KHOVANOV HOMOLOGY VIA 1-TANGLE DIAGRAMS IN THE ANNULUS

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Abstract. We show that the reduced Khovanov homology of an oriented link L in S^3 can be expressed as the homology of a chain complex constructed from a description of L as the closure of a 1-tangle diagram T in the annulus. Our chain complex is constructed using a cube of resolutions of T in a manner similar to ordinary Khovanov homology, but it is typically smaller than the ordinary Khovanov chain complex and has several unusual features, such as *long differentials* corresponding to pairs of successive saddles in the cube of resolutions. Our chain complex carries a natural filtration, which we use to construct a spectral sequence that converges to reduced Khovanov homology. Our results are part of a larger program to construct an analog of Khovanov homology for links in lens spaces by generalizing a symplectic interpretation of Khovanov homology due to Hedden, Herald, Hogancamp, and Kirk, and our chain complex was predicted by this program for the case when the lens space is S^3.

1. Introduction

Khovanov homology is a powerful invariant defined for oriented links in S^3 [8]. The Khovanov homology of an oriented link is a finitely-generated bigraded abelian group that categorifies its Jones polynomial [7]; roughly speaking, the relationship between the Khovanov homology of a link and its Jones polynomial is analogous to the relationship between the singular homology of a topological space and its Euler characteristic. The Jones polynomial of a link can be recovered from its Khovanov homology, but the Khovanov homology generally contains more information: it can sometimes distinguish between links with the same Jones polynomial, and Khovanov homology detects the unknot [9], but it is not known whether the same is true of the Jones polynomial.

Here we construct a new chain complex for the reduced Khovanov homology of a link via a description of the link as the closure of an oriented 1-tangle diagram T in the annulus $S^1 \times [0,1]$. We choose a basepoint $b_0 \in S^1$ and take the boundary of T to be the points $(b_0, 0)$ and $(b_0, 1)$ on the inner and outer bounding circles of the annulus. We close T with an unknotted overpass or underpass arc to obtain link diagrams T^+ and T^-. These link diagrams describe oriented links in S^3 that are unique up to isotopy, which for simplicity we also denote by T^+ and T^-. We construct a chain complex $(C_{T^\pm}, \partial_{T^\pm})$ for the link $T^\pm$ from a cube of resolutions of T. An example tangle diagram T and its cube of resolutions are shown in Figure 1. For this example the link diagrams T^+ and T^- describe the unknot and right trefoil, respectively.

The vertices of the cube of resolutions of T are planar tangles. Each planar tangle consists of an arc component, which we orient in the *outward* direction from $(b_0, 0)$ to $(b_0, 1)$, and some number of circle components. We assign a vector space to each planar tangle based on its number of circle components in a manner analogous to the usual Khovanov chain complex, and we take the direct sum of these vector spaces to construct the vector space $C_{T^\pm}$.

The edges of the cube of resolutions of T are saddles between pairs of planar tangles. The image of the arc component of a planar tangle under the projection $S^1 \times [0,1] \to S^1$ is an oriented loop based at b_0, and we define the *winding number* of the planar tangle to be the number of times this loop winds counterclockwise around S^1. A saddle either preserves the winding number or changes it by two.

To each saddle that preserves the winding number, we assign the same linear map as in ordinary reduced Khovanov homology, where the arc component plays the role of the marked circle component. We sum the linear maps for all saddles that preserve the winding number to obtain a differential ∂^0_T that squares to zero. In Figure 1, the saddles $T_{10} \to T_{11}$ and $T_{01} \to T_{11}$ preserve the winding number and hence contribute to ∂^0_T.

We next assign a linear map that we call a *long differential* to each pair of successive saddles for which one saddle lowers the winding number by two for T^+ or raises it by two for T^-, and one saddle connects a circle to the *left* side of the arc component, given its outward orientation. We sum the linear maps for all

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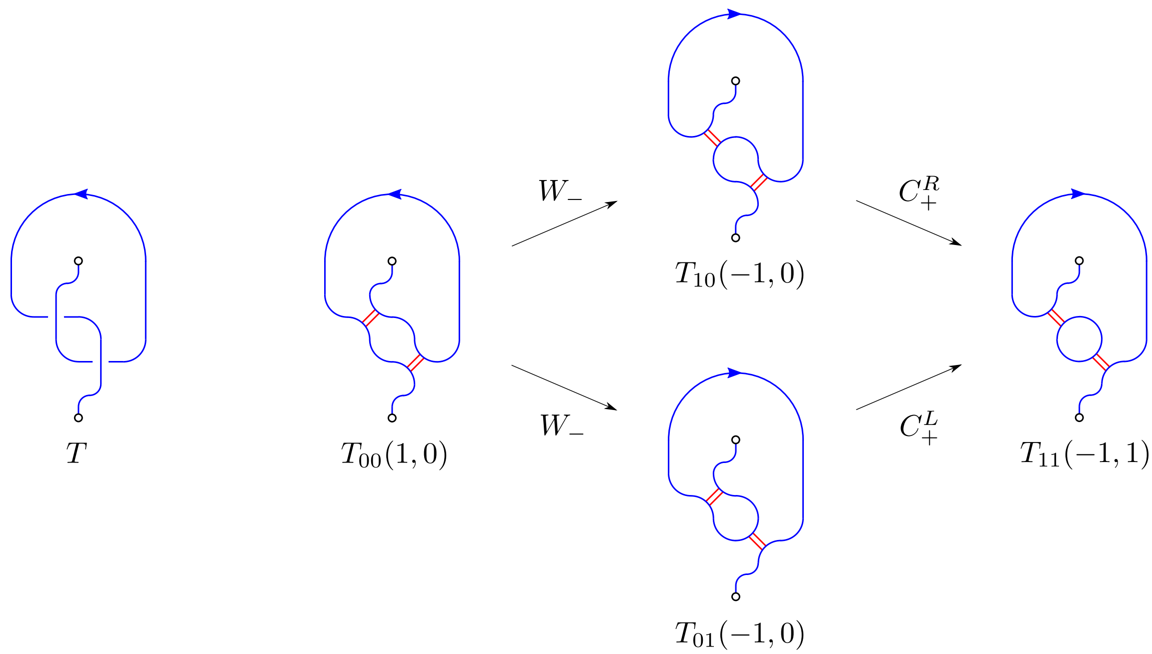

FIGURE 1. Example tangle diagram T and its cube of resolutions. For each planar tangle T_i in the cube of resolutions, we indicate its winding number n_i and circle number r_i as $T_i(n_i, r_i)$. The saddles in the cube of resolutions are labeled by their types, as described in Section 3.

such pairs of saddles to obtain a differential $\partial_T^\pm$ for $T^\pm$. In Figure 1, the saddle $T_{00} \to T_{01}$ lowers the winding number by two and the saddle $T_{01} \to T_{11}$ splits a circle from the left side of the arc component, so this pair of saddles contributes to ∂_T^+. There are no saddles that raise the winding number by two, so $\partial_T^- = 0$.

The total differential $\partial_{T^\pm}$ is then given by

$$\partial_{T^\pm} = \partial_T^0 + \partial_T^\pm.$$

We prove:

Theorem 1.1. *The chain complex $(C_{T^\pm}, \partial_{T^\pm})$ is homotopy equivalent to the chain complex for the reduced Khovanov homology of the link diagram $T^\pm$.*

Corollary 1.2. *The homology of the chain complex $(C_{T^\pm}, \partial_{T^\pm})$ is the reduced Khovanov homology of the link $T^\pm$.*

If the planar tangles in the cube of resolutions of T all have winding number zero, then it is possible to close T with an arc that does not cross T, and for such an arc $(C_{T^\pm}, \partial_{T^\pm}) = (C_{T^\pm}, \partial_T^0)$ is the usual Khovanov chain complex for the link diagram $T^\pm$. But in general our chain complex is smaller than the usual chain complex, since crossings between T and the closing arc are not included in the cube of resolutions. For example, a right trefoil has crossing number three, but we can compute its reduced Khovanov homology using the tangle diagram shown in Figure 1, which has only two crossings.

We also prove:

Theorem 1.3. *There is a spectral sequence with E_2 page given by the homology of $(C_{T^\pm}, \partial_T^0)$ and differential d_2 induced by $\partial_T^\pm$ that converges to the reduced Khovanov homology of the link $T^\pm$.*

The vector spaces C_{T^+} and C_{T^-} differ only in their bigrading structure, so if we ignore the bigradings the E_2 pages of the spectral sequences for T^+ and T^- are the same.

Khovanov homology was originally defined only for links in S^3, and an important open problem is to generalize Khovanov homology to links in arbitrary 3-manifolds. Khovanov homology has been extended to links in I-bundles over arbitrary surfaces by Asaeda, Przytycki, and Sikora [1], to links in $S^2 \times S^1$ by Rozansky [12], to links in $\mathbb{RP}^3$ by Gabrovšek [3], and to links in all connected sums of $S^2 \times S^1$ by Willis [13]. The results presented here are part of a larger program described in [2] to construct Khovanov homology for links in arbitrary lens spaces by generalizing a symplectic interpretation of Khovanov homology due to Hedden, Herald, Hogancamp, and Kirk [6]. This interpretation originated as an offshoot of their project to construct *pillowcase homology* [4, 5], a symplectic counterpart to Kronheimer and Mrowka's singular instanton link homology [9, 10, 11].

The setup for pillowcase homology is as follows. Given an oriented link L in S^3, one considers a Heegaard splitting

$$(S^3, L) = (B^3, T_0) \cup_{(S^2,4)} (B^3, T_1), \tag{1}$$

where the Heegaard surface $(S^2, 4)$ is a 2-sphere that transversely intersects L in four points, the handlebodies (B^3, T_0) and (B^3, T_1) are closed 3-balls containing 2-tangles T_0 and T_1, and T_0 is trivial. To the Heegaard surface $(S^2, 4)$ one associates the irreducible locus $R^*(S^2, 4)$ of the traceless $SU(2)$ character variety for the 2-sphere with four punctures, a symplectic manifold known as the *pillowcase*. To the handlebodies (B^3, T_0) and (B^3, T_1) one associates traceless $SU(2)$ character varieties $R^\natural_\pi(B^3, T_0)$ and $R^*(B^3, T_1)$. By pulling back $SU(2)$ representations along the inclusions $(S^2, 4) \hookrightarrow (B^3, T_0)$ and $(S^2, 4) \hookrightarrow (B^3, T_1)$, one obtains maps $R^\natural_\pi(B^3, T_0) \to R^*(S^2, 4)$ and $R^*(B^3, T_1) \to R^*(S^2, 4)$ of the corresponding character varieties. The images of these maps define Lagrangians L_0 and W_1 in $R^*(S^2, 4)$. Roughly speaking, the pillowcase homology of (S^3, L) is defined to be the Lagrangian Floer homology of the pair of Lagrangians (L_0, W_1).

In [6], Hedden, Herald, Hogancamp, and Kirk obtain a symplectic interpretation of reduced Khovanov homology by modifying the construction used to define pillowcase homology. Instead of working directly with the tangle T_1, they consider a cube of resolutions of a 2-tangle diagram in the disk obtained by projecting T_1 onto the plane. This cube of resolutions is used to construct an object (X_1, δ_1) in the twisted Fukaya category of $R^*(S^2, 4)$, an A_∞ category that can be thought of as an analog for Fukaya categories of the notion of a cochain complex for vector spaces. The object X_1 consists of shifted copies of Lagrangians corresponding to planar tangles at the vertices of the cube, and the morphism $\delta_1 : X \to X$ consists of maps of Lagrangians corresponding to saddles at the edges of the cube. The trivial tangle T_0 corresponds to an object $(W_0, 0)$ of the twisted Fukaya category.

The morphism spaces of the twisted Fukaya category have the structure of cochain complexes, so in particular the space of morphisms from $(W_0, 0)$ to (X_1, δ_1) is a cochain complex. Hedden, Herald, Hogancamp, and Kirk show this cochain complex is identical to the usual cochain complex for the reduced Khovanov homology $\mathrm{Khr}(L)$ of L, thus proving:

Theorem 1.4. *(Hedden–Herald–Hogancamp–Kirk* [6, Corollary 1.2]*) We have an isomorphism of bigraded vector spaces*

$$\mathrm{Khr}(L) \to H^*(\hom((W_0, 0), (X_1, \delta_1))),$$

where the hom *space is taken within the twisted Fukaya category of* $R^*(S^2, 4)$.

Theorem 1.4 shows that the Fukaya category of $R^*(S^2, 4)$ knows about Khovanov homology. Our strategy for generalizing Khovanov homology is based on generalizing this observation. Theorem 1.4 is formulated in terms of $R^*(S^2, 4)$ because this is the character variety of the Heegaard surface $(S^2, 4)$ for the Heegaard splitting (1) of (S^3, L). In general, one can split a 3-manifold Y containing a link L along a Heegaard surface (Σ, n) that intersects L in n points, and in light of Theorem 1.4 one might hope that the Fukaya category of the corresponding character variety $R^*(\Sigma, n)$ could provide clues as to how to generalize Khovanov homology to links in Y.

As a first step towards this goal, in [2] we consider the case of links in lens spaces. Given an oriented link L in a lens space Y, we consider a Heegaard splitting

$$(Y, L) = (U_0, T_0) \cup_{(T^2,2)} (U_1, T_1)$$

such that the Heegaard surface $(T^2, 2)$ is a 2-torus that transversely intersects L in two points, the handlebodies (U_0, T_0) and (U_1, T_1) are solid tori containing 1-tangles T_0 and T_1, and T_0 is trivial. To the Heegaard surface $(T^2, 2)$ we associate the irreducible locus $R^*(T^2, 2)$ of the traceless $SU(2)$ character variety for the torus with two punctures. We project (U_1, T_1) onto the plane to obtain a 1-tangle diagram T in the annulus, and we use a cube of resolutions of T to construct an object (X_1, δ_1) of the twisted Fukaya category of $R^*(T^2, 2)$. The handlebody (U_0, T_0) corresponds to an object $(W_0, 0)$ of the twisted Fukaya category.

The space of morphisms from $(W_0, 0)$ to (X_1, δ_1) has the structure of a cochain complex, and in [2] we use a partly conjectural description of the Fukaya category for $R^*(T^2, 2)$ to explicitly construct this cochain complex for the case of links in S^3. The result is the chain complex $(C_{T^\pm}, \partial_{T^\pm})$ that we consider here. The methods described in [2] can be used to construct chain complexes for links in lens spaces, but they do not guarantee that these chain complexes yield link invariants, and the purpose of the current paper is to prove this is in fact the case for the chain complex $(C_{T^\pm}, \partial_{T^\pm})$. Our Theorem 1.1 can thus be viewed as the analog for $R^*(T^2, 2)$ of Theorem 1.4 for $R^*(S^2, 4)$, but whereas $R^*(S^2, 4)$ yields the usual Khovanov chain complex, $R^*(T^2, 2)$ yields a chain complex that is only homotopy equivalent to the usual Khovanov chain complex. As we describe in Remark 7.3, our chain complex is also relevant to the Fukaya category of $R^*(S^2, 4)$.

One could perhaps view this successful prediction of a new chain complex for links in S^3 as evidence that our approach might yield invariants analogous to Khovanov homology for links in other lens spaces. In [2] we use this approach to explicitly construct chain complexes for some links in $S^2 \times S^1$ and we present results that suggest the cohomology is indeed a link invariant.

The paper is organized as follows. In Section 2, we define vector spaces and linear maps that are used to construct the chain complex $(C_{T^\pm}, \partial_{T^\pm})$. In Section 3, we explain how the chain complex $(C_{T^\pm}, \partial_{T^\pm})$ is constructed from the cube of resolutions of the tangle diagram T, and we use this chain complex to construct the spectral sequence described in Theorem 1.3. In Section 4, we illustrate our results using the example tangle diagram T shown in Figure 1. In Section 5, we introduce some notation that is useful for describing planar tangles and saddles. In Section 6, we express the differential ∂_T^+ in terms of commuting squares of saddles in the cube of resolutions of T. In Section 7, we outline the proof of Theorem 1.1. In Sections 8 – 11, as well as Appendices A and B, we fill in technical details needed to complete the proof.

2. Vector spaces and linear maps

Here we define vector spaces and linear maps that we will use to construct the chain complex $(C_{T^\pm}, \partial_{T^\pm})$. The vector spaces we consider are typically bigraded. Given a bigraded vector space V, we use the notation $v^{(h,q)}$ to indicate that a homogeneous vector $v \in V$ has bigrading (h, q). We refer to h as the *homological grading* and q as the *quantum grading*. We define the vector space $V[h_s, q_s]$ to be V with gradings shifted *upward* by (h_s, q_s), so if $v \in V$ is homogeneous with bigrading (h, q) then the corresponding vector $v \in V[h_s, q_s]$ is homogeneous with bigrading $(h + h_s, q + q_s)$. We define $\mathbb{F}$ to be the field of two elements, where $1 \in \mathbb{F}$ is assigned bigrading $(0, 0)$. We define a two-dimensional bigraded $\mathbb{F}$-vector space

$$A = \langle e^{(0,1)},\, x^{(0,-1)} \rangle.$$

We define the following $\mathbb{F}$-linear maps. We define *unit maps* $\eta^{(0,1)}$ and $\dot{\eta}^{(0,-1)}$:

$$\eta : \mathbb{F} \to A, \qquad \eta(1) = e,$$
$$\dot{\eta} : \mathbb{F} \to A, \qquad \dot{\eta}(1) = x.$$

We define *counit maps* $\epsilon^{(0,1)}$ and $\dot{\epsilon}^{(0,-1)}$:

$$\epsilon : A \to \mathbb{F}, \qquad \epsilon(e) = 0, \qquad \epsilon(x) = 1,$$
$$\dot{\epsilon} : A \to \mathbb{F}, \qquad \dot{\epsilon}(e) = 1, \qquad \dot{\epsilon}(x) = 0.$$

We define a *raising map* $\mathbb{1}_{ex}^{(0,2)}$ and a *lowering map* $\mathbb{1}_{xe}^{(0,-2)}$:

$$\mathbb{1}_{ex} : A \to A, \qquad \mathbb{1}_{ex}(e) = 0, \qquad \mathbb{1}_{ex}(x) = e,$$
$$\mathbb{1}_{xe} : A \to A, \qquad \mathbb{1}_{xe}(e) = x, \qquad \mathbb{1}_{xe}(x) = 0.$$

We define *projection maps* $\mathbb{1}_{ee}^{(0,0)}$ and $\mathbb{1}_{xx}^{(0,0)}$:

$$\mathbb{1}_{ee} : A \to A, \qquad \mathbb{1}_{ee}(e) = e, \qquad \mathbb{1}_{ee}(x) = 0,$$
$$\mathbb{1}_{xx} : A \to A, \qquad \mathbb{1}_{xx}(e) = 0, \qquad \mathbb{1}_{xx}(x) = x.$$

We define a *multiplication map* $m^{(0,-1)}$:

$$m : A \otimes A \to A, \qquad m(e \otimes e) = e, \qquad m(e \otimes x) = m(x \otimes e) = x, \qquad m(x \otimes x) = 0.$$

We define a *comultiplication map* $\Delta^{(0,-1)}$:

$$\Delta : A \to A \otimes A, \qquad \Delta(e) = e \otimes x + x \otimes e, \qquad \Delta(x) = x \otimes x.$$

The graded vector space A, together with the multiplication m, comultiplication Δ, unit η, and counit ϵ, gives Khovanov's Frobenius algebra. For notational simplicity, given an $\mathbb{F}$-vector space V we often identify V with $\mathbb{F} \otimes V$ and $V \otimes \mathbb{F}$.

3. Chain complex $(C_{T^\pm}, \partial_{T^\pm})$

Consider an oriented tangle diagram T in the annulus $S^1 \times [0,1]$. We choose a point $b_0 \in S^1$ and take the boundary of T to be the points $(b_0, 0)$ and $(b_1, 1)$ on the inner and outer bounding circles of the annulus. We close T with an unknotted overpass arc A^+ or underpass arc A^- that crosses T transversely to obtain an oriented link diagram $T^\pm := T \cup A^\pm$. The image of $A^\pm$ under the projection $S^1 \times [0,1] \to S^1$ is a loop based at b_0, and we choose $A^\pm$ such that this loop is contractible. We construct a bigraded chain complex $(C_{T^\pm}, \partial_{T^\pm})$ for the link diagram $T^\pm$ from a cube of resolutions of the tangle diagram T as follows.

First we describe the cube of resolutions. Let $m_+(T)$ and $m_-(T)$ denote the number of positive and negative crossings of T, and let $m(T) = m_+(T) + m_-(T)$ denote the total number of crossings. We can specify a planar resolution of T by specifying how each crossing is to be resolved. Define the 0-resolution, respectively 1-resolution, of a crossing such that the overpass turns left, respectively right. We fix an arbitrary ordering of the crossings and encode the data needed to specify a planar resolution as a binary string of length $m(T)$, so the k-th bit of the binary string specifies the resolution of the k-th crossing of T. Let $I = \{0,1\}^{m(T)}$ denote the set of binary strings of length $m(T)$. For each binary string $i \in I$, let T_i denote the corresponding planar resolution of T. We define the *resolution degree* $r(T_i)$ of the planar tangle T_i to be the number of 1's in the binary string i. For each pair of binary strings $i, j \in I$ such that $r(T_j) = r(T_i) + 1$, we define a saddle $T_i \to T_j$. The strings i and j are identical except for a single bit that is 0 in i but 1 in j, and the saddle $T_i \to T_j$ changes the resolution of the corresponding crossing of T from the 0-resolution in T_i to the 1-resolution in T_j.

Next we describe the bigraded vector space $C_{T^\pm}$, which is constructed from the planar tangles in the cube of resolutions of T. A planar tangle p consists of an arc component connecting the points $(b_0, 0)$ and $(b_0, 1)$ on the inner and outer boundary of the annulus $S^1 \times [0,1]$, together with some number of circle components. We orient the arc component of p in the *outward* direction from $(b_0, 0)$ to $(b_0, 1)$. The image of the arc component of p under the projection $S^1 \times [0,1] \to S^1$ is an oriented loop based at b_0, and we define the *winding number* $w(p)$ to be the number of times this loop winds *counterclockwise* around S^1. We define the *circle number* $c(p)$ to be the number of circle components of p. Let $a_+(A^\pm, T)$ and $a_-(A^\pm, T)$ denote the number of positive and negative crossings between $A^\pm$ and T. For each planar tangle T_i in the cube of resolutions of T, we define a corresponding bigraded vector space

$$C^\pm_{T_i} = A^{\otimes c(T_i)}[h^\pm(T, T_i),\, q^\pm(T, T_i)],$$

where the bigrading shift is given by

$$h^\pm(T, T_i) = -m_-(T) + \frac{1}{2}(a_+(A^\pm, T) - a_-(A^\pm, T) \pm w(T_i)) + r(T_i), \tag{2}$$

$$q^\pm(T, T_i) = m_+(T) - 2m_-(T) + \frac{3}{2}(a_+(A^\pm, T) - a_-(A^\pm, T) \pm w(T_i)) + r(T_i). \tag{3}$$

We define the bigraded vector space $C_{T^\pm}$ as

$$C_{T^\pm} = \bigoplus_{i \in I} C^\pm_{T_i}.$$

Next we describe the differential $\partial_{T^\pm}$, which is constructed from the saddles in the cube of resolutions of T. We classify the saddles as type $C^R_\pm$, $C^L_\pm$, $C^C_\pm$, or $W_\pm$, as shown in Figure 2. A saddle of type $C^R_\pm$, $C^L_\pm$, or $C^C_\pm$ splits $(+)$ or merges $(-)$ a circle component. A saddle of type $C^R_\pm$ or $C^L_\pm$ connects a circle component to the right (R) or left (L) side of the outward-oriented arc component. A saddle of type $C^C_\pm$ connects two circle components. A saddle of type $W_\pm$ raises $(+)$ or lowers $(-)$ the winding number by two.

We assign several linear maps to each saddle in the cube of resolutions. We denote a planar tangle in the cube of resolutions by p or q, and indicate that a planar tangle has winding number n and circle number r using the notation $p(n, r)$ or $q(n, r)$.

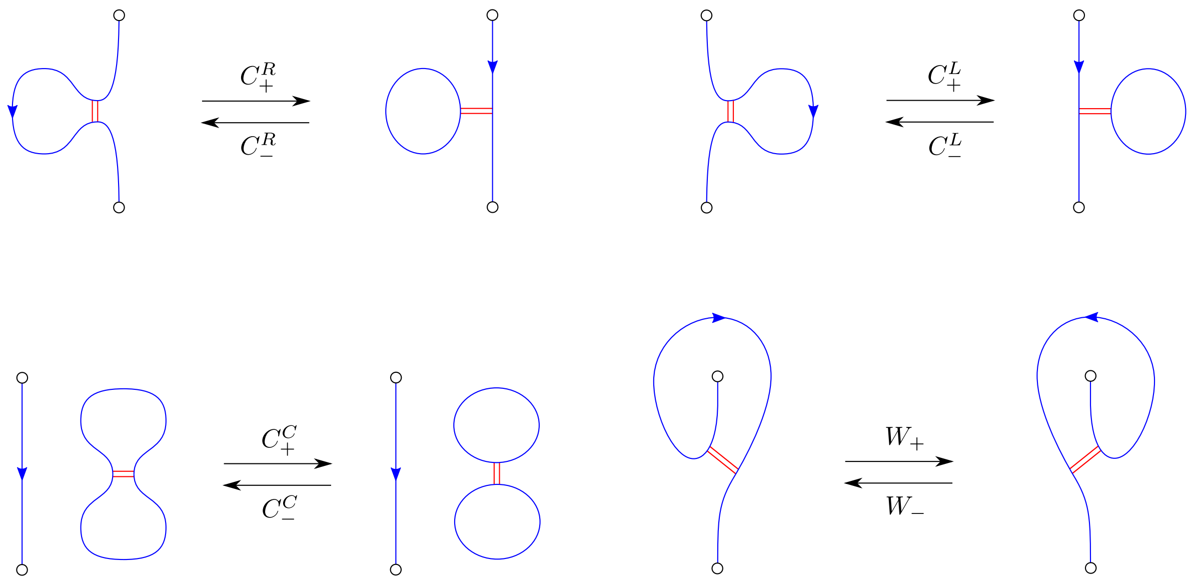

FIGURE 2. Example saddles of type $C_\pm^R$, $C_\pm^L$, $C_\pm^C$, and $W_\pm$.

For each saddle $s : p \to q$, we define a map $T(s) : C_p^\pm \to C_q^\pm$ as follows:

- If $s : p(n, r) \to q(n, r + 1)$ is type C_+^L or C_+^R, we define
$$T(s) = (\mathbb{1}_{A^{\otimes r}} \otimes \dot{\eta})[1, 1] : A^{\otimes r}[h, q] \to (A^{\otimes r} \otimes A)[h + 1, q + 1].$$
- If $s : p(n, r + 1) \to q(n, r)$ is type C_-^L or C_-^R, we define
$$T(s) = (\mathbb{1}_{A^{\otimes r}} \otimes \dot{\epsilon})[1, 1] : (A^{\otimes r} \otimes A)[h, q] \to A^{\otimes r}[h + 1, q + 1].$$
- If $s : p(n, r + 1) \to q(n, r + 2)$ is type C_+^C, we define
$$T(s) = (\mathbb{1}_{A^{\otimes r}} \otimes \Delta)[1, 1] : (A^{\otimes r} \otimes A)[h, q] \to (A^{\otimes r} \otimes A \otimes A)[h + 1, q + 1].$$
- If $s : p(n, r + 2) \to q(n, r + 1)$ is type C_-^C, we define
$$T(s) = (\mathbb{1}_{A^{\otimes r}} \otimes m)[1, 1] : (A^{\otimes r} \otimes A \otimes A)[h, q] \to (A^{\otimes r} \otimes A)[h + 1, q + 1].$$
- If s is type $W_\pm$, we define $T(s) = 0$.

We extend $T(s)$ by zero to obtain a map $T(s) : C_{T^\pm} \to C_{T^\pm}$.

For each saddle $s : p \to q$ and each orientation $\beta \in \{L, R\}$, we define a map $\tilde{T}^\beta(s) : C_p^\pm \to C_q^\pm$ as follows:

- If $s : p(n, r) \to q(n, r + 1)$ is type C_+^β, we define
$$\tilde{T}^\beta(s) = (\mathbb{1}_{A^{\otimes r}} \otimes \eta)[1, 1] : A^{\otimes r}[h, q] \to (A^{\otimes r} \otimes A)[h + 1, q + 1].$$
- If $s : p(n, r + 1) \to q(n, r)$ is type C_-^β, we define
$$\tilde{T}^\beta(s) = (\mathbb{1}_{A^{\otimes r}} \otimes \epsilon)[1, 1] : (A^{\otimes r} \otimes A)[h, q] \to A^{\otimes r}[h + 1, q + 1].$$
- If s is not type C_+^β or C_-^β, we define $\tilde{T}^\beta(s) = 0$.

We extend $\tilde{T}^\beta(s)$ by zero to obtain a map $\tilde{T}^\beta(s) : C_{T^\pm} \to C_{T^\pm}$.

For each saddle $s : p \to q$, we define a map $P(s) : C_p^- \to C_q^-$ as follows:

- If $s : p(n, r) \to q(n + 2, r)$ is type W_+, we define
$$P(s) = \mathbb{1}_{A^{\otimes r}}[0, -2] : A^{\otimes r}[h, q] \to A^{\otimes r}[h, q - 2].$$
- If s is not type W_+, we define $P(s) = 0$.

We extend $P(s)$ by zero to obtain a map $P(s) : C_{T^-} \to C_{T^-}$.

For each saddle $s : p \to q$, we define a map $Q(s) : C_p^+ \to C_q^+$ as follows:

- If $s : p(n, r) \to q(n-2, r)$ is type W_-, we define
$$Q(s) = \mathbb{1}_{A^{\otimes r}}[0, -2] : A^{\otimes r}[h, q] \to A^{\otimes r}[h, q-2].$$
- If s is not type W_-, we define $Q(s) = 0$.

We extend $Q(s)$ by zero to obtain a map $Q(s) : C_{T^+} \to C_{T^+}$.

In the above expressions, we have assumed a specific choice of ordering of the factors of A corresponding to the circle components of the planar tangles. For a different ordering of these factors we would need to modify the above expressions accordingly.

We define a map
$$\tilde{T}(s) = \tilde{T}^L(s) + \tilde{T}^R(s).$$
We define maps
$$T = \sum_s T(s), \qquad \tilde{T}^\beta = \sum_s \tilde{T}^\beta(s), \qquad P = \sum_s P(s), \qquad Q = \sum_s Q(s),$$
where the sums are over all saddles in the cube of resolutions of T. The bigradings of these maps are
$$(T)^{(1,0)}, \qquad (\tilde{T}^\beta)^{(1,2)}, \qquad (Q)^{(0,-2)}, \qquad (P)^{(0,-2)}.$$
We define maps
$$\partial_T^0 = T, \qquad \partial_T^+ = Q\tilde{T}^L + \tilde{T}^L Q, \qquad \partial_T^- = P\tilde{T}^L + \tilde{T}^L P,$$
which have bigrading $(1, 0)$. The differential $\partial_{T^\pm}$ for the link diagram $T^\pm$ is then given by
$$\partial_{T^\pm} = \partial_T^0 + \partial_T^\pm,$$
and has bigrading $(1, 0)$. We note that for ∂_T^0 the assignment of linear maps to saddles is the same as for the usual chain complex for reduced Khovanov homology, where the arc component plays the role of the marked circle component. The new feature of our chain complex is the term $\partial_T^\pm$, which describes *long differentials* corresponding to pairs of successive saddles.

Our chain complex can be viewed as a generalization of the usual chain complex for reduced Khovanov homology in the following sense. Consider a tangle diagram T such that $w(T_i) = 0$ for all $i \in I$. Then T can be closed with an arc A^0 that does not cross T and whose image under the projection $S^1 \times [0, 1] \to S^1$ is a contractible loop. Since A^0 can be viewed as either an overpass or underpass arc, we can take $A^+ = A^- = A^0$ and obtain a link diagram $T^+ = T^- = T \cup A^0$. There are no long differentials, so $\partial_{T^\pm} = \partial_T^0$, and equations (2) and (3) for the bigrading shift reduce to the usual expressions for Khovanov homology:

$$h^\pm(T, T_i) = -m_-(T) + r(T_i), \qquad q^\pm(T, T_i) = m_+(T) - 2m_-(T) + r(T_i). \tag{4}$$

It follows that $(C_{T^\pm}, \partial_{T^\pm}) = (C_{T^\pm}, \partial_T^0)$ is the usual chain complex for the reduced Khovanov homology of the link diagram $T^+ = T^-$, where the marked point is taken to be $(b_0, 0)$ or $(b_0, 1)$.

For a tangle diagram T of this special form, equation (4) shows that the homological grading $h^\pm(T, T_i)$ is just the resolution degree $r(T_i)$ shifted by a constant. But in general this is not the case, since equation (2) for $h^\pm(T, T_i)$ contains an additional term that depends on $w(T_i)$. In particular, both ∂_T^0 and $\partial_T^\pm$ increase the homological grading by one, but ∂_T^0 increases the resolution degree by one and $\partial_T^\pm$ increases the resolution degree by two. The fact that $(\partial_{T^\pm})^2 = (\partial_T^0 + \partial_T^\pm)^2 = 0$ must hold in each resolution degree thus implies

Lemma 3.1. *We have* $(\partial_T^0)^2 = 0$.

We can use the resolution degree to define a filtration $\mathcal{F}^0_{T^\pm} = C_{T^\pm} \supset \mathcal{F}^1_{T^\pm} \supset \mathcal{F}^2_{T^\pm} \supset \cdots$ of the chain complex $(C_{T^\pm}, \partial_{T^\pm})$ as follows:
$$\mathcal{F}^s_{T^\pm} := \bigoplus_{i \in I \mid r(T_i) \geq s} C^\pm_{T_i}.$$
This filtration, and the fact that $(C_{T^\pm}, \partial_T^0)$ is a chain complex by Lemma 3.1, yield the spectral sequence described in Theorem 1.3.

4. Example

We will now illustrate our results using the example tangle diagram T shown in Figure 1. The number of positive and negative crossings $m_+(T)$ and $m_-(T)$ of T are

$$m_+(T) = 2, \qquad m_-(T) = 0.$$

The cube of resolutions of T is shown in Figure 1 and reproduced here:

$$\begin{array}{ccccc} & & T_{10}(-1,0) & & \\ & \overset{W_-}{\nearrow} & & \overset{C_+^R}{\searrow} & \\ T_{00}(1,0) & & & & T_{11}(-1,1), \\ & \underset{W_-}{\searrow} & & \underset{C_+^L}{\nearrow} & \\ & & T_{01}(-1,0) & & \end{array}$$

where $T_i(n_i, r_i)$ indicates that the planar tangle T_i has winding number n_i and circle number r_i.

4.1. **Link diagram T^+.** Closing T with an overpass arc A^+ yields a link diagram $T^+ = T \cup A^+$ for the unknot. We have $a_+(A^+, T) - a_-(A^+, T) = -1$, so the chain complex $(C_{T^+}, \partial_{T^+})$ is

$$\begin{array}{ccc} & C^+_{T_{10}} = \mathbb{F}[0,0] & \\ & & \searrow^{\dot{\eta}[1,1]} \\ C^+_{T_{00}} = \mathbb{F}[0,2] & \xRightarrow{\eta[1,-1]} & C^+_{T_{11}} = A[1,1]. \\ & & \nearrow_{\dot{\eta}[1,1]} \\ & C^+_{T_{01}} = \mathbb{F}[0,0] & \end{array}$$

The differential is $\partial_{T^+} = \partial_T^0 + \partial_T^+$, where ∂_T^0 consists of the two arrows labeled $\dot{\eta}[1,1]$ and ∂_T^+ consists of the double arrow labeled $\eta[1,-1]$. Note that the pair of saddles $T_{00} \to T_{01}$ of type W_- and $T_{01} \to T_{11}$ of type C_+^L give the long differential $\eta[1,-1] : C^+_{T_{00}} \to C^+_{T_{11}}$. The cohomology of $(C_{T^+}, \partial_{T^+})$ is the reduced Khovanov homology for the unknot:

$$H^*(C_{T^+}, \partial_{T^+}) = \mathrm{Khr}(T^+) = \mathbb{F}[0,0].$$

We next consider the spectral sequence for $(C_{T^+}, \partial_{T^+})$. The E_2 page is given by

$$E_2 = H^*(C_{T^+}, \partial_T^0) = \mathbb{F}[0,0] \oplus \mathbb{F}[0,2] \oplus \mathbb{F}[1,2],$$

and the differential $d_2 : E_2 \to E_2$, which is induced by ∂_T^+, maps the generator in bigrading $(0,2)$ to the generator in bigrading $(1,2)$. The E_3 page is thus given by

$$E_3 = H^*(E_2, d_2) = \mathrm{Khr}(T^+) = \mathbb{F}[0,0],$$

and the differential $d_3 : E_3 \to E_3$ is zero, so the spectral sequence collapses at this page.

4.2. **Link diagram T^-.** Closing T with an underpass arc A^- yields a link diagram $T^- = T \cup A^-$ for the right trefoil. We have $a_+(A^-, T) - a_-(A^-, T) = 1$, so the chain complex $(C_{T^-}, \partial_{T^-})$ is

$$\begin{array}{ccc} & C^-_{T_{10}} = \mathbb{F}[2,6] & \\ & & \searrow^{\dot{\eta}[1,1]} \\ C^-_{T_{00}} = \mathbb{F}[0,2] & & C^-_{T_{11}} = A[3,7]. \\ & & \nearrow_{\dot{\eta}[1,1]} \\ & C^-_{T_{01}} = \mathbb{F}[2,6] & \end{array}$$

The differential is $\partial_{T^-} = \partial_T^0 + \partial_T^-$, where ∂_T^0 consists of the two arrows labeled $\dot{\eta}[1,1]$ and $\partial_T^- = 0$. The cohomology of $(C_{T^-}, \partial_{T^-})$ is the reduced Khovanov homology for the right trefoil:

$$H^*(C_{T^-}, \partial_{T^-}) = \mathrm{Khr}(T^-) = \mathbb{F}[0,2] \oplus \mathbb{F}[2,6] \oplus \mathbb{F}[3,8].$$

We next consider the spectral sequence for $(C_{T^-}, \partial_{T^-})$. The E_2 page is given by

$$E_2 = H^*(C_{T^-}, \partial_T^0) = \mathrm{Khr}(T^-) = \mathbb{F}[0,2] \oplus \mathbb{F}[2,6] \oplus \mathbb{F}[3,8],$$

and the differential $d_2 : E_2 \to E_2$ is zero, so the spectral sequence collapses at this page. Note that if we ignore bigradings, the E_2 pages of the spectral sequences for T^+ and T^- are the same.

5. Notation for planar tangles and saddles

We now introduce some notation that will be useful for describing planar tangles and saddles. Recall that a planar tangle in the annulus $S^1 \times [0,1]$ consists of an arc component connecting the points $y_0 := (b_0, 0)$ and $y_1 := (b_0, 1)$ on the inner and outer bounding circles, together with some number of circle components. We orient the arc component in the outward direction from y_0 to y_1. For the purpose of defining our notation we choose arbitrary orientations for the circle components, and view different choices of orientations as providing equivalent descriptions of the same underlying tangle.

The orientation of the arc component defines an ordering of its points, and we will use strings of inequalities such as $x_1 < \cdots < x_n$ to indicate that points $x_1, \cdots, x_n$ on the arc component are ordered in the stated manner. We use the notation $(x_1 < \cdots < x_n)$ to indicate that points $x_1, \cdots, x_n$ lie on a circle component and are encountered in the stated order as we move around the circle in the direction given by its chosen orientation. We can cyclically permute the ordering of points on a circle component, so

$$(x_1 < x_2 < \cdots < x_n) = (x_2 < \cdots < x_n < x_1). \tag{5}$$

We can flip the orientation of a circle component to obtain an equivalent description of the same planar tangle, so

$$(x_1 < x_2 < \cdots < x_n) = (x_n < \cdots < x_2 < x_1). \tag{6}$$

This notation is useful for describing saddles. The attaching sphere of a saddle $s : p \to q$ consists of two points, which we call the *attaching points* of the saddle. We indicate that an attaching point of the saddle s attaches to the right or left side of an oriented component using the notation sR or sL. For example, the planar tangles in the cube of resolutions in Figure 1 are described as

$$\begin{aligned} &T_{00}: \quad y_0 < s_1R < s_2R < s_1L < s_2L < y_1 && T_{10}: \quad y_0 < s_1L < s_2L < s_1R < s_2L < y_1 \\ &T_{01}: \quad y_0 < s_1R < s_2L < s_1R < s_2R < y_1 && T_{11}: \quad y_0 < s_1L < s_2R < y_1,\ (s_1R < s_2R), \end{aligned}$$

where we have oriented the circle component in T_{11} counterclockwise. Note that the saddles $T_{10} \to T_{11}$ and $T_{01} \to T_{11}$ split a circle component from the right and left sides of the arc component, respectively. Since the arc component is oriented, each saddle naturally assigns an orientation to this circle component, but because the saddles split the circle from opposite sides of the arc component, the two orientations are not consistent. This type of inconsistency is the reason we treat clockwise and counterclockwise orientations of circle components on equal footing.

We can adapt rules (5) and (6) for circle components to include the orientation data for the attaching points of saddles:

$$(x_1\alpha_1 < x_2\alpha_2 < \cdots < x_n\alpha_n) = (x_2\alpha_2 < \cdots < x_n\alpha_n < x_1\alpha_1) = (x_n\overline{\alpha}_n < \cdots < x_2\overline{\alpha}_2 < x_1\overline{\alpha}_1),$$

where $\alpha_i \in \{R, L\}$ and $\overline{\alpha}_i$ denotes the opposite orientation of α_i, so $\overline{R} = L$ and $\overline{L} = R$. Using these rules, we see that the circle component in the planar tangle T_{11} from Figure 1 could be denoted in any of the following ways:

$$(s_1R < s_2R) = (s_2R < s_1R) = (s_2L < s_1L) = (s_1L < s_2L).$$

We can use this notation to describe saddles of type $C_\pm^\alpha$, $C_\pm^C$, and $W_\pm$, as shown in Figure 2. For a saddle s of type C_+^α, which attaches to the *same* side of the arc component at its attaching points, the saddle splits off the segment of the arc component between its attaching points to form a new circle component:

$$y_0 < s\overline{\alpha} < x_1\beta_1 < \cdots < x_n\beta_n < s\overline{\alpha} < y_1 \qquad \longrightarrow \qquad y_0 < s\alpha < y_1,\ (x_1\beta_1 < \cdots < x_n\beta_n < s\alpha). \tag{7}$$

For a saddle of type C_-^α, we flip the direction of the arrow in (7).

For a saddle s of type C_+^C, which attaches to the *same* side of a circle component at its attaching points, the saddle splits off the segment of the circle component between its attaching points to form a new circle component:

$$(s\overline{\alpha} < x_1\beta_1 < \cdots < x_n\beta_n < s\overline{\alpha}) \longrightarrow (s\alpha),\ (x_1\beta_1 < \cdots < x_n\beta_n < s\alpha). \tag{8}$$

For a saddle of type C_-^C, we flip the direction of the arrow in (8).

For a saddle s of type $W_\pm$, which attaches to *opposite* sides of the arc component at its attaching points, the saddle flips the orientation of the segment of the arc component between its attaching points:

$$y_0 < s\alpha < x_1\beta_1 < \cdots < x_n\beta_n < s\overline{\alpha} < y_1 \longrightarrow y_0 < s\overline{\alpha} < x_n\overline{\beta}_n < \cdots < x_1\overline{\beta}_1 < s\alpha < y_1.$$

This segment forms a single loop around the annulus, so flipping its orientation changes the winding number by two. Given our convention that *counterclockwise* winding is *positive*, we see that if $\alpha = R$ then s *lowers* the winding number by two and hence is type W_-, and if $\alpha = L$ then s *raises* the winding number by two and hence is type W_+.

The following two lemmas describe restrictions on the possible pairs of saddles:

Lemma 5.1. *A pair of saddles s and t of the following form is not possible:*

$$s\alpha_1 < t\beta < t\overline{\beta} < s\alpha_2,$$

Proof. If we glue the attaching points of the saddle s together then the segment between them forms a circle, and the saddle t cannot attach to points on opposite sides of this circle. □

Lemma 5.2. *Pairs of saddles s and t of the following forms are not possible:*

$$s\overline{\alpha} < t\alpha < s\alpha < t\alpha, \qquad s\alpha < t\overline{\alpha} < s\overline{\alpha} < t\alpha, \qquad s\alpha < t\alpha < s\alpha < t\overline{\alpha}.$$

Proof. These planar tangles make up three corners of a square of saddles:

$$\begin{array}{ccc} s\overline{\alpha} < t\alpha < s\alpha < t\alpha & \overset{s}{\longleftrightarrow} & s\alpha < t\overline{\alpha} < s\overline{\alpha} < t\alpha \\ \updownarrow t & & \updownarrow t \\ s\overline{\alpha} < t\overline{\alpha},\ (s\alpha < t\overline{\alpha}) & \overset{s}{\longleftrightarrow} & s\alpha < t\alpha < s\alpha < t\overline{\alpha}. \end{array}$$

Such a square is not possible, since the saddles s and t connect the arc component to opposite sides of the circle component $(s\alpha < t\overline{\alpha})$ in the planar tangle in the bottom left corner. □

6. Squares of saddles

Recall that the differential $\partial_{T^+} = \partial_T^0 + \partial_T^+$ is the sum of a term ∂_T^0 corresponding to single saddles and a term ∂_T^+ corresponding to pairs of successive saddles in the cube of resolutions of T. Each pair of successive saddles belongs to a unique commuting square of saddles $\square$ of the following form:

$$\begin{array}{ccc} \square_{00} & \xrightarrow{\square_T} & \square_{01} \\ \downarrow \square_L & & \downarrow \square_R \\ \square_{10} & \xrightarrow{\square_B} & \square_{11}, \end{array}$$

where $\square_{00}$, $\square_{01}$, $\square_{10}$, and $\square_{11}$ indicate planar tangles at the corners of the square, and $\square_T$, $\square_B$, $\square_L$, and $\square_R$ indicate saddles at the top, bottom, left, and right sides of the square. The square contains two pairs of successive saddles $(\square_T, \square_R)$ and $(\square_L, \square_B)$, each of which contributes to ∂_T^+:

$$(\square_T, \square_R) \rightsquigarrow \tilde{T}^L(\square_R)Q(\square_T) + Q(\square_R)\tilde{T}^L(\square_T),$$

$$(\square_L, \square_B) \rightsquigarrow \tilde{T}^L(\square_B)Q(\square_L) + Q(\square_B)\tilde{T}^L(\square_L).$$

We define a map $\partial_T^+(\square)$ that gives the net contribution of the square $\square$ to ∂_T^+ by summing the contributions of $(\square_T, \square_R)$ and $(\square_L, \square_B)$:

$$\partial_T^+(\square) = \tilde{T}^L(\square_R)Q(\square_T) + Q(\square_R)\tilde{T}^L(\square_T) + \tilde{T}^L(\square_B)Q(\square_L) + Q(\square_B)\tilde{T}^L(\square_L).$$

We can then express ∂_T^+ as a sum over all the squares in the cube of resolutions of T:

$$\partial_T^+ = \sum_{\square} \partial_T^+(\square).$$

We will show that the map $\partial_T^+(\square)$ vanishes unless the square $\square$ is one of several special types. We say that a square $\square$ is *interleaved*, or type $I_\pm$, if it has one of the following forms:

$$\begin{array}{ccc} sR < tR < sL < tL & \xrightarrow[s]{W_-} & sL < tL < sR < tL \\ {\scriptstyle t}\big\downarrow{\scriptstyle W_-} & & {\scriptstyle t}\big\downarrow{\scriptstyle C_+^R} \\ sR < tL < sR < tR & \xrightarrow[s]{C_+^L} & sL < tR,\ (sR < tR) \\ & I_+ & \end{array} \qquad \begin{array}{ccc} sR < tL,\ (sL < tL) & \xrightarrow[s]{C_-^R} & sL < tR < sL < tL \\ {\scriptstyle t}\big\downarrow{\scriptstyle C_-^L} & & {\scriptstyle t}\big\downarrow{\scriptstyle W_-} \\ sR < tR < sL < tR & \xrightarrow[s]{W_-} & sL < tL < sR < tR \\ & I_- & \end{array}$$

For example, the cube of resolutions in Figure 1 is an interleaved square of type I_+. We say that a square $\square$ is *nested*, or type $N_\pm^\beta$, if it has one of the following forms:

$$\begin{array}{ccc} sR < t\beta < t\beta < sL & \xrightarrow[s]{W_-} & sL < t\overline{\beta} < t\overline{\beta} < sR \\ {\scriptstyle t}\big\downarrow{\scriptstyle C_+^{\overline{\beta}}} & & {\scriptstyle t}\big\downarrow{\scriptstyle C_+^{\beta}} \\ sR < t\overline{\beta} < sL,\ (t\overline{\beta}) & \xrightarrow[s]{W_-} & sL < t\beta < sR,\ (t\beta) \\ & N_+^\beta & \end{array} \qquad \begin{array}{ccc} sR < t\overline{\beta} < sL,\ (t\overline{\beta}) & \xrightarrow[s]{W_-} & sL < t\beta < sR,\ (t\beta) \\ {\scriptstyle t}\big\downarrow{\scriptstyle C_-^{\overline{\beta}}} & & {\scriptstyle t}\big\downarrow{\scriptstyle C_-^{\beta}} \\ sR < t\beta < t\beta < sL & \xrightarrow[s]{W_-} & sL < t\overline{\beta} < t\overline{\beta} < sR \\ & N_-^\beta & \end{array}$$

We say that a square $\square$ is *disjoint*, or type D, if it has one of the following forms:

$$\begin{array}{ccc} t\overline{\beta} < t\overline{\beta} < sR < sL & \xrightarrow[s]{W_-} & t\overline{\beta} < t\overline{\beta} < sL < sR \\ {\scriptstyle t}\big\downarrow{\scriptstyle C_+^{\beta}} & & {\scriptstyle t}\big\downarrow{\scriptstyle C_+^{\beta}} \\ t\beta < sR < sL,\ (t\beta) & \xrightarrow[s]{W_-} & t\beta < sL < sR,\ (t\beta) \end{array} \qquad \begin{array}{ccc} t\beta < sR < sL,\ (t\beta) & \xrightarrow[s]{W_-} & t\beta < sL < sR,\ (t\beta) \\ {\scriptstyle t}\big\downarrow{\scriptstyle C_-^{\beta}} & & {\scriptstyle t}\big\downarrow{\scriptstyle C_-^{\beta}} \\ t\overline{\beta} < t\overline{\beta} < sR < sL & \xrightarrow[s]{W_-} & t\overline{\beta} < t\overline{\beta} < sL < sR \end{array}$$

$$\begin{array}{ccc} sR < sL < t\overline{\beta} < t\overline{\beta} & \xrightarrow[s]{W_-} & sL < sR < t\overline{\beta} < t\overline{\beta} \\ {\scriptstyle t}\big\downarrow{\scriptstyle C_+^{\beta}} & & {\scriptstyle t}\big\downarrow{\scriptstyle C_+^{\beta}} \\ sR < sL < t\beta,\ (t\beta) & \xrightarrow[s]{W_-} & sL < sR < t\beta,\ (t\beta) \end{array} \qquad \begin{array}{ccc} sR < sL < t\beta,\ (t\beta) & \xrightarrow[s]{W_-} & sL < sR < t\beta,\ (t\beta) \\ {\scriptstyle t}\big\downarrow{\scriptstyle C_-^{\beta}} & & {\scriptstyle t}\big\downarrow{\scriptstyle C_-^{\beta}} \\ sR < sL < t\overline{\beta} < t\overline{\beta} & \xrightarrow[s]{W_-} & sL < sR < t\overline{\beta} < t\overline{\beta} \end{array}$$

Lemma 6.1. *We have $\partial_T^+(\square) = 0$ unless $\square$ is interleaved or nested.*

Proof. For a pair of successive saddles $(\square_T, \square_R)$ or $(\square_L, \square_B)$ in a square $\square$ to give a nonzero contribution to $\partial_T^+(\square)$, one saddle must be type W_- and one must be type $C_\pm^L$. We let s denote the saddle of type W_- and t denote the saddle of type $C_\pm^L$. One corner of $\square$ must therefore contain both $s\alpha < s\overline{\alpha}$ and $t\overline{\beta} < t\overline{\beta}$. We enumerate the six possible orderings of the attaching points of s and t for this corner and classify $\square$ for each ordering:

$t\overline{\beta} < t\overline{\beta} < s\alpha < s\overline{\alpha}$	D
$t\overline{\beta} < s\alpha < t\overline{\beta} < s\overline{\alpha}$	I_- if $\alpha = \beta = R$, I_+ if $\alpha = \beta = L$, not possible if $\alpha = \overline{\beta}$ by Lemma 5.2
$t\overline{\beta} < s\alpha < s\overline{\alpha} < t\overline{\beta}$	not possible by Lemma 5.1
$s\alpha < t\overline{\beta} < t\overline{\beta} < s\overline{\alpha}$	$N_\pm^{\overline{\beta}}$ if $\alpha = R$, $N_\pm^{\beta}$ if $\alpha = L$
$s\alpha < t\overline{\beta} < s\overline{\alpha} < t\overline{\beta}$	I_- if $\alpha = \overline{\beta} = R$, I_+ if $\alpha = \overline{\beta} = L$, not possible if $\alpha = \beta$ by Lemma 5.2
$s\alpha < s\overline{\alpha} < t\overline{\beta} < t\overline{\beta}$	D

If $\square$ is disjoint then $\partial_T^+(\square) = 0$, since the contributions from $(\square_T, \square_R)$ and $(\square_L, \square_B)$ cancel. $\square$

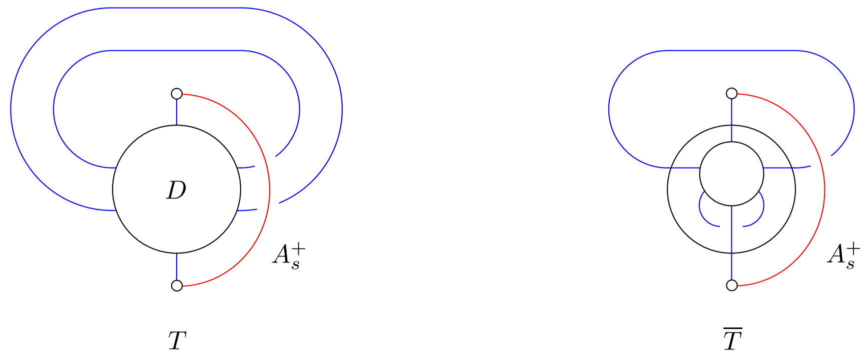

FIGURE 3. (Left) A tangle diagram T in standard position and the overpass arc A_s^+. All the crossings of T are contained in the disk D. (Right) We flip the outermost loop of T under the annulus and then perform an isotopy of the annulus to obtain a tangle diagram $\overline{T}$ in standard position with one additional crossing.

7. Proof of the main theorem

We are now ready to outline the proof of Theorem 1.1 from the Introduction, which we restate here:

Theorem 7.1. *The chain complex $(C_{T^\pm}, \partial_{T^\pm})$ is homotopy equivalent to the chain complex for the reduced Khovanov homology of the link diagram $T^\pm$.*

We construct the homotopy equivalence using the following lemma:

Lemma 7.2 (Reduction Lemma)**.** *If (C, ∂) is a chain complex such that $C = A \oplus B \oplus B$ and $\partial : C \to C$ has the form*

$$\partial = \begin{pmatrix} \partial_A & \beta_1 & \beta_2 \\ \alpha_1 & \partial_B & 0 \\ \alpha_2 & \mathbb{1}_B & \partial_B \end{pmatrix}$$

relative to this decomposition, then $(A, \partial_A + \beta_1\alpha_2)$ is a chain complex homotopy equivalent to (C, ∂).

Proof. The fact that $\partial^2 = 0$ implies that $(\partial_A + \beta_1\alpha_2)^2 = 0$, so $(A, \partial_A + \beta_1\alpha_2)$ is a chain complex. Define linear maps $F : C \to A$, $G : A \to C$ and $H : C \to C$ by

$$F = \begin{pmatrix} \mathbb{1}_A & 0 & \beta_1 \end{pmatrix}, \qquad G = \begin{pmatrix} \mathbb{1}_A \\ \alpha_2 \\ 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathbb{1}_B \\ 0 & 0 & 0 \end{pmatrix}.$$

The fact that $\partial^2 = 0$ implies that F and G are chain maps. We find

$$GF = \mathbb{1}_C + \partial H + H\partial, \qquad FG = \mathbb{1}_A,$$

so F and G are homotopy equivalences. □

We choose a disk D in the annulus $S^1 \times [0, 1]$ as shown in Figure 3 and say that a tangle diagram T is in *standard position* if all the crossings of T are contained in D. Given a tangle diagram T in standard position, we define the *loop number* $\ell(T)$ as the number of times T crosses the overpass arc A_s^+ shown in Figure 3.

Proof of Theorem 7.1. We prove the claim for T^+; the proof for T^- is similar. By performing an isotopy of the annulus and an isotopy of the overpass arc, we can reduce to the case where the tangle diagram T is in standard position and the overpass arc is A_s^+. The latter isotopy is always possible due to the condition we imposed in Section 3 that the image of the overpass arc under the projection $S^1 \times [0, 1] \to S^1$ must be a contractible loop, and the Khovanov chain complexes for the link diagrams before and after the isotopies are homotopy equivalent. We prove the claim by induction on the loop number k.

For the base case $k = 0$, note that if T is a tangle diagram in standard position with loop number $\ell(T) = 0$, then the chain complex $(C_{T^+}, \partial_{T^+}) = (C_{T^+}, \partial_T^0)$ is the Khovanov chain complex of $T^+ := T \cup A_s^+$ with marked point $(b_0, 0)$ or $(b_0, 1)$, so the claim is trivially true.

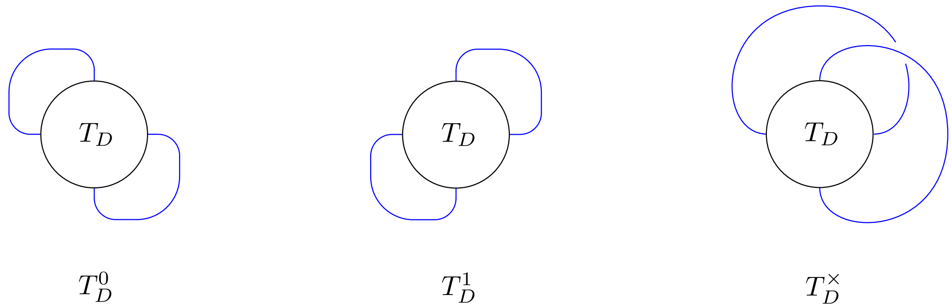

FIGURE 4. Link diagrams T_D^0, T_D^1, and $T_D^\times$ are constructed by closing a 2-tangle diagram T_D in the disk.

For the induction step, assume the claim is true for each tangle diagram T in standard position with loop number $\ell(T) < k$, and consider a tangle diagram T in standard position with loop number $\ell(T) = k$. As shown in Figure 3, we can flip the outermost loop of T under the annulus and then perform an isotopy of the annulus to obtain a tangle diagram $\overline{T}$ in standard position with one more crossing than T and loop number $\ell(\overline{T}) = k-1$. Since the link diagrams $T^+ := T \cup A_s^+$ and $\overline{T}^+ := \overline{T} \cup A_s^+$ describe isotopic links in S^3, the Khovanov chain complexes of T^+ and $\overline{T}^+$ are homotopy equivalent. The induction hypothesis implies the Khovanov chain complex of $\overline{T}^+$ is homotopy equivalent to $(C_{\overline{T}^+}, \partial_{\overline{T}^+})$. We will show that $(C_{\overline{T}^+}, \partial_{\overline{T}^+})$ satisfies the hypotheses of the Reduction Lemma 7.2, which we apply to obtain a homotopy equivalent chain complex $(C_{red}, \partial_{red})$. To complete the proof, we will show that $(C_{red}, \partial_{red}) = (C_{T^+}, \partial_{T^+})$, so $(C_{T^+}, \partial_{T^+})$ is a chain complex, and the string of homotopy equivalences implies that $(C_{T^+}, \partial_{T^+})$ is homotopy equivalent to the Khovanov chain complex of T^+. □

It remains to show that the *induced chain complex* $(C_{\overline{T}^+}, \partial_{\overline{T}^+})$ for $\overline{T}$ satisfies the hypotheses of the Reduction Lemma 7.2 and that $(C_{red}, \partial_{red}) = (C_{T^+}, \partial_{T^+})$.

Remark 7.3. We can use the notion of standard position to describe an interesting relationship between our chain complex and the symplectic interpretation of Khovanov homology due to Hedden, Herald, Hogancamp, and Kirk that was discussed in the Introduction. In [6] they consider link diagrams T_D^0 and T_D^1 obtained by closing a 2-tangle diagram T_D in the disk as shown in Figure 4. They use the tangle diagram T_D to construct an object of the twisted Fukaya category of the character variety $R^*(S^2, 4)$, which in turn is used to construct chain complexes for the link diagrams T_D^0 and T_D^1. These chain complexes turn out to be precisely the Khovanov chain complexes for T_D^0 and T_D^1.

Consider a 1-tangle diagram T in the annulus in standard position with loop number $\ell(T) = 1$. If we restrict T to the disk D shown in Figure 3, we obtain a 2-tangle diagram T_D in the disk. We close T with the overpass arc A_s^+ shown in Figure 3 to obtain the link diagram $T^+ := T \cup A_s^+$, which is identical to the link diagram $T_D^\times$ shown in Figure 4 that is obtained by closing T_D with a 2-tangle diagram with a single crossing. In [2] we apply methods described in [6] to construct a chain complex for $T_D^\times$ via the twisted Fukaya category of $R^*(S^2, 4)$, and we show that the resulting chain complex is precisely our chain complex $(C_{T^+}, \partial_{T^+})$, provided one properly assigns certain gradings to generators of the hom spaces of the Fukaya category. As explained in the Introduction, our chain complex $(C_{T^+}, \partial_{T^+})$ was predicted via the Fukaya category of $R^*(T^2, 2)$. The fact that the chain complexes for $T_D^\times$ and T^+ constructed via $R^*(S^2, 4)$ and $R^*(T^2, 2)$ are identical appears to be due to a close relationship between the Fukaya categories of these character varieties. This relationship is discussed in [2], which shows, for example, that $R^*(S^2, 4)$ is a symplectic submanifold of $R^*(T^2, 2)$.

8. INDUCED CHAIN COMPLEX $(C_{\overline{T}^+},\, \partial_{\overline{T}^+})$

We begin by analyzing the structure of the induced chain complex $(C_{\overline{T}^+},\, \partial_{\overline{T}^+})$ for the tangle diagram $\overline{T}$. We first consider the vector space $C_{\overline{T}^+}$. For each planar tangle p in the cube of resolutions for T, there are two *induced planar tangles* that we denote $[p]_0$ and $[p]_1$ in the cube of resolutions for $\overline{T}$, which are obtained from p by resolving the additional crossing of $\overline{T}$ using the 0-resolution or 1-resolution. Thus

$$C_{\overline{T}^+} = \bigoplus_{i\in I}\left(C^+_{[T_i]_0} \oplus C^+_{[T_i]_1}\right) = \bigoplus_{i\in I}\overline{C}^+_{T_i},$$

where we have grouped the vector spaces for each pair of induced planar tangles into a single space:

$$\overline{C}^+_p := C^+_{[p]_0} \oplus C^+_{[p]_1}.$$

For simplicity, we will initially ignore the bigrading structure of the various vector spaces. We denote the underlying ungraded vector space of a bigraded vector space V^+ as V. We will show that each vector space $\overline{C}_p$ can be decomposed as

$$\overline{C}_p = A_p \oplus B_p \oplus B_p,$$

where

$$A_p = C_p \tag{9}$$

and either $B_p = C_p$ or $B_p = 0$. We thus obtain a decomposition of $C_{\overline{T}^+}$:

$$C_{\overline{T}^+} = \bigoplus_p \overline{C}_p = A \oplus B \oplus B, \qquad A = \bigoplus_p A_p, \qquad B = \bigoplus_p B_p. \tag{10}$$

Next we consider the differential $\partial_{\overline{T}^+}$ for the induced chain complex, which is constructed from linear maps corresponding to saddles in the cube of resolutions of $\overline{T}$. The saddles are of two different types:

- For each planar tangle p in the cube of resolutions of T, there is a saddle $n_p : [p]_0 \to [p]_1$ in the cube of resolutions of $\overline{T}$ due to the additional crossing of $\overline{T}$. We call these *ancillary saddles*. We denote a linear map corresponding to an ancillary saddle using the subscript 10, as for example $Q_{10}(n_p) := Q(n_p)$. Examples of ancillary saddles are shown in Figures 6, 7, and 8.
- For each saddle $s : p \to q$ in the cube of resolutions of T, there are two *induced saddles* $[s]_0 : [p]_0 \to [q]_0$ and $[s]_1 : [p]_1 \to [q]_1$ in the cube of resolutions of $\overline{T}$. We denote linear maps corresponding to induced saddles $[s]_0$ and $[s]_1$ using the subscripts 00 and 11, as for example $Q_{00}(s) := Q([s]_0)$ and $Q_{11}(s) := Q([s]_1)$. Examples of induced saddles are shown in Figure 5.

It is useful to express $\partial_{\overline{T}^+}$ as

$$\partial_{\overline{T}^+} := \partial^0_{\overline{T}} + \partial^\pm_{\overline{T}} = \overline{\partial}^0 + \overline{\partial}^1 + \overline{\partial}^2,$$

where $\overline{\partial}^0$, $\overline{\partial}^1$, and $\overline{\partial}^2$ collect together the terms corresponding to different types of saddles.

We define $\overline{\partial}^0$ to be the part of $\partial^0_{\overline{T}}$ corresponding to the ancillary saddles:

$$\overline{\partial}^0 = \sum_p T_{10}(n_p),$$

where the sum is taken over all planar tangles in the cube of resolutions of T.

Given a saddle $s : p \to q$ in the cube of resolutions of T, we define a map

$$\overline{\partial}^1(s) = T_{00}(s) + T_{11}(s) + \tilde{T}^L_{10}(n_q)Q_{00}(s) + Q_{10}(n_q)\tilde{T}^L_{00}(s) + \tilde{T}^L_{11}(s)Q_{10}(n_p) + Q_{11}(s)\tilde{T}^L_{10}(n_p), \tag{11}$$

which collects the terms of $\partial^0_{\overline{T}}$ corresponding to the induced saddles of s, as well as the terms of $\partial^\pm_{\overline{T}}$ corresponding to pairs of successive saddles, one of which is an induced saddle of s and one of which is an ancillary saddle. We define

$$\overline{\partial}^1 = \sum_s \overline{\partial}^1(s),$$

where the sum is taken over all saddles in the cube of resolutions of T.

Given a pair of successive saddles (s, t) in the cube of resolutions of T, we define a map

$$\overline{\partial}^2(s,t) = \tilde{T}^L_{00}(t)Q_{00}(s) + Q_{00}(t)\tilde{T}^L_{00}(s) + \tilde{T}^L_{11}(t)Q_{11}(s) + Q_{11}(t)\tilde{T}^L_{11}(s), \tag{12}$$

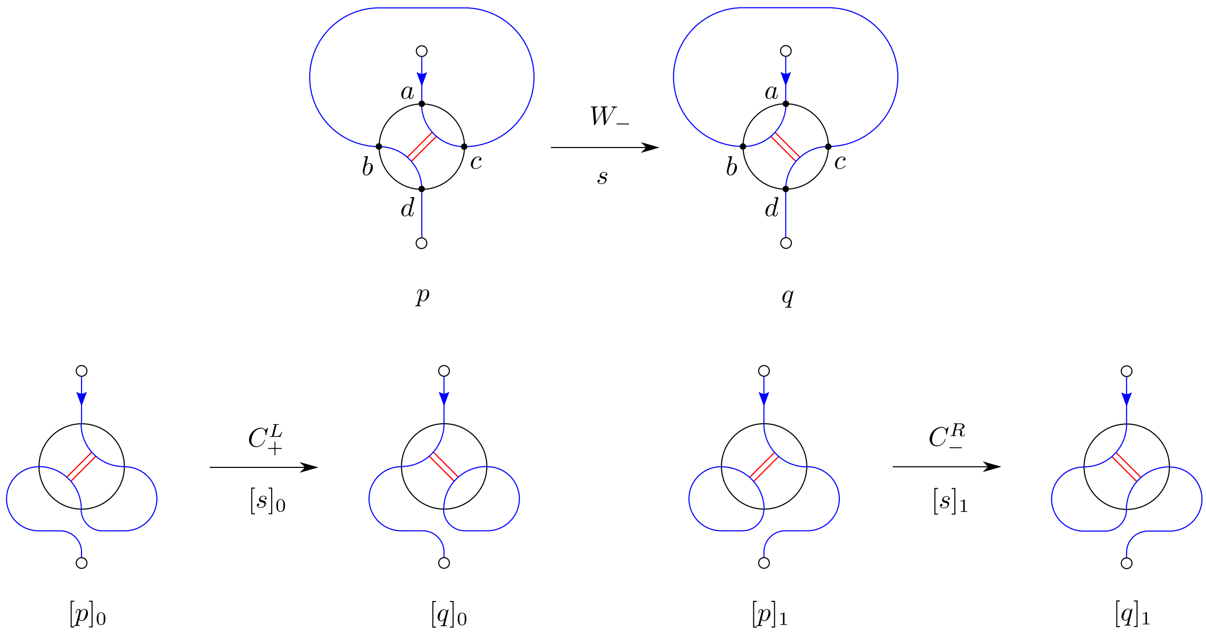

FIGURE 5. Example type W_- saddle $s : p \to q$ in the cube of resolutions of T and the corresponding induced saddles $[s]_0 : [p]_0 \to [q]_0$ and $[s]_1 : [p]_1 \to [q]_1$ in the cube of resolutions of $\overline{T}$.

which collects the terms of $\partial^{\pm}_{\overline{T}}$ corresponding to pairs of successive saddles, one of which is an induced saddle of s and one of which is an induced saddle of t. Given a square $\square$ in the cube of resolutions of T, we define a map

$$\overline{\partial}^2(\square) = \overline{\partial}^2(\square_T, \square_R) + \overline{\partial}^2(\square_L, \square_B), \tag{13}$$

which collects the terms of $\partial^{\pm}_{\overline{T}}$ corresponding to squares of saddles induced by $\square$. We define

$$\overline{\partial}^2 = \sum_{\square} \overline{\partial}^2(\square),$$

where the sum is taken over all squares in the cube of resolutions of T.

In Section 9, we compute the map $T_{10}(n_p)$ corresponding to the ancillary saddle n_p for each planar tangle p in the cube of resolutions of T and show that

$$\overline{\partial}^0 := \sum_p T_{10}(n_p) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mathbb{1}_B & 0 \end{pmatrix} \tag{14}$$

relative to the decomposition $C_{\overline{T}^+} = A \oplus B \oplus B$ in equation (10). We also compute the maps $Q_{10}(n_p)$ and $\tilde{T}^L_{10}(n_p)$ that contribute to $\overline{\partial}^1$.

In Section 10, we compute the map $\overline{\partial}^1(s)$ for each saddle s in the cube of resolutions of T and show it has the form

$$\overline{\partial}^1(s) = \begin{pmatrix} \partial_A(s) & \beta_1(s) & \beta_2(s) \\ \alpha_1(s) & \partial_B(s) & 0 \\ \alpha_2(s) & 0 & \partial_B(s) \end{pmatrix} \tag{15}$$

relative to the decomposition $C_{\overline{T}^+} = A \oplus B \oplus B$ in equation (10). We also compute the maps $Q_{00}(s)$, $Q_{11}(s)$, $\tilde{T}^L_{00}(s)$, and $\tilde{T}^L_{11}(s)$ that contribute to $\overline{\partial}^2$.

In Section 11, we compute the map $\overline{\partial}^2(\square)$ for each square $\square$ in the cube of resolutions of T and show it has the form

$$\overline{\partial}^2(\square) = \begin{pmatrix} \partial_A(\square) & 0 & \beta_2(\square) \\ \alpha_1(\square) & \partial_B(\square) & 0 \\ 0 & 0 & \partial_B(\square) \end{pmatrix}. \tag{16}$$

relative to the decomposition $C_{\overline{T}^+} = A \oplus B \oplus B$ in equation (10).

From equations (14), (15), and (16), it follows that ∂_T^+ has the form

$$\partial_{\overline{T}^+} = \begin{pmatrix} \partial_A & \beta_1 & \beta_2 \\ \alpha_1 & \partial_B & 0 \\ \alpha_2 & \mathbb{1}_B & \partial_B \end{pmatrix},$$

so we can apply the Reduction Lemma 7.2 to obtain a chain complex $(C_{red}, \partial_{red})$. From equation (9) it follows that

$$C_{red} := A = \sum_p A_p = \sum_p C_p = C_{T^+} \tag{17}$$

as ungraded vector spaces, and in Appendix A we show that equation (17) also holds when we include the bigrading structure. From equations (14), (15), and (16), it follows that

$$\partial_{red} := \partial_A + \beta_1\alpha_2 = \sum_s \partial_A(s) + \sum_\square \partial_{red}(\square), \tag{18}$$

where we have defined

$$\partial_{red}(\square) := \partial_A(\square) + \beta_1(\square_R)\alpha_2(\square_T) + \beta_1(\square_B)\alpha_2(\square_L). \tag{19}$$

In Section 10 we show

$$\partial_A(s) = T(s). \tag{20}$$

In Section 11 we show

$$\partial_{red}(\square) = \partial_T^+(\square). \tag{21}$$

We substitute equations (20) and (21) into equation (18) to obtain

$$\partial_{red} = \sum_s T(s) + \sum_\square \partial_T^+(\square) = \partial_T^0 + \partial_T^+ = \partial_{T^+}, \tag{22}$$

where we have expressed ∂_T^+ as a sum over squares as described in Section 6. Equations (17) and (22) show that $(C_{red}, \partial_{red}) = (C_{T^+}, \partial_{T^+})$, thus completing the proof of Theorem 7.1.

9. Induced planar tangles and ancillary saddles

Given a planar tangle diagram T in standard position, we let a and d denote the points where T intersects the disk D on the top and bottom, and we let b and c denote the points where the outermost loop of T intersects D on the left and right. We classify the planar tangles in the cube of resolutions of T into three types:

- We say a planar tangle p is type L if b and c lie on the arc component of p and $b < c$, so the outermost loop of p belongs to the arc component and is oriented clockwise.
- We say a planar tangle p is type R if b and c lie on the arc component of p and $c < b$, so the outermost loop of p belongs to the arc component and is oriented counterclockwise.
- We say a planar tangle p is type C if b and c lie on a circle component of p, so the outermost loop of p belongs to a circle component.

Examples of type L, R, and C planar tangles are shown in Figures 6, 7, and 8. For each planar tangle p in the cube of resolutions of T, we specify a decomposition $\overline{C}_p = A_p \oplus B_p \oplus B_p$ and we determine the maps $\tilde{T}^L_{10}(n_p)$ and $Q_{10}(n_p)$ corresponding to the ancillary saddle $n_p : [p]_0 \to [p]_1$ that contribute to $\overline{\partial}^1$. We will use the notation $p(L)$, $p(R)$, and $p(C)$ to indicate that a planar tangle p in the cube of resolutions of T is type L, R, or C.

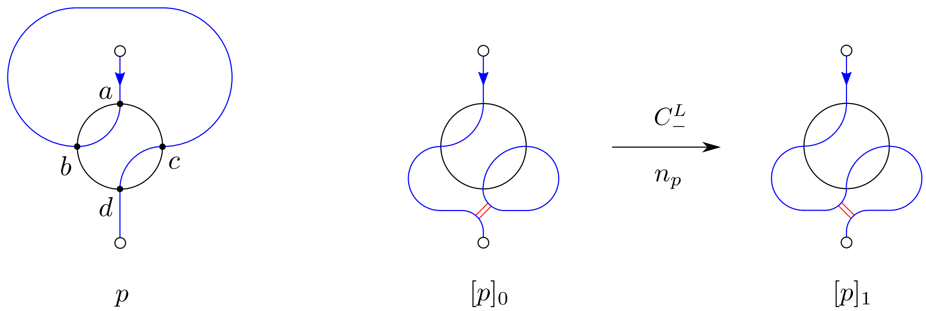

FIGURE 6. Example type L planar tangle p in the cube of resolutions of T and the corresponding induced planar tangles $[p]_0$ and $[p]_1$ and ancillary saddle $n_p : [p]_0 \to [p]_1$ in the cube of resolutions of $\overline{T}$.

9.1. **Type L planar tangles.** A planar tangle p of type L has the form $a < b < c < d$, as shown in Figure 6. The ancillary saddle $n_p : [p]_0 \to [p]_1$ merges a circle with the left side of the arc component of $[p]_0$, and is thus type C_-^L:

$$n_p : [p]_0 \to [p]_1 : \qquad a < b < n_pL,\ (d < c < n_pL) \qquad \longrightarrow \qquad a < b < n_pR < d < c < n_pR.$$

So as ungraded vector spaces we have

$$C_{[p]_0} = C_p \otimes A^{dc}, \qquad\qquad C_{[p]_1} = C_p,$$

where $C_p = A^{\otimes c(p)}$ and $c(p)$ is the circle number of p. The superscript on A^{dc} indicates that this factor corresponds to the additional circle component of $[p]_0$ that contains points d and c. We decompose $\overline{C}_p$ as

$$\overline{C}_p := C_{[p]_0} \oplus C_{[p]_1} = A_p \oplus B_p^1 \oplus B_p^2,$$

where

$$A_p := C_p = C_p \otimes x \subset C_{[p]_0}, \qquad B_p^1 := C_p = C_p \otimes e \subset C_{[p]_0}, \qquad B_p^2 := C_p = C_{[p]_1}. \tag{23}$$

The superscripts on B_p^1 and B_p^2 indicate the ordering of these summands in the decomposition of $\overline{C}_p$. In the definitions of A_p and B_p^1, we have used the fact that C_p is canonically isomorphic to the subspaces $C_p \otimes x$ and $C_p \otimes e$ of $C_{[p]_0} = C_p \otimes A^{dc}$.

Lemma 9.1. *For a planar tangle p of type L, the ancillary saddle $n_p : [p]_0 \to [p]_1$ contributes the maps*

$$T_{10}(n_p) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mathbb{1}_{C_p} & 0 \end{pmatrix}, \qquad\qquad \tilde{T}_{10}^L(n_p) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathbb{1}_{C_p} & 0 & 0 \end{pmatrix}.$$

Proof. Recall that the ancillary saddle n_p is type C_-^L. The corresponding linear maps $T_{10}(n_p)$ and $\tilde{T}_{10}^L(n_p)$ are nonzero only in the block $C_{[p]_0} = C_p \otimes A^{dc} \to C_{[p]_1} = C_p$, and their restrictions to that block are $\mathbb{1}_{C_p} \otimes \dot{\epsilon}$ and $\mathbb{1}_{C_p} \otimes \epsilon$, respectively. From the definitions of A_p, B_p^1, and B_p^2 in equation (23), it follows that $\mathbb{1}_{C_p} \otimes \dot{\epsilon}$ is the identity map $\mathbb{1}_{C_p}$ from $B_p^1 = C_p$ to $B_p^2 = C_p$, and $\mathbb{1}_{C_p} \otimes \epsilon$ is the identity map $\mathbb{1}_{C_p}$ from $A_p = C_p$ to $B_p^2 = C_p$. □

9.2. **Type R planar tangles.** A planar tangle p of type R has the form $a < c < b < d$, as shown in Figure 7. The ancillary saddle $n_p : [p]_0 \to [p]_1$ splits a circle from the right side of the arc component of $[p]_0$, and is thus type C_+^R:

$$n_p : [p]_0 \to [p]_1 : \qquad a < c < n_pL < d < b < n_pL \qquad \longrightarrow \qquad a < c < n_pR,\ (d < b < n_pR).$$

So as ungraded vector spaces we have

$$C_{[p]_0} = C_p, \qquad\qquad C_{[p]_1} = C_p \otimes A^{db},$$

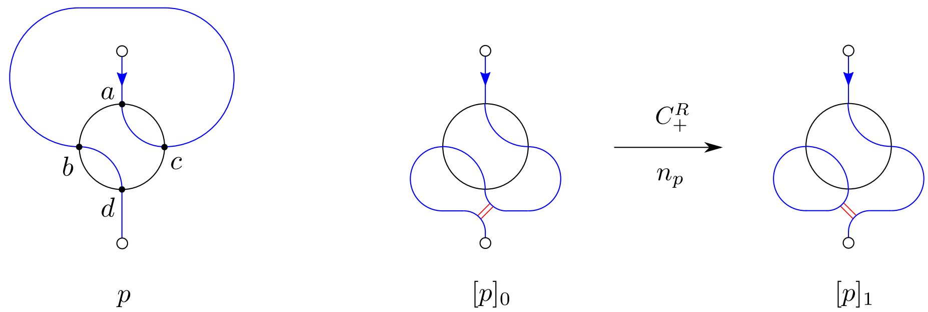

FIGURE 7. Example type R planar tangle p in the cube of resolutions of T and the corresponding induced planar tangles $[p]_0$ and $[p]_1$ and ancillary saddle $n_p : [p]_0 \to [p]_1$ in the cube of resolutions of $\overline{T}$.

where $C_p = A^{\otimes c(p)}$ and $c(p)$ is the circle number of p. The superscript on A^{db} indicates that this factor corresponds to the additional circle component of $[p]_1$ that contains points d and b. We decompose $\overline{C}_p$ as

$$\overline{C}_p := C_{[p]_0} \oplus C_{[p]_1} = A_p \oplus B_p^1 \oplus B_p^2,$$

where

$$A_p := C_p = C_p \otimes e \subset C_{[p]_1}, \qquad B_p^1 := C_p = C_{[p]_0}, \qquad B_p^2 := C_p = C_p \otimes x \subset C_{[p]_1}. \tag{24}$$

The superscripts on B_p^1 and B_p^2 indicate the ordering of these summands in the decomposition of $\overline{C}_p$. In the definitions of A_p and B_p^2, we have used the fact that C_p is canonically isomorphic to the subspaces $C_p \otimes x$ and $C_p \otimes e$ of $C_{[p]_1} = C_p \otimes A^{db}$.

Lemma 9.2. *For p of type R, the ancillary saddle n_p contributes the map*

$$T_{10}(n_p) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mathbb{1}_{C_p} & 0 \end{pmatrix}.$$

Proof. Recall that the ancillary saddle n_p is type C_+^R. The corresponding linear map $T_{10}(n_p)$ is nonzero only in the block $C_{[p]_0} = C_p \to C_{[p]_1} = C_p \otimes A^{db}$, and its restriction to that block is $\mathbb{1}_{C_p} \otimes \dot{\eta}$. From the definitions of A_p and B_p^1 in equation (23), it follows that $\mathbb{1}_{C_p} \otimes \dot{\eta}$ is the identity map $\mathbb{1}_{C_p}$ from $B_p^1 = C_p$ to $B_p^2 = C_p$. □

9.3. **Type C planar tangles.** A planar tangle p of type C has the form $a < d$, $(b < c)$, as shown in Figure 8. The ancillary saddle n_p lowers the winding number by two, and is thus type W_-:

$$n_p : [p]_0 \to [p]_1 : \qquad a < d < n_pR < c < b < n_pL \qquad \longrightarrow \qquad a < d < n_pL < b < c < n_pR.$$

If p has circle number $c(p) = r + 1$ then $[p]_0$ and $[p]_1$ have circle number $c([p]_0) = c([p]_1) = r$. We let A^{bc} denote the vector space factor corresponding to the additional circle component of p that contains b and c, and we define a vector space $W_p = A^{\otimes r}$ corresponding to the remaining circle components of p. As ungraded vector spaces, we have

$$C_p = W_p \otimes A^{bc}, \qquad C_{[p]_0} = W_p = W_p \otimes e \subset C_p, \qquad C_{[p]_1} = W_p = W_p \otimes x \subset C_p.$$

We have used the fact that W_p is canonically isomorphic to the subspaces $W_p \otimes e$ and $W_p \otimes e$ of $W_p \otimes A^{bc}$ to identify $C_{[p]_0}$ and $C_{[p]_1}$ with subspaces of C_p. We decompose $\overline{C}_p$ as

$$\overline{C}_p := C_{[p]_0} \oplus C_{[p]_1} = A_p \oplus B_p^1 \oplus B_p^2,$$

where

$$A_p := C_p = W_p \otimes A^{bc}, \qquad B_p^1 := 0, \qquad B_p^2 := 0. \tag{25}$$

Since we are viewing $C_{[p]_0}$ and $C_{[p]_1}$ as subspaces of $C_p = A_p$, we can define a map

$$\mathbb{1}_{C_{[p]_1} C_{[p]_0}} : A_p \to A_p, \qquad \mathbb{1}_{C_{[p]_1} C_{[p]_0}} = \mathbb{1}_{W_p} \otimes \mathbb{1}_{xe}.$$

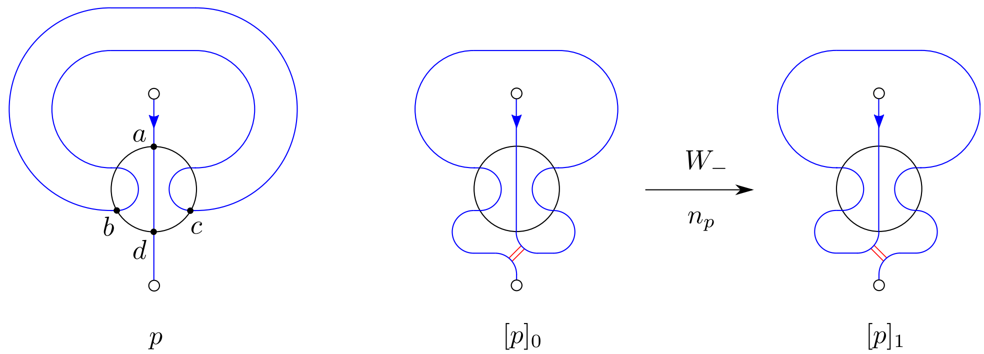

FIGURE 8. Example type C planar tangle p in the cube of resolutions of T and the corresponding induced planar tangles $[p]_0$ and $[p]_1$ and ancillary saddle $n_p : [p]_0 \to [p]_1$ in the cube of resolutions of $\overline{T}$.

We extend $\mathbb{1}_{C_{[p]_1} C_{[p]_0}}$ by zero to obtain a map $A \to A$, where $A = \bigoplus_p A_p$.

Lemma 9.3. *For a planar tangle p of type C, the ancillary saddle $n_p : [p]_0 \to [p]_1$ contributes the map*

$$Q_{10}(n_p) = \begin{pmatrix} \mathbb{1}_{C_{[p]_1} C_{[p]_0}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. Recall that the ancillary saddle n_p is of type W_-. The corresponding linear map $Q_{10}(n_p)$ is nonzero only in the block $C_{[p]_0} = W_p \otimes e \to C_{[p]_1} = W_p \otimes x$, and its restriction to that block is $\mathbb{1}_{W_p} \otimes \mathbb{1}_{xe} = \mathbb{1}_{C_{[p]_1 [p]_0}}$. The result now follows from the definitions of A_p, B_p^1, and B_p^2 in equation (25). □

9.4. **Summary.** Lemmas 9.1, 9.2, and 9.3, together with the definitions of A_p, B_p^1, and B_p^2 in equations (23), (24), and (25), prove

Lemma 9.4. *We have*

$$\overline{\partial}^0 := \sum_p T_{10}(n_p) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mathbb{1}_B & 0 \end{pmatrix}.$$

10. TERMS $\overline{\partial}^1(s)$

For each saddle $s : p \to q$ in the cube of resolutions of T, we compute the map $\overline{\partial}^1(s)$ defined in equation (11), which collects the terms of $\partial^0_{\overline{T}}$ corresponding to the induced saddles of s, as well as the terms of $\partial^{\pm}_{\overline{T}}$ corresponding to pairs of successive saddles, one of which is an induced saddle of s and one of which is an ancillary saddle. The form of the map $\overline{\partial}^1(s)$ depends on the relative positions of the points b and c and the attaching points of s, so we divide the saddle types $W_\pm$, $C^\beta_\pm$, and $C^C_\pm$ into subtypes according to the ordering of these points as described in Table 1. The following three lemmas describe restrictions on the subtypes of saddles that are actually possible:

Lemma 10.1. *Type $W1_\pm$ saddles are not possible.*

Proof. We will prove the claim for a type $W1_\pm$ saddle $s : p(L) \to q(L)$. We have

$$\begin{array}{llcl} s : p \to q : & a < b < c < s\alpha < s\overline{\alpha} < d & \longrightarrow & a < b < c < s\overline{\alpha} < s\alpha < d \\ [s]_0 : [p]_0 \to [q]_0 : & a < b,\ (c < s\alpha < s\overline{\alpha} < d) & \longrightarrow & a < b,\ (c < s\overline{\alpha} < s\alpha < d). \end{array}$$

So the saddle $[s]_0$ attaches to opposite sides of the circle component $(c < s\alpha < s\overline{\alpha} < d)$ in $[p]_0$, which is not possible. The proof for $s : p(R) \to q(R)$ is similar. □

Lemma 10.2. *Type $W2_+$ saddles $p(R) \to q(L)$ and type $W2_-$ saddles $p(L) \to q(R)$ are not possible.*

$W1_\pm$:	$x_1 < x_2 < sL < sR$	$\longleftrightarrow$	$x_1 < x_2 < sR < sL$	(not possible by Lemma 10.1)
$W2_\pm$:	$sL < x_1 < x_2 < sR$	$\longleftrightarrow$	$sR < x_2 < x_1 < sL$	(restricted by Lemma 10.2)
$W3_\pm$:	$sL < sR < x_1 < x_2$	$\longleftrightarrow$	$sR < sL < x_1 < x_2$	
$W4_\pm$:	$sL < sR$, $(x_1 < x_2)$	$\longleftrightarrow$	$sR < sL$, $(x_1 < x_2)$	
$C1^\beta_\pm$:	$x_1 < x_2 < s\overline{\beta} < s\overline{\beta}$	$\longleftrightarrow$	$x_1 < x_2 < s\beta$, $(s\beta)$	
$C2^\beta_\pm$:	$s\overline{\beta} < x_1 < x_2 < s\overline{\beta}$	$\longleftrightarrow$	$s\beta$, $(x_1 < x_2 < s\beta)$	(restricted by Lemma 10.3)
$C3^\beta_\pm$:	$s\overline{\beta} < s\overline{\beta} < x_1 < x_2$	$\longleftrightarrow$	$s\beta < x_1 < x_2$, $(s\beta)$	
$C4^\beta_\pm$:	$s\overline{\beta} < s\overline{\beta}$, $(x_1 < x_2)$	$\longleftrightarrow$	$s\beta$, $(s\beta)$, $(x_1 < x_2)$	
$C1^C_\pm$:	$(x_1 < x_2 < s\overline{\beta} < s\overline{\beta})$	$\longleftrightarrow$	$(s\beta)$, $(x_1 < x_2 < s\beta)$	
$C2^C_\pm$:	$(x_1 < x_2)$, $(s\overline{\beta} < s\overline{\beta})$	$\longleftrightarrow$	$(x_1 < x_2)$, $(s\beta)$, $(s\beta)$	
$C3^C_\pm$:	$x_1 < x_2$, $(s\overline{\beta} < s\overline{\beta})$	$\longleftrightarrow$	$x_1 < x_2$, $(s\beta)$, $(s\beta)$.	

TABLE 1. Saddle subtypes. Subscripts $+$ and $-$ correspond to saddles $p \to q$ and $p \leftarrow q$. The pair (x_1, x_2) is either (b, c) or (c, b). Saddles of type $W1_\pm$ are not possible by Lemma 10.1, and saddles of type $W2_\pm$ and $C2^\beta_\pm$ are restricted by Lemmas 10.2 and 10.3.

Proof. A type $W2_+$ saddle $p \to q$ flips the orientation of the outermost loop of p from clockwise to counterclockwise, so p must be type L. A type $W2_-$ saddle $p \to q$ flips the orientation of the outermost loop of p from counterclockwise to clockwise, so p must be type R. □

Lemma 10.3. *Type $C2^L_+$ saddles $p(R) \to q(C)$, type $C2^L_-$ saddles $p(C) \to q(R)$, type $C2^R_+$ saddles $p(L) \to q(C)$, and type $C2^R_-$ saddles $p(C) \to q(L)$ are not possible.*

Proof. For a type $C2^\beta_+$ saddle $s : p \to q$, the planar tangle q has a circle component $(x_1 < x_2 < s\beta)$. The saddle s must attach to the *outside* of this circle, so $\beta = L$ if the circle is oriented clockwise and $\beta = R$ if the circle is oriented counterclockwise. The points x_1 and x_2 lie on the boundary of the disk D and the attaching point $s\beta$ lies inside the disk D, so if the circle is oriented clockwise then $(x_1 < x_2 < s\beta) = (b < c < sL)$ and if the circle is oriented counterclockwise then $(x_1 < x_2 < s\beta) = (c < b < sR)$. The proof for a type $C2^\beta_-$ saddle is similar. □

For each of the saddle subtypes in Table 1, we compute the types of the corresponding induced saddles. For example, consider a type $W2_-$ saddle $s : p(R) \to q(L)$, as shown in Figure 5:

$s : p \to q$:	$a < sR < c < b < sL < d$	$\longrightarrow$	$a < sL < b < c < sR < d$
$[s]_0 : [p]_0 \to [q]_0$:	$a < sR < c < d < sR < b$	$\longrightarrow$	$a < sL < b$, $(c < d < sL)$
$[s]_1 : [p]_1 \to [q]_1$:	$a < sR < c$, $(b < d < sR)$	$\longrightarrow$	$a < sL < b < d < sL < c$.

We see that $[s]_0$ is type C^L_+ and $[s]_1$ is type C^R_-. We perform similar calculations for the remaining saddle subtypes and display the results in Table 2.

Using the types of the induced saddles summarized in Table 2, we compute the map $\overline{\partial}^1(s)$ described in equation (11) for each subtype of saddle. We verify that $\overline{\partial}^1(s)$ has the form given in equation (15) and we read off the blocks $\partial_A(s)$, $\beta_1(s)$, and $\alpha_2(s)$. These calculations are straightforward but somewhat lengthy, so we postpone these calculations to Appendix B and summarize the results in Table 2. The expressions for $\partial_A(s)$ in Table 2 prove

Lemma 10.4. *For each saddle s in the cube of resolutions of T, the map $\overline{\partial}^1(s)$ has the form given in equation (15) with $\partial_A(s) = T(s)$.*

s	$p \to q$	$[s]_0$	$[s]_1$	$\partial_A(s)$	$\beta_1(s)$	$\alpha_2(s)$
$W2_-$	$p(R) \to q(L)$	C^L_+	C^R_-	0	$Q(s)$	$Q(s)$
$W2_+$	$p(L) \to q(R)$	C^L_-	C^R_+	0	0	0
$W3_\pm$	$p(L) \to q(L)$	$W_\pm$	$W_\pm$	0	0	0
$W3_\pm$	$p(R) \to q(R)$	$W_\pm$	$W_\pm$	0	0	0
$W4_\pm$	$p(C) \to q(C)$	$W_\pm$	$W_\pm$	0	0	0
$C1^\beta_\pm$	$p(L) \to q(L)$	$C^C_\pm$	$C^{\overline{\beta}}_\pm$	$T(s)$	$\tilde{T}(s)$	0
$C1^\beta_\pm$	$p(R) \to q(R)$	$C^{\overline{\beta}}_\pm$	$C^C_\pm$	$T(s)$	0	$\tilde{T}(s)$
$C2^L_+$	$p(L) \to q(C)$	C^R_-	W_-	$T(s)$	$\tilde{T}^L(s)$	0
$C2^L_-$	$p(C) \to q(L)$	C^R_+	W_+	$T(s)$	0	0
$C2^R_+$	$p(R) \to q(C)$	W_+	C^L_-	$T(s)$	0	0
$C2^R_-$	$p(C) \to q(R)$	W_-	C^L_+	$T(s)$	0	$\tilde{T}^R(s)$
$C3^\beta_\pm$	$p(L) \to q(L)$	$C^\beta_\pm$	$C^\beta_\pm$	$T(s)$	0	0
$C3^\beta_\pm$	$p(R) \to q(R)$	$C^\beta_\pm$	$C^\beta_\pm$	$T(s)$	0	0
$C4^\beta_\pm$	$p(C) \to q(C)$	$C^\beta_\pm$	$C^\beta_\pm$	$T(s)$	0	0
$C1^C_\pm$	$p(C) \to q(C)$	$C^\sigma_\pm$	$C^{\overline{\sigma}}_\pm$	$T(s)$	0	0
$C2^C_\pm$	$p(C) \to q(C)$	$C^C_\pm$	$C^C_\pm$	$T(s)$	0	0
$C3^C_\pm$	$p(L) \to q(L)$	$C^C_\pm$	$C^C_\pm$	$T(s)$	0	0
$C3^C_\pm$	$p(R) \to q(R)$	$C^C_\pm$	$C^C_\pm$	$T(s)$	0	0

TABLE 2. Induced saddle types. For each saddle $s : p \to q$, we list the types of the induced saddles $[s]_0$ and $[s]_1$ and the blocks $\partial_A(s)$, $\beta_1(s)$, and $\alpha_2(s)$ of $\overline{\partial}^1(s)$.

11. SQUARES OF INDUCED SADDLES

Recall that in equation (13) we defined a map $\overline{\partial}^2(\square)$, which collects the terms of $\partial^\pm_T$ corresponding to pairs of successive saddles that are both induced by saddles in $\square$. In this section we prove:

Lemma 11.1. *For each square $\square$ in the cube of resolutions of T, the map $\overline{\partial}^2(\square)$ has the form given in equation (16), and we have*

$$\partial_{red}(\square) := \partial_A(\square) + \beta_1(\square_R)\alpha_2(\square_T) + \beta_1(\square_B)\alpha_2(\square_L) = \partial^\pm_T(\square).$$

We consider separately the cases in which the square is interleaved or nested and cases in which it is not interleaved or nested.

11.1. **Squares that are interleaved or nested.** The form of the map $\overline{\partial}^2(\square)$ depends on the relative positions of the points b and c and the attaching points of the saddles that make up the square, so we divide interleaved and nested squares into subtypes according to the ordering of these points as described in Tables 3 and 4. The squares marked $-$ are not possible, since they would contain a saddle of type $W1_-$, which by Lemma 10.1 is not possible. In enumerating these subtypes we have also applied Lemmas 10.2 and 10.3, which impose restrictions on the possible saddles of type $W2_\pm$ and $C2^\beta_\pm$.

We prove Lemma 11.1 by treating each subtype of square in Tables 3 and 4 as a separate case. We prove three representative cases in detail:

Lemma 11.2. *Lemma 11.1 holds for squares of type $I3_+$.*

I_+	$\square_{00}$	I_-	$\square_{11}$
$-$	$c < b < sR < tR < sL < tL$	$-$	$b < c < sL < tL < sR < tR$
$-$	$sR < c < b < tR < sL < tL$	$-$	$sL < b < c < tL < sR < tR$
$I3_+$:	$sR < tR < c < b < sL < tL$	$I3_-$:	$sL < tL < b < c < sR < tR$
$I4_+$:	$sR < tR < sL < c < b < tL$	$I4_-$:	$sL < tL < sR < b < c < tR$
$I5_+$:	$sR < tR < sL < tL < x_1 < x_2$	$I5_-$:	$sL < tL < sR < tR < x_1 < x_2$
$I6_+$:	$sR < tR < sL < tL$, $(b < c)$	$I6_-$:	$sL < tL < sR < tR$, $(b < c)$

TABLE 3. Subtypes of interleaved squares. For I_+ we define subtypes according to the form of the planar tangle $\square_{00}$. For I_- we define subtypes according to the form of the planar tangle $\square_{11}$. The pair (x_1, x_2) is either (b, c) or (c, b).

N^β_+	$\square_{00}$	N^β_-	$\square_{11}$
$-$	$c < b < sR < t\beta < t\beta < sL$	$-$	$b < c < sL < t\overline{\beta} < t\overline{\beta} < sR$
$N2^\beta_+$:	$sR < c < b < t\beta < t\beta < sL$	$N2^\beta_-$:	$sL < b < c < t\overline{\beta} < t\overline{\beta} < sR$
$N3^L_+$:	$sR < tL < c < b < tL < sL$	$N3^L_-$:	$sL < tR < b < c < tR < sR$
$N4^\beta_+$:	$sR < t\beta < t\beta < c < b < sL$	$N4^\beta_-$:	$sL < t\overline{\beta} < t\overline{\beta} < b < c < sR$
$N5^\beta_+$:	$sR < t\beta < t\beta < sL < x_1 < x_2$	$N5^\beta_-$:	$sL < t\overline{\beta} < t\overline{\beta} < sR < x_1 < x_2$
$N6^\beta_+$:	$sR < t\beta < t\beta < sL$, $(b < c)$	$N6^\beta_-$:	$sL < t\overline{\beta} < t\overline{\beta} < sR$, $(b < c)$

TABLE 4. Subtypes of nested squares. For N^β_+ we define subtypes according to the form of the planar tangle $\square_{00}$. For N^β_- we define subtypes according to the form of the planar tangle $\square_{11}$. The pair (x_1, x_2) is either (b, c) or (c, b).

Proof. A square $\square$ of type $I3_+$ has the form

$$\begin{array}{ccc} sR < tR < c < b < sL < tL & \xrightarrow[s]{W2_-} & sL < b < c < tL < sR < tL \\ {\scriptstyle t}\big\downarrow{\scriptstyle W2_-} & & {\scriptstyle t}\big\downarrow{\scriptstyle C1^R_+} \\ sR < tL < sR < b < c < tR & \xrightarrow[s]{C3^L_+} & sL < b < c < tR,\ (sR < tR). \end{array}$$

According to Table 2, none of the saddles induced by $\square_T$, $\square_B$, $\square_R$, or $\square_L$ are type W_-, so $\overline{\partial}^2(\square) = 0$. Thus $\overline{\partial}^2(\square)$ has the form given in equation (16) with

$$\partial_A(\square) = 0. \tag{26}$$

According to Table 2, we have

$$\beta_1(\square_R)\alpha_2(\square_T) = \tilde{T}(\square_R)Q(\square_T), \qquad \beta_1(\square_B)\alpha_2(\square_L) = 0. \tag{27}$$

From equations (26) and (27), it follows that

$$\partial_{red}(\square) := \partial_A(\square) + \beta_1(\square_R)\alpha_2(\square_T) + \beta_1(\square_B)\alpha_2(\square_L) = \tilde{T}(\square_R)Q(\square_T).$$

Since the saddle $\square_R$ is type $C1^R_+$, we have

$$\tilde{T}(\square_R) := \tilde{T}^L(\square_R) + \tilde{T}^R(\square_R) = \tilde{T}^R(\square_R),$$

and from the definitions of the maps $\tilde{T}^R(\square_R)$, $Q(\square_T)$, $\tilde{T}^L(\square_B)$, $Q(\square_L)$, it follows that

$$\tilde{T}^R(\square_R)Q(\square_T) = \tilde{T}^L(\square_B)Q(\square_L).$$

Thus

$$\partial_{red}(\square) = \tilde{T}^L(\square_B)Q(\square_L) = \partial_T^+(\square).$$

$\square$

Lemma 11.3. *Lemma 11.1 holds for squares of type $I5_+$.*

Proof. A square $\square$ of type $I5_+$ has the form

$$\begin{array}{ccc} sR < tR < sL < tL < x_1 < x_2 & \xrightarrow[s]{W3_-} & sL < tL < sR < tL < x_1 < x_2 \\ t\Big\downarrow W3_- & & t\Big\downarrow C3_+^R \\ sR < tL < sR < tR < x_1 < x_2 & \xrightarrow[s]{C3_+^L} & sL < tR < x_1 < x_2,\ (sR < tR). \end{array}$$

According to Table 2, the induced saddles $(\square_T)_0$, $(\square_T)_1$, $(\square_L)_0$, and $(\square_L)_1$ are type W_-, the induced saddles $(\square_R)_0$ and $(\square_R)_1$ are type C_+^R, and the induced saddles $(\square_B)_0$ and $(\square_B)_1$ are type C_+^L. It follows that

$$\overline{\partial}^2(\square) = \tilde{T}_{00}^L(\square_B)Q_{00}(\square_L) + \tilde{T}_{11}^L(\square_B)Q_{11}(\square_L). \tag{28}$$

The maps $Q_{00}(\square_L)$, $Q_{11}(\square_L)$, $\tilde{T}_{00}^L(\square_B)$, and $\tilde{T}_{11}^L(\square_B)$, are computed in Lemmas B.3 and B.11 in Appendix B. We substitute these maps into equation (28) to obtain

$$\overline{\partial}^2(\square) = \begin{pmatrix} \partial_A(\square) & 0 & 0 \\ 0 & \partial_B(\square) & 0 \\ 0 & 0 & \partial_B(\square) \end{pmatrix},$$

where

$$\partial_A(\square) = \partial_B(\square) = \tilde{T}^L(\square_B)Q(\square_L). \tag{29}$$

So $\overline{\partial}^2(\square)$ has the form given in equation (16). According to Table 2, we have

$$\beta_1(\square_R)\alpha_2(\square_T) = 0, \qquad \beta_1(\square_B)\alpha_2(\square_L) = 0. \tag{30}$$

From equations (29) and (30), it follows that

$$\partial_{red}(\square) := \partial_A(\square) + \beta_1(\square_R)\alpha_2(\square_T) + \beta_1(\square_B)\alpha_2(\square_L) = \tilde{T}^L(\square_B)Q(\square_L) = \partial_T^+(\square).$$

$\square$

Lemma 11.4. *Lemma 11.1 holds for squares of type $N5_+^\beta$.*

Proof. A square $\square$ of type $N5_+^\beta$ has the form

$$\begin{array}{ccc} sR < t\beta < t\beta < sL < x_1 < x_2 & \xrightarrow[s]{W3_-} & sL < t\overline{\beta} < t\overline{\beta} < sR < x_1 < x_2 \\ t\Big\downarrow C3_+^{\overline{\beta}} & & t\Big\downarrow C3_+^{\beta} \\ sR < t\overline{\beta} < sL < x_1 < x_2,\ (t\overline{\beta}) & \xrightarrow[s]{W3_-} & sL < t\beta < sR < x_1 < x_2,\ (t\beta). \end{array}$$

According to Table 2, the induced saddles $(\square_T)_0$, $(\square_T)_1$, $(\square_B)_0$, and $(\square_B)_1$ are type W_-, the induced saddles $(\square_R)_0$ and $(\square_R)_1$ are type C_+^β, and the induced saddles $(\square_L)_0$ and $(\square_L)_1$ are type $C_+^{\overline{\beta}}$. It follows that

$$\overline{\partial}^2(\square) = \tilde{T}_{00}^L(\square_R)Q_{00}(\square_T) + \tilde{T}_{11}^L(\square_R)Q_{11}(\square_T) + Q_{00}(\square_B)\tilde{T}_{00}^L(\square_L) + Q_{11}(\square_B)\tilde{T}_{11}^L(\square_L). \tag{31}$$

The maps $Q_{00}(\square_T)$, $Q_{11}(\square_T)$, $Q_{00}(\square_B)$, $Q_{11}(\square_B)$, $\tilde{T}_{00}^L(\square_R)$, $\tilde{T}_{11}^L(\square_R)$, $\tilde{T}_{00}^L(\square_L)$, and $\tilde{T}_{11}^L(\square_L)$ are computed in Lemmas B.3 and B.11 in Appendix B. We substitute these maps into equation (31) to obtain

$$\overline{\partial}^2(\square) = \begin{pmatrix} \partial_A(\square) & 0 & 0 \\ 0 & \partial_B(\square) & 0 \\ 0 & 0 & \partial_B(\square) \end{pmatrix},$$

where

$$\partial_A(\square) = \partial_B(\square) = \tilde{T}^L(\square_R)Q(\square_T) + Q(\square_B)\tilde{T}^L(\square_L). \tag{32}$$

$\Box$	$\Box_T$	$\Box_R$	$\Box_L$	$\Box_B$	$\partial_A(\Box)$	$\beta_1(\Box_R)\alpha_2(\Box_T)$	$\beta_1(\Box_B)\alpha_2(\Box_L)$
$I3_+$	$W2_-$	$C1^R_+$	$W2_-$	$C3^L_+$	0	$\tilde{T}(\Box_R)Q(\Box_T)$	0
$I4_+$	$W3_-$	$C2^R_+$	$W2_-$	$C2^L_+$	0	0	$\tilde{T}^L(\Box_B)Q(\Box_L)$
$I5_+$	$W3_-$	$C3^R_+$	$W3_-$	$C3^L_+$	$\tilde{T}^L(\Box_B)Q(\Box_L)$	0	0
$I6_+$	$W4_-$	$C4^R_+$	$W4_-$	$C4^L_+$	$\tilde{T}^L(\Box_B)Q(\Box_L)$	0	0
$I3_-$	$C3^R_-$	$W2_-$	$C1^L_-$	$W2_-$	0	0	$Q(\Box_B)\tilde{T}(\Box_L)$
$I4_-$	$C2^R_-$	$W2_-$	$C2^L_-$	$W3_-$	0	$Q(\Box_R)\tilde{T}^R(\Box_T)$	0
$I5_-$	$C3^R_-$	$W3_-$	$C3^L_-$	$W3_-$	$Q(\Box_B)\tilde{T}^L(\Box_L)$	0	0
$I6_-$	$C4^R_-$	$W4_-$	$C4^L_-$	$W4_-$	$Q(\Box_B)\tilde{T}^L(\Box_L)$	0	0
$N2^\beta_+$	$W2_-$	$C3^\beta_+$	$C1^{\overline{\beta}}_+$	$W2_-$	0	0	$Q(\Box_B)\tilde{T}(\Box_L)$
$N3^L_+$	$W2_-$	$C2^L_+$	$C2^R_+$	$W4_-$	0	$\tilde{T}^L(\Box_R)Q(\Box_T)$	0
$N4^\beta_+$	$W2_-$	$C1^\beta_+$	$C3^{\overline{\beta}}_+$	$W2_-$	0	$\tilde{T}(\Box_R)Q(\Box_T)$	0
$N5^\beta_+$	$W3_-$	$C3^\beta_+$	$C3^{\overline{\beta}}_+$	$W3_-$	$\tilde{T}^L(\Box_R)Q(\Box_T)+Q(\Box_B)\tilde{T}^L(\Box_L)$	0	0
$N6^\beta_+$	$W4_-$	$C4^\beta_+$	$C4^{\overline{\beta}}_-$	$W4_-$	$\tilde{T}^L(\Box_R)Q(\Box_T)+Q(\Box_B)\tilde{T}^L(\Box_L)$	0	0
$N2^\beta_-$	$W2_-$	$C1^\beta_-$	$C3^{\overline{\beta}}_-$	$W2_-$	0	$\tilde{T}(\Box_R)Q(\Box_T)$	0
$N3^L_-$	$W4_-$	$C2^L_-$	$C2^R_-$	$W2_-$	0	0	$Q(\Box_B)\tilde{T}^R(\Box_L)$
$N4^\beta_-$	$W2_-$	$C3^\beta_-$	$C1^{\overline{\beta}}_-$	$W2_-$	0	0	$Q(\Box_B)\tilde{T}(\Box_L)$
$N5^\beta_-$	$W3_-$	$C3^\beta_-$	$C3^{\overline{\beta}}_-$	$W3_-$	$\tilde{T}^L(\Box_R)Q(\Box_T)+Q(\Box_B)\tilde{T}^L(\Box_L)$	0	0
$N6^\beta_-$	$W4_-$	$C4^\beta_-$	$C4^{\overline{\beta}}_-$	$W4_-$	$\tilde{T}^L(\Box_R)Q(\Box_T)+Q(\Box_B)\tilde{T}^L(\Box_L)$	0	0

TABLE 5. Interleaved and nested squares in the cube of resolutions of T. For each square $\Box$, we list the terms $\Box$ contributes to $\partial_{red}(\Box) = \partial_A(\Box) + \beta_1(\Box_R)\alpha_2(\Box_T) + \beta_1(\Box_B)\alpha_2(\Box_L)$.

So $\overline{\partial}^2(\Box)$ has the form given in equation (16). Note that if $\beta = L$ then the first term in equation (32) vanishes and if $\beta = R$ then the second term in equation (32) vanishes. According to Table 2, we have

$$\beta_1(\Box_R)\alpha_2(\Box_T) = 0, \qquad \beta_1(\Box_B)\alpha_2(\Box_L) = 0. \tag{33}$$

From equations (32) and (33), it follows that

$$\partial_{red}(\Box) := \partial_A(\Box) + \beta_1(\Box_R)\alpha_2(\Box_T) + \beta_1(\Box_B)\alpha_2(\Box_L) = \tilde{T}^L(\Box_R)Q(\Box_T) + Q(\Box_B)\tilde{T}^L(\Box_L) = \partial^+_T(\Box).$$

□

The remaining cases are similar. The results of the calculations for each case are summarized in Table 5.

11.2. **Squares that are not interleaved or nested.** Recall from Lemma 6.1 that $\partial^+_T(\Box) = 0$ if $\Box$ is not interleaved or nested, so for such squares Lemma 11.1 follows from the following two lemmas:

Lemma 11.5. *For each square $\Box$ in the cube of resolutions of T that is not interleaved or nested, we have*

$$\beta_1(\Box_R)\alpha_2(\Box_T) + \beta_1(\Box_B)\alpha_2(\Box_L) = 0.$$

Proof. Using the results summarized in Table 2, we enumerate the types of pairs of successive saddles (s,t) in the cube of resolutions of T that could yield a nonzero term $\beta_1(t)\alpha_2(s)$, and we display the results in Table 6. For each such pair, we enumerate the types of squares to which the pair could belong.

For example, consider a pair of successive saddles (s,t), where s is type $W2_-$ and t is type $C1^R_+$:

$$p_1(R) \xrightarrow[s]{W2_-} p_2(L) \xrightarrow[t]{C1^R_+} p_3(L).$$

$p_1 \xrightarrow[s]{} p_2 \xrightarrow[t]{} p_3$	$\beta_1(t)\alpha_2(s)$	$\square$
$p_1(R) \xrightarrow[s]{W2_-} p_2(L) \xrightarrow[t]{C1_+^L} p_3(L)$	$\tilde{T}(t)Q(s)$	$N4_+^L$, D
$p_1(R) \xrightarrow[s]{W2_-} p_2(L) \xrightarrow[t]{C1_+^R} p_3(L)$	$\tilde{T}(t)Q(s)$	$I3_+$, $N4_+^R$, D
$p_1(R) \xrightarrow[s]{W2_-} p_2(L) \xrightarrow[t]{C1_-^\beta} p_3(L)$	$\tilde{T}(t)Q(s)$	$N2_-^\beta$, D
$p_1(R) \xrightarrow[s]{C1_+^\beta} p_2(R) \xrightarrow[t]{W2_-} p_3(L)$	$Q(t)\tilde{T}(s)$	$N2_+^{\overline{\beta}}$, D
$p_1(R) \xrightarrow[s]{C1_-^L} p_2(R) \xrightarrow[t]{W2_-} p_3(L)$	$Q(t)\tilde{T}(s)$	$I3_-$, $N4_-^R$, D
$p_1(R) \xrightarrow[s]{C1_-^R} p_2(R) \xrightarrow[t]{W2_-} p_3(L)$	$Q(t)\tilde{T}(s)$	$N4_-^L$, D
$p_1(R) \xrightarrow[s]{W2_-} p_2(L) \xrightarrow[t]{C2_+^L} p_3(C)$	$\tilde{T}^L(t)Q(s)$	$I4_+$, $N3_+^L$
$p_1(C) \xrightarrow[s]{C2_-^R} p_2(R) \xrightarrow[t]{W2_-} p_3(L)$	$Q(t)\tilde{T}^R(s)$	$I4_-$, $N3_-^L$

TABLE 6. Pairs of successive of saddles (s,t) in the cube of resolutions of T that yield a nonzero term $\beta_1(t)\alpha_2(s)$. For each such pair (s,t), we compute $\beta_1(t)\alpha_2(s)$ and enumerate the squares to which the pair (s,t) could belong.

According to Table 2, the pair (s,t) yields the term

$$\beta_1(t)\alpha_2(s) = \tilde{T}(t)Q(s).$$

Given that s is type $W2_-$ and t is type $C1_+^R$, there are three possibilities for the orderings of the attaching points of the saddles s and t in the planar tangle $p_2(L)$, which we enumerate in the second column of the following table:

$p_1(R) = \square_{00}$	$p_2(L)$	$\square$
$sR < tR < tR < c < b < sL$	$sL < b < c < tL < tL < sR$	$N4_+^R$
$sR < tR < c < b < sL < tL$	$sL < b < c < tL < sR < tL$	$I3_+$
$sR < c < b < sL < tL < tL$	$sL < b < c < sR < tL < tL$	D

We describe the corresponding planar tangle $p_1(R) = \square_{00}$ for each ordering in the first column of the table, and use this description to classify the corresponding square, as indicated in the third column.

We repeat this calculation for each of the remaining pairs of saddles (s,t) that yield a nonzero term $\beta_1(t)\alpha_2(s)$ and display the results in Table 6. In each case, we find that (s,t) must belong to a square that is interleaved, nested, or disjoint. If the square $\square$ is disjoint, then

$$\beta_1(\square_R)\alpha_2(\square_T) + \beta_1(\square_B)\alpha_2(\square_L) = 0,$$

since the two terms cancel. $\square$

Lemma 11.6. *For each square $\square$ in the cube of resolutions of T that is not interleaved or nested, the map $\overline{\partial}^2(\square)$ has the form given in equation (16) with $\partial_A(\square) = 0$.*

Proof. Recall from equations (12) and (13) that given a square $\square$ in the cube of resolutions of T, the corresponding map $\overline{\partial}^2(\square)$ is given by

$$\overline{\partial}^2(\square) := \overline{\partial}^2(\square_R, \square_T) + \overline{\partial}^2(\square_B, \square_L),$$

where

$$\overline{\partial}^2(s,t) := \tilde{T}_{00}^L(t)Q_{00}(s) + Q_{00}(t)\tilde{T}_{00}^L(s) + \tilde{T}_{11}^L(t)Q_{11}(s) + Q_{11}(t)\tilde{T}_{11}^L(s)$$

collects the terms in $\partial_T^{\pm}$ corresponding to the pairs of induced saddles $([s]_0, [t]_0)$ and $([s]_1, [t]_1)$. We determine which pairs of saddles (s,t) in the cube of resolutions of T could yield a nonzero map $\overline{\partial}^2(s,t)$, and for each such pair we enumerate the squares to which (s,t) could belong. We show that for each of these squares the map $\overline{\partial}^2(\square)$ has the form given in equation (16), and if $\partial_A(\square)$ is nonzero then the square must be interleaved or nested.

For $\overline{\partial}^2(s,t)$ to be nonzero, at least one of the pairs $([s]_0, [t]_0)$ or $([s]_1, [t]_1)$ must contain one saddle of type W_- and one saddle of type $C^L_{\pm}$. According to Table 2, if a saddle u in the cube of resolutions of T induces a saddle $[u]_0$ or $[u]_1$ of type W_-, then u must be type $W3_-$, $W4_-$, $C2^L_+$, or $C2^R_-$. We consider these cases separately in Tables 7 – 11. For each case, we enumerate the saddles v in the cube of resolutions of T that induce a saddle $[v]_0$ or $[v]_1$ of type $C^L_{\pm}$ and could pair with the saddle $[u]_0$ or $[u]_1$ of type W_- to yield a nonzero map $\overline{\partial}^2(s,t)$, where (s,t) is either (u,v) or (v,u).

For example, consider a saddle $s : p_1(L) \to p_2(L)$ of type $W3_-$. According to Table 2, the induced saddles $[s]_0$ and $[s]_1$ are type W_-. We can obtain a nonzero term $\overline{\partial}^2(s,t)$ by pairing s with a saddle $t : p_2(L) \to p_3(L)$ of type $C3^L_+$, since according to Table 2 the induced saddles $[t]_0$ and $[t]_1$ are type C^L_+:

$$\overline{\partial}^2(s,t) = \tilde{T}^L_{00}(t)Q_{00}(s) + \tilde{T}^L_{11}(t)Q_{11}(s). \tag{34}$$

The maps $Q_{00}(s)$, $Q_{11}(s)$, $\tilde{T}^L_{00}(t)$, and $\tilde{T}^L_{11}(t)$ are computed in Lemmas B.3 and B.11 in Appendix B. We substitute these maps into equation (34) to obtain

$$\overline{\partial}^2(s,t) = \begin{pmatrix} \partial_A(s,t) & 0 & 0 \\ 0 & \partial_{B^1}(s,t) & 0 \\ 0 & 0 & \partial_{B^2}(s,t) \end{pmatrix},$$

where

$$\partial_A(s,t) = \partial_{B^1}(s,t) = \partial_{B^2}(s,t) = \tilde{T}^L(t)Q(s).$$

Given that s is type $W3_-$ and t is type $C3^L_+$, there are six possibilities for the orderings of the attaching points of the saddles s and t in the planar tangle $p_2(L)$, which we enumerate in the second column of the following table:

$p_1(L) = \square_{00}$	$p_2(L)$	$\square$
$sR < sL < tR < tR < b < c$	$sL < sR < tR < tR < b < c$	D
$sR < tL < sL < tR < b < c$	$sL < tR < sR < tR < b < c$	not possible by Lemma 5.2
$tR < sR < sL < tR < b < c$	$tR < sL < sR < tR < b < c$	not possible by Lemma 5.1
$sR < tL < tL < sL < b < c$	$sL < tR < tR < sR < b < c$	$N5^L_+$
$tR < sR < tL < sL < b < c$	$tR < sL < tR < sR < b < c$	$I5_+$
$tR < tR < sR < sL < b < c$	$tR < tR < sL < sR < b < c$	D

We describe the corresponding planar tangle $p_1(L) = \square_{00}$ for each ordering in the first column of the table, and use this description to classify the corresponding square, as indicated in the third column.

If a pair of saddles (s,t) belongs to a disjoint square $\square$, then

$$\overline{\partial}^2(\square) := \overline{\partial}^2(\square_R, \square_T) + \overline{\partial}^2(\square_B, \square_L) = 0,$$

since the two terms cancel, so $\overline{\partial}^2(\square)$ is of the form given in equation (16) with $\partial_A(\square) = 0$.

We note that in certain cases, such as the pairs of saddles (s,t) in Tables 10 and 11, the only nonvanishing blocks of $\overline{\partial}^2(s,t)$ are $\alpha_1(s,t)$ or $\beta_2(s,t)$. In such cases we do not enumerate the squares to which (s,t) could belong, since we already know that for each such square the contribution of $\overline{\partial}^2(s,t)$ to $\overline{\partial}^2(\square)$ is of the form given in equation (16) with $\partial_A(s,t) = 0$. $\square$

$p_1 \xrightarrow[s]{} p_2 \xrightarrow[t]{} p_3$	nonzero blocks of $\overline{\partial}^2(s,t)$	$\square$
$p_1(L) \xrightarrow[s]{W3_-} p_2(L) \xrightarrow[t]{W2_+} p_3(R)$	$\alpha_1(s,t)$	$-$
$p_1(R) \xrightarrow[s]{W2_-} p_2(L) \xrightarrow[t]{W3_-} p_3(L)$	$\partial_{B^1}(s,t) = Q(t)Q(s)$	D
$p_1(L) \xrightarrow[s]{W3_-} p_2(L) \xrightarrow[t]{C1^R_\pm} p_3(L)$	$\partial_{B^2}(s,t) = \tilde{T}^R(t)Q(s)$	D
$p_1(L) \xrightarrow[s]{C1^R_\pm} p_2(L) \xrightarrow[t]{W3_-} p_3(L)$	$\partial_{B^2}(s,t) = Q(t)\tilde{T}^R(s)$	D
$p_1(L) \xrightarrow[s]{W3_-} p_2(L) \xrightarrow[t]{C3^L_+} p_3(L)$	$\partial_A(s,t) = \partial_{B^1}(s,t) = \partial_{B^2}(s,t) = \tilde{T}^L(t)Q(s)$	$I5_+$, $N5^L_+$, D
$p_1(L) \xrightarrow[s]{W3_-} p_2(L) \xrightarrow[t]{C3^L_-} p_3(L)$	$\partial_A(s,t) = \partial_{B^1}(s,t) = \partial_{B^2}(s,t) = \tilde{T}^L(t)Q(s)$	$N5^L_-$, D
$p_1(L) \xrightarrow[s]{C3^L_+} p_2(L) \xrightarrow[t]{W3_-} p_3(L)$	$\partial_A(s,t) = \partial_{B^1}(s,t) = \partial_{B^2}(s,t) = Q(t)\tilde{T}^L(s)$	$N5^R_+$, D
$p_1(L) \xrightarrow[s]{C3^L_-} p_2(L) \xrightarrow[t]{W3_-} p_3(L)$	$\partial_A(s,t) = \partial_{B^1}(s,t) = \partial_{B^2}(s,t) = Q(t)\tilde{T}^L(s)$	$I5_-$, $N5^R_-$, D

TABLE 7. Nonzero maps $\overline{\partial}^2(s,t)$ for which the induced saddle of type W_- is induced by a saddle $p(L) \to q(L)$ of type $W3_-$. For each pair of saddles (s,t) we describe the nonzero blocks of $\overline{\partial}^2(s,t)$ and enumerate the squares to which the pair (s,t) could belong.

$p_1 \xrightarrow[s]{} p_2 \xrightarrow[t]{} p_3$	nonzero blocks of $\overline{\partial}^2(s,t)$	$\square$
$p_1(R) \xrightarrow[s]{W3_-} p_2(R) \xrightarrow[t]{W2_-} p_3(L)$	$\partial_{B^2}(s,t) = Q(t)Q(s)$	D
$p_1(L) \xrightarrow[s]{W2_+} p_2(R) \xrightarrow[t]{W3_-} p_3(R)$	$\alpha_1(s,t)$	$-$
$p_1(R) \xrightarrow[s]{W3_-} p_2(R) \xrightarrow[t]{C1^R_\pm} p_3(R)$	$\partial_{B^1}(s,t) = \tilde{T}^R(t)Q(s)$	D
$p_1(R) \xrightarrow[s]{C1^R_\pm} p_2(R) \xrightarrow[t]{W3_-} p_3(R)$	$\partial_{B^1}(s,t) = Q(t)\tilde{T}^R(s)$	D
$p_1(R) \xrightarrow[s]{W3_-} p_2(R) \xrightarrow[t]{C2^R_+} p_3(C)$	$\beta_2(s,t)$	$-$
$p_1(C) \xrightarrow[s]{C2^R_-} p_2(R) \xrightarrow[t]{W3_-} p_3(R)$	$\partial_A(s,t) = Q(t)\tilde{T}^R(s)$	D
$p_1(R) \xrightarrow[s]{W3_-} p_2(R) \xrightarrow[t]{C3^L_+} p_3(R)$	$\partial_A(s,t) = \partial_{B^1}(s,t) = \partial_{B^2}(s,t) = \tilde{T}^L(t)Q(s)$	$I5_+$, $N5^L_+$, D
$p_1(R) \xrightarrow[s]{W3_-} p_2(R) \xrightarrow[t]{C3^L_-} p_3(R)$	$\partial_A(s,t) = \partial_{B^1}(s,t) = \partial_{B^2}(s,t) = \tilde{T}^L(t)Q(s)$	$N5^L_-$, D
$p_1(R) \xrightarrow[s]{C3^L_+} p_2(R) \xrightarrow[t]{W3_-} p_3(R)$	$\partial_A(s,t) = \partial_{B^1}(s,t) = \partial_{B^2}(s,t) = Q(t)\tilde{T}^L(s)$	$N5^R_+$, D
$p_1(R) \xrightarrow[s]{C3^L_-} p_2(R) \xrightarrow[t]{W3_-} p_1(R)$	$\partial_A(s,t) = \partial_{B^1}(s,t) = \partial_{B^2}(s,t) = Q(t)\tilde{T}^L(s)$	$I5_-$, $N5^R_-$, D

TABLE 8. Nonzero maps $\overline{\partial}^2(s,t)$ for which the induced saddle of type W_- is induced by a saddle $p(R) \to q(R)$ of type $W3_-$. For each pair of saddles (s,t) we list the nonzero blocks of $\overline{\partial}^2(s,t)$ and enumerate the squares to which the pair (s,t) could belong.

$p_1 \xrightarrow[s]{} p_2 \xrightarrow[t]{} p_3$	nonzero blocks of $\overline{\partial}^2(s,t)$	$\square$
$p_1(C) \xrightarrow[s]{W4_-} p_2(C) \xrightarrow[t]{C2^R_-} p_3(R)$	$\partial_A(s,t) = \tilde{T}^R(s)Q(t)$	D
$p_1(R) \xrightarrow[s]{C2^R_+} p_2(C) \xrightarrow[t]{W4_-} p_3(C)$	$\beta_2(s,t)$	$-$
$p_1(C) \xrightarrow[s]{W4_-} p_2(C) \xrightarrow[t]{C4^L_+} p_3(C)$	$\partial_A(s,t) = \tilde{T}^L(t)Q(s)$	$I6_+$, $N6^L_+$, D
$p_1(C) \xrightarrow[s]{W4_-} p_2(C) \xrightarrow[t]{C4^L_-} p_3(C)$	$\partial_A(s,t) = \tilde{T}^L(t)Q(s)$	$N6^L_-$, D
$p_1(C) \xrightarrow[s]{C4^L_+} p_2(C) \xrightarrow[t]{W4_-} p_3(C)$	$\partial_A(s,t) = Q(t)\tilde{T}^L(s)$	$N6^R_+$, D
$p_1(C) \xrightarrow[s]{C4^L_-} p_2(C) \xrightarrow[t]{W4_-} p_3(C)$	$\partial_A = Q(t)\tilde{T}^L(s)$	$I6_-$, $N6^R_-$, D
$p_1(C) \xrightarrow[s]{W4_-} p_2(C) \xrightarrow[t]{C1^C_\pm} p_3(C)$	$\partial_A(s,t) = \tilde{T}^{L_0}(t)Q(s)\mathbb{1}_{C_{[p_1]_0}} + \tilde{T}^{L_1}(t)Q(s)\mathbb{1}_{C_{[p_1]_1}}$	D
$p_1(C) \xrightarrow[s]{C1^C_\pm} p_2(C) \xrightarrow[t]{W4_-} p_3(C)$	$\partial_A(s,t) = Q(t)\tilde{T}^{L_0}(s)\mathbb{1}_{C_{[p_1]_0}} + Q(t)\tilde{T}^{L_1}(s)\mathbb{1}_{C_{[p_1]_1}}$	D

Table 9. Nonzero maps $\overline{\partial}^2(s,t)$ for which the induced saddle of type W_- is induced by a saddle $p(C) \to q(C)$ of type $W4_-$. For each pair of saddles (s,t) we list the nonzero blocks of $\overline{\partial}^2(s,t)$ and enumerate the squares to which the pair (s,t) could belong.

$p_1 \xrightarrow[s]{} p_2 \xrightarrow[t]{} p_3$	nonzero blocks of $\overline{\partial}^2(s,t)$
$p_1(L) \xrightarrow[s]{C1^R_\pm} p_2(L) \xrightarrow[t]{C2^L_+} p_3(C)$	$\beta_2(s,t)$
$p_1(L) \xrightarrow[s]{C2^L_+} p_2(C) \xrightarrow[t]{C2^R_-} p_3(R)$	$\beta_2(s,t)$
$p_1(L) \xrightarrow[s]{C3^L_\pm} p_2(L) \xrightarrow[t]{C2^L_+} p_3(C)$	$\beta_2(s,t)$
$p_1(L) \xrightarrow[s]{C2^L_+} p_2(C) \xrightarrow[t]{C4^L_\pm} p_3(C)$	$\beta_2(s,t)$
$p_1(L) \xrightarrow[s]{C2^L_+} p_2(C) \xrightarrow[t]{C1^C_\pm} p_3(C)$	$\beta_2(s,t)$

Table 10. Nonzero maps $\overline{\partial}^2(s,t)$ for which the induced saddle of type W_- is induced by a saddle $p(L) \to q(C)$ of type $C2^L_+$. For each pair of saddles (s,t) we list the nonzero blocks of $\overline{\partial}^2(s,t)$.

Appendix A. Bigradings

Here we prove:

Lemma A.1. *For each planar tangle p in the cube of resolutions of a tangle diagram T, we have $A_p = C^+_p$ as bigraded vector spaces.*

Proof. By performing an isotopy of the annulus and an isotopy of the overpass arc, we can reduce to the case where the tangle diagram T is in standard position and the overpass arc is the arc A^+_s shown in Figure 3. Note that the bigrading shift in C^+_p given in equations (2) and (3) is unchanged under these isotopies. Consider the oriented link diagram $T^+ = T \cup A^+_s$. Recall that $a_+(A^+_s, T)$ and $a_-(A^+_s, T)$ denote the number

$p_1 \xrightarrow[s]{} p_2 \xrightarrow[t]{} p_3$	nonzero blocks of $\overline{\partial}^2(s,t)$
$p_1(C) \xrightarrow[s]{C2^R_-} p_2(R) \xrightarrow[t]{W2_-} p_3(L)$	$\alpha_1(s,t)$
$p_1(C) \xrightarrow[s]{C2^R_-} p_2(R) \xrightarrow[t]{C1^R_\pm} p_3(R)$	$\alpha_1(s,t)$
$p_1(C) \xrightarrow[s]{C2^R_-} p_2(R) \xrightarrow[t]{C3^L_\pm} p_3(R)$	$\alpha_1(s,t)$
$p_1(C) \xrightarrow[s]{C4^L_\pm} p_2(C) \xrightarrow[t]{C2^R_-} p_3(R)$	$\alpha_1(s,t)$
$p_1(C) \xrightarrow[s]{C1^C_\pm} p_2(C) \xrightarrow[t]{C2^R_-} p_3(R)$	$\alpha_1(s,t)$

TABLE 11. Nonzero maps $\overline{\partial}^2(s,t)$ for which the induced saddle of type W_- is induced by a saddle $p(C) \to q(R)$ of type $C2^R_-$. For each pair of saddles (s,t) we list the nonzero blocks of $\overline{\partial}^2(s,t)$.

of positive and negative crossings between A_s^+ and T. The loop number $\ell(T)$ is given by

$$\ell(T) = a_+(A_s^+, T) + a_-(A_s^+, T). \tag{35}$$

Let $n_+(T^+)$ and $n_-(T^+)$ denote the total number of positive and negative crossings of the link diagram T^+:

$$n_\pm(T^+) = m_\pm(T) + a_\pm(A_s^+, T). \tag{36}$$

Using equations (35) and (36), we can express equations (2) and (3) for the bigrading shift in C_p^+ as

$$h^+(T,p) = -n_-(T^+) + \frac{1}{2}(\ell(T) + w(p)) + r(p), \tag{37}$$

$$q^+(T,p) = n_+(T^+) - 2n_-(T^+) + \frac{1}{2}(\ell(T) + 3w(p)) + r(p). \tag{38}$$

The loop number of $\overline{T}$ and the number of positive and negative crossings of $\overline{T}^+$ are given by

$$\ell(\overline{T}) = \ell(T) - 1, \qquad n_\pm(\overline{T}^+) = n_\pm(T^+). \tag{39}$$

The resolution degrees of the planar tangles $[p]_0$ and $[p]_1$ induced by p are

$$r([p]_0) = r(p), \qquad r([p]_1) = r(p) + 1. \tag{40}$$

Consider the case that p is type L. The outermost loop of p is oriented clockwise, so

$$w([p]_0) = w(p) + 1. \tag{41}$$

Substituting equations (39), (40), and (41) into equations (37) and (38), we find

$$h^+(\overline{T}, [p]_0) = h^+(T,p), \qquad q^+(\overline{T}, [p]_0) = q^+(T,p) + 1. \tag{42}$$

As an ungraded vector space $A_p = C_p$ is identified as the subspace $C_p \otimes x \subset C_p \otimes A^{dc} = C_{[p]_0}$, so from equation (42) and the fact that x has bigrading $(0,-1)$, it follows that $A_p = C_p^+$ as bigraded vector spaces.

Consider the case that p is type R. The outermost loop of p is oriented counterclockwise, so

$$w([p]_1) = w(p) - 1. \tag{43}$$

Substituting equations (39), (40), and (43) into equations (37) and (38), we find

$$h^+(\overline{T}, [p]_1) = h^+(T,p), \qquad q^+(\overline{T}, [p]_1) = q^+(T,p) - 1. \tag{44}$$

As an ungraded vector space $A_p = C_p$ is identified as the subspace $C_p \otimes e \subset C_p \otimes A^{db} = C_{[p]_1}$, so from equation (44) and the fact that and e has bigrading $(0,1)$ it follows that $A_p = C_p^+$ as bigraded vector spaces.

Consider the case that p is type C. The winding numbers of the planar tangles induced by p are given by equation (41) for $[p]_0$ and equation (43) for $[p]_1$, so equations (42) and (44) hold for $[p]_0$ and $[p]_1$. As an ungraded vector space $A = C_p$ is identified as $W_p \otimes A^{bc}$, where $C_{[p]_0} = W_p \otimes e$ and $C_{[p]_1} = W_p \otimes x$, so from equations (42) and (44) and the bigradings of e and x it follows that $A_p = C_p^+$ as bigraded vector spaces. □

Appendix B. Induced saddles

For each saddle $s : p \to q$ in the cube of resolutions of T, we compute the map $\overline{\partial}^1(s)$ defined in equation (11), which collects the terms of ∂_T^0 corresponding to the induced saddles of s, as well as the terms of ∂_T^+ corresponding to pairs of successive saddles, one of which is an induced saddle of s and one of which is an ancillary saddle:

$$\overline{\partial}^1(s) = T_{00}(s) + T_{11}(s) + \tilde{T}^L_{10}(n_q)Q_{00}(s) + Q_{10}(n_q)\tilde{T}^L_{00}(s) + \tilde{T}^L_{11}(s)Q_{10}(n_p) + Q_{11}(s)\tilde{T}^L_{10}(n_p).$$

The terms of $\overline{\partial}^1(s)$ involving pairs of saddles correspond to the following square of induced and ancillary saddles:

$$\begin{array}{ccc} C_{[p]_0} & \xrightarrow[{[s]_0}]{} & C_{[q]_0} \\ {\scriptstyle n_p}\downarrow & & {\scriptstyle n_q}\downarrow \\ C_{[p]_1} & \xrightarrow[{[s]_1}]{} & C_{[q]_1}, \end{array}$$

and by Lemma 6.1 give a nonzero contribution to $\overline{\partial}^1(s)$ only if this square is interleaved or nested. We also compute the maps $Q_{00}(s)$, $\tilde{T}^L_{00}(s)$, $Q_{11}(s)$, and $\tilde{T}^L_{11}(s)$ due to the induced saddles $[s]_0$ and $[s]_1$ of s that contribute to $\overline{\partial}^2$. These results are straightforward calculations that are very similar to the calculations used to prove Lemmas 9.1, 9.2, and 9.3, so we will be somewhat brief.

Lemma B.1. *For a type $W2_-$ saddle $s : p(R) \to q(L)$, the map $\overline{\partial}^1(s)$ has the form given in equation (15), where*

$$\partial_A(s) = 0, \qquad \beta_1(s) = Q(s), \qquad \alpha_2(s) = Q(s).$$

The saddle s gives the following map that contributes to $\overline{\partial}^2$:

$$\tilde{T}^L_{00}(s) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & Q(s) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. We have the following square of saddles, which is neither interleaved nor nested:

$$\begin{array}{ccc} C_{[p]_0} & \xrightarrow[{[s]_0}]{C^L_+} & C_{[q]_0} \\ {\scriptstyle n_p}\downarrow{\scriptstyle C^R_+} & & {\scriptstyle n_q}\downarrow{\scriptstyle C^L_-} \\ C_{[p]_1} & \xrightarrow[{[s]_1}]{C^R_-} & C_{[q]_1}. \end{array}$$

We have the following vector spaces and linear maps:

$$\begin{array}{lll} C_p = A^{\otimes c(p)}, & C_q = C_p, & Q(s) = \mathbb{1}_{C_p}, \\ C_{[p]_0} = C_p, & C_{[q]_0} = C_p \otimes A^{dc}, & T_{00}(s) = C_p \otimes \dot{\eta},\ \tilde{T}^L_{00}(s) = C_p \otimes \eta, \\ C_{[p]_1} = C_p \otimes A^{db}, & C_{[q]_1} = C_p, & T_{11}(s) = C_p \otimes \dot{\epsilon}\,. \end{array}$$

Using the decompositions of C_p and C_q defined in equations (24) and (23), we find that

$$\overline{\partial}^1(s) = T_{00}(s) + T_{11}(s) = \begin{pmatrix} 0 & Q(s) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Q(s) & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & Q(s) & 0 \\ 0 & 0 & 0 \\ Q(s) & 0 & 0 \end{pmatrix},$$

and the saddle $[s]_0$ gives the map $\tilde{T}^L_{00}(s)$ stated in the lemma. □

Lemma B.2. *For a type $W2_+$ saddle $s : p(L) \to q(R)$, the map $\overline{\partial}^1(s)$ has the form given in equation (15), where*

$$\partial_A(s) = 0, \qquad \beta_1(s) = 0, \qquad \alpha_2(s) = 0.$$

The saddle s gives the following map that contributes to $\overline{\partial}^2$:

$$\tilde{T}^L_{00}(s) = \begin{pmatrix} 0 & 0 & 0 \\ P(s) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. We have the following square of saddles, which is neither interleaved nor nested:

$$\begin{array}{ccc} C_{[p]_0} & \xrightarrow[{[s]_0}]{C^L_-} & C_{[q]_0} \\ {\scriptstyle n_p}\big\downarrow{\scriptstyle C^L_-} & & {\scriptstyle n_q}\big\downarrow{\scriptstyle C^R_+} \\ C_{[p]_1} & \xrightarrow[{[s]_1}]{C^R_+} & C_{[q]_1}. \end{array}$$

A calculation similar to the one used in the proof of Lemma B.1 shows that

$$\overline{\partial}^1(s) = T_{00}(s) + T_{11}(s) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & P(s) & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & P(s) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & P(s) & 0 \\ 0 & 0 & P(s) \end{pmatrix},$$

and the saddle $[s]_0$ gives the map $\tilde{T}^L_{00}(s)$ stated in the lemma. □

Lemma B.3. *For a type $W3_\pm$ saddle $s : p(L) \to q(L)$ or $s : p(R) \to q(R)$, the map $\overline{\partial}^1(s)$ has the form given in equation (15), where*

$$\partial_A(s) = 0, \qquad \beta_1(s) = 0, \qquad \alpha_2(s) = 0.$$

A type $W3_-$ saddle $s : p(L) \to q(L)$ gives the following maps that contribute to $\overline{\partial}^2$:

$$Q_{00}(s) = \begin{pmatrix} Q(s) & 0 & 0 \\ 0 & Q(s) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Q_{11}(s) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Q(s) \end{pmatrix}.$$

A type $W3_-$ saddle $s : p(R) \to q(R)$ gives the following maps that contribute to $\overline{\partial}^2$:

$$Q_{00}(s) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & Q(s) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Q_{11}(s) = \begin{pmatrix} Q(s) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Q(s) \end{pmatrix}.$$

A type $W3_+$ saddle does not give any maps that contribute to $\overline{\partial}^2$.

Proof. For a saddle $s : p(L) \to q(L)$, we have the following square of saddles, which is neither interleaved nor nested:

$$\begin{array}{ccc} C_{[p]_0} & \xrightarrow[{[s]_0}]{W_\pm} & C_{[q]_0} \\ {\scriptstyle n_p}\big\downarrow{\scriptstyle C^L_-} & & {\scriptstyle n_q}\big\downarrow{\scriptstyle C^L_-} \\ C_{[p]_1} & \xrightarrow[{[s]_1}]{W_\pm} & C_{[q]_1}. \end{array}$$

Thus $\overline{\partial}^1(s) = 0$. A calculation similar to the one used in the proof of Lemma B.1 shows that for a saddle s of type $W3_-$ the saddles $[s]_0$ and $[s]_1$ give the maps $Q_{00}(s)$ and $Q_{11}(s)$ stated in the lemma. The proof for a saddle $s : p(R) \to q(R)$ is similar. □

Lemma B.4. *For a type $W4_\pm$ saddle $s : p(C) \to q(C)$, the map $\overline{\partial}^1(s)$ has the form given in equation (15), where*

$$\partial_A(s) = 0, \qquad \beta_1(s) = 0, \qquad \alpha_2(s) = 0.$$

A type $W4_-$ saddle s gives the following maps that contribute to $\overline{\partial}^2$:

$$Q_{00}(s) = \begin{pmatrix} Q(s)\mathbb{1}_{C_{[p]_0}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Q_{11}(s) = \begin{pmatrix} Q(s)\mathbb{1}_{C_{[p]_1}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A type $W4_+$ saddle does not give any maps that contribute to $\overline{\partial}^2$.

Proof. We have the following square of saddles, which is neither interleaved nor nested:

$$\begin{array}{ccc} C_{[p]_0} & \xrightarrow[{[s]_0}]{W_\pm} & C_{[q]_0} \\ n_p \big\downarrow W_- & & n_q \big\downarrow W_- \\ C_{[p]_1} & \xrightarrow[{[s]_1}]{W_\pm} & C_{[q]_1}. \end{array}$$

Thus $\overline{\partial}^1(s) = 0$. A calculation similar to the one used in the proof of Lemma B.1 shows that for a saddle s of type $W4_-$ the saddles $[s]_0$ and $[s]_1$ give the maps $Q_{00}(s)$ and $Q_{11}(s)$ stated in the lemma. □

Lemma B.5. *For a type $C1^\beta_\pm$ saddle $s : p(L) \to q(L)$, the term $\overline{\partial}^1(s)$ has the form given in equation (15), where*

$$\partial_A(s) = T(s), \qquad \beta_1(s) = \tilde{T}(s), \qquad \alpha_2(s) = 0.$$

The saddle s gives the following map that contributes to $\overline{\partial}^2$:

$$\tilde{T}^L_{11}(s) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \tilde{T}^R(s) \end{pmatrix}.$$

Proof. We have the following square of saddles, which is neither interleaved nor nested:

$$\begin{array}{ccc} C_{[p]_0} & \xrightarrow[{[s]_0}]{C^C_\pm} & C_{[q]_0} \\ n_p \big\downarrow C^L_- & & n_q \big\downarrow C^L_- \\ C_{[p]_1} & \xrightarrow[{[s]_1}]{C^{\overline{\beta}}_\pm} & C_{[q]_1}. \end{array}$$

For a saddle s of type $C1^\beta_+$, we have the following vector spaces and linear maps:

$$\begin{aligned} &C_p = A^{\otimes c(p)}, && C_q = C_p \otimes A, && T(s) = \mathbb{1}_{C_p} \otimes \dot{\eta},\ \tilde{T}(s) = \mathbb{1}_{C_p} \otimes \eta, \\ &C_{[p]_0} = C_p \otimes A^{dc}, && C_{[q]_0} = C_q \otimes A^{dc} = C_p \otimes A \otimes A^{dc}, && T_{00}(s) = \mathbb{1}_{C_p} \otimes \Delta, \\ &C_{[p]_1} = C_p, && C_{[q]_1} = C_q = C_p \otimes A, && T_{11}(s) = \mathbb{1}_{C_p} \otimes \dot{\eta},\ \tilde{T}_{11}(s) = \mathbb{1}_{C_p} \otimes \eta. \end{aligned}$$

Note that

$$T_{00}(s) = \mathbb{1}_{C_p} \otimes \Delta = \mathbb{1}_{C_p} \otimes \dot{\eta} \otimes \mathbb{1}_A + \mathbb{1}_{C_p} \otimes \eta \otimes \mathbb{1}_{xe} = T(s) \otimes \mathbb{1}_A + \tilde{T}(s) \otimes \mathbb{1}_{xe}.$$

Using the decompositions of C_p and C_q given in equation (23), we find that

$$\overline{\partial}^1(s) = T_{00}(s) + T_{11}(s) = \begin{pmatrix} T(s) & \tilde{T}(s) & 0 \\ 0 & T(s) & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T(s) \end{pmatrix} = \begin{pmatrix} T(s) & \tilde{T}(s) & 0 \\ 0 & T(s) & 0 \\ 0 & 0 & T(s) \end{pmatrix},$$

and the saddle $[s]_1$ gives the map $\tilde{T}^L_{11}(s)$ stated in the lemma, where we have used the fact that $[s]_1$ attaches to the opposite side of the arc component as s. The proof for a saddle of type $C1^\beta_-$ is similar. □

Lemma B.6. *For a type $C1^\beta_\pm$ saddle $s : p(R) \to q(R)$, the term $\overline{\partial}^1(s)$ has the form given in equation (15), where*

$$\partial_A(s) = T(s), \qquad \beta_1(s) = 0, \qquad \alpha_2(s) = \tilde{T}(s).$$

The saddle s gives the following map that contributes to $\overline{\partial}^2$:

$$\tilde{T}^L_{00}(s) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \tilde{T}^R(s) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. We have the following square of saddles, which is neither interleaved nor nested:

$$\begin{array}{ccc} C_{[p]_0} & \xrightarrow[{[s]_0}]{C^{\overline{\beta}}_{\pm}} & C_{[q]_0} \\ {\scriptstyle n_p}\Big\downarrow{\scriptstyle C^R_+} & & {\scriptstyle n_q}\Big\downarrow{\scriptstyle C^R_+} \\ C_{[p]_1} & \xrightarrow[{[s]_1}]{C^C_{\pm}} & C_{[q]_1}. \end{array}$$

A calculation similar to the one used in the proof of Lemma B.5 shows that

$$\overline{\partial}^1(s) = T_{00}(s) + T_{11}(s) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & T(s) & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} T(s) & 0 & 0 \\ 0 & 0 & 0 \\ \tilde{T}(s) & 0 & T(s) \end{pmatrix} = \begin{pmatrix} T(s) & 0 & 0 \\ 0 & T(s) & 0 \\ \tilde{T}(s) & 0 & T(s) \end{pmatrix},$$

and the saddle $[s]_0$ gives the map $\tilde{T}^L_{00}(s)$ stated in the lemma, where we have used the fact that $[s]_0$ attaches to the opposite side of the arc component as s. □

Lemma B.7. *For a type $C2^L_+$ saddle $s : p(L) \to q(C)$, the map $\overline{\partial}^1(s)$ has the form given in equation (15), where*

$$\partial_A(s) = T(s), \qquad \beta_1(s) = \tilde{T}^L(s), \qquad \alpha_2(s) = 0.$$

The saddle s gives the following map that contributes to $\overline{\partial}^2$:

$$Q_{11}(s) = \begin{pmatrix} 0 & 0 & T(s) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. We have the following square of saddles, which is interleaved of type I_-:

$$\begin{array}{ccc} C_{[p]_0} & \xrightarrow[{[s]_0}]{C^R_-} & C_{[q]_0} \\ {\scriptstyle n_p}\Big\downarrow{\scriptstyle C^L_-} & & {\scriptstyle n_q}\Big\downarrow{\scriptstyle W_-} \\ C_{[p]_1} & \xrightarrow[{[s]_1}]{W_-} & C_{[q]_1}. \end{array}$$

We have the following vector spaces and linear maps:

$$\begin{aligned} &C_p = A^{\otimes c(p)} = W_q, && C_q = C_p \otimes A^{bc}, && T(s) = \mathbb{1}_{C_p} \otimes \dot{\eta},\ \tilde{T}^L(s) = \mathbb{1}_{C_p} \otimes \eta, \\ &C_{[p]_0} = C_p \otimes A^{dc}, && C_{[q]_0} = C_p = C_p \otimes e, && T_{00}(s) = \mathbb{1}_{C_p} \otimes \dot{\epsilon}, \\ &C_{[p]_1} = C_p, && C_{[q]_1} = C_p = C_p \otimes x, && Q_{11}(s) = \mathbb{1}_{C_p}. \end{aligned}$$

Using the decompositions of C_p and C_q given in equations (23) and (25), we find that

$$\begin{aligned} \overline{\partial}^1(s) &= T_{00}(s) + Q_{11}(s)\tilde{T}^L_{10}(n_p) \\ &= \begin{pmatrix} 0 & \tilde{T}^L(s) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & T(s) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathbb{1}_{C_p} & 0 & 0 \end{pmatrix} = \begin{pmatrix} T(s) & \tilde{T}^L(s) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

and the saddle $[s]_1$ gives the map $Q_{11}(s)$ stated in the lemma. □

Lemma B.8. *For a a type $C2^L_-$ saddle $s : p(C) \to q(L)$, the map $\overline{\partial}^1(s)$ has the form given in equation (15), where*

$$\partial_A(s) = T(s), \qquad \beta_1(s) = 0, \qquad \alpha_2(s) = 0.$$

The saddle s does not give any maps that contribute to $\overline{\partial}^2$.

Proof. We have the following square of saddles, which is neither interleaved nor nested:

$$\begin{array}{ccc} C_{[p]_0} & \xrightarrow[{[s]_0}]{C^R_+} & C_{[q]_0} \\ {\scriptstyle n_p}\big\downarrow{\scriptstyle W_-} & & {\scriptstyle n_q}\big\downarrow{\scriptstyle C^L_-} \\ C_{[p]_1} & \xrightarrow[{[s]_1}]{W_+} & C_{[q]_1}. \end{array}$$

A calculation similar to the one used in the proof of Lemma B.7 shows that

$$\overline{\partial}^1(s) = T_{00}(s) = \begin{pmatrix} T(s) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

□

Lemma B.9. *For a type $C2^R_+$ saddle $s : p(R) \to q(C)$, the map $\overline{\partial}^1(s)$ has the form given in equation (15), where*

$$\partial_A(s) = T(s), \qquad \beta_1(s) = 0, \qquad \alpha_2(s) = 0.$$

The saddle s gives the following map that contributes to $\overline{\partial}^1(s)$:

$$\tilde{T}^L_{11}(s) = \begin{pmatrix} 0 & 0 & T(s) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. We have the following square of saddles, which is neither interleaved nor nested:

$$\begin{array}{ccc} C_{[p]_0} & \xrightarrow[{[s]_0}]{W_+} & C_{[q]_0} \\ {\scriptstyle n_p}\big\downarrow{\scriptstyle C^R_+} & & {\scriptstyle n_q}\big\downarrow{\scriptstyle W_-} \\ C_{[p]_1} & \xrightarrow[{[s]_1}]{C^L_-} & C_{[q]_1}. \end{array}$$

A calculation similar to the one used in the proof of Lemma B.7 shows that

$$\overline{\partial}^1(s) = T_{11}(s) = \begin{pmatrix} T(s) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the saddle $[s]_1$ gives the map $\tilde{T}^L_{11}(s)$ stated in the lemma. □

Lemma B.10. *For a type $C2^R_-$ saddle $s : p(C) \to q(R)$, the map $\overline{\partial}^1(s)$ has the form given in equation (15), where*

$$\partial_A(s) = T(s), \qquad \beta_1(s) = 0, \qquad \alpha_2(s) = \tilde{T}^R(s).$$

The saddle s gives the following maps that contribute to $\overline{\partial}^2$:

$$Q_{00}(s) = \begin{pmatrix} 0 & 0 & 0 \\ T(s) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{T}^L_{11}(s) = \begin{pmatrix} \tilde{T}^R(s) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. We have the following square of saddles, which is interleaved of type I_+:

$$\begin{array}{ccc} C_{[p]_0} & \xrightarrow[{[s]_0}]{W_-} & C_{[q]_0} \\ {\scriptstyle n_p}\big\downarrow{\scriptstyle W_-} & & {\scriptstyle n_q}\big\downarrow{\scriptstyle C^R_+} \\ C_{[p]_1} & \xrightarrow[{[s]_1}]{C^L_+} & C_{[q]_1}. \end{array}$$

We have the following vector spaces and linear maps:

$$\begin{array}{lll} C_p = C_q \otimes A^{bc}, & C_q = A^{\otimes c(q)} = W_p, & T(s) = \mathbb{1}_{C_q} \otimes \dot{\epsilon},\ \tilde{T}^R(s) = \mathbb{1}_{C_q} \otimes \epsilon, \\ C_{[p]_0} = C_q = C_q \otimes e, & C_{[q]_0} = C_q, & Q_{00}(s) = \mathbb{1}_{C_q}, \\ C_{[p]_1} = C_q = C_q \otimes x, & C_{[q]_1} = C_q \otimes A^{db}, & T_{11}(s) = \mathbb{1}_{C_q} \otimes \dot{\eta},\ \tilde{T}^L_{11}(s) = \mathbb{1}_{C_q} \otimes \eta. \end{array}$$

Using the decompositions of C_p and C_q given in equations (25) and (24), we find that

$$\begin{aligned} \overline{\partial}^1(s) &= T_{11}(s) + \tilde{T}^L_{11}(s) Q_{10}(n_p) \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \tilde{T}^R(s) & 0 & 0 \end{pmatrix} + \begin{pmatrix} \tilde{T}^R(s) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1}_{C_{[p]_0} C_{[p]_1}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} T(s) & 0 & 0 \\ 0 & 0 & 0 \\ \tilde{T}^R(s) & 0 & 0 \end{pmatrix}, \end{aligned}$$

where we have used the fact that

$$\tilde{T}^R(s) \mathbb{1}_{C_{[p]_0} C_{[p]_1}} = (\mathbb{1}_{C_q} \otimes \epsilon)(\mathbb{1}_{C_q} \otimes \mathbb{1}_{xe}) = \mathbb{1}_{C_q} \otimes \dot{\epsilon} = T(s),$$

and the saddles $[s]_0$ and $[s]_1$ give the maps $Q_{00}(s)$ and $\tilde{T}^L_{11}(s)$ stated in the lemma. □

Lemma B.11. *For a type $C3^\beta_\pm$ saddle $s : p(L) \to q(L)$ or $s : p(R) \to q(R)$, the map $\overline{\partial}^1(s)$ has the form given in equation (15), where*

$$\partial_A(s) = T(s), \qquad \beta_1(s) = 0, \qquad \alpha_2(s) = 0.$$

A type $C3^\beta_\pm$ saddle $s : p(L) \to q(L)$ gives the following maps that contribute to $\overline{\partial}^2$:

$$\tilde{T}^L_{00}(s) = \begin{pmatrix} \tilde{T}^L(s) & 0 & 0 \\ 0 & \tilde{T}^L(s) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{T}^L_{11}(s) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \tilde{T}^L(s) \end{pmatrix}.$$

A type $C3^\beta_\pm$ saddle $s : p(R) \to q(R)$ gives the following maps that contribute to $\overline{\partial}^2$:

$$\tilde{T}^L_{00}(s) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \tilde{T}^L(s) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{T}^L_{11}(s) = \begin{pmatrix} \tilde{T}^L(s) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \tilde{T}^L \end{pmatrix}.$$

Proof. For $s : p(L) \to q(L)$, we have the following square of saddles, which is neither interleaved nor nested:

$$\begin{array}{ccc} C_{[p]_0} & \xrightarrow[{[s]_0}]{C^\beta_\pm} & C_{[q]_0} \\ {\scriptstyle n_p}\big\downarrow{\scriptstyle C^L_-} & & {\scriptstyle n_q}\big\downarrow{\scriptstyle C^L_-} \\ C_{[p]_1} & \xrightarrow[{[s]_1}]{C^\beta_\pm} & C_{[q]_1}. \end{array}$$

A calculation shows:

$$\overline{\partial}^1(s) = T_{00}(s) + T_{11}(s) = \begin{pmatrix} T(s) & 0 & 0 \\ 0 & T(s) & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T(s) \end{pmatrix} = \begin{pmatrix} T(s) & 0 & 0 \\ 0 & T(s) & 0 \\ 0 & 0 & T(s) \end{pmatrix},$$

and the saddles $[s]_0$ and $[s]_1$ give the maps $\tilde{T}^L_{00}(s)$ and $\tilde{T}^L_{11}(s)$ stated in the lemma. The argument for $s : p(R) \to q(R)$ is similar. Note that if $\beta = R$ then $\tilde{T}^L(s) = 0$. □

Lemma B.12. *For a type $C4^\beta_\pm$ saddle $s : p(C) \to q(C)$, the map $\overline{\partial}^1(s)$ has the form given in equation (15), where*

$$\partial_A(s) = T(s), \qquad \beta_1(s) = 0, \qquad \alpha_2(s) = 0.$$

The saddle s gives the following maps that contribute to $\overline{\partial}^2$:

$$\tilde{T}^L_{00}(s) = \begin{pmatrix} \tilde{T}^L(s)\mathbb{1}_{C_{[p]_0}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{T}^L_{11}(s) = \begin{pmatrix} \tilde{T}^L(s)\mathbb{1}_{C_{[p]_1}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. We have the following square of saddles, which is disjoint:

$$\begin{array}{ccc} C_{[p]_0} & \xrightarrow[{[s]_0}]{C^{\beta}_{\pm}} & C_{[q]_0} \\ {\scriptstyle n_p}\downarrow{\scriptstyle W_-} & & {\scriptstyle n_q}\downarrow{\scriptstyle W_-} \\ C_{[p]_1} & \xrightarrow[{[s]_1}]{C^{\beta}_{\pm}} & C_{[q]_1}. \end{array}$$

A calculation shows that

$$\overline{\partial}^1(s) = T_{00}(s) + T_{11}(s) = \begin{pmatrix} T(s)\mathbb{1}_{C_{[p]_0}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} T(s)\mathbb{1}_{C_{[p]_1}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} T(s) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the saddles $[s]_0$ and $[s]_1$ give the maps $\tilde{T}^L_{00}(s)$ and $\tilde{T}^L_{11}(s)$ stated in the lemma. Note that if $\beta = R$ then $\tilde{T}^L(s) = 0$. □

Lemma B.13. *For a type $C1^C_{\pm}$ saddle $s : p(C) \to q(C)$, the map $\overline{\partial}^1(s)$ has the form given in equation (15), where*

$$\partial_A(s) = T(s), \qquad \beta_1(s) = 0, \qquad \alpha_2(s) = 0.$$

The saddle s gives the following maps that contribute to $\overline{\partial}^2$:

$$\tilde{T}^L_{00}(s) = \begin{pmatrix} \tilde{T}^{L_0}(s)\mathbb{1}_{C_{[p]_0}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{T}^L_{11}(s) = \begin{pmatrix} \tilde{T}^{L_1}(s)\mathbb{1}_{C_{[p]_1}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where the maps $\tilde{T}^{L_0}(s)$ and $\tilde{T}^{L_1}(s)$ are described in the proof.

Proof. Define $\sigma = R$ if s splits or merges a circle *inside* the circle component of p containing b and c, and $\sigma = L$ otherwise. We have the following square of saddles, which is nested of type $N^{\overline{\sigma}}_{\pm}$:

$$\begin{array}{ccc} C_{[p]_0} & \xrightarrow[{[s]_0}]{C^{\sigma}_{\pm}} & C_{[q]_0} \\ {\scriptstyle n_p}\downarrow{\scriptstyle W_-} & & {\scriptstyle n_q}\downarrow{\scriptstyle W_-} \\ C_{[p]_1} & \xrightarrow[{[s]_1}]{C^{\overline{\sigma}}_{\pm}} & C_{[q]_1}. \end{array}$$

Consider a type $C1^C_+$ saddle $s : p(C) \to q(C)$. The vector spaces and linear maps are

$$\begin{aligned} &C_p = W_p \otimes A^{bc} && C_q = W_p \otimes A \otimes A^{bc} && T(s) = \mathbb{1}_{W_p} \otimes \Delta, \\ &C_{[p]_0} = W_p \otimes e && C_{[q]_0} = W_p \otimes A \otimes e && T_{00}(s) = T^{\bullet}(s)\mathbb{1}_{C_{[p]_0}}, \tilde{T}^L_{00}(s) = \tilde{T}^{L_0}(s)\mathbb{1}_{C_{[p]_0}}, \\ &C_{[p]_1} = W_p \otimes x && C_{[q]_1} = W_p \otimes A \otimes x && T_{11}(s) = T^{\bullet}(s)\mathbb{1}_{C_{[p]_1}}, \tilde{T}^L_{11}(s) = \tilde{T}^{L_1}(s)\mathbb{1}_{C_{[p]_1}}, \end{aligned}$$

where $W_p = A^{\otimes(c(p)-1)}$ and we have defined

$$\begin{aligned} &T^{\bullet}(s) = \mathbb{1}_{W_p} \otimes \dot{\eta} \otimes \mathbb{1}_A, && T^{\circ}(s) = \mathbb{1}_{W_p} \otimes \eta \otimes \mathbb{1}_A, \\ &\tilde{T}^{L_0}(s) = \delta_{\sigma,L} T^{\circ}(s), && \tilde{T}^{L_1}(s) = \delta_{\overline{\sigma},L} T^{\circ}(s). \end{aligned}$$

Using the decompositions of C_p and C_q given in equation (25), we find that

$$\overline{\partial}^1(s) = T_{00}(s) + T_{11}(s) + Q_{10}(n_q)\tilde{T}^L_{00}(s) + \tilde{T}^L_{11}(s)Q_{10}(n_p) = \begin{pmatrix} T(s) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

where we have used the fact that

$$\begin{aligned} T^{\bullet}(s)\mathbb{1}_{C_{[p]_0}} + T^{\bullet}(s)\mathbb{1}_{C_{[p]_1}} + \mathbb{1}_{C_{[q]_1}C_{[q]_0}}\tilde{T}^{L_0}(s)\mathbb{1}_{C_{[p]_0}} + \tilde{T}^{L_1}(s)\mathbb{1}_{C_{[p]_1}}\mathbb{1}_{C_{[p]_1}C_{[p]_0}} = \\ \mathbb{1}_{W_p} \otimes \dot{\eta} \otimes \mathbb{1}_{ee} + \mathbb{1}_{W_p} \otimes \dot{\eta} \otimes \mathbb{1}_{xx} + (\delta_{\sigma,L} + \delta_{\overline{\sigma},L})\mathbb{1}_{W_p} \otimes \eta \otimes \mathbb{1}_{xe} = \mathbb{1}_{W_p} \otimes \Delta = T(s), \end{aligned}$$

and the saddles $[s]_0$ and $[s]_1$ give the maps $\tilde{T}^L_{00}(s)$ and $\tilde{T}^L_{11}(s)$ stated in the lemma. The argument for a saddle of type $C1^C_-$ is similar. □

Lemma B.14. *For a type* $C2^C_\pm$ *saddle* $s : p(C) \to q(C)$*, the map* $\overline{\partial}^1(s)$ *has the form given in equation (15), where*

$$\partial_A(s) = T(s), \qquad \beta_1(s) = 0, \qquad \alpha_2(s) = 0.$$

The saddle s *does not give any maps that contribute to* $\overline{\partial}^2$*.*

Proof. We have the following square of saddles, which is neither interleaved nor nested:

$$\begin{array}{ccc} C_{[p]_0} & \xrightarrow[{[s]_0}]{C^C_\pm} & C_{[q]_0} \\ {\scriptstyle n_p}\big\downarrow{\scriptstyle W_-} & & {\scriptstyle n_q}\big\downarrow{\scriptstyle W_-} \\ C_{[p]_1} & \xrightarrow[{[s]_1}]{C^C_\pm} & C_{[q]_1}. \end{array}$$

A calculation shows that

$$\overline{\partial}^1(s) = T_{00}(s) + T_{11}(s) = \begin{pmatrix} T(s)\mathbb{1}_{C_{[p]_0}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} T(s)\mathbb{1}_{C_{[p]_1}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} T(s) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

□

Lemma B.15. *For a type* $C3^C_\pm$ *saddle* $s : p(L) \to q(L)$ *or* $s : p(R) \to q(R)$*, the map* $\overline{\partial}^1(s)$ *has the form given in equation (15), where*

$$\partial_A(s) = T(s), \qquad \beta_1(s) = 0, \qquad \alpha_2(s) = 0.$$

The saddle s *does not give any maps that contribute to* $\overline{\partial}^2$*.*

Proof. For a saddle $s : p(L) \to q(L)$, we have the following square of saddles, which is neither interleaved nor nested:

$$\begin{array}{ccc} C_{[p]_0} & \xrightarrow[{[s]_0}]{C^C_\pm} & C_{[q]_0} \\ {\scriptstyle n_p}\big\downarrow{\scriptstyle C^L_-} & & {\scriptstyle n_q}\big\downarrow{\scriptstyle C^L_-} \\ C_{[p]_1} & \xrightarrow[{[s]_1}]{C^C_\pm} & C_{[q]_1}. \end{array}$$

A calculation shows that

$$\overline{\partial}^1(s) = T_{00}(s) + T_{11}(s) = \begin{pmatrix} T(s) & 0 & 0 \\ 0 & T(s) & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T(s) \end{pmatrix} = \begin{pmatrix} T(s) & 0 & 0 \\ 0 & T(s) & 0 \\ 0 & 0 & T(s) \end{pmatrix}.$$

The proof for a saddle $s : p(R) \to q(R)$ is similar. □

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