MODULI SPACES OF HECKE MODIFICATIONS FOR RATIONAL AND ELLIPTIC CURVES

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Abstract. We propose definitions of complex manifolds $\mathcal{P}_M(X, m, n)$ that could potentially be used to construct the symplectic Khovanov homology of n-stranded links in lens spaces. The manifolds $\mathcal{P}_M(X, m, n)$ are defined as moduli spaces of Hecke modifications of rank 2 parabolic bundles over an elliptic curve X. To characterize these spaces, we describe all possible Hecke modifications of all possible rank 2 vector bundles over X, and we use these results to define a canonical open embedding of $\mathcal{P}_M(X, m, n)$ into $M^s(X, m+n)$, the moduli space of stable rank 2 parabolic bundles over X with trivial determinant bundle and $m+n$ marked points. We explicitly compute $\mathcal{P}_M(X, 1, n)$ for $n = 0, 1, 2$. For comparison, we present analogous results for the case of rational curves, for which a corresponding complex manifold $\mathcal{P}_M(\mathbb{CP}^1, 3, n)$ is isomorphic for n even to a space $\mathcal{Y}(S^2, n)$ defined by Seidel and Smith that can be used to compute the symplectic Khovanov homology of n-stranded links in S^3.

Contents

1. Introduction

Khovanov homology is a powerful invariant for distinguishing links in S^3 [15]. Khovanov homology can be viewed as a categorification of the Jones polynomial [12]: one can recover the Jones polynomial of a link from its Khovanov homology, but the Khovanov homology generally contains more information. For example, Khovanov homology can sometimes distinguish distinct links that have the same Jones polynomial, and Khovanov homology detects the unknot [16], but it is not known whether the Jones polynomial has this property. The Khovanov homology of a link can be obtained in a purely algebraic fashion by computing the homology of a chain complex constructed from a generic planar projection of the link. The Khovanov homology can also be obtained in a geometric fashion by

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Heegaard-splitting S^3 into two 3-balls in such a way that the intersection of the link with each 3-ball consists of r unknotted arcs. Each 3-ball determines a Lagrangian in a symplectic manifold $\mathcal{Y}(S^2, 2r)$ known as the Seidel–Smith space, and the Lagrangian Floer homology of the pair of Lagrangians yields the Khovanov homology of the link (modulo grading) [1, 24]. Recently Witten has outlined gauge theory interpretations of Khovanov homology and the Jones polynomial in which the Seidel–Smith space is viewed as a moduli space of solutions to the Bogomolny equations [30, 29, 31].

Little is known about how Khovanov homology could be generalized to describe links in 3-manifolds other than S^3, but such results would be of great interest. As a first step towards this goal, one might consider the problem of generalizing Khovanov homology to links in 3-manifolds with Heegaard genus 1; that is, lens spaces. In analogy with the Seidel–Smith approach to Khovanov homology, one could Heegaard-split a lens space into two solid tori, each containing r unknotted arcs, and compute the Lagrangian Floer homology of a pair of Lagrangians intersecting in a symplectic manifold $\mathcal{Y}(T^2, 2r)$ that generalizes the Seidel–Smith space $\mathcal{Y}(S^2, 2r)$.

In this paper we propose candidates for the manifold $\mathcal{Y}(T^2, 2r)$. In outline, our approach is as follows. First, using a result due to Kamnitzer [13], we reinterpret the Seidel–Smith space $\mathcal{Y}(S^2, 2r)$ as a moduli space $\mathcal{H}(\mathbb{CP}^1, 2r)$ of equivalence classes of sequences of Hecke modifications of rank 2 holomorphic vector bundles over a rational curve. Roughly speaking, a *Hecke modification* is a way of locally modifying a holomorphic vector bundle near a point to obtain a new vector bundle. We show that there is a close relationship between Hecke modifications and parabolic bundles, and we use this relationship to reinterpret the Kamnitzer space $\mathcal{H}(\mathbb{CP}^1, 2r)$ as a moduli space $\mathcal{P}_M(\mathbb{CP}^1, 2r)$ of isomorphism classes of parabolic bundles with marking data. The space $\mathcal{P}_M(\mathbb{CP}^1, 2r)$ has two natural generalizations $\mathcal{P}_M(X, 1, 2r)$ and $\mathcal{P}_M(X, 3, 2r)$ to the case of an elliptic curve X, and we propose these spaces as candidates for $\mathcal{Y}(T^2, 2r)$.

To explain our approach in detail, we first introduce some additional spaces. Given a rank 2 holomorphic vector bundle E over a curve C, we define a set $\mathcal{H}^{tot}(C, E, n)$ of equivalence classes of sequences of n Hecke modifications of E. As is well-known, the set $\mathcal{H}^{tot}(C, E, n)$ has the structure of a complex manifold that is (noncanonically) isomorphic to $(\mathbb{CP}^1)^n$, where each factor of $\mathbb{CP}^1$ corresponds to a single Hecke modification. The Kamnitzer space $\mathcal{H}(\mathbb{CP}^1, 2r)$ is then defined to be the open submanifold of $\mathcal{H}^{tot}(\mathbb{CP}^1, \mathcal{O} \oplus \mathcal{O}, 2r)$ consisting of equivalence classes of sequences of Hecke modifications for which the terminal vector is semistable.

Example 1.1. For $r = 1$, we have that

$$\mathcal{H}^{tot}(\mathbb{CP}^1, \mathcal{O} \oplus \mathcal{O}, 2) = (\mathbb{CP}^1)^2, \qquad \mathcal{H}(\mathbb{CP}^1, 2) = (\mathbb{CP}^1)^2 - \{(a, a) \mid a \in \mathbb{CP}^1\}.$$

To generalize the Kamnitzer space to curves of higher genus, we want to define moduli spaces of sequences of Hecke modifications in which the initial vector bundle in the sequence is allowed to vary. We define such spaces via the use of parabolic bundles. For any curve C and any natural numbers m and n, we define a moduli space $\mathcal{P}^{tot}_M(C, m, n)$ of marked parabolic bundles. We prove:

Theorem 1.2. *The moduli space $\mathcal{P}^{tot}_M(C, m, n)$ naturally has the structure of a complex manifold isomorphic to a $(\mathbb{CP}^1)^n$-bundle over $M^s(C, m)$.*

Here the complex manifold $M^s(C, m)$ is the moduli space of stable rank 2 parabolic bundles over a curve C with trivial determinant bundle and m marked points. Roughly speaking, the space $\mathcal{P}^{tot}_M(C, m, n)$ describes isomorphism classes of sequences of n Hecke modifications in which the initial vector bundle in the sequence is allowed to range over isomorphism classes that are parameterized by $M^s(C, m)$. By imposing a condition analogous to the semistability condition used to define the Kamnitzer space $\mathcal{H}(\mathbb{CP}^1, 2r)$, we identify an open submanifold $\mathcal{P}_M(C, m, n)$ of $\mathcal{P}^{tot}_M(C, m, n)$. We prove that $\mathcal{P}^{tot}_M(\mathbb{CP}^1, n) := \mathcal{P}^{tot}_M(\mathbb{CP}^1, 3, n)$ is isomorphic to $\mathcal{H}^{tot}(\mathbb{CP}^1, \mathcal{O} \oplus \mathcal{O}, n)$ and $\mathcal{P}_M(\mathbb{CP}^1, 2r) := \mathcal{P}_M(\mathbb{CP}^1, 3, 2r)$ is isomorphic to to $\mathcal{H}(\mathbb{CP}^1, 2r)$. Thus the Seidel–Smith space $\mathcal{Y}(S^2, 2r)$, the Kamnitzer space $\mathcal{H}(\mathbb{CP}^1, 2r)$, and the space of marked parabolic bundles $\mathcal{P}_M(\mathbb{CP}^1, 2r)$ are all isomorphic. However, unlike $\mathcal{Y}(S^2, 2r)$ or $\mathcal{H}(\mathbb{CP}^1, 2r)$, the space $\mathcal{P}_M(\mathbb{CP}^1, 2r)$ naturally generalizes to the case of elliptic curves.

Although our primary motivation for introducing the spaces $\mathcal{P}^{tot}_M(\mathbb{CP}^1, n)$ and $\mathcal{P}_M(\mathbb{CP}^1, n)$ is to facilitate generalization, they are also useful for proving canonical versions of certain results for $\mathbb{CP}^1$. For example, we prove a canonical version of the noncanonical isomorphism $\mathcal{H}^{tot}(\mathbb{CP}^1, \mathcal{O} \oplus \mathcal{O}, n) \to (\mathbb{CP}^1)^n$ for $\mathcal{P}^{tot}_M(\mathbb{CP}^1, n)$:

Theorem 1.3. *There is a canonical isomorphism $\mathcal{P}^{tot}_M(\mathbb{CP}^1, n) \to (M^{ss}(\mathbb{CP}^1, 4))^n$.*

Here the complex manifold $M^{ss}(\mathbb{CP}^1, 4) \cong \mathbb{CP}^1$ is the moduli space of semistable rank 2 parabolic bundles over $\mathbb{CP}^1$ with trivial determinant bundle and 4 marked points. We also prove:

Theorem 1.4. *There is a canonical open embedding $\mathcal{P}_M(\mathbb{CP}^1, m, n) \to M^s(\mathbb{CP}^1, m + n)$.*

Corollary 1.5. *There is a canonical open embedding $\mathcal{P}_M(\mathbb{CP}^1, 2r) \to M^s(\mathbb{CP}^1, 2r + 3)$.*

Here the complex manifold $M^s(\mathbb{CP}^1, m+n)$ is the moduli space of stable rank 2 parabolic bundles over $\mathbb{CP}^1$ with trivial determinant bundle and $m+n$ marked points. For $r = 1, 2$ we have verified that the embedding of $\mathcal{P}_M(\mathbb{CP}^1, 2r)$ into $M^s(\mathbb{CP}^1, 2r+3)$ agrees with a (noncanonical) embedding due to Woodward of the Seidel–Smith space $\mathcal{Y}(S^2, 2r)$ into $M^s(\mathbb{CP}^1, 2r+3)$, and we conjecture that the agreement holds for all r.

We next proceed to the case of elliptic curves. We show that our reinterpretation $\mathcal{P}_M(\mathbb{CP}^1, 2r)$ of the Seidel–Smith space $\mathcal{Y}(S^2, 2r)$ has two natural generalizations to the case of an elliptic curve X, namely $\mathcal{P}_M(X, 1, 2r)$ and $\mathcal{P}_M(X, 3, 2r)$. We prove an elliptic-curve analog to Theorem 1.3:

Theorem 1.6. *There is a canonical isomorphism $\mathcal{P}_M^{tot}(X, 1, n) \to (M^{ss}(X))^{n+1}$.*

Here the complex manifold $M^{ss}(X) \cong \mathbb{CP}^1$ is the moduli space of semistable rank 2 vector bundles over an elliptic curve X with trivial determinant bundle. Comparing Theorems 1.3 and 1.6, we see that $\mathcal{P}_M^{tot}(\mathbb{CP}^1, n)$ is (noncanonically) isomorphic to $(\mathbb{CP}^1)^n$, whereas $\mathcal{P}_M^{tot}(X, 1, n)$ is (noncanonically) isomorphic to $(\mathbb{CP}^1)^{n+1}$. The extra factor of $\mathbb{CP}^1$ for $\mathcal{P}_M^{tot}(X, 1, n)$ can be understood from Theorem 1.2, which states that $\mathcal{P}_M^{tot}(\mathbb{CP}^1, n) = \mathcal{P}_M^{tot}(\mathbb{CP}^1, 3, n)$ is a $(\mathbb{CP}^1)^n$-bundle over $M^s(\mathbb{CP}^1, 3)$ and $\mathcal{P}_M^{tot}(X, 1, n)$ is a $(\mathbb{CP}^1)^n$-bundle over $M^s(X, 1)$. But $M^s(\mathbb{CP}^1, 3)$ is a single point, whereas $M^s(X, 1)$ is isomorphic to $\mathbb{CP}^1$. We use Theorem 1.6 to explicitly compute $\mathcal{P}_M(X, 1, n)$ for $n = 0, 1, 2$:

Theorem 1.7. *The space $\mathcal{P}_M(X, 1, n)$ for $n = 0, 1, 2$ is given by*

$$\mathcal{P}_M(X, 1, 0) = \mathbb{CP}^1, \qquad \mathcal{P}_M(X, 1, 1) = (\mathbb{CP}^1)^2 - g(X), \qquad \mathcal{P}_M(X, 1, 2) = (\mathbb{CP}^1)^3 - f(X),$$

where $g : X \to (\mathbb{CP}^1)^2$ and $f : X \to (\mathbb{CP}^1)^3$ are holomorphic embeddings defined in Sections 5.5.2 and 5.5.3.

We also generalize the embedding result of Theorem 1.4 to the case of elliptic curves:

Theorem 1.8. *There is a canonical open embedding $\mathcal{P}_M(X, m, n) \to M^s(X, m+n)$.*

Here the complex manifold $M^s(X, m+n)$ is the moduli space of stable rank 2 parabolic bundles on X with trivial determinant bundle and $m+n$ marked points. In order to prove Theorems 1.6, 1.7, and 1.8, we construct a list of all possible Hecke modifications of all possible rank 2 vector bundles on an elliptic curve X.

We conclude by discussing possible applications of our results to the problem of generalizing symplectic Khovanov homology to lens spaces. We observe that the embedding results of Theorems 1.4 and 1.8 could be related to a conjectural spectral sequence from symplectic Khovanov homology to symplectic instanton homology, which would generalize a spectral sequence due to Kronheimer and Mrowka from Khovanov homology to singular instanton homology [17]. Based on such considerations, we make the following conjectures:

Conjecture 1.9. *The space $\mathcal{P}_M(C, 3, 2r)$ is the correct generalization of the Seidel–Smith space $\mathcal{Y}(S^2, 2r)$ to a curve C of arbitrary genus.*

Conjecture 1.10. *Given a curve C of arbitrary genus, there is a canonical open embedding $\mathcal{P}_M(C, m, n) \to M^s(C, m+n)$.*

The paper is organized as follows. In Section 2 we introduce some terminology specific to this paper. In Section 3 we review the relevant background material on Hecke modifications, discuss the relationship between Hecke modifications and parabolic bundles, and define moduli spaces of marked parabolic bundles $\mathcal{P}_M^{tot}(C, m, n)$ and $\mathcal{P}_M(C, m, n)$. In Section 4 we describe Hecke modifications of rank 2 vector bundles on rational curves and show that the Seidel–Smith space $\mathcal{Y}(S^2, 2r)$ can be reinterpreted as the space of marked parabolic bundles $\mathcal{P}_M(\mathbb{CP}^1, 2r) = \mathcal{P}_M(\mathbb{CP}^1, 3, 2r)$. In Section 5 we describe Hecke modifications of rank 2 vector bundles on elliptic curves and discuss possible elliptic-curve generalizations of the Seidel–Smith space. In Section 6 we consider possible applications of our results to topology. In Appendix A and B we review the material we will need regarding vector bundles and parabolic bundles on curves.

2. Terminology

Here we introduce some terminology specific to this paper.

Definition 2.1. Given a rank 2 holomorphic vector bundle E over a curve C, we define the *instability degree* of E to be $\deg L - \deg E/L$, where $L \subset E$ is a line subbundle of maximal degree.

The degree of the proper subbundles of a vector bundle E on a curve C is bounded above (see for example [22, Lemma 3.21]), so the notion of instability degree is well-defined. The instability degree is positive for unstable bundles, 0 for strictly semistable bundles, and negative for stable bundles.

Definition 2.2. Given a rank 2 holomorphic vector bundle E over a curve C and a point $p \in C$, we say that a line $\ell_p \in \mathbb{P}(E_p)$ in the fiber E_p over p is *bad* if there is a line subbundle $L \subset E$ of maximal degree such that $L_p = \ell_p$, and *good* otherwise.

Example 2.3. For the trivial bundle $\mathcal{O} \oplus \mathcal{O}$ over $\mathbb{CP}^1$, all lines are bad.

Definition 2.4. Given a rank 2 holomorphic vector bundle E over a curve C and points $p_1, \cdots, p_n \in C$, we say that lines $\ell_{p_i} \in \mathbb{P}(E_{p_i})$ for $i = 1, \cdots, n$ are *bad in the same direction* if there is a line subbundle $L \subset E$ of maximal degree such that $L_{p_i} = \ell_{p_i}$ for $i = 1, \cdots, n$.

3. Hecke modifications and parabolic bundles

3.1. **Hecke modifications at a single point.** A fundamental concept for us is the notion of a Hecke modification of a rank 2 holomorphic vector bundle. This notion is described in [13, 14]. Here we consider the case of a single Hecke modification.

Definition 3.1. Let $\pi_E : E \to C$ be a rank 2 holomorphic vector bundle over a curve C. A *Hecke modification* $E \xleftarrow[p]{\alpha} F$ of E at a point $p \in C$ is a rank 2 holomorphic vector bundle $\pi_F : F \to C$ and a bundle map $\alpha : F \to E$ that satisfies the following two conditions:

1. The induced maps on fibers $\alpha_q : F_q \to E_q$ are isomorphisms for all points $q \in C$ such that $q \neq p$.
2. We also impose a condition on the behavior of α near p. We require that there is an open neighborhood $U \subset C$ of p, local coordinates $\xi : U \to V$ for $V \subset \mathbb{C}$ such that $\xi(p) = 0$, and local trivializations $\psi_E : \pi_E^{-1}(U) \to U \times \mathbb{C}^2$ and $\psi_F : \pi_F^{-1}(U) \to U \times \mathbb{C}^2$ of E and F over U such that the following diagram commutes:

$$\begin{tikzcd}
\pi_E^{-1}(U) \arrow[d, "\psi_E", "\cong"'] & \pi_F^{-1}(U) \arrow[l, "\alpha"'] \arrow[d, "\psi_F", "\cong"'] \\
U \times \mathbb{C}^2 & U \times \mathbb{C}^2, \arrow[l]
\end{tikzcd}$$

where the bottom horizontal arrow is

$$(\psi_E \circ \alpha \circ \psi_F^{-1})(q, v) = (q, \bar{\alpha}(\xi(q))v)$$

and $\bar{\alpha} : V \to M(2, \mathbb{C})$ has the form

$$\bar{\alpha}(z) = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}.$$

It follows directly from Definition 3.1 that $\det F = (\det E) \otimes \mathcal{O}(-p)$ and $\deg F = \deg E - 1$. There is a natural notion of equivalence of Hecke modifications:

Definition 3.2. We say that two Hecke modifications $E \xleftarrow[p]{\alpha} F$ and $E \xleftarrow[p]{\alpha'} F'$ of E at a point $p \in C$ are *equivalent* if there is an isomorphism $\phi : F \to F'$ such that $\alpha = \alpha' \circ \phi$.

Definition 3.3. We define the *total space of Hecke modifications* $\mathcal{H}^{tot}(C, E; p)$ to be the set of equivalence classes of Hecke modifications of a rank 2 vector bundle $\pi_E : E \to C$ at a point $p \in C$.

As is well-known, the set $\mathcal{H}^{tot}(C, E; p)$ naturally has the structure of a complex manifold that is (noncanonically) isomorphic to $\mathbb{CP}^1$. A canonical version of this statement is:

Theorem 3.4. *There is a canonical isomorphism $\mathcal{H}^{tot}(C, E; p) \to \mathbb{P}(E_p)$, $[E \xleftarrow[p]{\alpha} F] \mapsto \mathrm{im}(\alpha_p : F_p \to E_p)$.*

It is also useful to think about Hecke modifications in terms of sheaves of sections. Consider a rank 2 vector bundle E and a line $\ell_p \in \mathbb{P}(E_p)$. Let $\mathcal{E}$ be the sheaf of sections of E, and define a subsheaf $\mathcal{F}$ of $\mathcal{E}$ whose set of sections over an open set $U \subset C$ is given by

$$\mathcal{F}(U) = \{s \in \mathcal{E}(U) \mid p \in U \implies s(p) \in \ell_p\}.$$

Define F to be the vector bundle whose sheaf of sections is $\mathcal{F}$, and define $\alpha : F \to E$ to be the bundle map corresponding to the inclusion of sheaves $\mathcal{F} \to \mathcal{E}$. Then $[E \xleftarrow[p]{\alpha} F] \in \mathcal{H}^{tot}(C, E; p)$ corresponds to $\ell_p \in \mathbb{P}(E_p)$ under the isomorphism described in Theorem 3.4. We have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathbb{C}_p \longrightarrow 0,$$

where $\mathbb{C}_p$ is a skyscraper sheaf supported at the point p. It is important to note, however, that the usual notion of equivalence of extensions differs from the notion of equivalence of Hecke modifications given in Definition 3.2.

3.2. **Sequences of Hecke modifications.** We would now like to generalize the notion of a Hecke modification of a vector bundle at a single point $p \in C$ to the notion of a sequence of Hecke modifications at distinct points $(p_1, \cdots, p_n) \in C^n$.

Definition 3.5. Let $\pi_E : E \to C$ be a rank 2 holomorphic vector bundle over a curve C. A *sequence of Hecke modifications* $E \xleftarrow[p_1]{\alpha_1} E_1 \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_n]{\alpha_n} E_n$ of E at distinct points $(p_1, p_2, \cdots, p_n) \in C^n$ is a collection of rank 2 holomorphic vector bundles $\pi_{E_i} : E_i \to C$ and Hecke modifications $E_{i-1} \xleftarrow[p_i]{\alpha_i} E_i$ for $i = 1, 2, \cdots, n$, where $E_0 := E$.

Definition 3.6. Two sequences of Hecke modifications $E \xleftarrow[p_1]{\alpha_1} E_1 \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_n]{\alpha_n} E_n$ and $E \xleftarrow[p_1]{\alpha_1'} E_1' \xleftarrow[p_2]{\alpha_2'} \cdots \xleftarrow[p_n]{\alpha_n'} E_n'$ are *equivalent* if there are isomorphisms $\phi_i : E_i \to E_i'$ such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[d, "="'] & E_1 \arrow[l, "\alpha_1"'] \arrow[d, "\cong"', "\phi_1"] & E_2 \arrow[l, "\alpha_2"'] \arrow[d, "\cong"', "\phi_2"] & \cdots \arrow[l, "\alpha_3"'] & E_n \arrow[l, "\alpha_n"'] \arrow[d, "\cong"', "\phi_n"] \\ E & E_1' \arrow[l, "\alpha_1'"'] & E_2' \arrow[l, "\alpha_2'"'] & \cdots \arrow[l, "\alpha_3'"'] & E_n'. \arrow[l, "\alpha_n'"'] \end{tikzcd}$$

Definition 3.7. We define the *total space of Hecke modifications* $\mathcal{H}^{tot}(C, E; p_1, \cdots, p_n)$ to be the set of equivalence classes of sequences of Hecke modifications of the rank 2 vector bundle $\pi_E : E \to C$ at points $(p_1, \cdots, p_n) \in C^n$. For simplicity, we will often suppress the dependence on $p_1, \cdots, p_n$ and denote this space as $\mathcal{H}^{tot}(C, E, n)$.

Definition 3.8. We say that an isomorphism of vector bundles $\phi : E \to E'$ is an *isomorphism of equivalence classes of sequences of Hecke modifications* $\phi : [E \xleftarrow[p_1]{\alpha_1} E_1 \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_n]{\alpha_n} E_n] \to [E' \xleftarrow[p_1]{\alpha_1'} E_1' \xleftarrow[p_2]{\alpha_2'} \cdots \xleftarrow[p_n]{\alpha_n'} E_n']$ if

$$[E' \xleftarrow[p_1]{\beta_1} E_1 \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_n]{\alpha_n} E_n] = [E' \xleftarrow[p_1]{\alpha_1'} E_1' \xleftarrow[p_2]{\alpha_2'} \cdots \xleftarrow[p_n]{\alpha_n'} E_n'],$$

where $\beta_1 := \phi \circ \alpha_1$, or equivalently, if there are isomorphisms $\phi_i : E_i \to E_i'$ such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[d, "\cong"', "\phi"] & E_1 \arrow[l, "\alpha_1"'] \arrow[d, "\cong"', "\phi_1"] & E_2 \arrow[l, "\alpha_2"'] \arrow[d, "\cong"', "\phi_2"] & \cdots \arrow[l, "\alpha_3"'] & E_n \arrow[l, "\alpha_n"'] \arrow[d, "\cong"', "\phi_n"] \\ E' & E_1' \arrow[l, "\alpha_1'"'] & E_2' \arrow[l, "\alpha_2'"'] & \cdots \arrow[l, "\alpha_3'"'] & E_n'. \arrow[l, "\alpha_n'"'] \end{tikzcd}$$

In what follows, it will be useful to reinterpret equivalence classes of sequences of Hecke modifications in terms of parabolic bundles. The relevant background material on parabolic bundles is discussed in Appendix B. For our purposes here, a rank 2 parabolic bundle over a curve C consists of a rank 2 holomorphic vector bundle $\pi_E : E \to C$, a parameter $\mu > 0$, called the *weight*, and a choice of line $\ell_{p_i} \in \mathbb{P}(E_{p_i})$ in the fiber $E_{p_i} = \pi_E^{-1}(p_i)$ over the point $p_i \in C$ for a finite number of distinct points $(p_1, \cdots, p_n) \in C^n$. The data of just the marked points and lines, without the weight, is referred to as a *quasi-parabolic structure* on E. The additional data of the weight allows us to define the notions of stable, semistable, and unstable parabolic bundles.

Definition 3.9. We define $\mathcal{P}^{tot}(C, E; p_1, \cdots, p_n) = \mathbb{P}(E_{p_1}) \times \cdots \times \mathbb{P}(E_{p_n})$ to be the set of all quasi-parabolic structures with marked points $(p_1, \cdots, p_n) \in C^n$ on a rank 2 holomorphic vector bundle $\pi_E : E \to C$. For simplicity, we will often suppress the dependence on $p_1, \cdots, p_n$ and denote this space as $\mathcal{P}^{tot}(C, E, n)$.

Since $\mathbb{P}(E_{p_i})$ is (noncanonically) isomorphic to $\mathbb{CP}^1$, the set $\mathcal{P}^{tot}(C, E; p_1, \cdots, p_n)$ naturally has the structure of a complex manifold that is (noncanonically) isomorphic to $(\mathbb{CP}^1)^n$. We have the following generalization of Theorem 3.4, which allows us to reinterpret Hecke modifications in terms of parabolic bundles:

Theorem 3.10. *There is a canonical isomorphism* $\mathcal{H}^{tot}(C, E; p_1, \cdots, p_n) \to \mathcal{P}^{tot}(C, E; p_1, \cdots, p_n)$ *given by*

$$[E \xleftarrow[p_1]{\alpha_1} E_1 \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_n]{\alpha_n} E_n] \mapsto (E, \ell_{p_1}, \cdots, \ell_{p_n}),$$

where $\ell_{p_i} := \mathrm{im}((\alpha_1 \circ \cdots \circ \alpha_i)_{p_i} : (E_i)_{p_i} \to E_{p_i})$.

Under our reinterpretation of Hecke modifications in terms of parabolic bundles, an isomorphism of equivalence classes of sequences of Hecke modifications corresponds to an isomorphism of parabolic bundles:

Definition 3.11. We say that an isomorphism of vector bundles $\phi : E \to E'$ is an *isomorphism of parabolic bundles* $\phi : (E, \ell_{p_1}, \cdots, \ell_{p_n}) \to (E', \ell_{p_1}', \cdots, \ell_{p_n}')$ if $\phi(\ell_{p_i}) = \ell_{p_i}'$ for $i = 1, \cdots, n$.

Theorem 3.12. *Two equivalence classes of sequences of Hecke modifications are isomorphic if and only if their corresponding parabolic bundles are isomorphic.*

Proof. This is a direct consequence of Theorem 3.10 and Definitions 3.8 and 3.11. □

For many applications, given an equivalence class $[E \xleftarrow[p_1]{\alpha_1} E_1 \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_n]{\alpha_n} E_n]$ of sequences of Hecke modifications we will be interested only in the isomorphism class of the terminal vector bundle E_n, and it is useful to have a means of extracting this information from the corresponding parabolic bundle $(E, \ell_{p_1}, \cdots, \ell_{p_n})$:

Definition 3.13. Let $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ be a parabolic bundle over a curve C. We define the *Hecke transform* $H(E, \ell_{p_1}, \cdots, \ell_{p_n})$ of E to be the vector bundle F that is constructed as follows. Let $\mathcal{E}$ be the sheaf of sections of E. Define a subsheaf $\mathcal{F}$ of $\mathcal{E}$ whose set of sections over an open set $U \subset C$ is given by

$$\mathcal{F}(U) = \{s \in \mathcal{E}(U) \mid p_i \in U \implies s(p_i) \in \ell_{p_i} \text{ for } i = 1, \cdots, n\}.$$

Now define F to be the vector bundle whose sheaf of sections is $\mathcal{F}$.

In particular, $H(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is isomorphic to E_n. We will often want to pick out an open subset of $\mathcal{P}^{tot}(C, E, n)$ by using the Hecke transform to impose a semistability condition:

Definition 3.14. Given a rank 2 holomorphic vector bundle E over a curve C and distinct points $(p_1, \cdots, p_n) \in C^n$, we define $\mathcal{P}(C, E, n)$ to be the subset of $\mathcal{P}^{tot}(C, E, n)$ consisting of parabolic bundles $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ such that $H(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is semistable:

$$\mathcal{P}(C, E, n) = \{(E, \ell_{p_1}, \cdots, \ell_{p_n}) \in \mathcal{P}^{tot}(C, E, n) \mid H(E, \ell_{p_1}, \cdots, \ell_{p_n}) \text{ is semistable}\}.$$

For simplicity, we are suppressing the dependence of $\mathcal{P}(C, m, n)$ on $p_1, \cdots, p_n$ in the notation.

Theorem 3.15. *The set $\mathcal{P}(C, E, n)$ is an open submanifold of $\mathcal{P}^{tot}(C, E, n)$.*

Proof. This follows from the fact that semistability is an open condition. □

We have generalized the notion of a Hecke modification to the case of multiple points $p_1, \cdots, p_n$ by considering sequences of Hecke modifications, for which the points must be ordered. For most of our purposes we could equally well use an alternative generalization, described in [13], for which the the points need not be ordered. Though we will not use it here, we briefly describe this alternative generalization and show how it relates to parabolic bundles:

Definition 3.16. Let $\pi_E : E \to C$ be a rank 2 holomorphic vector bundle over a curve C. A *simultaneous Hecke modification* $E \xleftarrow[\{p_1, \cdots, p_n\}]{\alpha} F$ of E at a set of distinct points $\{p_1, p_2, \cdots, p_n\} \subset C$ is a rank 2 holomorphic vector bundle $\pi_F : F \to C$ and a bundle map $\alpha : F \to E$ that satisfies the following two conditions:

(1) The induced map on fibers $\alpha_q : E_q \to F_q$ is an isomorphism for all points $q \notin \{p_1, \cdots, p_n\}$.
(2) Condition (2) of Definition 3.1, which constrains the local behavior of α near a Hecke-modification point, holds at each of the points $p_1, \cdots, p_n$.

Definition 3.17. Two simultaneous Hecke modifications $E \xleftarrow[\{p_1, \cdots, p_n\}]{\alpha} F$ and $E \xleftarrow[\{p_1, \cdots, p_n\}]{\alpha'} F'$ are *equivalent* if there is an isomorphism $\phi : F \to F'$ such that $\alpha = \alpha' \circ \phi$.

Definition 3.18. We define the *total space of simultaneous Hecke modifications* $\overline{\mathcal{H}}^{tot}(C, E, \{p_1, \cdots, p_n\})$ to be the set of equivalence classes of simultaneous Hecke modifications of the rank 2 vector bundle $\pi_E : E \to C$ at the set of points $\{p_1, \cdots, p_n\} \subset C$. For simplicity, we will often suppress the dependence on $\{p_1, \cdots, p_n\}$ and denote this space as $\overline{\mathcal{H}}^{tot}(C, E, n)$.

We can define a set of parabolic bundles $\overline{\mathcal{P}}^{tot}(C, E, n)$ for which the marked points are unordered. We can define an isomorphism $\overline{\mathcal{H}}^{tot}(C, E, n) \to \overline{\mathcal{P}}^{tot}(C, E, n)$ by

$$[E \xleftarrow[\{p_1, \cdots, p_n\}]{\alpha} F] \mapsto (E, \ell_{p_1}, \cdots, \ell_{p_n}),$$

where $\ell_{p_i} = \mathrm{im}(\alpha_{p_i} : F_{p_i} \to E_{p_i})$. Since the Hecke transform does not depend on the ordering of the points, it is well-defined on parabolic bundles in $\overline{\mathcal{P}}^{tot}(C, E, n)$, and we have that $H(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is isomorphic to F.

3.3. **Moduli spaces of marked parabolic bundles.** So far we have considered spaces of isomorphism classes of sequences of Hecke modifications in which the initial vector bundle in the sequence is held fixed. But in what follows we will want to generalize these spaces so the initial vector bundle is allowed to range over the isomorphism classes in a moduli space of vector bundles. Translating into the language of parabolic bundles, such spaces are equivalent to spaces of isomorphism classes of parabolic bundles in which the underlying vector bundles are allowed to range over the isomorphism classes in a moduli space of vector bundles. However, there is a problem with defining such spaces arising from the fact that vector bundles may have nontrivial automorphisms.

To illustrate the problem, consider the space $\mathcal{P}^{tot}(\mathbb{CP}^1, \mathcal{O}\oplus\mathcal{O}; p_1)$, which is (noncanonically) isomorphic to $\mathbb{CP}^1$. We might want to reinterpret this space as a moduli space of isomorphism classes of parabolic bundles of the form (E, ℓ_{p_1}) for $[E] \in M^{ss}(\mathbb{CP}^1)$, where $M^{ss}(\mathbb{CP}^1)$, the moduli space of semistable rank 2 vector bundles over $\mathbb{CP}^1$ with trivial determinant bundle, consists of the single point $[\mathcal{O}\oplus\mathcal{O}]$. But $\mathrm{Aut}(\mathcal{O}\oplus\mathcal{O}) = GL(2,\mathbb{C})$, and for any pair of parabolic bundles $(\mathcal{O}\oplus\mathcal{O}, \ell_{p_1})$ and $(\mathcal{O}\oplus\mathcal{O}, \ell'_{p_1})$ there is an automorphism $\phi \in \mathrm{Aut}(\mathcal{O}\oplus\mathcal{O})$ such that $\phi(\ell_{p_1}) = \ell'_{p_1}$. It follows that all parabolic bundles of the form $(\mathcal{O}\oplus\mathcal{O}, \ell_{p_1})$ are isomorphic and our proposed moduli space collapses to a point, when what we wanted was $\mathbb{CP}^1$.

To remedy the problem, we will add marking data to eliminate the nontrivial automorphisms. In particular, since stable parabolic bundles have no nontrivial automorphisms, we make the following definition:

Definition 3.19. Given a curve C and distinct points $(q_1, \cdots, q_m, p_1, \cdots, p_n) \in C^{m+n}$, we define the *total space of marked parabolic bundles* $\mathcal{P}^{tot}_M(C, m, n)$ to be the set of isomorphism classes of parabolic bundles of the form $(E, \ell_{q_1}, \cdots, \ell_{q_m}, \ell_{p_1}, \cdots, \ell_{p_n})$ such that $[E, \ell_{q_1}, \cdots, \ell_{q_m}] \in M^s(C, m)$:

$$\mathcal{P}^{tot}_M(C, m, n) = \{[E, \ell_{q_1}, \cdots, \ell_{q_m}, \ell_{p_1}, \cdots, \ell_{p_n}] \mid [E, \ell_{q_1}, \cdots, \ell_{q_m}] \in M^s(C, m)\}.$$

For simplicity, we are suppressing the dependence of $\mathcal{P}^{tot}_M(C, m, n)$ on $q_1, \cdots, q_m, p_1, \cdots, p_n$ in the notation.

Here the complex manifold $M^s(C, m)$ is the moduli space of stable rank 2 parabolic bundles over C with trivial determinant bundle and m marked points. We will refer to the lines $\ell_{q_1}, \cdots, \ell_{q_m}$ as *marking lines*, since their purpose is to add additional structure to E so as to eliminate nontrivial automorphisms. We will refer to the lines $\ell_{p_1}, \cdots, \ell_{p_n}$ as *Hecke lines*, since their purpose is to parameterize Hecke modifications at the points $p_1, \cdots, p_n$. Because we have defined $\mathcal{P}^{tot}_M(C, m, n)$ in terms of stable parabolic bundles, which have no nontrivial automorphisms, the collapsing phenomenon described above does not occur, and we have the following result:

Theorem 3.20. *The set $\mathcal{P}^{tot}_M(C, m, n)$ naturally has the structure of a complex manifold isomorphic to a $(\mathbb{CP}^1)^n$-bundle over $M^s(C, m)$.*

The base manifold $M^s(C, m)$ constitutes the moduli space over which the isomorphism classes of vector bundles with marking data range, and the $(\mathbb{CP}^1)^n$ fibers correspond to a space of Hecke modifications $\mathbb{CP}^1$ for each of the points $p_1, \cdots, p_n$. We will prove Theorem 3.20 by constructing $\mathcal{P}^{tot}_M(C, m, n)$ from a universal $\mathbb{CP}^1$-bundle, which we first describe for the case $m = 0$:

Lemma 3.21. *There is a universal $\mathbb{CP}^1$-bundle $P \to C \times M^s(C)$, which has the the property that for any complex manifold S and any $\mathbb{CP}^1$-bundle $Q \to C \times S$, the bundle Q is isomorphic to the pullback of P along $1_C \times f_Q$ for a unique map $f_Q : S \to M^s(C)$.*

Lemma 3.21 is proven in [5]. One way to understand this result is as follows. Let $M^s_{r,d}(C)$ denote the moduli space of stable vector bundles of rank r and degree d on a curve C. As discussed in [9], one can define a corresponding moduli stack $\mathrm{Bun}^s_{r,d}(C)$ and a $\mathbb{G}_m$-gerbe $\pi : \mathrm{Bun}^s_{r,d}(C) \to \mathrm{Hom}(-, M^s_{r,d}(C))$; this is a morphism of stacks for which all the fibers are isomorphic to $B\mathbb{G}_m$. The stack $\mathrm{Hom}(-, C) \times \mathrm{Bun}^s_{r,d}(C)$ carries a universal rank 2 vector bundle $\mathcal{E}$. One can show (see [8, Corollary 3.12]) that if $\gcd(r, d) = 1$ then $M^s_{r,d}(C)$ is a fine moduli space and $\mathcal{E}$ descends to a universal vector bundle $E \to C \times M^s_{r,d}(C)$, which can be viewed as a generalization of the Poincaré line bundle $L \to C \times \mathrm{Jac}(C)$ for the case $r = 1$, $d = 0$. By projectivizing E, we also get a universal $\mathbb{CP}^1$-bundle $\mathbb{P}(E) \to C \times M^s_{r,d}(C)$. If $\gcd(r, d) \neq 1$, then $M^s_{r,d}(C)$ is not a fine moduli space and $C \times M^s_{r,d}(C)$ does not carry a universal vector bundle. It is still possible, however, to use $\mathcal{E}$ to construct a universal $\mathbb{CP}^{r-1}$-bundle $P \to C \times M^s_{r,d}(C)$, only now this $\mathbb{CP}^{r-1}$-bundle is not the projectivization of a universal vector bundle. One way to make this result plausible is to note that whereas a stable vector bundle has automorphism group $\mathbb{C}^\times$, consisting of automorphisms that scale the fibers by a constant factor, the projectivization of a stable vector bundle has trivial automorphism group, consisting of just the identity automorphism.

Similar results hold for moduli spaces of stable vector bundles for which the determinant bundle is a fixed line bundle. In particular, the space $M^s(C)$ of stable rank 2 vector bundles with trivial determinant bundle is not a fine moduli space and $C \times M^s(C)$ does not carry a universal vector bundle; nonetheless, it does carry a universal $\mathbb{CP}^1$-bundle $P \to C \times M^s(C)$. We will use this universal $\mathbb{CP}^1$-bundle to construct $\mathcal{P}^{tot}_M(C, m, n)$ for the case $m = 0$:

Proof of Theorem 3.20. First consider the case $m = 0$, and note that $M^s(C, 0) = M^s(C)$. Given a point $p \in C$, let $P(p) \to M^s(C)$ denote the pullback of the universal $\mathbb{CP}^1$-bundle $P \to C \times M^s(C)$ described in Lemma 3.21 along the inclusion $M^s(C) \to C \times M^s(C)$, $[E] \mapsto (p, [E])$. Given distinct points $(p_1, \cdots, p_n) \in C^n$, we can pull back the $(\mathbb{CP}^1)^n$-bundle $P(p_1) \times \cdots \times P(p_n) \to (M^s(C))^n$ along the diagonal map $M^s(C) \to (M^s(C))^n$, $[E] \mapsto ([E], \cdots, [E])$ to obtain $\mathcal{P}^{tot}_M(C, 0, n)$.

The proof for $m > 0$ is the same. One can define a moduli stack corresponding to $M^s(C, m)$ that carries a universal rank 2 parabolic bundle [9]. Using a numerical condition analogous to the condition $\gcd(r, d) = 1$ for vector bundles (see [9, Example 5.7] and [6, Proposition 3.2]), one can show that $M^s(C, m)$ is a fine moduli space for $m > 0$ and the universal parabolic bundle on the moduli stack descends to a universal parabolic bundle on $C \times M^s(C, m)$. We can projectivize this latter bundle and use it to construct $\mathcal{P}^{tot}_M(C, m, n)$ in the same manner as for the $m = 0$ case. □

We will often want to pick out an open subset of $\mathcal{P}^{tot}_M(C, m, n)$ by imposing a semistability condition:

Definition 3.22. Given a curve C and distinct points $(q_1, \cdots, q_m, p_1, \cdots, p_n) \in C^{m+n}$, we define $\mathcal{P}_M(C, m, n)$ to be the subset of $\mathcal{P}^{tot}_M(C, m, n)$ consisting of points $[E, \ell_{q_1}, \cdots, \ell_{q_m}, \ell_{p_1}, \cdots, \ell_{p_n}]$ such that $H(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is semistable:

$$\mathcal{P}_M(C, m, n) = \{[E, \ell_{q_1}, \cdots, \ell_{q_m}, \ell_{p_1}, \cdots, \ell_{p_n}] \in \mathcal{P}^{tot}_M(C, m, n) \mid H(E, \ell_{p_1}, \cdots, \ell_{p_n}) \text{ is semistable}\}.$$

For simplicity, we are suppressing the dependence of $\mathcal{P}_M(C, m, n)$ on $q_1, \cdots, q_m, p_1, \cdots, p_n$ in the notation.

Theorem 3.23. *The set $\mathcal{P}_M(C, m, n)$ is an open submanifold of $\mathcal{P}^{tot}_M(C, m, n)$.*

Proof. This follows from the fact that semistability is an open condition. □

We can interpret marked parabolic bundles in terms of Hecke modifications as follows:

Definition 3.24. We define a *sequence of Hecke modifications of a parabolic bundle* $(E, \ell_{q_1}, \cdots, \ell_{q_m})$ to be a sequence of Hecke modifications of the underlying vector bundle E.

Definition 3.25. We say that two sequences of Hecke modifications

$$(E, \ell_{q_1}, \cdots, \ell_{q_m}) \xleftarrow[p_1]{\alpha_1} E_1 \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_n]{\alpha_n} E_n, \qquad (E', \ell'_{q_1}, \cdots, \ell'_{q_m}) \xleftarrow[p_1]{\alpha'_1} E'_1 \xleftarrow[p_2]{\alpha'_2} \cdots \xleftarrow[p_n]{\alpha'_n} E'_n$$

are *equivalent* if there are isomorphisms $\phi_i : E_i \to E'_i$ for $i = 0, \cdots, n$ such that the following diagram commutes:

$$\begin{tikzcd}
(E, \ell_{q_1}, \cdots, \ell_{q_m}) \arrow[d, "\cong"', "\phi_0"] & E_1 \arrow[l, "\alpha_1"'] \arrow[d, "\cong"', "\phi_1"] & E_2 \arrow[l, "\alpha_2"'] \arrow[d, "\cong"', "\phi_2"] & \cdots \arrow[l, "\alpha_3"'] & E_n \arrow[l, "\alpha_n"'] \arrow[d, "\cong"', "\phi_n"] \\
(E', \ell'_{q_1}, \cdots, \ell'_{q_m}) & E'_1 \arrow[l, "\alpha'_1"'] & E'_2 \arrow[l, "\alpha'_2"'] & \cdots \arrow[l, "\alpha'_3"'] & E'_n. \arrow[l, "\alpha'_n"']
\end{tikzcd}$$

The space $\mathcal{P}^{tot}_M(C, m, n)$ can then be interpreted as a moduli space of equivalence classes of sequences of Hecke modifications of parabolic bundles, and the space $\mathcal{P}_M(C, m, n)$ can be interpreted as the subspace $\mathcal{P}^{tot}_M(C, m, n)$ consisting of equivalence classes of sequences for which the terminal vector bundles are semistable. We will not use these interpretations here, since it is simpler to work directly with the marked parabolic bundles.

4. Rational curves

4.1. **Vector bundles on rational curves.** Grothendieck showed that all rank 2 holomorphic vector bundles on (smooth projective) rational curves are decomposable [7]; that is, they have the form $\mathcal{O}(n) \oplus \mathcal{O}(m)$ for integers n and m. The instability degree of $\mathcal{O}(n) \oplus \mathcal{O}(m)$ is $|n - m|$, so the bundle $\mathcal{O}(n) \oplus \mathcal{O}(m)$ is strictly semistable if $n = m$ and unstable otherwise. There are no stable rank 2 vector bundles on rational curves.

4.2. **List of all possible single Hecke modifications.** Here we present a list of all possible Hecke modifications at a point $p \in \mathbb{CP}^1$ of all possible rank 2 vector bundles on $\mathbb{CP}^1$. We will parameterize Hecke modifications of a vector bundle E at a point p in terms of lines $\ell_p \in \mathbb{P}(E_p)$, as described in Theorem 3.4. Since we are always free to tensor a Hecke modification with a line bundle, it suffices to consider vector bundles of nonnegative degree.

Theorem 4.1. *Consider the vector bundle $\mathcal{O}(n) \oplus \mathcal{O}$ for $n \geq 1$ (unstable, instability degree n). The possible Hecke modifications are*

$$\mathcal{O}(n) \oplus \mathcal{O} \leftarrow \begin{cases} \mathcal{O}(n) \oplus \mathcal{O}(-1) & \text{if } \ell_p = \mathcal{O}(n)_p \text{ (a bad line)}, \\ \mathcal{O}(n-1) \oplus \mathcal{O} & \text{otherwise (a good line)}. \end{cases}$$

Proof. (1) The case $\ell_p = \mathcal{O}(n)_p$. A Hecke modification $\alpha : \mathcal{O}(n) \oplus \mathcal{O} \to \mathcal{O}(n) \oplus \mathcal{O}(-1)$ corresponding to ℓ_p is

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix},$$

where $f : \mathcal{O}(-1) \to \mathcal{O}$ is the unique (up to rescaling by a constant) nonzero morphism such that $f_p = 0$ on the fibers over p.

(2) The case $\ell_p \neq \mathcal{O}(n)_p$. Since $n \geq 1$, we can choose a section t of $\mathcal{O}(n)$ such that $t(p) \neq 0$. Choose a section $s = (at, b)$ of $\mathcal{O}(n) \oplus \mathcal{O}$ for $a, b \in \mathbb{C}$ such that $s(p) \neq 0$ and $s \in \ell_p$. A Hecke modification $\alpha : \mathcal{O}(n) \oplus \mathcal{O} \to \mathcal{O}(n-1) \oplus \mathcal{O}$ corresponding to ℓ_p is

$$\alpha = \begin{pmatrix} f & at \\ 0 & b \end{pmatrix},$$

where $f : \mathcal{O}(n-1) \to \mathcal{O}(n)$ is the unique (up to rescaling by a constant) nonzero morphism such that $f_p = 0$ on the fibers over p. □

Theorem 4.2. *Consider the vector bundle $\mathcal{O} \oplus \mathcal{O}$ (strictly semistable, instability degree 0). The possible Hecke modifications are*

$$\mathcal{O} \oplus \mathcal{O} \leftarrow \mathcal{O} \oplus \mathcal{O}(-1) \qquad \text{for all } \ell_p \text{ (all lines are bad).}$$

Proof. Define a section $s = (a, b)$ of $\mathcal{O} \oplus \mathcal{O}$ for $a, b \in \mathbb{C}$ such that $s(p) \neq 0$ and $s(p) \in \ell_p$. If $b = 0$, then a Hecke modification $\alpha : \mathcal{O} \oplus \mathcal{O}(-1) \to \mathcal{O} \oplus \mathcal{O}$ corresponding to ℓ_p is

$$\alpha = \begin{pmatrix} a & 0 \\ 0 & f \end{pmatrix},$$

and if $b \neq 0$, then a Hecke modification $\alpha : \mathcal{O} \oplus \mathcal{O}(-1) \to \mathcal{O} \oplus \mathcal{O}$ corresponding to ℓ_p is

$$\alpha = \begin{pmatrix} a & f \\ b & 0 \end{pmatrix},$$

where $f : \mathcal{O}(-1) \to \mathcal{O}$ is the unique (up to rescaling by a constant) nonzero morphism such that $f_p = 0$ on the fibers over p. □

4.2.1. *Observations.* From this list, we make the following observations:

Lemma 4.3. *The following results hold for Hecke modifications of a rank 2 vector bundle E over $\mathbb{CP}^1$:*

(1) *A Hecke modification of E changes the instability degree by ± 1.*
(2) *Hecke modification of E corresponding to a line $\ell_p \in \mathbb{P}(E_p)$ changes the instability degree by -1 if ℓ_p is a good line and $+1$ if ℓ_p is a bad line.*
(3) *A generic Hecke modification of E changes the instability degree by -1 unless E has the minimum possible instability degree 0, in which case all Hecke modifications of E change the instability degree by $+1$.*

4.3. **Moduli spaces $\mathcal{P}_M^{tot}(\mathbb{CP}^1, m, n)$ and $\mathcal{P}_M(\mathbb{CP}^1, m, n)$.** Our goal is to define a moduli space of Hecke modifications that is isomorphic to the Seidel–Smith space $\mathcal{Y}(S^2, 2r)$. Kamnitzer showed that such a space can be defined as follows:

Definition 4.4 (Kamnitzer [13]). Given distinct points $(p_1, \cdots, p_{2r}) \in (\mathbb{CP}^1)^{2r}$, define the *Kamnitzer space* $\mathcal{H}(\mathbb{CP}^1, 2r)$ to be the subset of $\mathcal{H}^{tot}(\mathbb{CP}^1, \mathcal{O} \oplus \mathcal{O}, 2r)$ consisting of equivalence classes of sequences of Hecke modifications $\mathcal{O} \oplus \mathcal{O} \xleftarrow[p_1]{\alpha_1} E_1 \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_{2r}]{\alpha_{2r}} E_{2r}$ such that E_{2r} is semistable.

In particular, the condition that E_{2r} must be semistable implies that $E_{2r} = \mathcal{O}(-r) \oplus \mathcal{O}(-r)$.

Theorem 4.5 (Kamnitzer [13]). *The Kamnitzer space $\mathcal{H}(\mathbb{CP}^1, 2r)$ has the structure of a complex manifold isomorphic to the Seidel–Smith space $\mathcal{Y}(S^2, 2r)$.*

We will describe the Seidel–Smith space $\mathcal{Y}(S^2, 2r)$ and Kamnitzer's isomorphism in Section 4.6. We can use the results of Section 3.2 to reinterpret the Kamnitzer space $\mathcal{H}(\mathbb{CP}^1, 2r)$ in terms of parabolic bundles:

Definition 4.6. Define $\mathcal{P}^{tot}(\mathbb{CP}^1, n) := \mathcal{P}^{tot}(\mathbb{CP}^1, \mathcal{O} \oplus \mathcal{O}, n)$ and $\mathcal{P}(\mathbb{CP}^1, 2r) := \mathcal{P}(\mathbb{CP}^1, \mathcal{O} \oplus \mathcal{O}, 2r)$.

Theorem 4.7. *There is a canonical isomorphism $\mathcal{H}(\mathbb{CP}^1, 2r) \to \mathcal{P}(\mathbb{CP}^1, 2r)$.*

Proof. This follows from restricting the domain and range of the canonical isomorphism $\mathcal{H}^{tot}(\mathbb{CP}^1, \mathcal{O} \oplus \mathcal{O}, 2r) \to \mathcal{P}^{tot}(\mathbb{CP}^1, \mathcal{O} \oplus \mathcal{O}, 2r)$ described in Theorem 3.10. □

We can also reinterpret the spaces $\mathcal{P}^{tot}(\mathbb{CP}^1, n)$ and $\mathcal{P}(\mathbb{CP}^1, 2r)$ in terms of the moduli spaces of marked parabolic bundles that we defined in Section 3.3. In what follows, we will choose a global trivialization $\mathcal{O} \oplus \mathcal{O} \to \mathbb{CP}^1 \times \mathbb{C}^2$ and identify all the fibers of $\mathcal{O} \oplus \mathcal{O}$ with $\mathbb{C}^2$. We can then identify lines $\ell_p \in \mathbb{P}(E_p)$ with points in $\mathbb{CP}^1$ and speak of lines in different fibers as being equal or unequal. In Appendix A we define a moduli space $M^{ss}(\mathbb{CP}^1)$ of semistable rank 2 vector bundles with trivial determinant bundle, and in Appendix B we define a moduli space $M^s(\mathbb{CP}^1, m)$ of stable rank 2 parabolic bundles with trivial determinant bundle and m marked points. From the fact that $M^{ss}(\mathbb{CP}^1) = \{[\mathcal{O} \oplus \mathcal{O}]\}$ and $\mathrm{Aut}(\mathcal{O} \oplus \mathcal{O}) = GL(2, \mathbb{C})$, we obtain the following results:

Theorem 4.8. *The moduli space $M^s(\mathbb{CP}^1, 3)$ consists of the single point $[\mathcal{O} \oplus \mathcal{O}, \ell_{q_1}, \ell_{q_2}, \ell_{q_3}]$, where $\ell_{q_1}, \ell_{q_2}, \ell_{q_3}$ are any three distinct lines. Given any two stable parabolic bundles of the form $(\mathcal{O} \oplus \mathcal{O}, \ell_{q_1}, \ell_{q_2}, \ell_{q_3})$ and $(\mathcal{O} \oplus \mathcal{O}, \ell'_{q_1}, \ell'_{q_2}, \ell'_{q_3})$, there is a unique (up to rescaling by a constant) automorphism $\phi \in \mathrm{Aut}(\mathcal{O} \oplus \mathcal{O})$ such that $\phi(\ell_{q_i}) = \ell'_{q_i}$ for $i = 1, 2, 3$.*

Corollary 4.9. *There is an isomorphism $M^s(\mathbb{CP}^1, 3) \to M^{ss}(\mathbb{CP}^1)$, $[\mathcal{O} \oplus \mathcal{O}, \ell_{q_1}, \ell_{q_2}, \ell_{q_3}] \mapsto [\mathcal{O} \oplus \mathcal{O}]$.*

These results motivate the following definitions of "marked" versions of $\mathcal{P}^{tot}(\mathbb{CP}^1, n)$ and $\mathcal{P}(\mathbb{CP}^1, 2r)$:

Definition 4.10. Define $\mathcal{P}^{tot}_M(\mathbb{CP}^1, n) := \mathcal{P}^{tot}_M(\mathbb{CP}^1, 3, n)$ and $\mathcal{P}_M(\mathbb{CP}^1, n) := \mathcal{P}_M(\mathbb{CP}^1, 3, n)$.

The marked and unmarked versions of these spaces are easily seen to be isomorphic:

Theorem 4.11. *The spaces $\mathcal{P}^{tot}_M(\mathbb{CP}^1, n)$ and $\mathcal{P}^{tot}(\mathbb{CP}^1, n)$ are (noncanonically) isomorphic.*

Proof. Choose three distinct lines ℓ_{q_1}, ℓ_{q_2}, ℓ_{q_3}, and define an isomorphism $\mathcal{P}^{tot}(\mathbb{CP}^1, n) \to \mathcal{P}^{tot}_M(\mathbb{CP}^1, n)$ by

$$[\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_n}] \mapsto [\mathcal{O} \oplus \mathcal{O}, \ell_{q_1}, \ell_{q_2}, \ell_{q_3}, \ell_{p_1}, \cdots, \ell_{p_n}].$$

The isomorphism is not canonical, since it depends on the choice of lines ℓ_{q_1}, ℓ_{q_2}, ℓ_{q_3} ☐

Theorem 4.12. *The spaces $\mathcal{P}_M(\mathbb{CP}^1, n)$ and $\mathcal{P}(\mathbb{CP}^1, n)$ are (noncanonically) isomorphic*

Proof. This follows from restricting the domain and range of the isomorphism $\mathcal{P}^{tot}(\mathbb{CP}^1, n) \to \mathcal{P}^{tot}_M(\mathbb{CP}^1, n)$ described in Theorem 4.11 ☐

Our primary motivation for defining $\mathcal{P}^{tot}_M(\mathbb{CP}^1, n)$ is to draw a parallel with the case of elliptic curves, which we consider in Section 5. But the space $\mathcal{P}^{tot}_M(\mathbb{CP}^1, n)$ also has an advantage over $\mathcal{P}^{tot}(\mathbb{CP}^1, n)$ in that we can use the marking lines to render certain constructions canonical. For example, we can define a canonical version of the noncanonical isomorphism $\mathcal{P}^{tot}(\mathbb{CP}^1, n) = \mathcal{P}^{tot}(\mathbb{CP}^1, \mathcal{O} \oplus \mathcal{O}, n) \to (\mathbb{CP}^1)^n$:

Lemma 4.13. *Fix a parabolic bundle $(E, \ell_{q_1}, \ell_{q_2}, \ell_{q_3})$ such that $[E, \ell_{q_1}, \ell_{q_2}, \ell_{q_3}] \in M^s(\mathbb{CP}^1, 3)$ and a point $p \in \mathbb{CP}^1$. There is a canonical isomorphism $\mathbb{P}(E_p) \to M^{ss}(\mathbb{CP}^1, 4) \cong \mathbb{CP}^1$ given by*

$$\ell_p \mapsto [E, \ell_{q_1}, \ell_{q_2}, \ell_{q_3}, \ell_p].$$

Proof. This follows from the fact that $(E, \ell_{q_1}, \ell_{q_2}, \ell_{q_3})$ is stable, so the lines $\ell_{q_1}, \ell_{q_2}, \ell_{q_3}$ are all distinct under the global trivialization of E. ☐

Theorem 4.14. *There is a canonical isomorphism $h : \mathcal{P}^{tot}_M(\mathbb{CP}^1, n) \to (M^{ss}(\mathbb{CP}^1, 4))^n$.*

Proof. Define maps $h_i : \mathcal{P}^{tot}_M(\mathbb{CP}^1, n) \to M^{ss}(\mathbb{CP}^1, 4)$ for $i = 1, \cdots, n$ by

$$h_i([E, \ell_{q_1}, \ell_{q_2}, \ell_{q_3}, \ell_{p_1}, \cdots, \ell_{p_n}]) = [E, \ell_{q_1}, \ell_{q_2}, \ell_{q_3}, \ell_{p_i}].$$

Then $h := (h_1, \cdots, h_n)$ is an isomorphism by Theorem 3.20 and Lemma 4.13. ☐

Remark 4.15. Definition 3.22 for $\mathcal{P}_M(C, m, n)$ implies that $\mathcal{P}_M(\mathbb{CP}^1, m, n) = \varnothing$ for odd n, since there are no semistable rank 2 vector bundles of odd degree on $\mathbb{CP}^1$. We could alternatively define $\mathcal{P}_M(\mathbb{CP}^1, m, n)$ by requiring that $H(E, \ell_{p_1}, \cdots, \ell_{p_n})$ have the minimal possible instability degree, which is 0 for n even and 1 for n odd. This condition is equivalent to semistability for n even, but is a distinct condition for n odd, and gives a nonempty space.

4.4. **Embedding $\mathcal{P}_M(\mathbb{CP}^1, m, n) \to M^s(\mathbb{CP}^1, m+n)$.** We will now describe a canonical open embedding of the space $\mathcal{P}_M(\mathbb{CP}^1, m, n)$ into the space of stable parabolic bundles $M^s(\mathbb{CP}^1, m+n)$. We first need two Lemmas:

Lemma 4.16. *Given a parabolic bundle $(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_n})$ over $\mathbb{CP}^1$, if $\ell_{p_1}, \cdots, \ell_{p_n}$ are bad in the same direction then $H(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_n}) = \mathcal{O} \oplus \mathcal{O}(-n)$.*

Proof. Since $\ell_{p_1}, \cdots, \ell_{p_n}$ are bad in the same direction, we have that $\ell_{p_1} = \cdots = \ell_{p_n}$ under a global trivialization of $\mathcal{O} \oplus \mathcal{O}$ in which all the fibers are identified with $\mathbb{C}^2$. An explicit sequence of Hecke modifications with $\ell_{p_1} = \cdots = \ell_{p_n}$ is given by

$$\mathcal{O} \oplus \mathcal{O} \xleftarrow[p_1]{\alpha_1} \mathcal{O} \oplus \mathcal{O}(-1) \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_n]{\alpha_n} \mathcal{O} \oplus \mathcal{O}(-n),$$

where $\mathcal{O} \oplus \mathcal{O} \xleftarrow[p_1]{\alpha_1} \mathcal{O} \oplus \mathcal{O}(-1)$ is a Hecke modification corresponding to the line ℓ_{p_1} and for $i = 2, \cdots, n$ we define

$$\alpha_i = \begin{pmatrix} 1 & 0 \\ 0 & f_i \end{pmatrix},$$

where $f_i : \mathcal{O}(-i) \to \mathcal{O}(-i+1)$ is the unique (up to rescaling by a constant) nonzero morphism such that $(f_i)_{p_i} = 0$ on the fibers over p_i. □

Lemma 4.17. *Given a parabolic bundle $(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_n})$ over $\mathbb{CP}^1$, if $H(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_n})$ is semistable then $(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_n})$ is semistable.*

Proof. We will prove the contrapositive, so assume that $(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_n})$ is unstable. It follows that more $n/2$ of the lines are bad in the same direction. Let s denote the number of such lines, and choose a permutation $\sigma \in \Sigma_n$ such that the first s points of $(\sigma(p_1), \cdots, \sigma(p_n))$ correspond to these lines. By Lemma 4.16 we have that $H(\mathcal{O} \oplus \mathcal{O}, \ell_{\sigma(p_1)}, \cdots, \ell_{\sigma(p_s)}) = \mathcal{O} \oplus \mathcal{O}(-s)$, which has instability degree s. Lemma 4.3 states that a single Hecke modification changes the instability degree by ± 1, so $H(\mathcal{O} \oplus \mathcal{O}, \ell_{\sigma(p_1)}, \cdots, \ell_{\sigma(p_n)}) = H(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_n})$ has instability degree at least $s - (n - s) = 2s - n > 0$, and is thus unstable. □

Remark 4.18. The converse to Lemma 4.17 does not always hold; for example, consider the semistable parabolic bundle $(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}, \ell_{p_4})$ for points $p_i = [1 : \mu_i] \in \mathbb{CP}^1$, where

$$\ell_{p_1} = [1:0], \qquad \ell_{p_2} = [0:1], \qquad \ell_{p_3} = [1:1], \qquad \ell_{p_4} = [(\mu_3 - \mu_1)(\mu_4 - \mu_2) : (\mu_3 - \mu_2)(\mu_4 - \mu_1)].$$

One can show that $H(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}, \ell_{p_4}) = \mathcal{O}(-3) \oplus \mathcal{O}(-1)$, which is unstable.

Theorem 4.19. *There is a canonical open embedding $\mathcal{P}_M(\mathbb{CP}^1, m, n) \to M^s(\mathbb{CP}^1, m+n)$.*

Proof. Take $[E, \ell_{q_1}, \cdots, \ell_{q_m}, \ell_{p_1}, \cdots, \ell_{p_n}] \in \mathcal{P}_M(\mathbb{CP}^1, m, n)$; note that $E = \mathcal{O} \oplus \mathcal{O}$. Since $(E, \ell_{q_1}, \cdots, \ell_{q_m})$ is stable, fewer than $m/2$ of the lines $\ell_{q_1}, \cdots, \ell_{q_m}$ are equal under the global trivialization of E. Since $H(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is semistable, it follows from Lemma 4.17 that $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is semistable, so at most $n/2$ of the lines $\ell_{p_1}, \cdots, \ell_{p_n}$ are equal under the global trivialization of E. It follows that fewer than $(m+n)/2$ of the lines $\ell_{q_1}, \cdots, \ell_{q_m}, \ell_{p_1}, \cdots, \ell_{p_n}$ are equal, so $(E, \ell_{q_1}, \cdots, \ell_{q_m}, \ell_{p_1}, \cdots, \ell_{p_n})$ is stable. So $\mathcal{P}_M(\mathbb{CP}^1, m, n)$ is a subset of $M^s(\mathbb{CP}^1, m+n)$. Specifically, the set $\mathcal{P}_M(\mathbb{CP}^1, m, n)$ consists of points $[E, \ell_{q_1}, \cdots, \ell_{q_m}, \ell_{p_1}, \cdots, \ell_{p_n}] \in M^s(\mathbb{CP}^1, m+n)$ such that $(E, \ell_{q_1}, \cdots, \ell_{q_m})$ is stable and $H(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is semistable. Since stability and semistability are open conditions, we have that $\mathcal{P}_M(\mathbb{CP}^1, m, n)$ is an open subset of $M^s(\mathbb{CP}^1, m+n)$. □

4.5. **Examples.** We can generalize the Kamnitzer space $\mathcal{H}(\mathbb{CP}^1, n)$ to allow for both even and odd n, in analogy with the generalization described in Remark 4.15:

Definition 4.20. Given distinct points $(p_1, \cdots, p_n) \in (\mathbb{CP}^1)^n$, define the *Kamnitzer space* $\mathcal{H}(\mathbb{CP}^1, n)$ to be the subset of $\mathcal{H}^{tot}(\mathbb{CP}^1, \mathcal{O} \oplus \mathcal{O}, n)$ consisting of equivalence classes of sequences of Hecke modifications $\mathcal{O} \oplus \mathcal{O} \xleftarrow[p_1]{\alpha_1} E_1 \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_n]{\alpha_n} E_n$ such that E_n has the minimum possible instability degree (0 for n even, 1 for n odd.)

Here we compute Kamnitzer space $\mathcal{H}(\mathbb{CP}^1, n)$ for $n = 0, 1, 2, 3$.

4.5.1. *Calculate $\mathcal{H}(\mathbb{CP}^1, 0)$.* We have

$$\mathcal{H}(\mathbb{CP}^1, 0) = \mathcal{H}^{tot}(\mathbb{CP}^1, 0) = \{\mathcal{O} \oplus \mathcal{O}\}.$$

4.5.2. *Calculate $\mathcal{H}(\mathbb{CP}^1, 1)$.* All Hecke modifications of $\mathcal{O} \oplus \mathcal{O}$ give $\mathcal{O} \oplus \mathcal{O}(-1)$, which has instability degree 1, so

$$\mathcal{H}(\mathbb{CP}^1, 1) = \mathcal{H}^{tot}(\mathbb{CP}^1, 1) = \mathbb{CP}^1.$$

4.5.3. *Calculate* $\mathcal{H}(\mathbb{CP}^1,2)$. A sequence of two Hecke modifications of $\mathcal{O}\oplus\mathcal{O}$ must have one of two forms:

$$\mathcal{O}\oplus\mathcal{O}\xleftarrow[p_1]{\alpha_1}\mathcal{O}\oplus\mathcal{O}(-1)\xleftarrow[p_2]{\alpha_2}\mathcal{O}(-1)\oplus\mathcal{O}(-1),\qquad\qquad \mathcal{O}\oplus\mathcal{O}\xleftarrow[p_1]{\alpha_1}\mathcal{O}\oplus\mathcal{O}(-1)\xleftarrow[p_2]{\alpha_2}\mathcal{O}\oplus\mathcal{O}(-2).$$

In the first case the terminal bundle $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ is semistable, whereas in the second case the terminal bundle $\mathcal{O}\oplus\mathcal{O}(-2)$ is unstable. So $\mathcal{H}(\mathbb{CP}^1,2)$ is the complement in $\mathcal{H}^{tot}(\mathbb{CP}^1,2)$ of sequences of Hecke modifications of the second form. As we showed in the proof of Lemma 4.16, the resulting space is

$$\mathcal{H}(\mathbb{CP}^1,2)=(\mathbb{CP}^1)^2-\{(a,a)\mid a\in\mathbb{CP}^1\}.$$

4.5.4. *Calculate* $\mathcal{H}(\mathbb{CP}^1,3)$. Now consider a sequence of three Hecke modifications of $\mathcal{O}\oplus\mathcal{O}$. The only sequences for which the terminal bundle does not have instability degree 1 are of the form

$$\mathcal{O}\oplus\mathcal{O}\xleftarrow[p_1]{\alpha_1}\mathcal{O}\oplus\mathcal{O}(-1)\xleftarrow[p_2]{\alpha_2}\mathcal{O}\oplus\mathcal{O}(-2)\xleftarrow[p_3]{\alpha_3}\mathcal{O}\oplus\mathcal{O}(-3).$$

So $\mathcal{H}(\mathbb{CP}^1,3)$ is the complement in $\mathcal{H}^{tot}(\mathbb{CP}^1,3)$ of sequences of Hecke modifications of this form. As we showed in the proof of Lemma 4.16, the resulting space is

$$\mathcal{H}(\mathbb{CP}^1,3)=(\mathbb{CP}^1)^3-\{(a,a,a)\mid a\in\mathbb{CP}^1\}.$$

4.6. **The Seidel–Smith space $\mathcal{Y}(S^2,2r)$.** Here we compare the embedding of $\mathcal{P}_M(\mathbb{CP}^1,2r)$ into $M^s(\mathbb{CP}^1,2r+3)$ that we defined in Theorem 4.19 with an embedding of the Seidel–Smith space $\mathcal{Y}(S^2,2r)$ into $M^s(\mathbb{CP}^1,2r+3)$ due to Woodward. We begin by defining the Seidel–Smith space $\mathcal{Y}(S^2,2r)$.

Definition 4.21. We define the *Slodowy slice* S_{2r} to be the subspace of $\mathfrak{gl}(2r,\mathbb{C})$ consisting of matrices with 2×2 identity matrices I on the superdiagonal, arbitrary 2×2 matrices in the left column, and zeros everywhere else.

Example 4.22. Elements of S_6 have the form

$$\begin{pmatrix} Y_1 & I & 0\\ Y_2 & 0 & I\\ Y_3 & 0 & 0\end{pmatrix},$$

where Y_1, Y_2, and Y_3 are arbitrary 2×2 complex matrices.

Definition 4.23. Define a map $\chi: S_{2r}\to\mathbb{C}^{2r}/\Sigma_{2r}$ that sends a matrix to the multiset of the roots of its characteristic polynomial, where a root of multiplicity m occurs m times in the multiset.

Definition 4.24. Given distinct points $(\mu_1,\cdots,\mu_{2r})\in\mathbb{C}^{2r}$, define the *Seidel–Smith space* $\mathcal{Y}(S^2,2r)$ to be the fiber $\chi^{-1}(\{\mu_1,\cdots,\mu_{2r}\})$. For simplicity, we are suppressing the dependence of $\mathcal{Y}(S^2,2r)$ on $\mu_1,\cdots,\mu_{2r}$ in the notation. This space was introduced in [24], which denotes $\mathcal{Y}(S^2,2r)$ by $\mathcal{Y}_r$.

The Seidel–Smith space $\mathcal{Y}(S^2,2r)$ naturally has the structure of a complex manifold, in fact a smooth complex affine variety. In what follows, it will be useful to define local coordinates $\xi:U\to V$ on $\mathbb{CP}^1$, where $U=\{[1:z]\mid z\in\mathbb{C}\}\subset\mathbb{CP}^1$, $V=\mathbb{C}$, and $\xi([1:z])=z$. We define points $p_i:=\xi^{-1}(\mu_i)\in\mathbb{CP}^1$ corresponding to μ_i for $i=1,\cdots,2r$.

4.6.1. *Kamnitzer isomorphism* $\mathcal{H}(\mathbb{CP}^1,2r)\to\mathcal{Y}(S^2,2r)$. Here we describe an isomorphism due to Kamnitzer from the space of Hecke modifications $\mathcal{H}(\mathbb{CP}^1,2r)$ to the Seidel–Smith space $\mathcal{Y}(S^2,2r)$.

Define global meromorphic sections s_n of $\mathcal{O}(n)$ such that $\operatorname{div}s_n=n\cdot[\infty]$. For each rank 2 vector bundle $E=\mathcal{O}(n)\oplus\mathcal{O}(m)$, define standard meromorphic sections

$$e^1_E=(s_n,0),\qquad\qquad e^2_E=(0,s_m),$$

and define a standard local trivialization $\psi_E:\pi_E^{-1}(U)\to U\times\mathbb{C}^2$ of E over U by

$$e^1_E(p)\mapsto(p,(1,0)),\qquad\qquad e^2_E(p)\mapsto(p,(0,1)).$$

Consider an element of $\mathcal{H}(\mathbb{CP}^1,2r)$:

$$[E_0\xleftarrow[p_1]{\alpha_1}E_1\xleftarrow[p_2]{\alpha_2}\cdots\xleftarrow[p_{2r}]{\alpha_{2r}}E_{2r}],\tag{1}$$

where $E_0=\mathcal{O}\oplus\mathcal{O}$. Define rank 2 free $\mathbb{C}[z]$-modules L_i for $i=0,\cdots,2r$ as spaces of sections of E_i over U:

$$L_i=\Gamma(U,E_i)=\mathbb{C}[z]\cdot\{e^1_{E_i},e^2_{E_i}\}.$$

The sequence of Hecke modifications (1) then yields a sequence of $\mathbb{C}[z]$-module morphisms $\bar{\alpha}_i$:

$$L_0\xleftarrow{\bar{\alpha}_1}L_1\xleftarrow{\bar{\alpha}_2}\cdots\xleftarrow{\bar{\alpha}_{2r}}L_{2r}.$$

We can also view $\bar{\alpha}_i$ as a holomorphic map $\bar{\alpha}_i : V \to M(2,\mathbb{C})$, defined as in Definition 3.1 such that

$$(\psi_{E_{i-1}} \circ \alpha_i \circ \psi_{E_i}^{-1})(q,v) = (q, \bar{\alpha}_i(\xi(q))v).$$

Define an $2r$-dimensional complex vector space V by

$$V = \operatorname{coker}(\bar{\alpha}_1 \circ \bar{\alpha}_2 \circ \cdots \circ \bar{\alpha}_{2r}) = L_0/(\bar{\alpha}_1 \circ \bar{\alpha}_2 \circ \cdots \circ \bar{\alpha}_{2r})(L_{2r}).$$

One can show that an ordered basis for V is given by

$$(z^{r-1}e^1_{E_0},\, z^{r-1}e^2_{E_0},\, \cdots,\, ze^1_{E_0},\, ze^2_{E_0},\, e^1_{E_0},\, e^2_{E_0}).$$

Note that z acts $\mathbb{C}$-linearly on V, and thus defines a $2r \times 2r$ complex matrix A relative to this basis.

Theorem 4.25 (Kamnitzer [13]). *We have an isomorphism $\mathcal{H}(\mathbb{CP}^1, 2r) \to \mathcal{Y}(S^2, 2r)$ given by*

$$[E_0 \xleftarrow[p_1]{\alpha_1} E_1 \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_{2r}]{\alpha_{2r}} E_{2r}] \mapsto A.$$

To perform calculations, it is useful to have explicit expressions for the maps $\bar{\alpha}_i$. For each vector bundle E and point $p = [1 : \mu] \in U$, we use the standard trivialization $\psi_E : \pi^{-1}(E) \to U \times \mathbb{C}^2$ to identify $\mathbb{P}(E_p)$ with $\mathbb{CP}^1$. For each line $\ell_p \in \mathbb{P}(E_p) = \mathbb{CP}^1$, we give a holomorphic map $\bar{\alpha} : V \to M(2,\mathbb{C})$ that describes a Hecke modification $\alpha : F \to E$ corresponding to ℓ_p:

Hecke modifications of $\mathcal{O}(n) \oplus \mathcal{O}$ for $n \geq 1$:

$$\ell_p = [1:0]: \qquad \mathcal{O}(n) \oplus \mathcal{O} \xleftarrow[p]{\alpha} \mathcal{O}(n) \oplus \mathcal{O}(-1), \qquad \bar{\alpha}(z) = \begin{pmatrix} 1 & 0 \\ 0 & z-\mu \end{pmatrix},$$

$$\ell_p = [\lambda:1]: \qquad \mathcal{O}(n) \oplus \mathcal{O} \xleftarrow[p]{\alpha} \mathcal{O}(n-1) \oplus \mathcal{O}, \qquad \bar{\alpha}(z) = \begin{pmatrix} z-\mu & \lambda \\ 0 & 1 \end{pmatrix}.$$

Hecke modifications of $\mathcal{O} \oplus \mathcal{O}$:

$$\ell_p = [1:0]: \qquad \mathcal{O} \oplus \mathcal{O} \xleftarrow[p]{\alpha} \mathcal{O} \oplus \mathcal{O}(-1), \qquad \bar{\alpha}(z) = \begin{pmatrix} 1 & 0 \\ 0 & z-\mu \end{pmatrix},$$

$$\ell_p = [\lambda:1]: \qquad \mathcal{O} \oplus \mathcal{O} \xleftarrow[p]{\alpha} \mathcal{O} \oplus \mathcal{O}(-1), \qquad \bar{\alpha}(z) = \begin{pmatrix} \lambda & z-\mu \\ 1 & 0 \end{pmatrix}.$$

4.6.2. *Woodward embedding $\mathcal{Y}(S^2, 2r) \to M^s(\mathbb{CP}^1, 2r+3)$.* Here we describe an embedding due to Woodward [32] of the Seidel–Smith space $\mathcal{Y}(S^2, 2r)$ into the space of stable rank 2 parabolic bundles $M^s(\mathbb{CP}^1, 2r+3)$. We first make the following definition:

Definition 4.26. Given distinct points $(p_1, \cdots, p_n) \in (\mathbb{CP}^1)^n$, define a subspace $\mathcal{P}^{ss}(\mathbb{CP}^1, n)$ of $\mathcal{P}^{tot}(\mathbb{CP}^1, \mathcal{O}\oplus\mathcal{O}, n)$ consisting of semistable parabolic bundles $(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_n})$.

In particular, $\mathcal{P}^{ss}(\mathbb{CP}^1, n)$ consists of parabolic bundles $(\mathcal{O}\oplus\mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_n})$ for which at most $n/2$ of the lines are equal to any given line under a global trivialization of $\mathcal{O}\oplus\mathcal{O}$. Given distinct points $(q_1, q_2, q_3, p_1, \cdots, p_n) \in (\mathbb{CP}^1)^{n+3}$ and distinct lines $\ell_{q_1}, \ell_{q_2}, \ell_{q_3} \in \mathbb{CP}^1$, we can define an embedding $\mathcal{P}^{ss}(\mathbb{CP}^1, n) \to M^s(\mathbb{CP}^1, n+3)$,

$$(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_n}) \mapsto [\mathcal{O} \oplus \mathcal{O}, \ell_{q_1}, \ell_{q_2}, \ell_{q_3}, \ell_{p_1}, \cdots, \ell_{p_n}].$$

We will define an embedding $\mathcal{Y}(S^2, 2r) \to \mathcal{P}^{ss}(\mathbb{CP}^1, 2r)$. Composing with $\mathcal{P}^{ss}(\mathbb{CP}^1, 2r) \to M^s(\mathbb{CP}^1, 2r+3)$ will then yield the Woodward embedding. We first define some vectors. Define vectors $x, y \in \mathbb{C}^2$ by

$$x = (1,0), \qquad y = (0,1).$$

Define vectors $x_1, \cdots, x_r, y_1, \cdots, y_r \in \mathbb{C}^{2r}$ by

$$x_1 = (x,\, 0,\, \cdots,\, 0), \qquad x_2 = (0,\, x,\, 0,\, \cdots,\, 0), \qquad \cdots, \qquad x_r = (0,\, \cdots,\, 0,\, x),$$
$$y_1 = (y,\, 0,\, \cdots,\, 0), \qquad y_2 = (0,\, y,\, 0,\, \cdots,\, 0), \qquad \cdots, \qquad y_r = (0,\, \cdots,\, 0,\, y).$$

Define vectors $x(\mu), y(\mu) \in \mathbb{C}^{2r}$ by

$$x(\mu) = (\mu^{r-1}x,\, \mu^{r-2}x,\, \cdots,\, \mu x,\, x) = \mu^{r-1}x_1 + \mu^{r-2}x_2 + \cdots + \mu x_{r-1} + x_r,$$
$$y(\mu) = (\mu^{r-1}y,\, \mu^{r-2}y,\, \cdots,\, \mu y,\, y) = \mu^{r-1}y_1 + \mu^{r-2}y_2 + \cdots + \mu y_{r-1} + y_r.$$

We use the vectors to define a subspace $W(s,t)$ of $\mathbb{C}^{2r}$, and we calculate its dimension:

Definition 4.27. Given $(s,t) \in \mathbb{C}^2$, define a subspace $W(s,t) = \mathbb{C} \cdot \{sx(\mu) + ty(\mu) \mid \mu \in \mathbb{C}\}$ of $\mathbb{C}^{2r}$.

Lemma 4.28. *For $(s,t) \in \mathbb{C}^2 - \{0\}$, we have that* $\dim W(s,t) = r$.

Proof. Define a vector $w(s,t,\mu) \in \mathbb{C}^{2r}$ by

$$w(s,t,\mu) := sx(\mu) + ty(\mu) = \mu^{r-1}(sx_1 + ty_1) + \cdots + \mu(sx_{r-1} + ty_{r-1}) + (sx_r + ty_r). \tag{2}$$

Let $S \subset \mathbb{C}^{2r}$ denote the span of the linearly independent vectors $\{sx_1 + ty_1, \cdots, sx_r + ty_r\}$. Clearly $W(s,t) \subseteq S$. Form an $r \times r$ matrix V whose i-th row vector consists of the components of $w(s,t,i)$ relative to the ordered basis $(sx_1 + ty_1, \cdots, sx_r + ty_r)$ of S. From equation (2), it follows that the (i,j) matrix element of V is given by

$$V_{ij} = (i)^{r-j}.$$

So V is a Vandermonde matrix corresponding to the distinct integers $(1, 2, \cdots, r)$, and thus has nonzero determinant. It follows that the vectors $\{w(s,t,1), \cdots, w(s,t,r)\}$ are linearly independent, hence $W(s,t) = S$ and $\dim W(s,t) = \dim S = r$. □

We are now ready to define the Woodward embedding. Take a matrix $A \in \mathcal{Y}(S^2, 2r)$. Let $v(\mu) \in \mathbb{C}^{2r}$ be a left-eigenvector of A with eigenvalue μ:

$$v(\mu)A = \mu v(\mu).$$

Given the form of A, it follows that

$$v(\mu) = X(\mu)x(\mu) + Y(\mu)y(\mu)$$

for some $X(\mu), Y(\mu) \in \mathbb{C}$. Since $A \in \mathcal{Y}(S^2, 2r)$, the eigenvalues of A are $\mu_1, \cdots, \mu_{2r} \in \mathbb{C}$. Define lines $\ell_{p_i} \in \mathbb{CP}^1$ for $i = 1, \cdots, 2r$ by

$$\ell_{p_i} = [X(\mu_i) : Y(\mu_i)].$$

Theorem 4.29 (Woodward [32])**.** *We have an embedding* $\mathcal{Y}(S^2, 2r) \to \mathcal{P}^{ss}(\mathbb{CP}^1, 2r)$*,* $A \mapsto (\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_{2r}})$*.*

Proof. A priori the codomain of the map is $\mathcal{P}^{tot}(\mathbb{CP}^1, 2r)$, so we need to show that the image is in fact contained in $\mathcal{P}^{ss}(\mathbb{CP}^1, 2r)$. Note that if $\ell_{p_i} = [s:t]$ then $v(\mu_i) \in W(s,t)$. Since the eigenvalues $\mu_1, \cdots, \mu_{2r}$ are distinct, the eigenvectors $\{v(\mu_1), \cdots, v(\mu_{2r})\}$ are linearly independent, so the maximum number of eigenvectors that can live in $W(s,t)$ is $\dim W(s,t) = r$ by Lemma 4.28. So at most r of the lines $\ell_{p_1}, \cdots, \ell_{p_{2r}}$ can be equal to any given line $[s:t]$ in $\mathbb{CP}^1$, and thus $(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_{2r}})$ is semistable. □

Lemma 4.17 states that we have an embedding $\mathcal{P}(\mathbb{CP}^1, 2r) \to \mathcal{P}^{ss}(\mathbb{CP}^1, 2r)$. We can precompose this embedding with the canonical isomorphism $\mathcal{H}(\mathbb{CP}^1, 2r) \to \mathcal{P}(\mathbb{CP}^1, 2r)$ described in Theorem 4.7 to obtain an embedding $\mathcal{H}(\mathbb{CP}^1, 2r) \to \mathcal{P}^{ss}(\mathbb{CP}^1, 2r)$, and we obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{H}(\mathbb{CP}^1, 2r) \arrow[r] \arrow[d, "\cong"'] & \mathcal{P}^{ss}(\mathbb{CP}^1, 2r) \arrow[d] \\
\mathcal{P}_M(\mathbb{CP}^1, 2r) \arrow[r] & M^{ss}(\mathbb{CP}^1, 2r+3),
\end{tikzcd}$$

where the bottom horizontal arrow is the embedding described in Theorem 4.19. It is interesting to compare the embedding $\mathcal{H}(\mathbb{CP}^1, 2r) \to \mathcal{P}^{ss}(\mathbb{CP}^1, 2r)$ to the embedding $\mathcal{Y}(S^2, 2r) \to \mathcal{P}^{ss}(\mathbb{CP}^1, 2r)$ from Theorem 4.29. We make the following conjecture:

Conjecture 4.30. *There is a commutative diagram*

$$\begin{tikzcd}
\mathcal{H}(\mathbb{CP}^1, 2r) \arrow[r] \arrow[d, "\cong"] & \mathcal{P}^{ss}(\mathbb{CP}^1, 2r) \arrow[d, "\cong"] \\
\mathcal{Y}(S^2, 2r) \arrow[r] & \mathcal{P}^{ss}(\mathbb{CP}^1, 2r),
\end{tikzcd}$$

where the left downward arrow is the Kamnitzer isomorphism and the right downward arrow is the map on parabolic bundles induced by $\phi \in \mathrm{Aut}(\mathcal{O} \oplus \mathcal{O}) = GL(2, \mathbb{C})$*, where*

$$\phi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Theorem 4.31. *Conjecture 4.30 holds for* $r = 1$ *and* $r = 2$*.*

Proof. This can be shown by a direct calculation. □

5. Elliptic curves

5.1. **Vector bundles on elliptic curves.** Vector bundles on elliptic curves have been classified by Atiyah [2]:

Definition 5.1. Define $\mathcal{E}(r,d)$ to be the set of isomorphism classes of indecomposable vector bundles of rank r and degree d on an elliptic curve X.

The set $\mathcal{E}(r,d)$ naturally has the structure of a complex manifold, and we have the following result:

Theorem 5.2 (Atiyah [2]). *There are isomorphisms* $\mathrm{Jac}(X) \to \mathcal{E}(r,d)$ *for all* r *and* d.

In particular, $\mathcal{E}(1,d)$ is the set of isomorphism classes of line bundles of degree d, and the isomorphism $\mathrm{Jac}(X) \to \mathcal{E}(1,d)$ is given by $[L] \mapsto [L \otimes \mathcal{O}(d \cdot e)]$ for a choice of basepoint $e \in X$. Here we summarize the facts we will need regarding line bundles and rank 2 vector bundles on elliptic curves. Results that are well-known will be stated without proof; full proofs can be found in [2, 11, 26].

Definition 5.3. We say that a degree 0 line bundle L is *2-torsion* if $L^2 = \mathcal{O}$.

There are four 2-torsion line bundles on an elliptic curve. We will denote the 2-torsion line bundles by L_i for $i = 1,2,3,4$, with the convention that $L_1 = \mathcal{O}$.

Definition 5.4. Given line bundles L and M on an elliptic curve, an *extension* of L by M is an exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow L \longrightarrow 0,$$

where E is a rank 2 vector bundle.

Lemma 5.5. *Given line bundles L and M on an elliptic curve, equivalence classes of extensions of L by M are classified by* $\mathrm{Ext}^1(L,M) = H^0(L \otimes M^{-1})$.

Lemma 5.6 (Teixidor [26, Lemma 4.5]). *If $[E] \in \mathcal{E}(2,d)$, then $h^0(E) = 0$ if $d < 0$ and $h^0(E) = d$ if $d > 0$, where* $h^0(E) := \dim H^0(E)$.

We will now list the rank 2 vector bundles on an elliptic curve X. Up to tensoring with a line bundle, we have the following vector bundles:

5.1.1. *Rank 2 decomposable vector bundles.* Decomposable bundles have the form $L_1 \oplus L_2$, where L_1 and L_2 are line bundles. The instability degree of $L_1 \oplus L_2$ is $|\deg L_1 - \deg L_2|$, so $L_1 \oplus L_2$ is strictly semistable if $\deg L_1 = \deg L_2$ and unstable otherwise. The proof of the following result is straightforward:

Lemma 5.7. *Let $E = L_1 \oplus L_2$, where L_1 and L_2 are line bundles such that $\deg L_1 > \deg L_2$. At a point $p \in X$ the line $(L_1)_p$ is bad, and all other lines in $\mathbb{P}(E_p)$ are good.*

A semistable decomposable bundle must have even degree, so after tensoring with a suitable line bundle it has the form $L \oplus L^{-1}$, where L is a degree 0 line bundle. There are two subclasses of such bundles: the four bundles $L_i \oplus L_i$, and bundles $L \oplus L^{-1}$ such that $L^2 \neq \mathcal{O}$. These two subclasses of semistable decomposable bundles have very different properties:

Lemma 5.8. *The bundle $L_i \oplus L_i$ has no good lines, and* $\mathrm{Aut}(L_i \oplus L_i) = GL(2,\mathbb{C})$.

Lemma 5.9. *Let $E = L \oplus L^{-1}$, where L is a degree 0 line bundle such that $L^2 \neq \mathcal{O}$. The automorphism group $\mathrm{Aut}(E)$ is the subgroup of $GL(2,\mathbb{C})$ matrices of the form*

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}.$$

At a point $p \in X$ the lines L_p and $(L^{-1})_p$ are bad, and all other lines in $\mathbb{P}(E_p)$ are good. Given a pair of good lines $\ell_p, \ell'_p \in \mathbb{P}(E_p)$, there is a unique (up to rescaling by a constant) automorphism $\phi \in \mathrm{Aut}(E)$ such that $\phi(\ell_p) = \phi(\ell'_p)$.

The proofs of Lemmas 5.8 and 5.9 are straightforward and have been omitted.

5.1.2. *Rank 2 degree 0 indecomposable bundles.* There is a unique indecomposable bundle F_2 that can be obtained via an extension of $\mathcal{O}$ by $\mathcal{O}$:

$$0 \longrightarrow \mathcal{O} \xrightarrow{\alpha} F_2 \xrightarrow{\beta} \mathcal{O} \longrightarrow 0. \tag{3}$$

The bundle F_2 is strictly semistable, and hence has instability degree 0. The map $\mathrm{Jac}(X) \to \mathcal{E}(2,0)$, $[L] \mapsto [F_2 \otimes L]$ is an isomorphism, so in particular $F_2 \otimes L = F_2$ if and only if $L = \mathcal{O}$.

Lemma 5.10. *If L is a degree 0 line bundle, then*

$$\operatorname{Hom}(L, F_2) = \begin{cases} \mathbb{C}\cdot\alpha & \text{if } L = \mathcal{O}, \\ 0 & \text{otherwise.} \end{cases} \qquad \operatorname{Hom}(F_2, L) = \begin{cases} \mathbb{C}\cdot\beta & \text{if } L = \mathcal{O}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Apply $\operatorname{Hom}(L, -)$ to the short exact sequence (3) to obtain

$$0 \longrightarrow \operatorname{Hom}(L, \mathcal{O}) \xrightarrow{\alpha_*} \operatorname{Hom}(L, F_2) \xrightarrow{\beta_*} \operatorname{Hom}(L, \mathcal{O}) \xrightarrow{\delta} \operatorname{Ext}^1(L, \mathcal{O}). \tag{4}$$

If $L \neq \mathcal{O}$, then $\operatorname{Hom}(L, \mathcal{O}) = 0$ and the long exact sequence (4) implies that $\operatorname{Hom}(L, F_2) = 0$. So assume $L = \mathcal{O}$. Then $\operatorname{Hom}(L, \mathcal{O}) = \mathbb{C}\cdot 1_{\mathcal{O}}$. To prove that $\operatorname{Hom}(L, F_2) = \mathbb{C}\cdot\alpha$, it suffices to show that $\delta(1_{\mathcal{O}}) \neq 0$. Assume for contradiction that this is not the case. Then the long exact sequence (4) implies that there is a morphism $f \in \operatorname{Hom}(L, F_2)$ such that $\beta_*(f) = \beta \circ f = 1_{\mathcal{O}}$. It follows that the short exact sequence (3) splits, contradiction.

The claim regarding $\operatorname{Hom}(F_2, L)$ can be proven in a similar manner by applying $\operatorname{Hom}(-, L)$ to the short exact sequence (3). □

Lemma 5.11. *Given a point $q \in X$, there are nonzero sections t_0 and t_1 of $F_2 \otimes \mathcal{O}(q)$ such that*

1. $H^0(F_2 \otimes \mathcal{O}(q)) = \mathbb{C}\cdot\{t_0, t_1\}$,
2. $\operatorname{div} t_0 = 0$ *and* $\operatorname{div} t_1 = q$,
3. $t_1(p) \in \mathcal{O}(q)_p$ *for all* $p \in X$, *where* $\mathcal{O}(q) \to F_2 \otimes \mathcal{O}(q)$ *is the unique degree 1 line subbundle of* $F_2 \otimes \mathcal{O}(q)$,
4. $\{t_0(p), t_1(p)\}$ *are linearly independent for all* $p \in X$ *such that* $p \neq q$.

Proof. Tensoring $\alpha : \mathcal{O} \to F_2$ with $\mathcal{O}(q)$ and precomposing with the unique (up to rescaling by a constant) nonzero morphism $\mathcal{O} \to \mathcal{O}(q)$, we obtain a section t_1 of $F_2 \otimes \mathcal{O}(q)$ such that $\operatorname{div} t_1 = q$ and $t_1(p) \in \mathcal{O}(q)_p$ for all $p \in X$. By Lemma 5.6 we have that $h^0(F_2 \otimes \mathcal{O}(q)) = 2$, so we can choose a section t_0 of $F_2 \otimes \mathcal{O}(q)$ linearly independent from t_1.

We claim that $\operatorname{div} t_0 = 0$. Assume for contradiction that this is not the case. We obtain a subbundle $\mathcal{O}(\operatorname{div} t_0) \to F_2 \otimes \mathcal{O}(q)$, and by semistability of $F_2 \otimes \mathcal{O}(q)$ it follows that $\operatorname{div} t_0 = p$ for some $p \in X$. We thus obtain a subbundle $\mathcal{O}(p) \to F_2 \otimes \mathcal{O}(q)$, hence a subbundle $\mathcal{O}(p - q) \to F_2$. But this contradicts Lemma 5.10 unless $p = q$, in which case t_0 and t_1 are linearly dependent.

We claim $t_0(p)$ and $t_1(p)$ are linearly independent at all points $p \in X$ such that $p \neq q$. Assume for contradiction that they are linearly dependent at some point p distinct from q. Then we can choose a nonzero section $s = at_0 + bt_1$ of $F_2 \otimes \mathcal{O}(q)$ for $a, b \in \mathbb{C}$ such that $s(p) = 0$. We thus obtain a subbundle $\mathcal{O}(\operatorname{div} s) \to F_2 \otimes \mathcal{O}(q)$. We have that $p \in \operatorname{div} s$, so semistability of $F_2 \otimes \mathcal{O}(q)$ implies that $\operatorname{div} s = p$. We thus obtain a subbundle $\mathcal{O}(p) \to F_2 \otimes \mathcal{O}(q)$, hence a subbundle $\mathcal{O}(p - q) \to F_2$. But this contradicts Lemma 5.10. □

Lemma 5.12. *The automorphism group* $\operatorname{Aut}(F_2)$ *is the subgroup of* $GL(2, \mathbb{C})$ *matrices of the form*

$$\begin{pmatrix} A & B \\ 0 & A \end{pmatrix}.$$

At a point $p \in X$ the line $\mathcal{O}_p$ is bad, where $\mathcal{O} \to F_2$ is the unique degree 0 subbundle of F_2, and all other lines in $\mathbb{P}((F_2)_p)$ are good. Given a pair of good lines $\ell_p, \ell'_p \in \mathbb{P}((F_2)_p)$, there is a unique (up to rescaling by a constant) automorphism $\phi \in \operatorname{Aut}(E)$ such that $\phi(\ell_p) = \phi(\ell'_p)$.

Proof. Apply $\operatorname{Hom}(-, F_2)$ to the short exact sequence (3) to obtain

$$0 \longrightarrow \operatorname{Hom}(\mathcal{O}, F_2) \xrightarrow{\beta^*} \operatorname{Hom}(F_2, F_2) \xrightarrow{\alpha^*} \operatorname{Hom}(\mathcal{O}, F_2). \tag{5}$$

Note that $\alpha^*(1_{F_2}) = \alpha$, so Lemma 5.10 implies that α^* is surjective and thus the sequence (5) is in fact short exact. It follows that $\operatorname{Hom}(F_2, F_2) = \mathbb{C}\cdot\{1_{F_2}, \eta\}$, where $\eta := \beta^*(\alpha) = \alpha \circ \beta$. Note that $\eta \circ \eta = 0$, so we can define an injective group homomorphism $\operatorname{Aut}(F_2) \to GL(2, \mathbb{C})$ by

$$A\, 1_{F_2} + B\,\eta \mapsto \begin{pmatrix} A & B \\ 0 & A \end{pmatrix}.$$

The fact that $\mathcal{O}_p$ is the unique bad line of $(F_2)_p$ follows from Lemma 5.10. Given good lines $\ell_p, \ell'_p \in \mathbb{P}((F_2)_p)$, choose nonzero vectors $v, v', w \in (F_2)_p$ such that $v \in \ell_p$, $v' \in \ell'_p$, and $w \in \mathcal{O}_p$. Since $\ell_p \neq \mathcal{O}_p$, it follows that $\{v, w\}$ is a basis for $(F_2)_p$. Define $a, b \in \mathbb{C}$ such that $v' = av + bw$; note that since $\ell'_p \neq \mathcal{O}_p$ we have that $a \neq 0$. Define $c \in \mathbb{C}$ such that $\eta_p(v) = cw$; note that since $\eta_p(w) = 0$ and $\eta_p \neq 0$, we have that $c \neq 0$. Then $v' = \phi_p(v)$, where $\phi = a1_F + (b/c)\eta \in \operatorname{Aut}(F_2)$. Hence $\phi(\ell_p) = \ell'_p$, and ϕ is clearly unique up to rescaling by a constant. □

5.1.3. *Rank 2 degree 1 indecomposable bundles.* Given a point $p \in X$, there is a unique degree 1 indecomposable bundle $G_2(p)$ that can be obtained via an extension of $\mathcal{O}(p)$ by $\mathcal{O}$:

$$0 \longrightarrow \mathcal{O} \longrightarrow G_2(p) \longrightarrow \mathcal{O}(p) \longrightarrow 0.$$

The bundle $G_2(p)$ is stable, with instability degree -1. The map $\mathcal{E}(1,1) \to \mathcal{E}(2,1)$, $[\mathcal{O}(p)] \mapsto [G_2(p)]$ is an isomorphism, with inverse isomorphism given by $\det : \mathcal{E}(2,1) \to \mathcal{E}(1,1)$, $[E] \to [\det E]$. It follows that for any degree 0 divisor D on X we have that

$$G_2(p + 2D) = G_2(p) \otimes \mathcal{O}(D),$$

and in particular $G_2(p) \otimes L \cong G_2(p)$ if and only if $L^2 = \mathcal{O}$.

Lemma 5.13. *We have that* $\mathrm{Aut}(G_2(p)) = \mathbb{C}^\times$ *consists only of trivial automorphisms that scale the fibers by a constant factor.*

Proof. This follows from the fact that $G_2(p)$ is stable. $\square$

Lemma 5.14. *Any degree 0 line bundle L is a subbundle of $G_2(p)$ via a unique (up to rescaling by a constant) inclusion map $L \to G_2(p)$.*

Proof. Let L be a degree 0 line bundle. By Lemma 5.6 we have that $h^0(G_2(p) \otimes L^{-1}) = 1$, hence $G_2(p) \otimes L^{-1}$ has a nonzero section s. We thus obtain a subbundle $\mathcal{O}(\operatorname{div} s) \to G_2(p) \otimes L^{-1}$. By stability of $G_2(p) \otimes L^{-1}$, we must have $\operatorname{div} s = 0$. Tensoring with L, we obtain a subbundle $L \to G_2(p)$. The claim regarding uniqueness follows from the fact that $h^0(G_2(p) \otimes L^{-1}) = 1$. $\square$

Corollary 5.15. *All lines of $G_2(p)$ are bad.*

Proof. This is shown in Theorem 5.19. $\square$

5.2. **List of all possible single Hecke modifications.** Here we present a list of all possible Hecke modifications at a point $p \in X$ of all possible rank 2 vector bundles on X, up to tensoring with a line bundle. We will parameterize Hecke modifications of a vector bundle E at a point p in terms of lines $\ell_p \in \mathbb{P}(E_p)$, as described in Theorem 3.4. Since we are always free to tensor a Hecke modification with a line bundle, it suffices to consider vector bundles of nonnegative degree.

To construct the list, we will often use the following strategy. By tensoring E with a line bundle of sufficiently high degree if necessary, we can assume without loss of generality that E is generated by global sections. Consider a Hecke modification $\alpha : F \to E$ of E at p corresponding to a line $\ell_p := \operatorname{im} \alpha_p \in \mathbb{P}(E_p)$. Since we have assumed E is generated by global sections, there is a section s of E such that $s(p) \neq 0$ and $s(p) \in \ell_p$. We then get a subbundle $\mathcal{O}(\operatorname{div} s) \to E$ and a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(\operatorname{div} s) \arrow[r] \arrow[d, "="] & F \arrow[r] \arrow[d, "\alpha"] & L \otimes \mathcal{O}(-p) \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \mathcal{O}(\operatorname{div} s) \arrow[r] & E \arrow[r] & L \arrow[r] & 0,
\end{tikzcd}$$

where f is the unique (up to rescaling by a constant) nonzero morphism $L \otimes \mathcal{O}(-p) \to L$. Thus F is an extension of $L \otimes \mathcal{O}(-p)$ by $\mathcal{O}(\operatorname{div} s)$, and we can often use this information to determine F.

5.2.1. *Rank 2 bundles of degree greater than 1.*

Theorem 5.16. *Consider a bundle of the form $L \oplus \mathcal{O}$ for L a line bundle of degree greater than 1 (unstable, instability degree $\deg L$). The possible Hecke modifications are*

$$L \oplus \mathcal{O} \leftarrow \begin{cases} L \oplus \mathcal{O}(-p) & \textit{if } \ell_p = L_p \textit{ (a bad line)}, \\ (L \otimes \mathcal{O}(-p)) \oplus \mathcal{O} & \textit{otherwise (a good line)}. \end{cases}$$

Proof. (1) The case $\ell_p = L_p$. A Hecke modification $\alpha : L \oplus \mathcal{O}(-p) \to L \oplus \mathcal{O}$ corresponding to ℓ_p is

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix},$$

where f is the unique (up to rescaling by a constant) nonzero morphism $\mathcal{O}(-p) \to \mathcal{O}$.

(2) The case $\ell_p \neq L_p$. Since $\deg L > 1$, we can choose a nonzero section t of L such that $t(p) \neq 0$. Since t is nonvanishing at p, we can choose a section $s = (at, b)$ of $L \oplus \mathcal{O}$ for $a, b \in \mathbb{C}$ such that $s(p) \neq 0$ and $s(p) \in \ell_p$. A Hecke modification $\alpha : (L \otimes \mathcal{O}(-p)) \oplus \mathcal{O} \to L \oplus \mathcal{O}$ corresponding to ℓ_p is

$$\alpha = \begin{pmatrix} f & at \\ 0 & b \end{pmatrix},$$

where f is the unique (up to rescaling by a constant) nonzero morphism $L \otimes \mathcal{O}(-p) \to L$. □

5.2.2. *Rank 2 bundles of degree 1.*

Theorem 5.17. *Consider the bundle $\mathcal{O}(q) \oplus \mathcal{O}$ with $q \neq p$ (unstable, instability degree 1). The possible Hecke modifications are*

$$\mathcal{O}(q) \oplus \mathcal{O} \leftarrow \begin{cases} \mathcal{O}(q) \oplus \mathcal{O}(-p) & \text{if } \ell_p = \mathcal{O}(q)_p \text{ (a bad line)}, \\ \mathcal{O}(q-p) \oplus \mathcal{O} & \text{otherwise (a good line).} \end{cases}$$

Proof. One can prove this result by using the fact that $\mathcal{O}(q)$ has a section t such that $t(p) \neq 0$ and writing down explicit Hecke modifications, as in the proof of Theorem 5.16. □

Theorem 5.18. *Consider the bundle $\mathcal{O}(p) \oplus \mathcal{O}$ (unstable, instability degree 1). The possible Hecke modifications are*

$$\mathcal{O}(p) \oplus \mathcal{O} \leftarrow \begin{cases} \mathcal{O}(p) \oplus \mathcal{O}(-p) & \text{if } \ell_p = \mathcal{O}(p)_p \text{ (a bad line)}, \\ \mathcal{O} \oplus \mathcal{O} & \text{if } \ell_p = \mathcal{O}_p \text{ (a good line)}, \\ F_2 & \text{otherwise (a good line).} \end{cases}$$

Proof. (1) The case $\ell_p = \mathcal{O}(p)_p$. A Hecke modification $\alpha : \mathcal{O}(p) \oplus \mathcal{O}(-p) \to \mathcal{O}(p) \oplus \mathcal{O}$ is given by

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix},$$

where f is the unique (up to rescaling by a constant) nonzero morphism $\mathcal{O}(-p) \to \mathcal{O}$.

(2) The case $\ell_p = \mathcal{O}_p$. A Hecke modification $\alpha : \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(p) \oplus \mathcal{O}$ is given by

$$\alpha = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix},$$

where t is the unique (up to rescaling by a constant) nonzero morphism $\mathcal{O} \to \mathcal{O}(p)$.

(3) The case $\ell_p \neq \mathcal{O}(p)_p$ and $\ell_p \neq \mathcal{O}_p$. Pick a point $q \in X$ such that $q \neq p$. Choose a nonzero section t_0 of $\mathcal{O}(p+q)$ such that $t_0(q) \neq 0$ and $t_0(p) \neq 0$. Choose a nonzero section t_1 of $\mathcal{O}(q)$. Note that $\operatorname{div} t_1 = q$. Since $t_0(p) \neq 0$ and $t_1(p) \neq 0$, we can define a section $s = (at_0, bt_1)$ of $\mathcal{O}(p+q) \oplus \mathcal{O}(q)$ for $a, b \in \mathbb{C}$ such that $s(p) \neq 0$ and $s(p) \in \ell_p$. Since $\ell_p \neq \mathcal{O}(p)_p$ and $\ell_p \neq \mathcal{O}_p$, it follows that $a \neq 0$ and $b \neq 0$, thus $\operatorname{div} s = 0$ and we obtain a subbundle $\mathcal{O}(\operatorname{div} s) = \mathcal{O} \to \mathcal{O}(p+q) \oplus \mathcal{O}(q)$, $1 \mapsto s$. Thus we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d, "="] & F \arrow[r] \arrow[d, "\alpha"] & \mathcal{O}(2q) \arrow[r] \arrow[d] & 0, \\
0 \arrow[r] & \mathcal{O} \arrow[r] & \mathcal{O}(p+q) \oplus \mathcal{O}(q) \arrow[r] & \mathcal{O}(2q+p) \arrow[r] & 0.
\end{tikzcd}$$

The bundle F cannot split, since there are no nonzero morphisms $\mathcal{O}(2q) \to \mathcal{O}(p+q) \oplus \mathcal{O}(q)$, hence F is indecomposable. Since $\det F = \mathcal{O}(2q)$ it follows that $F = F_2 \otimes \mathcal{O}(q) \otimes L$ for a 2-torsion line bundle L. We can compose α with projection onto the second summand of $\mathcal{O}(p+q) \oplus \mathcal{O}(q)$ to obtain a nonzero morphism $F \to \mathcal{O}(q)$, so Lemma 5.10 implies $L = \mathcal{O}$ and $F = F_2 \otimes \mathcal{O}(q)$. □

Theorem 5.19. *Consider the bundle $G_2(p)$ (stable, instability degree -1). There is a canonical isomorphism $\mathbb{P}(G_2(p)_p) \to M^{ss}(X) \cong \mathbb{CP}^1$ given by*

$$\ell_p \mapsto [H(G_2(p), \ell_p)].$$

All lines of $G_2(p)$ are bad.

Proof. By Lemma 5.14, any degree 0 line bundle L is a subbundle of $G_2(p)$ via a unique (up to rescaling by a constant) inclusion map $L \to G_2(p)$. Thus we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, "="] & F \arrow[r] \arrow[d, "\alpha"] & L^{-1} \arrow[r] \arrow[d] & 0, \\ 0 \arrow[r] & L \arrow[r] & G_2(p) \arrow[r] & L^{-1} \otimes \mathcal{O}(p) \arrow[r] & 0. \end{tikzcd}$$

Note that $\operatorname{Ext}^1(L^{-1}, L) = H^0(L^{-2})$. If $L^2 \neq \mathcal{O}$ then $H^0(L^{-2}) = 0$, so F splits, thus $F = L \oplus L^{-1}$.

Now suppose $L^2 = \mathcal{O}$. We claim that F is indecomposable; assume for contradiction that this is not the case. Then $F = L \oplus L$, so $\alpha : F \to G_2(p)$ gives a map $\mathcal{O} \oplus \mathcal{O} \to G_2(p) \otimes L$ that is an isomorphism away from p, so we obtain two linearly independent sections of $G_2(p) \otimes L$. But by Lemma 5.6 we have that $h^0(G_2(p) \otimes L) = 1$, contradiction. It follows that F is indecomposable. Since $\det F = \mathcal{O}$, it follows that $F = F_2 \otimes M$ for a 2-torsion line bundle M. Since we have a nonzero morphism $L \to F$, Lemma 5.10 implies that $M = L$ and $F = F_2 \otimes L$.

The above considerations show that we have a surjection $\operatorname{Jac}(X) \to M^{ss}(X)$, $[L] \mapsto [F]$. The vector bundle F is isomorphic to $H(G_2(p), \ell_p)$, where $\ell_p \in \mathbb{P}(G_2(p)_p)$ is the line corresponding to $[G_2(p)_p \xleftarrow[p]{\alpha} F] \in \mathcal{H}^{tot}(X, G_2(p); p)$ under the canonical isomorphism described in Theorem 3.4, and we have a commutative diagram

$$\begin{tikzcd} \operatorname{Jac}(X) \arrow[r] \arrow[d] & M^{ss}(X) \\ \mathbb{P}(G_2(p)_p). \arrow[ur] & \end{tikzcd}$$

Here $\operatorname{Jac}(X) \to \mathbb{P}(G_2(p)_p)$ is given by $[L] \mapsto L_p$ and $\mathbb{P}(G_2(p)_p) \to M^{ss}(X)$ is given by $\ell_p \mapsto [H(G_2(p), \ell_p)]$. Since $\operatorname{Jac}(X) \to M^{ss}(X)$ is surjective, we have that $\operatorname{Jac}(X) \to \mathbb{P}(G_2(p)_p)$ is surjective and $\mathbb{P}(G_2(p)_p) \to M^{ss}(X)$ is an isomorphism. The surjectivity of $\operatorname{Jac}(X) \to \mathbb{P}(G_2(p)_p)$ implies that all lines of $\mathbb{P}(G_2(p)_p)$ are bad. Since $G_2(p) = G_2(q) \otimes M$ for a suitable degree 0 line bundle M, all lines of $G_2(p)$ are bad. ☐

5.2.3. *Rank 2 bundles of degree 0.*

Theorem 5.20. *Consider the bundle $\mathcal{O} \oplus \mathcal{O}$ (strictly semistable, instability degree 0). The possible Hecke modifications are*

$$\mathcal{O} \oplus \mathcal{O} \leftarrow \mathcal{O} \oplus \mathcal{O}(-p) \qquad \text{for all } \ell_p \text{ (all lines are bad).}$$

Proof. We can choose a section s of $\mathcal{O} \oplus \mathcal{O}$ such that $s(p) \neq 0$ and $s = \ell_p$. We thus obtain a subbundle $\mathcal{O} \to \mathcal{O} \oplus \mathcal{O}$, $1 \mapsto s$ and a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d, "="] & F \arrow[r] \arrow[d, "\alpha"] & \mathcal{O}(-p) \arrow[r] \arrow[d] & 0, \\ 0 \arrow[r] & \mathcal{O} \arrow[r] & \mathcal{O} \oplus \mathcal{O} \arrow[r] & \mathcal{O} \arrow[r] & 0. \end{tikzcd}$$

Since $\operatorname{Ext}^1(\mathcal{O}(-p), \mathcal{O}) = H^0(\mathcal{O}(-p)) = 0$, we have that F splits, thus $F = \mathcal{O} \oplus \mathcal{O}(-p)$. Alternatively, one can write down explicit Hecke modifications, as in the proof of Theorem 5.16. ☐

Theorem 5.21. *Consider a bundle of the form $L \oplus L^{-1}$, where L is a degree 0 line bundle such that $L^2 \neq \mathcal{O}$ (strictly semistable, instability degree 0). The possible Hecke modifications are*

$$L \oplus L^{-1} \leftarrow \begin{cases} L \oplus (L^{-1} \otimes \mathcal{O}(-p)) & \text{if } \ell_p = L_p \text{ (a bad line)}, \\ (L \otimes \mathcal{O}(-p)) \oplus L^{-1} & \text{if } \ell_p = (L^{-1})_p \text{ (a bad line)}, \\ G_2(p) \otimes \mathcal{O}(-p) & \text{otherwise (a good line)}. \end{cases}$$

Proof. For $\ell_p = L_p$ or $\ell_p = (L^{-1})_p$, we can write down explicit Hecke modifications, as in the proof of Theorem 5.16. So assume $\ell_p \neq L_p$ and $\ell_p \neq (L^{-1})_p$. Choose a point $e \in X$ such that $(L \oplus L^{-1}) \otimes \mathcal{O}(e) = \mathcal{O}(q_1) \oplus \mathcal{O}(q_2)$ for points $q_1, q_2 \in X$ distinct from p. Since $L^2 \neq \mathcal{O}$, it follows that $q_1 \neq q_2$. Note that $q_1 + q_2 = 2e$. Let t_k be the unique (up to rescaling by a constant) nonzero section of $\mathcal{O}(q_k)$; note that $\operatorname{div} t_k = q_k$. We can define a section $s = (at_1, bt_2)$ of $\mathcal{O}(q_1) \oplus \mathcal{O}(q_2)$ for $a, b \in \mathbb{C}$ such that $s(p) \neq 0$ and $s(p) \in \ell_p$. Since $\ell_p \neq L_p$ and $\ell_p \neq (L^{-1})_p$, it follows that $a \neq 0$

and $b \neq 0$, thus $\operatorname{div} s = 0$. We thus obtain a subbundle $\mathcal{O}(\operatorname{div} s) = \mathcal{O} \to \mathcal{O}(q_1) \oplus \mathcal{O}(q_2)$, $1 \mapsto s$ and a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d, "="] & F \arrow[r] \arrow[d, "\alpha"] & \mathcal{O}(q_1 + q_2 - p) \arrow[r] \arrow[d] & 0, \\
0 \arrow[r] & \mathcal{O} \arrow[r] & \mathcal{O}(q_1) \oplus \mathcal{O}(q_2) \arrow[r] & \mathcal{O}(q_1 + q_2) \arrow[r] & 0.
\end{tikzcd}$$

There are no nonzero morphisms $\mathcal{O}(q_1 + q_2 - p) \to \mathcal{O}(q_1) \oplus \mathcal{O}(q_2)$, so F cannot split. Since $\det F = \mathcal{O}(q_1 + q_2 - p)$, we have that $F = G_2(q_1 + q_2 - p) = G_2(2(e - p) + p) = G_2(p) \otimes \mathcal{O}(e - p)$. □

Theorem 5.22. *Consider the bundle F_2 (strictly semistable, instability degree 0). The possible Hecke modifications are*

$$F_2 \leftarrow \begin{cases} \mathcal{O} \oplus \mathcal{O}(-p) & \text{if } \ell_p = \mathcal{O}_p \text{ (a bad line)}, \\ G_2(p) \otimes \mathcal{O}(-p) & \text{otherwise (a good line)}, \end{cases}$$

where $\mathcal{O} \to F_2$ is the unique degree 0 line subbundle of F_2.

Proof. (1) The case $\ell_p = \mathcal{O}_p$. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d, "="] & F \arrow[r] \arrow[d, "\alpha"] & \mathcal{O}(-p) \arrow[r] \arrow[d] & 0, \\
0 \arrow[r] & \mathcal{O} \arrow[r] & F_2 \arrow[r] & \mathcal{O} \arrow[r] & 0.
\end{tikzcd}$$

Since $\operatorname{Ext}^1(\mathcal{O}(-p), \mathcal{O}) = H^0(\mathcal{O}(-p)) = 0$, we have that F splits, thus $F = \mathcal{O} \oplus \mathcal{O}(-p)$.

(2) The case $\ell_p \neq \mathcal{O}_p$. Pick a point $q \in X$ such that $q \neq p$. Choose sections t_0 and t_1 of $F_2 \otimes \mathcal{O}(q)$ as in Lemma 5.11. We can define a section $s = at_0 + bt_1$ of $F_2 \otimes \mathcal{O}(q)$ for $a, b \in \mathbb{C}$ such that $s(p) \neq 0$ and $s(p) \in \ell_p$. Since $\ell_p \neq \mathcal{O}_p$ it follows that $a \neq 0$, thus $\operatorname{div} s = 0$. We thus obtain a subbundle $\mathcal{O}(\operatorname{div} s) = \mathcal{O} \to F_2 \otimes \mathcal{O}(q)$, $1 \mapsto s$ and a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d, "="] & F \arrow[r] \arrow[d, "\alpha"] & \mathcal{O}(2q - p) \arrow[r] \arrow[d] & 0, \\
0 \arrow[r] & \mathcal{O} \arrow[r] & F_2 \otimes \mathcal{O}(q) \arrow[r] & \mathcal{O}(2q) \arrow[r] & 0.
\end{tikzcd}$$

We claim that F cannot split. Assume for contradiction that F splits, thus $F = \mathcal{O} \oplus \mathcal{O}(2q - p)$. Then we can precompose α with the inclusion $\mathcal{O}(2q - p) \to \mathcal{O} \oplus \mathcal{O}(2q - p)$ to obtain a nonzero morphism $\mathcal{O}(2q - p) \to F_2 \otimes \mathcal{O}(q)$, contradicting Lemma 5.10. Since F does not split and $\det F = \mathcal{O}(2q - p)$, we have that $F = G_2(2q - p) = G_2(2(q - p) + p) = G_2(p) \otimes \mathcal{O}(q - p)$. □

5.2.4. *Observations.* From this list, we make the following observations:

Lemma 5.23. *The following results hold for Hecke modifications of a rank 2 vector bundle E on an elliptic curve:*

1. *A Hecke modification of E changes the instability degree by ± 1.*
2. *Hecke modification of E corresponding to a line $\ell_p \in \mathbb{P}(E_p)$ changes the instability degree by -1 if ℓ_p is a good line and $+1$ if ℓ_p is a bad line.*
3. *A generic Hecke modification of E changes the instability degree by -1 unless E has the minimum possible instability degree -1, in which case all Hecke modifications of E change the instability degree by $+1$.*

5.3. **Moduli spaces $\mathcal{P}_M^{tot}(X, m, n)$ and $\mathcal{P}_M(X, m, n)$.** In Section 4.3 we defined a total space of marked parabolic bundles $\mathcal{P}_M^{tot}(\mathbb{CP}^1, n) = \mathcal{P}_M^{tot}(\mathbb{CP}^1, 3, n)$ for rational curves, and we showed that the Seidel–Smith space $\mathcal{Y}(S^2, 2r)$ could be reinterpreted as the subspace $\mathcal{P}_M(\mathbb{CP}^1, 2r) = \mathcal{P}_M(\mathbb{CP}^1, 3, 2r)$ of $\mathcal{P}_M^{tot}(\mathbb{CP}^1, 2r)$. We now want to generalize the spaces $\mathcal{P}_M^{tot}(\mathbb{CP}^1, n)$ and $\mathcal{P}_M(\mathbb{CP}^1, 2r)$ to the case of an elliptic curve X. There are obvious candidates: namely, the spaces $\mathcal{P}_M^{tot}(X, m, n)$ and $\mathcal{P}_M(X, m, n)$ for some value of m, which should be chosen to obtain the correct generalization. One possibility is to use same value $m = 3$ that we used for rational curves. But $m = 1$ is also yields a reasonable generalization, as can be understood from the following considerations.

In Appendix A we define a moduli space $M^{ss}(C)$ of semistable rank 2 vector bundles over a curve C with trivial determinant bundle, and in Appendix B we define a moduli space $M^s(C, m)$ of stable rank 2 parabolic bundles over a curve C with trivial determinant bundle and m marked points. Recall that for rational curves we chose $m = 3$ marking lines because we wanted $\mathcal{P}_M^{tot}(\mathbb{CP}^1, m, n)$ to be isomorphic to the space of parabolic bundles $\mathcal{P}^{tot}(\mathbb{CP}^1, \mathcal{O} \oplus \mathcal{O}, n)$ in which the underlying vector bundle is $\mathcal{O} \oplus \mathcal{O}$, and

$$\mathcal{P}_M^{tot}(\mathbb{CP}^1, 3, 0) = M^s(\mathbb{CP}^1, 3) = M^{ss}(\mathbb{CP}^1) = \{[\mathcal{O} \oplus \mathcal{O}]\}.$$

For an elliptic curve X, however, the corresponding spaces $M^s(X, 3)$ and $M^{ss}(X)$ are *not* isomorphic: the space $M^s(X, 3)$ is a complex manifold of dimension 3, whereas $M^{ss}(X)$ is isomorphic to $\mathbb{CP}^1$. Instead we have the following results, which can be viewed as elliptic-curve analogs to Theorem 4.8 and Corollary 4.9 for rational curves:

Theorem 5.24. *The moduli space $M^s(X, 1)$ consists of points $[E, \ell_q]$, where ℓ_q is a good line and either $E = F_2 \otimes L_i$ or $E = L \oplus L^{-1}$ for L a degree 0 line bundle such that $L^2 \neq \mathcal{O}$. Given any two parabolic bundles of the form (E, ℓ_q) and (E, ℓ'_q) representing points of $M^s(X, 1)$, there is a unique (up to rescaling by a constant) automorphism $\phi \in \operatorname{Aut}(E)$ such that $\phi(\ell_q) = \ell'_q$.*

Proof. If $[E, \ell_q] \in M^s(X, 1)$ then E is semistable, $\det E = \mathcal{O}$, and ℓ_q is a good line. Since E is semistable and $\det E = \mathcal{O}$, it must be $L_i \oplus L_i$, $F_2 \otimes L_i$, or $L \oplus L^{-1}$ for L a degree 0 line bundle such that $L^2 \neq \mathcal{O}$. But Lemma 5.8 states that $L_i \oplus L_i$ has no good lines, so E cannot be $L_i \oplus L_i$. Lemmas 5.9 and 5.12 show that the remaining two possibilities for E do have good lines and also prove the statement regarding unique automorphisms. □

Corollary 5.25. *The map $M^s(X, 1) \to M^{ss}(X)$, $[E, \ell_q] \mapsto [E]$ is an isomorphism.*

From these results, we see that there are *two* natural generalizations of $\mathcal{P}_M^{tot}(\mathbb{CP}^1, n)$ to an elliptic curve. The generalization of $\mathcal{P}_M^{tot}(\mathbb{CP}^1, 3, 0) = M^{ss}(\mathbb{CP}^1)$ is

$$\mathcal{P}_M^{tot}(X, 1, 0) = M^s(X, 1) = M^{ss}(X),$$

which would lead us to choose $m = 1$ marking lines. The generalization of $\mathcal{P}_M^{tot}(\mathbb{CP}^1, 3, 0) = M^s(\mathbb{CP}^1, 3)$ is

$$\mathcal{P}_M^{tot}(X, 3, 0) = M^s(X, 3),$$

which would lead us to choose $m = 3$ marking lines. We will address the question of which of these values of m yields the correct generalization of the Seidel–Smith space in Section 6.

From Theorem 3.20, we have that $\mathcal{P}_M^{tot}(X, 1, n)$ is a $(\mathbb{CP}^1)^n$-bundle over $M^s(X, 1) \cong \mathbb{CP}^1$. We will show that this bundle is trivial. To prove this result, we will use the marking line of $\mathcal{P}_M^{tot}(X, 1, n)$ to canonically identify $\mathbb{P}(E_p)$ with $M^{ss}(X) \cong \mathbb{CP}^1$ for $[E, \ell_{q_1}, \ell_{p_1}, \cdots, \ell_{p_n}] \in \mathcal{P}_M^{tot}(X, 1, n)$:

Lemma 5.26. *Fix a parabolic bundle (E, ℓ_q) such that $[E, \ell_q] \in M^s(X, 1)$, a point $p \in X$ such that $p \neq q$, and a point $e \in X$ such that $p + q = 2e$. There is a canonical isomorphism $\mathbb{P}(E_p) \to M^{ss}(X)$ given by*

$$\ell_p \mapsto [H(E, \ell_q, \ell_p) \otimes \mathcal{O}(e)].$$

Proof. Theorem 5.24 implies that ℓ_q is a good line and either $E = F_2 \otimes L_i$ or $E = L \oplus L^{-1}$ for L a degree 0 line bundle such that $L^2 \neq \mathcal{O}$. From Theorems 5.21 and 5.22, it follows that

$$H(E, \ell_q) = G_2(q) \otimes \mathcal{O}(-q) = G_2(p + 2(q - e)) \otimes \mathcal{O}(-q) = G_2(p) \otimes \mathcal{O}(-e).$$

The result now follows from Theorem 5.19. □

Lemma 5.26 can be viewed as the elliptic-curve analog to Lemma 4.13 for rational curves. To perform calculations, it will be useful to explicitly evaluate the map $\mathbb{P}(E_p) \to M^{ss}(X)$ for bad lines $\ell_p \in \mathbb{P}(E_p)$. In general, we prove:

Lemma 5.27. *Fix distinct points $p, q \in X$ and a point $e \in X$ such that $p + q = 2e$. If ℓ_q is a good line, then*

$$H(L \oplus L^{-1}, \ell_q, L_p) \otimes \mathcal{O}(e) = M \oplus M^{-1}, \text{ where } M = L \otimes \mathcal{O}(p - e) = L \otimes \mathcal{O}(e - q),$$
$$H(L \oplus L^{-1}, \ell_q, (L^{-1})_p) \otimes \mathcal{O}(e) = M \oplus M^{-1}, \text{ where } M = L \otimes \mathcal{O}(q - e) = L \otimes \mathcal{O}(e - p),$$
$$H(L \oplus L^{-1}, L_q, (L^{-1})_p) \otimes \mathcal{O}(e) = M \oplus M^{-1}, \text{ where } M = L \otimes \mathcal{O}(q - e) = L \otimes \mathcal{O}(e - p),$$
$$H(F_2, \ell_q, \mathcal{O}_p) \otimes \mathcal{O}(e) = M \oplus M^{-1}, \text{ where } M = \mathcal{O}(p - e) = \mathcal{O}(e - q).$$

Proof. These results are straightforward calculations using the list of Hecke modifications in Section 5.2. As an example, we will prove the result involving F_2. From Theorem 5.22 we have that

$$H(F_2, \mathcal{O}_p) = \mathcal{O} \oplus \mathcal{O}(-p).$$

Since ℓ_q is a good line, the bundle $H(F_2, \mathcal{O}_p, \ell_q) = H(F_2, \ell_q, \mathcal{O}_p)$ must be semistable, and the result now follows from Theorem 5.17. □

Theorem 5.28. *There is a canonical isomorphism $h : \mathcal{P}_M^{tot}(X, 1, n) \to (M^{ss}(X))^{n+1}$.*

Proof. Define $h_0 : \mathcal{P}_M^{tot}(X, 1, n) \to M^{ss}(X)$ by

$$h_0([E, \ell_{q_1}, \ell_{p_1}, \cdots, \ell_{p_n}]) = [E].$$

For $i = 1, \cdots, n$, choose a point $e_i \in X$ such that $q_1 + p_i = 2e_i$ and define $h_i : \mathcal{P}_M^{tot}(X, 1, n) \to M^{ss}(X)$ by

$$h_i([E, \ell_{q_1}, \ell_{p_1}, \cdots, \ell_{p_n}]) = [H(E, \ell_{q_1}, \ell_{p_i}) \otimes \mathcal{O}(e_i)].$$

Then $h := (h_0, h_1, \cdots, h_n)$ is an isomorphism by Theorem 3.20, Corollary 5.25, and Lemma 5.26. $\square$

For $n = 1$, the isomorphism $h : \mathcal{P}_M^{tot}(X, 1, n) \to (M^{ss}(X))^{n+1}$ appears to be closely related to an isomorphism $M^{ss}(X, 2) \to (\mathbb{CP}^1)^2$ defined in [28], and our definition of h was motivated by this isomorphism.

5.4. **Embedding $\mathcal{P}_M(X, m, n) \to M^s(X, m + n)$.** We will now describe a canonical open embedding of the space $\mathcal{P}_M(X, m, n)$ into the space of stable parabolic bundles $M^s(X, m + n)$. We first need two Lemmas:

Lemma 5.29. *Let $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ be a parabolic bundle over an elliptic curve X such that E is semistable. If the lines $\ell_{p_1}, \cdots, \ell_{p_n}$ are bad in the same direction then $H(E, \ell_{p_1}, \cdots, \ell_{p_n})$ has instability degree n.*

Proof. Up to tensoring with a line bundle, the bundle E has one of three forms:

(1) $E = \mathcal{O} \oplus \mathcal{O}$. Since $\ell_{p_1}, \cdots, \ell_{p_n}$ are bad in the same direction, we have that $\ell_{p_1} = \cdots = \ell_{p_n}$ under a global trivialization of E in which all the fibers are identified with $\mathbb{C}^2$. A sequence of Hecke modifications with $\ell_{p_1} = \cdots = \ell_{p_n}$ is given by

$$\mathcal{O} \oplus \mathcal{O} \xleftarrow[p_1]{\alpha_1} \mathcal{O} \oplus \mathcal{O}(-p_1) \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_n]{\alpha_n} \mathcal{O} \oplus \mathcal{O}(-p_1 - \cdots - p_n).$$

Here $\mathcal{O} \oplus \mathcal{O} \xleftarrow[p_1]{\alpha_1} \mathcal{O} \oplus \mathcal{O}(-p_1)$ is a Hecke modification corresponding to ℓ_{p_1}, and for $i = 2, \cdots, n$ we define

$$\alpha_i = \begin{pmatrix} 1 & 0 \\ 0 & f_i \end{pmatrix},$$

where f_i is the unique (up to rescaling by a constant) morphism from $\mathcal{O}(-p_1 - \cdots - p_i)$ to $\mathcal{O}(-p_1 - \cdots - p_{i-1})$. Thus $H(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_n}) = \mathcal{O} \oplus \mathcal{O}(-p_1 - \cdots - p_n)$ has instability degree n.

(2) $E = F_2$. Then $\ell_{p_i} = \mathcal{O}_{p_i}$ for $i = 1, \cdots, n$. A sequence of Hecke modifications with $\ell_{p_i} = \mathcal{O}_{p_i}$ for $i = 1, \cdots, n$ is given by

$$F_2 \xleftarrow[p_1]{\alpha_1} \mathcal{O} \oplus \mathcal{O}(-p_1) \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_n]{\alpha_n} \mathcal{O} \oplus \mathcal{O}(-p_1 - \cdots - p_n),$$

where $F_2 \xleftarrow[p_1]{\alpha_1} \mathcal{O} \oplus \mathcal{O}(-p_1)$ is a Hecke modification corresponding to $\ell_{p_1} = \mathcal{O}_{p_1}$ and α_i is as above for $i = 1, \cdots, n$. Thus $H(F_2, \mathcal{O}_{p_1}, \cdots, \mathcal{O}_{p_n}) = \mathcal{O} \oplus \mathcal{O}(-p_1 - \cdots - p_n)$ has instability degree n.

(3) $E = L \oplus L^{-1}$ for a degree 0 line bundle L such that $L^2 \neq \mathcal{O}$. Then either $\ell_{p_i} = L_{p_i}$ for $i = 1, \cdots, n$ or $\ell_{p_i} = (L^{-1})_{p_i}$ for $i = 1, \cdots, n$. A sequence of Hecke modifications with $\ell_{p_i} = L_{p_i}$ for $i = 1, \cdots, n$ is given by

$$L \oplus L^{-1} \xleftarrow[p_1]{\alpha_1} L \oplus (L^{-1} \otimes \mathcal{O}(-p_1)) \xleftarrow[p_2]{\alpha_2} \cdots \xleftarrow[p_n]{\alpha_n} L \oplus (L^{-1} \otimes \mathcal{O}(-p_1 - \cdots - p_n)),$$

where

$$\alpha_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \otimes f_i \end{pmatrix}$$

and f_i is as above. Thus $H(L \oplus L^{-1}, L_{p_1}, \cdots, L_{p_n}) = L \oplus (L^{-1} \otimes \mathcal{O}(-p_1 - \cdots - p_n))$ has instability degree n. We can write down a similar sequence of Hecke modifications to show that $H(L \oplus L^{-1}, (L^{-1})_{p_1}, \cdots, (L^{-1})_{p_n}) = (L \otimes \mathcal{O}(-p_1 - \cdots - p_n)) \oplus L^{-1}$ has instability degree n. $\square$

Using Lemma 5.29 in place of Lemma 4.16, the proofs of Lemma 4.17 and Theorem 4.19 for rational curves carry over to the case of elliptic curves. We thus obtain:

Lemma 5.30. *Let $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ be a parabolic bundle over an elliptic curve X such that E is semistable. If $H(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is semistable $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is semistable.*

Theorem 5.31. *There is a canonical open embedding $\mathcal{P}_M(X, m, n) \to M^s(X, m + n)$.*

5.5. **Examples.** Here we compute the space $\mathcal{P}_M(X,1,n)$ for $n=0,1,2$. We first make some definitions:

Definition 5.32. The *Abel-Jacobi isomorphism* $X \to \mathrm{Jac}(X)$ is given by $p \mapsto [\mathcal{O}(p-e)]$ for a choice of basepoint $e \in X$.

Definition 5.33. We define a map $\pi : \mathrm{Jac}(X) \to M^{ss}(X)$, $[L] \mapsto [L \oplus L^{-1}]$.

Note that π is surjective and $\pi(L) = \pi(L^{-1})$, so $\pi : \mathrm{Jac}(X) \cong X \to M^{ss}(X) \cong \mathbb{CP}^1$ is a 2:1 branched cover with four branch points $[L_i \oplus L_i]$ corresponding to the four 2-torsion line bundles L_i.

Definition 5.34. Given a degree 0 divisor D on an elliptic curve X, define the *translation map* $\tau_D : \mathrm{Jac}(X) \to \mathrm{Jac}(X)$, $[L] \mapsto [L \otimes \mathcal{O}(D)]$.

5.5.1. *Calculate* $\mathcal{P}_M(X,1,0)$. We have that

$$\mathcal{P}_M(X,1,0) = \mathcal{P}_M^{tot}(X,1,0) = M^s(X,1) = M^{ss}(X) = \mathbb{CP}^1.$$

Note that the embedding $\mathcal{P}_M(X,1,0) \to M^s(X,1)$ defined in Theorem 5.31 is an isomorphism.

5.5.2. *Calculate* $\mathcal{P}_M(X,1,1)$.

Theorem 5.35. *The map* $g : \mathrm{Jac}(X) \to (M^{ss}(X))^2$, $g = (\pi, \pi \circ \tau_{p_1 - e_1})$ *is injective and has image the complement of* $h(\mathcal{P}_M(X,1,1))$*, where* $h : \mathcal{P}_M^{tot}(X,1,1) \to (M^{ss}(X))^2$ *is the isomorphism described in Theorem 5.28.*

Proof. First we show that g has image the complement of $h(\mathcal{P}_M(X,1))$ in $(M^{ss}(X))^2$ Take a point $[E, \ell_{q_1}, \ell_{p_1}] \in \mathcal{P}_M^{tot}(X,1,1)$. From Theorems 5.21, 5.22, and 5.24, it follows that $H(E, \ell_{p_1}) = G_2(p_1) \otimes \mathcal{O}(-p_1)$ is stable if ℓ_{p_1} is a good line, and $H(E, \ell_{p_1})$ is unstable if ℓ_{p_1} is a bad line. So the complement of $\mathcal{P}_M(X,1,1)$ in $\mathcal{P}_M^{tot}(X,1,1)$ consists of isomorphism classes $[E, \ell_{q_1}, \ell_{p_1}]$ such that ℓ_{p_1} is a bad line, and is thus given by the union of the sets

$$\begin{aligned}
S_1 &= \{[L \oplus L^{-1}, \ell_{q_1}, L_{p_1}] \mid [L] \in \mathrm{Jac}(X),\ L^2 \neq \mathcal{O}\},\\
S_2 &= \{[L \oplus L^{-1}, \ell_{q_1}, (L^{-1})_{p_1}] \mid [L] \in \mathrm{Jac}(X),\ L^2 \neq \mathcal{O}\},\\
S_3 &= \{[F_2 \otimes L_i, \ell_{q_1}, (L_i)_{p_1}] \mid i = 1,2,3,4\},
\end{aligned}$$

where in each case ℓ_{q_1} is a good line. From Lemma 5.27, it follows that the complement of $h(\mathcal{P}_M(X,1,1))$ in $(M^{ss}(X))^2$ is given by the union of the sets

$$\begin{aligned}
h(S_1) &= \{(\pi([L]),\ (\pi \circ \tau_{p_1 - e_1})([L])) \mid [L] \in \mathrm{Jac}(X),\ L^2 \neq \mathcal{O}\},\\
h(S_2) &= \{(\pi([L]),\ (\pi \circ \tau_{e_1 - p_1})([L])) \mid [L] \in \mathrm{Jac}(X),\ L^2 \neq \mathcal{O}\},\\
h(S_3) &= \{(\pi([L_i]),\ (\pi \circ \tau_{p_1 - e_1})([L_i])) \mid i = 1,2,3,4\}.
\end{aligned}$$

Note that

$$(\pi([L]),\ (\pi \circ \tau_{e_1 - p_1})([L])) = (\pi([L^{-1}]),\ (\pi \circ \tau_{p_1 - e_1})([L^{-1}])),$$

so $h(S_1) = h(S_2)$, and we have that

$$h(S_1) \cup h(S_2) \cup h(S_3) = \{(\pi([L]),\ (\pi \circ \tau_{p_1 - e_1})([L])) \mid [L] \in \mathrm{Jac}(X)\} = \operatorname{im} g.$$

So the image of g is the complement of $h(\mathcal{P}_M(X,1,1))$ in $(M^{ss}(X))^2$

Next we show that g is injective. If $g(L) = g(L')$, then projection onto the first factor of $(M^{ss}(X))^2$ gives $\pi(L) = \pi(L')$, hence either $L' = L$ or $L' = L^{-1}$. Suppose $L' = L^{-1}$. Then projection onto the second factor of $(M^{ss}(X))^2$ gives $\pi(L \otimes \mathcal{O}(p_1 - e_1)) = \pi(L^{-1} \otimes \mathcal{O}(p_1 - e_1))$, hence either $L \otimes \mathcal{O}(p_1 - e_1) = L^{-1} \otimes \mathcal{O}(p_1 - e_1)$ or $L \otimes \mathcal{O}(p_1 - e_1) = L \otimes \mathcal{O}(e_1 - p_1)$. The first case implies $L = L^{-1}$. The second case implies $2p_1 = 2e_1$, but we chose e_1 such that $p_1 + q_1 = 2e_1$, hence $p_1 = q_1$, contradiction. Thus $L' = L$, so g is injective. ☐

If we use the Abel-Jacobi isomorphism to identify X and $\mathrm{Jac}(X)$, the (canonical) isomorphism $h : \mathcal{P}_M^{tot}(X,1,1) \to (M^{ss}(X))^2$ to identify $\mathcal{P}_M^{tot}(X,1,1)$ and $(M^{ss}(X))^2$, and the (noncanonical) isomorphism $M^{ss}(X) \cong \mathbb{CP}^1$ to identify $M^{ss}(X)$ and $\mathbb{CP}^1$, we find that

$$\mathcal{P}_M(X,1,1) = (\mathbb{CP}^1)^2 - g(X).$$

Remark 5.36. Using results from the proof of Theorem 5.35, it is straightforward to show that

$$M^{ss}(X,2) = \mathcal{P}_M^{tot}(X,1,1) = (\mathbb{CP}^1)^2, \qquad M^s(X,2) = \mathcal{P}_M(X,1,1) = (\mathbb{CP}^1)^2 - g(X).$$

These calculations reproduce the results of [28] for $M^{ss}(X,2)$ and $M^s(X,2)$.

5.5.3. *Calculate* $\mathcal{P}_M(X,1,2)$. The same method that we used to prove Theorem 5.35 can be used to calculate $\mathcal{P}_M(X,1,2)$:

Theorem 5.37. *The map* $f : \mathrm{Jac}(X) \to (M^{ss}(X))^3$, $f = (\pi, \pi \circ \tau_{p_1 - e_1}, \pi \circ \tau_{p_2-e_2})$ *is injective and has image the complement of* $h(\mathcal{P}_M(X,1,2))$, *where* $h : \mathcal{P}_M^{tot}(X,1,2) \to (M^{ss}(X))^3$ *is the isomorphism described in Theorem 5.28.*

If we use the Abel-Jacobi isomorphism to identify X and $\mathrm{Jac}(X)$, the (canonical) isomorphism $h : \mathcal{P}_M^{tot}(X,1,2) \to (M^{ss}(X))^3$ to identify $\mathcal{P}_M^{tot}(X,1,2)$ and $(M^{ss}(X))^3$, and the (noncanonical) isomorphism $M^{ss}(X) \cong \mathbb{CP}^1$ to identify $M^{ss}(X)$ and $\mathbb{CP}^1$, we find that

$$\mathcal{P}_M(X,1,2) = (\mathbb{CP}^1)^3 - f(X).$$

6. Possible applications to topology

Here we briefly outline some possible applications of our results to topology. We have proposed complex manifolds $\mathcal{P}_M(X,1,2r)$ and $\mathcal{P}_M(X,3,2r)$ as candidates for a space $\mathcal{Y}(T^2,2r)$ that generalizes the Seidel–Smith space $\mathcal{Y}(S^2,2r)$ and that could potentially be used to construct symplectic Khovanov homology for lens spaces. The following tasks remain to be done to complete the construction:

(1) We need to define a suitable symplectic form on $\mathcal{P}_M(X,m,2r)$. One possibility is to pull back the canonical symplectic form on $M^s(X,2r+m)$ using the open embedding $\mathcal{P}_M(X,m,2r) \to M^s(X,2r+m)$.
(2) We need to find a suitable action of the mapping class group $\mathrm{MCG}_{2r}(T^2)$ on $\mathcal{P}_M(X,m,2r)$ that is defined up to Hamiltonian isotopy. Such an action might be obtained via symplectic monodromy by viewing $\mathcal{P}_M(X,m,2r)$ as the fiber of a larger space that fibers over the moduli space of genus 1 curves with marked points. Such an approach would be analogous to the way Seidel and Smith obtain an action of the braid group on the Seidel–Smith space via monodromy around loops in the configuration space [23], and similar methods are used to define mapping class group actions for constructing Reshetikhin-Turaev-Witten invariants [3].
(3) We need to define suitable Lagrangians L_r in $\mathcal{P}_M(X,m,2r)$ corresponding to r unknotted arcs in a solid torus. For $\mathcal{P}_M(X,1,2r)$ we would expect L_r to be homeomorphic to $S^1 \times (S^2)^r$, and for $\mathcal{P}_M(X,3,2r)$ we would expect L_r to be homeomorphic to $S^3 \times (S^2)^r$. Perhaps such Lagrangians can be constructed in a manner analogous to Seidel–Smith by viewing $\mathcal{P}_M(X,m,2r)$ as the fiber of a larger space that fibers over the configuration space $\mathrm{Conf}_{2r}(X)$ of $2r$ unordered points in X and looking for vanishing cycles as points are successively brought together in pairs.
(4) We need to prove that the Lagrangian Floer homology of a knot K in a lens space is Y invariant under different Heegaard splittings of (Y,K) into solid tori.
(5) We need to verify that our construction of symplectic Khovanov homology reproduces ordinary Khovanov homology for the case of knots in S^3.

Several of our results appear to be related to a possible connection between Khovanov homology and symplectic instanton homology. Roughly speaking, symplectic instanton homology is defined as follows. Given a knot K in a 3-manifold Y, one Heegaard-splits (Y,K) along a Heegaard surface Σ to obtain handlebodies U_1 and U_2. Each handlebody U_i contains a portion of the knot $A_i := U_i \cap K$ consisting of r arcs that pairwise connect points $p_1, \cdots, p_{2r}$ in Σ. To the marked surface $(\Sigma, p_1, \cdots, p_{2r})$ one associates a character variety $R(\Sigma,2r)$, which has the structure of a symplectic manifold, and to the handlebody pairs (U_i, A_i) one associates Lagrangians $L_i \subset R(\Sigma,2r)$. The symplectic instanton homology of (Y,K) is then defined to be the Lagrangian Floer homology of the pair of Lagrangians (L_1,L_2).

In fact, there are several technical difficulties that must be overcome in order to get a well-defined homology theory. For example, one needs to introduce a *framing* in order to eliminate singularities in the character variety $R(\Sigma,2r)$. One way to introduce a framing is by replacing the knot K with $K \cup \Theta$, where Θ is the theta graph shown in Figure 1(a); this approach is described in [10]. We Heegaard-split $(Y, K \cup \Theta)$ along a Heegaard surface Σ that is chosen to transversely intersect each edge e_i of the theta graph in a single point q_i. The marked Heegaard surface is now $(\Sigma, q_1, q_2, q_3, p_1, \cdots, p_{2r})$, corresponding to the character variety $R(\Sigma, 2r+3)$, and the handlebody pairs are now $(U_i, A_i \cup \epsilon_i)$, where ϵ_i is the epsilon graph shown in Figure 1(b). The character variety $R(\Sigma,2r+3)$ has the structure of a symplectic manifold that is symplectomorphic to the moduli space of stable parabolic bundles $M^s(C,2r+3)$, where C is any complex curve homeomorphic to Σ. (The space $M^s(C,2r+3)$ has a canonical symplectic form.)

Symplectic instanton homology can be viewed as a symplectic replacement for singular instanton homology, a knot homology theory defined using gauge theory, and the two theories are conjectured to be isomorphic. This is an example of an Atiyah-Floer conjecture; such conjectures broadly relate Floer-theoretic invariants defined using gauge theory to corresponding invariants defined using symplectic topology. Kronheimer and Mrowka constructed a spectral sequence from Khovanov homology to singular instanton homology [17], and the embedding $\mathcal{P}_M(\mathbb{CP}^1,2r) \to M^s(\mathbb{CP}^1,2r+3)$ described in Theorem 4.4 suggests that it may be possible to construct an analogous spectral sequence from symplectic Khovanov homology to symplectic instanton homology. (This idea for constructing a

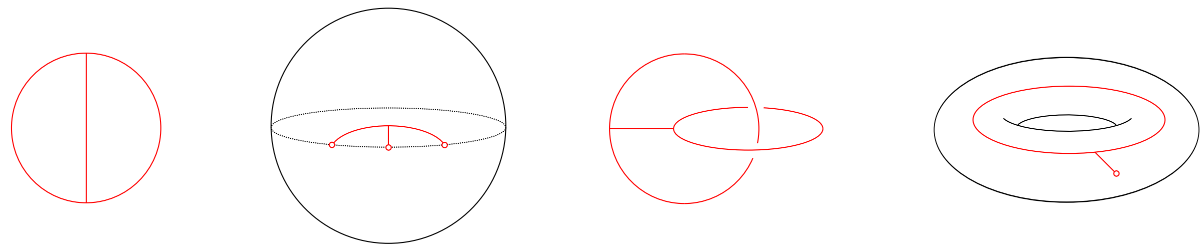

FIGURE 1. (a) The graph Θ. (b) The graph ϵ in B^3. (c) The graph $D_p \subset S^3$ for $p = 1$. (d) The graph σ in $S^1 \times D^2$.

spectral sequence was suggested to the author by Ivan Smith and Chris Woodward.) If so, perhaps the fact that we have an embedding $\mathcal{P}_M(X, 3, 2r) \to M^s(X, 2r + 3)$, as described in Theorem 5.4, is evidence that the correct generalization of the Seidel–Smith space is $\mathcal{P}_M(X, 3, 2r)$. Indeed, a calculation of the Lagrangian intersection $L_1 \cap L_2$ in the traceless character variety $R(T^2, 3)$ for (S^3, Θ) yields a single point, a space whose cohomology is the correct Khovanov homology for the empty knot, and calculations of the Lagrangian intersections in the traceless character variety $R(T^2, 5)$ for $(S^3, \text{unknot} \cup \Theta)$ and $(S^3, \text{trefoil} \cup \Theta)$ yield S^2 and $\mathbb{RP}^3 \amalg S^2$, spaces whose cohomology gives the correct Khovanov homology for the unknot and trefoil. Based on these speculations, we make the following conjectures:

Conjecture 6.1. *The space $\mathcal{P}_M(C, 3, 2r)$ is the correct generalization of the Seidel–Smith space $\mathcal{Y}(S^2, 2r)$ to a curve C of arbitrary genus.*

Conjecture 6.2. *Given a curve C of arbitrary genus, there is a canonical open embedding $\mathcal{P}_M(C, m, n) \to M^s(C, m+n)$.*

On the other hand, perhaps the embedding $\mathcal{P}_M(X, 1, 2r) \to M^s(X, 2r + 1)$ described in Theorem 5.4 is related to a spectral sequence from a Khovanov-like knot homology theory to symplectic instanton homology defined with a novel framing. Rather than using a theta graph, perhaps for the lens space $L(p, q)$ one could introduce a framing specific to that lens space by using a p-linked dumbbell graph D_p, as shown in Figure 1(c) for the case $p = 1$. There is a unique edge e_1 of the dumbbell graph that connects the two vertices, and one can choose a Heegaard surface Σ that transversely intersects e_1 in a single point q_1. The marked Heegaard surface is now $(\Sigma, q_1, p_1, \cdots, p_{2r})$, corresponding to the character variety $R(\Sigma, 2r + 1)$, and the handlebody pairs (U_i, A_i) are now $(U_i, A_i \cup \sigma_i)$, where σ_i is the sigma graph shown in Figure 1(d). The character variety $R(\Sigma, 2r + 1)$ has the structure of a symplectic manifold that is symplectomorphic to the moduli space of stable parabolic bundles $M^s(X, 2r + 1)$, which is the codomain of the embedding $\mathcal{P}_M(X, 1, 2r) \to M^s(X, 2r + 1)$.

APPENDIX A. VECTOR BUNDLES

Here we briefly review some results on holomorphic vector bundles and their moduli spaces that we will use throughout the paper. Some useful references on vector bundles are [18, 22, 26, 27].

Definition A.1. The *slope* of a holomorphic vector bundle E over a curve C is $\operatorname{slope} E := (\deg E)/(\operatorname{rank} E) \in \mathbb{Q}$.

Definition A.2. A holomorphic vector bundle E over a curve C is *stable* if $\operatorname{slope} F < \operatorname{slope} E$ for any proper subbundle $F \subset E$, *semistable* if $\operatorname{slope} F \leq \operatorname{slope} E$ for any proper subbundle $F \subset E$, *strictly semistable* if it is semistable but not stable, and *unstable* if there is a proper subbundle $F \subset E$ such that $\operatorname{slope} F > \operatorname{slope} E$.

If E is a stable vector bundle, then $\operatorname{Aut}(E) = \mathbb{C}^\times$ consists only of trivial automorphisms that scale the fibers by a constant factor.

Definition A.3. Given a semistable vector bundle E, a *Jordan-Hölder filtration* of E is a filtration

$$F_0 = 0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = E$$

of E by subbundles $F_i \subset E$ for $i = 0, \cdots, n$ such that the composition factors F_i/F_{i-1} are stable and $\operatorname{slope} F_i/F_{i-1} = \operatorname{slope} E$ for $i = 1, \cdots, n$.

Every semistable vector bundle E admits a Jordan-Hölder filtration. The filtration is not unique, but the composition factors F_i/F_{i-1} for $i = 1, \cdots, n$ are independent (up to permutation) of the choice of filtration.

Definition A.4. Given a semistable holomorphic vector bundle E over a curve C, the *associated graded* vector bundle $\operatorname{gr} E$ is defined to be

$$\operatorname{gr} E = \bigoplus_{i=1}^{n} F_i/F_{i-1},$$

where $F_0 = 0 \subset F_1 \subset \cdots \subset F_n = E$ is a Jordan-Hölder filtration of E.

The bundle $\operatorname{gr} E$ is independent (up to isomorphism) of the choice of filtration, and $\operatorname{slope}(\operatorname{gr} E) = \operatorname{slope} E$.

Definition A.5. Two semistable vector bundles are said to be *S-equivalent* if their associated graded bundles are isomorphic.

Example A.6. In Section 5.1 we define a strictly semistable rank 2 vector bundle F_2 and a stable rank 2 vector bundle $G_2(p)$ over an elliptic curve X. A Jordan-Hölder filtration of F_2 is $\mathcal{O} \subset F_2$, and the associated graded bundle is $\operatorname{gr} F_2 = \mathcal{O} \oplus \mathcal{O}$. It follows that F_2 and $\mathcal{O} \oplus \mathcal{O}$ are S-equivalent. A Jordan-Hölder filtration of $G_2(p)$ is just $G_2(p)$, and the associated graded bundle is $\operatorname{gr} G_2(p) = G_2(p)$.

Isomorphic bundles are S-equivalent. For rational curves, S-equivalent bundles are isomorphic, but this is not true in general. For example, on an elliptic curve the bundles F_2 and $\mathcal{O} \oplus \mathcal{O}$ are S-equivalent but not isomorphic.

Definition A.7. We define $M^{ss}(C)$ (respectively $M^s(C)$) to be the moduli space of semistable (respectively stable) rank 2 holomorphic vector bundles over curve C with trivial determinant bundle, mod S-equivalence. This space is defined in [25]; see also [20].

Remark A.8. An alternative way of interpreting $M^{ss}(C)$ is as the space of flat $SU(2)$-connections on a trivial rank 2 complex vector bundle $E \to C$, mod gauge transformations. Yet another way of interpreting the space $M^{ss}(C)$ is as the character variety $R(C)$ of conjugacy classes of group homomorphisms $\pi_1(C) \to SU(2)$. We will not use these interpretations here.

The moduli space $M^s(C)$ has the structure of a complex manifold of dimension $3(g-1)$, where g is the genus of the curve C. The space $M^s(C)$ carries a canonical symplectic form, which is obtained by interpreting $M^s(C)$ as a Hamiltonian reduction of a space of $SU(2)$-connections.

Example A.9. For rational curves, the bundle $\mathcal{O}\oplus\mathcal{O}$ is the unique semistable rank 2 bundle with trivial determinant bundle, and there are no stable rank 2 bundles, so

$$M^{ss}(\mathbb{CP}^1) = \{pt\} = \{[\mathcal{O} \oplus \mathcal{O}]\}, \qquad M^s(\mathbb{CP}^1) = \varnothing.$$

Example A.10. For an elliptic curve X, semistable rank 2 bundles with trivial determinant bundle have the form $L \oplus L^{-1}$, where L is a degree 0 line bundle, or $F_2 \otimes L_i$, where L_i for $i = 1, \cdots, 4$ are the four 2-torsion line bundles. The bundles $L_i \oplus L_i$ and $F_2 \otimes L_i$ are S-equivalent. The bundles $L \oplus L^{-1}$ and $L^{-1} \oplus L$ are isomorphic, hence S-equivalent. There are no stable rank 2 bundles with trivial determinant bundle. As shown in [27], we have that

$$M^{ss}(X) = \{[L \oplus L^{-1}] \mid [L] \in \operatorname{Jac}(X)\} = \mathbb{CP}^1, \qquad M^s(X) = \varnothing.$$

Appendix B. Parabolic bundles

Here we briefly review some results on parabolic bundles and their moduli spaces that we will use throughout the paper. Some useful references on parabolic bundles are [19, 21].

B.1. **Definition of a parabolic bundle.**

The concept of a parabolic bundle was introduced in [19]:

Definition B.1. A *parabolic bundle* of rank r on a curve C consists of following data:

1. A rank r holomorphic vector bundle $\pi_E : E \to C$.
2. Distinct marked points $(p_1, \cdots, p_n) \in C^n$.
3. For each marked point p_i, a flag of vector spaces $E^j_{p_i}$ in the fiber $E_{p_i} = \pi_E^{-1}(p_i)$ over the point p_i:
$$E^0_{p_i} = 0 \subset E^1_{p_i} \subset E^2_{p_i} \subset \cdots \subset E^{s_i}_{p_i} = E_{p_i}.$$
4. For each marked point p_i, a strictly decreasing list of *weights* $\lambda^j_{p_i} \in \mathbb{R}$:
$$\lambda^1_{p_i} > \lambda^2_{p_i} > \cdots > \lambda^{s_i}_{p_i}.$$

We refer the data of the marked points, the flags, and the weights as a *parabolic structure* on E. We refer to the data of just the marked points and flags, without the weights, as a *quasi-parabolic structure* on E. We define the *multiplicity* of the weight $\lambda^j_{p_i}$ to be $m^j_{p_i} := \dim(E^j_{p_i}) - \dim(E^{j-1}_{p_i})$. The definition of a parabolic bundle given in [19] differs slightly from our definition, in that the marked points are unordered and the weights are required to lie in the range $[0, 1)$.

Definition B.2. Two parabolic bundles with underlying vector bundles E and F are *isomorphic* if the marked points and weights for the two bundles are the same, and there is a bundle isomorphism $\alpha : E \to F$ that carries each flag of E to the corresponding flag of F; that is, $\alpha(E^j_{p_i}) = F^j_{p_i}$ for $j = 1, \cdots, s_i$ and $i = 1, \cdots, n$.

Definition B.3. The *rank* of a parabolic bundle is the rank of its underlying vector bundle.

Definition B.4. The *parabolic degree* and *parabolic slope* of a parabolic bundle $\mathcal{E}$ with underlying vector bundle E are defined to be

$$\operatorname{pdeg} \mathcal{E} = \deg E + \sum_{i=1}^{n} \sum_{j=1}^{s_i} m^j_{p_i} \lambda^j_{p_i} \in \mathbb{R}, \qquad \operatorname{pslope} \mathcal{E} = (\operatorname{pdeg} \mathcal{E})/(\operatorname{rank} \mathcal{E}) \in \mathbb{Q}.$$

We will not need the full generality of the concept of a parabolic bundle; rather, we will consider only parabolic line bundles and rank 2 parabolic bundles of a certain restricted form.

First we consider parabolic line bundles. For such bundles there is no flag data, so the parabolic structure is specified by a list of marked points $p_1, \cdots, p_n$ and a list of weights $\lambda^1_{p_1}, \cdots, \lambda^1_{p_n}$. We fix a parameter $\mu > 0$ and restrict to the case $\lambda^1_{p_i} \in \{\pm\mu\}$ for $i = 1, \cdots, n$. A parabolic line bundle of this form thus consists of the data $(L, \sigma_{p_1}, \cdots, \sigma_{p_n})$, where $\pi_L : L \to C$ is a holomorphic line bundle and $\sigma_{p_i} \in \{\pm 1\}$. The parabolic degree and parabolic slope of a parabolic line bundle $(L, \sigma_{p_1}, \cdots, \sigma_{p_n})$ are given by

$$\operatorname{pdeg}(L, \sigma_{p_1}, \cdots, \sigma_{p_n}) = \operatorname{pslope}(L, \sigma_{p_1}, \cdots, \sigma_{p_n}) = \deg L + \mu \sum_{i=1}^{n} \sigma_{p_i} = \operatorname{slope} L + \mu \sum_{i=1}^{n} \sigma_{p_i}.$$

Next we consider rank 2 parabolic bundles. We fix a parameter $\mu > 0$ and restrict to the case $s_i = 2$, $m^1_{p_i} = m^2_{p_i} = 1$, and $\lambda^1_{p_i} = -\lambda^2_{p_i} = \mu$ for $i = 1, \cdots, n$. A rank 2 parabolic bundle of this form thus consists of the data $(E, \ell_{p_1}, \cdots, \ell_{p_n})$, where $\pi_E : E \to C$ is a rank 2 holomorphic vector bundle and $\ell_{p_i} \in \mathbb{P}(E_{p_i})$ is a line in the fiber $E_{p_i} = \pi_E^{-1}(p_i)$ over the point p_i for $i = 1, \cdots, n$. The parabolic slope and parabolic degree of a rank 2 parabolic bundle $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ are given by

$$\operatorname{pdeg}(E, \ell_{p_1}, \cdots, \ell_{p_n}) = \deg E, \qquad \operatorname{pslope}(E, \ell_{p_1}, \cdots, \ell_{p_n}) = \operatorname{slope} E.$$

B.2. **Stable, semistable, and unstable parabolic bundles.** Consider a rank 2 parabolic bundle $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ and a line subbundle $L \subset E$. There are induced parabolic structures on the line bundles L and E/L given by $(L, \sigma_{p_1}, \cdots, \sigma_{p_n})$ and $(E/L, -\sigma_{p_1}, \cdots, -\sigma_{p_n})$, where

$$\sigma_{p_i} = \begin{cases} +1 & \text{if } L_{p_i} = \ell_{p_i}, \\ -1 & \text{if } L_{p_i} \neq \ell_{p_i}. \end{cases}$$

Definition B.5. Given a rank 2 parabolic bundle $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ and a line subbundle $L \subset E$, we say that the induced parabolic bundle $(L, \sigma_{p_1}, \cdots, \sigma_{p_n})$ is a *parabolic subbundle* of $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ and the induced parabolic bundle $(E/L, -\sigma_{p_1}, \cdots, -\sigma_{p_n})$ is a *parabolic quotient bundle* of $(E, \ell_{p_1}, \cdots, \ell_{p_n})$.

Definition B.6. A rank 2 parabolic bundle $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is said to be *decomposable* if there exists a decomposition $E = L \oplus L'$ for line bundles L and L' such that $\ell_{p_i} \in \{L_{p_i}, L'_{p_i}\}$ for $i = 1, \cdots, n$. For a rank 2 decomposable parabolic bundle $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ we write

$$(E, \ell_{p_1}, \cdots, \ell_{p_n}) = (L, \sigma_{p_1}, \cdots, \sigma_{p_n}) \oplus (L', \sigma'_{p_1}, \cdots, \sigma'_{p_n}),$$

where $(L, \sigma_{p_1}, \cdots, \sigma_{p_n})$ and $(L', \sigma'_{p_1}, \cdots, \sigma'_{p_n})$ are the induced parabolic structures on L and L'.

Definition B.7. A rank 2 parabolic bundle is *stable* if its parabolic slope is strictly greater than the parabolic slope of any of its proper parabolic subbundles, *semistable* if its parabolic slope is greater than or equal than the parabolic slope of any of its proper parabolic subbundles, *strictly semistable* if it is semistable but not stable, and *unstable* if it has a proper parabolic subbundle of strictly greater slope.

If $\mathcal{E}$ is a stable parabolic bundle, then $\operatorname{Aut}(\mathcal{E}) = \mathbb{C}^\times$ consists only of trivial automorphisms that scale the fibers of the underlying vector bundle by a constant factor.

Theorem B.8. *If the rank 2 parabolic bundle $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is semistable and $\mu < 1/2n$, then E is semistable.*

Proof. We will prove the contrapositive, so assume that E is unstable. Then there is a line subbundle $L \subset E$ such that $\operatorname{slope} L > \operatorname{slope} E$. Consider the parabolic structure $(L, \sigma_{p_1}, \cdots, \sigma_{p_n})$ induced on L by $(E, \ell_{p_1}, \cdots, \ell_{p_n})$. We have that

$$\operatorname{pslope}(L, \sigma_{p_1}, \cdots, \sigma_{p_n}) - \operatorname{pslope}(E, \ell_{p_1}, \cdots, \ell_{p_n}) = \operatorname{slope} L + \mu \sum_{i=1}^{n} \sigma_{p_i} - \operatorname{slope} E. \tag{6}$$

Since slope L is an integer, slope E is an integer or half-integer, and slope $L >$ slope E, it follows that slope $L -$ slope $E \geq 1/2$. From equation (6) and the assumption that $\mu < 1/2n$, it follows that

$$\operatorname{pslope}(L, \sigma_{p_1}, \cdots, \sigma_{p_n}) - \operatorname{pslope}(E, \ell_{p_1}, \cdots, \ell_{p_n}) \geq 1/2 - n\mu > 0,$$

so $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is unstable. □

Throughout this paper we will always assume $\mu \ll 1$, by which we mean that μ is always chosen to be sufficiently small such that Theorem B.8 holds under whatever circumstances we are considering.

Consider a rank 2 vector bundle E over a curve C. If E is unstable, then Theorem B.8 implies that the parabolic bundle $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is unstable. If E is a semistable, then the stability of the parabolic bundle $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ can be characterized as follows:

Theorem B.9. *Consider a rank 2 parabolic bundle of the form $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ with E semistable. Let m be the maximum number of lines that are bad in the same direction. Such a parabolic bundle is stable if and only if $m < n/2$, semistable if and only if $m \leq n/2$, and unstable if and only if $m > n/2$. In particular, if n is odd then stability and semistability are equivalent.*

Example B.10. As a special case of Theorem B.9, consider parabolic bundles $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ over $\mathbb{CP}^1$ with underlying vector bundle $E = \mathcal{O} \oplus \mathcal{O}$. We can globally trivialize E and identify all the fibers with $\mathbb{C}^2$. All lines of E are bad, and lines are bad in the same direction if and only if they are equal under the global trivialization. Let m denote the maximum number of lines ℓ_{p_i} equal to any given line in $\mathbb{CP}^1$. From Theorem B.9, we find that $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is stable if $m < n/2$, semistable if $m \leq n/2$, and unstable if $m > n/2$. For example, (E, ℓ_{p_1}) is unstable, $(E, \ell_{p_1}, \ell_{p_2})$ is strictly semistable if the lines are distinct and unstable otherwise, and $(E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3})$ is stable if the lines are distinct and unstable otherwise.

B.3. **S-equivalent semistable parabolic bundles.** There is Jordan-Hölder theorem for parabolic bundles, which asserts that any semistable parabolic bundle of parabolic degree 0 has a filtration in which quotients of successive parabolic bundles (composition factors) in the filtration are stable with parabolic slope 0 (see [19, Remark 1.16]). The filtration is not unique, but the composition factors are unique up to permutation. It follows that one can define an associated graded bundle of a semistable parabolic bundle of parabolic degree 0 that is unique up to isomorphism.

We will need the concept of an associated graded parabolic bundle only for the case of semistable rank 2 parabolic bundles. If such a parabolic bundle $\mathcal{E}$ is stable, then its associated graded parabolic bundle $\operatorname{gr}\mathcal{E}$ is just $\mathcal{E}$. Now consider a strictly semistable parabolic bundle $\mathcal{E} = (E, \ell_{p_1}, \cdots, \ell_{p_n})$. Under our standard assumption that $\mu \ll 1$, it follows from Theorem B.8 that E is semistable. The associated graded parabolic bundle $\operatorname{gr}\mathcal{E}$ is given by

$$\operatorname{gr}(E, \ell_{p_1}, \cdots, \ell_{p_n}) = (L, \sigma_{p_1}, \cdots, \sigma_{p_n}) \oplus (E/L, -\sigma_{p_1}, \cdots, -\sigma_{p_n}),$$

where $L \subset E$ is a line subbundle such that slope $L =$ slope E, and $(L, \sigma_{p_1}, \cdots, \sigma_{p_n})$ and $(E/L, -\sigma_{p_1}, \cdots, -\sigma_{p_n})$ are the induced parabolic structures on L and E/L. Note that

$$\operatorname{pslope}(\operatorname{gr}(E, \ell_{p_1}, \cdots, \ell_{p_n})) = \operatorname{pslope}(E, \ell_{p_1}, \cdots, \ell_{p_n}) = \operatorname{slope} E.$$

Definition B.11. We say that two semistable rank 2 parabolic bundles are *S-equivalent* if their associated graded bundles are isomorphic.

Isomorphic parabolic bundles are S-equivalent. Here we give an example to show that the converse does not always hold:

Example B.12. Consider parabolic bundles over $\mathbb{CP}^1$ with underlying vector bundle $E = \mathcal{O} \oplus \mathcal{O}$. We can globally trivialize E and identify all the fibers with $\mathbb{C}^2$. Let A, B, and C be distinct lines in $\mathbb{CP}^1$, and consider the two strictly semistable parabolic bundles

$$\mathcal{E} := (E,\, \ell_{p_1} = A,\, \ell_{p_2} = A,\, \ell_{p_3} = B,\, \ell_{p_4} = C), \qquad \mathcal{E}' := (E,\, \ell'_{p_1} = B,\, \ell'_{p_2} = C,\, \ell'_{p_3} = A,\, \ell'_{p_4} = A).$$

The bundles $\mathcal{E}$ and $\mathcal{E}'$ are not isomorphic but are S-equivalent, since the associated graded bundles of both bundles are isomorphic to

$$(\mathcal{O},\, \sigma_{p_1} = 1,\, \sigma_{p_2} = 1,\, \sigma_{p_3} = -1,\, \sigma_{p_4} = -1) \oplus (\mathcal{O},\, \sigma_{p_1} = -1,\, \sigma_{p_2} = -1,\, \sigma_{p_3} = 1,\, \sigma_{p_4} = 1).$$

B.4. **Moduli spaces of rank 2 parabolic bundles.**

Definition B.13. We define $M^{ss}(C, n)$ (respectively $M^s(C, n)$), to be the moduli space of semistable (respectively stable) rank 2 parabolic bundles of the form $(E, \ell_{p_1}, \cdots, \ell_{p_n})$ such that E has trivial determinant bundle, mod S-equivalence. In particular, $M^{ss}(C, 0) = M^{ss}(C)$ and $M^s(C, 0) = M^s(C)$. As always, we assume that $\mu \ll 1$. This space is defined in [19]; see also [4].

Remark B.14. An alternative way of interpreting $M^{ss}(C,n)$ is as the space of flat $SU(2)$-connections on a trivial rank 2 complex vector bundle $E \to C - \{p_1, \cdots, p_n\}$, where the holonomy around each puncture point p_i is conjugate to $\mathrm{diag}(e^{2\pi i\mu}, e^{-2\pi i\mu})$, mod $SU(2)$ gauge transformations. Yet another way of interpreting the space $M^{ss}(C,n)$ is as the character variety $R(C,n)$ of conjugacy classes of group homomorphisms $\pi_1(C - \{p_1, \cdots, p_n\}) \to SU(2)$ that take loops around the marked points to matrices conjugate to $\mathrm{diag}(e^{2\pi i\mu}, e^{-2\pi i\mu})$. Note that $\mu = 1/4$ corresponds to the traceless character variety. We will not use these interpretations here.

For rational μ, the moduli space $M^s(C,n)$ has the structure of a complex manifold of dimension $3(g-1)+n$, where g is the genus of the curve C, and $M^s(C,n)$ is compact for n odd. The space $M^s(C,n)$ carries a canonical symplectic form, which is obtained by viewing $M^s(C,n)$ as a Hamiltonian reduction of a space of $SU(2)$-connections with prescribed singularities.

Example B.15. Let $C = \mathbb{CP}^1$ be a rational curve. If we fix $n \leq 3$, then all rank 2 parabolic bundles of the form $(\mathcal{O} \oplus \mathcal{O}, \ell_{p_1}, \cdots, \ell_{p_n})$ for which all the lines are distinct are isomorphic. From this fact, together with the results of Example B.10, we find that

$$M^{ss}(\mathbb{CP}^1, 0) = \{pt\}, \qquad M^{ss}(\mathbb{CP}^1, 1) = \varnothing, \qquad M^{ss}(\mathbb{CP}^1, 2) = \{pt\}, \qquad M^{ss}(\mathbb{CP}^1, 3) = \{pt\},$$
$$M^{s}(\mathbb{CP}^1, 0) = \varnothing, \qquad M^{s}(\mathbb{CP}^1, 1) = \varnothing, \qquad M^{s}(\mathbb{CP}^1, 2) = \varnothing, \qquad M^{s}(\mathbb{CP}^1, 3) = \{pt\}.$$

Using the cross-ratio and considerations of S-equivalence as described in Example B.12, one can show

$$M^{ss}(\mathbb{CP}^1, 4) = \mathbb{CP}^1, \qquad M^{s}(\mathbb{CP}^1, 4) = \mathbb{CP}^1 - \{3 \text{ points}\}.$$

Example B.16. Let X be an elliptic curve. From Corollary 5.25, Example A.10, and Theorem B.9, we have that

$$M^{ss}(X, 0) = \mathbb{CP}^1, \qquad M^{s}(X, 0) = \varnothing, \qquad M^{ss}(X, 1) = \mathbb{CP}^1, \qquad M^{s}(X, 1) = \mathbb{CP}^1.$$

In [28] it is shown that

$$M^{ss}(X, 2) = (\mathbb{CP}^1)^2, \qquad M^{s}(X, 2) = (\mathbb{CP}^1)^2 - g(X),$$

where $g : X \to (\mathbb{CP}^1)^2$ is a holomorphic embedding. We also derive this result in Section 5.5.2.

Remark B.17. Throughout this paper we assume $\mu \ll 1$, but for some applications one wants to take $\mu = 1/4$ in order to interpret $M^{ss}(C,n)$ as a traceless character variety, as described in Remark B.14. In general $M^{ss}(C,n)$ depends on μ; for example, for $0 \leq n \leq 4$ the space $M^{ss}(\mathbb{CP}^1, n)$ is the same for $\mu \ll 1/4$ and $\mu = 1/4$, but, as shown in [23], for $n = 5$ we have

$$M^{ss}(\mathbb{CP}^1, 5) = \begin{cases} \mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2 & \text{for } \mu \ll 1, \\ \mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2 & \text{for } \mu = 1/4. \end{cases}$$

The dependence of $M^{ss}(C,n)$ on μ is discussed in [6].

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