu, \;\; Q_0'(0) = 0. \end{cases}$$

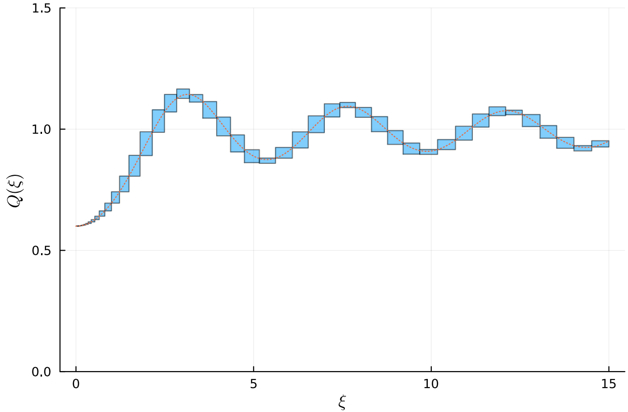
- Use numerical ODE solver, e.g. `RK4`, `TSit5` or `Vern7`.



Computing Q_0 - Rigorously

$$\begin{cases} Q_0'' + \frac{1}{\xi} Q_0' - Q_0 + Q_0^3 = 0, \\ Q_0(0) = \mu, \;\; Q_0'(0) = 0. \end{cases}$$

- Use rigorous numerical ODE solver, e.g. `CAPD` library.
- Taylor expand at $\xi = 0$.
- More next lecture!



Computing Q_∞ - Linear equation

$$Q'' + \frac{1}{\xi} Q' - Q = 0$$

$$\xi^2 Q'' + \xi Q' - \xi^2 Q = 0$$

- Modified Bessel equation with parameter zero
- Regular singular point at $\xi = 0$
- Irregular singular point at $\xi = \infty$
- Solutions: $K_0(\xi) \sim \xi^{-\frac{1}{2}} e^{-\xi}$ and $I_0(\xi) \sim \xi^{-\frac{1}{2}} e^{\xi}$ (modified Bessel functions)

Computing Q_∞ - Approximately

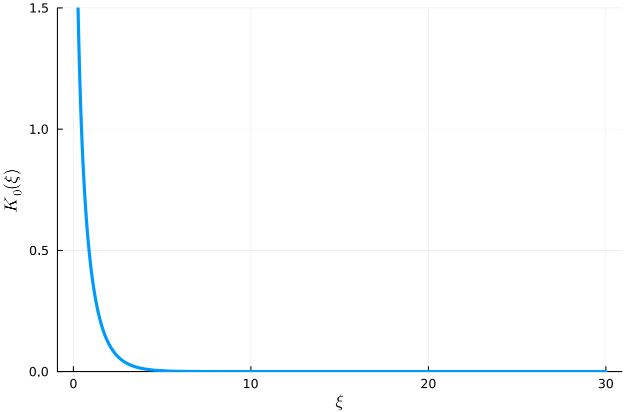
$$\begin{cases} Q_\infty'' + \frac{1}{\xi} Q_\infty' - Q_\infty + Q_\infty^3 = 0, \\ \lim_{\xi \to \infty} Q_\infty(\xi) = 0. \end{cases}$$

Q_∞ small $\to$ ignore Q_∞^3

$$\begin{cases} Q_\infty'' + \frac{1}{\xi} Q_\infty' - Q_\infty = 0, \\ \lim_{\xi \to \infty} Q_\infty(\xi) = 0. \end{cases}$$

- $K_0(\xi) \sim \xi^{-\frac{1}{2}} e^{-\xi}$
- $I_0(\xi) \sim \xi^{-\frac{1}{2}} e^{\xi}$

$$Q_\infty(\xi) \approx \gamma K_0(\xi)$$



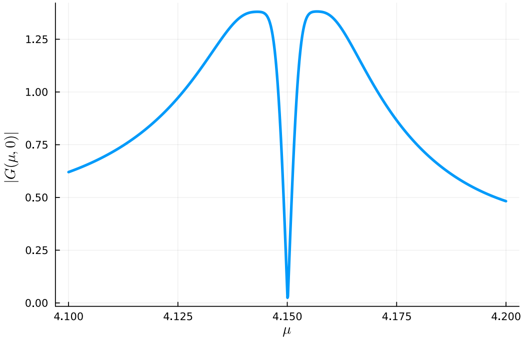
Computing Q_∞ - Rigorously

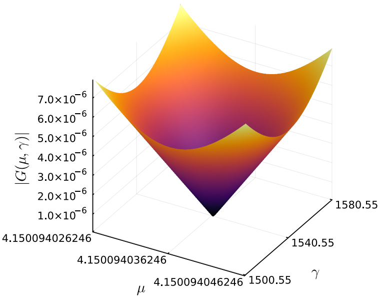
$$Q_\infty(\xi) = \gamma K_0(\xi) + K_0(\xi) I_I(\gamma, \xi) + I_0(\xi) I_K(\gamma, \xi)$$

- Initial bounds with contraction argument
- Improved bounds by bootstrapping
- More next lecture!

Finding approximate numerical solution

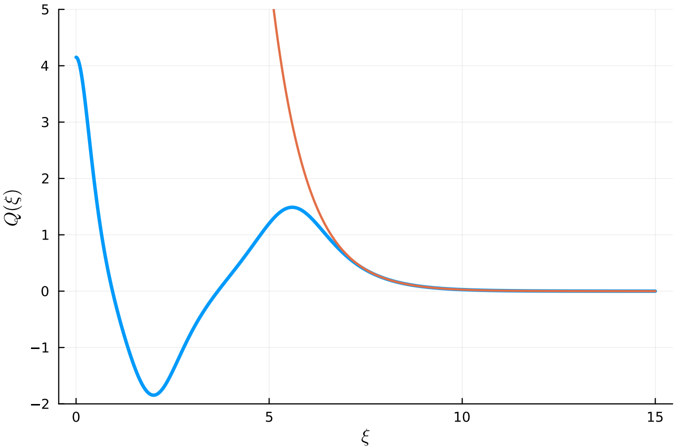
$$G(\mu,\gamma) = \left(Q_0(\mu;\xi_1) - Q_\infty(\gamma;\xi_1),\, Q_0'(\mu;\xi_1) - Q_\infty'(\gamma;\xi_1)\right)$$





Finding approximate numerical solution

$$G(4.150094036246, 1540.55)$$
$$\approx (2.5204639319664074 \cdot 10^{-10}, -2.942687336480402 \cdot 10^{-10})$$



Next lecture

- Computing Q_0 and Q_∞ rigorously
- Rigorously verifying an approximate solution (interval Newton method)
- Handling the full equation
- Results for CGL