COMPUTER BOUNDS FOR KRONHEIMER–MROWKA FOAM EVALUATION

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ABSTRACT. Kronheimer and Mrowka recently suggested a possible approach towards a new proof of the four color theorem. Their approach is based on a functor $J^\sharp$, which they define using gauge theory, from the category of webs and foams to the category of $\mathbb{F}$-vector spaces, where $\mathbb{F}$ is the field of two elements. They also consider a possible combinatorial replacement $J^\flat$ for $J^\sharp$. Of particular interest is the relationship between the dimension of $J^\flat(K)$ for a web K and the number of Tait colorings $\mathrm{Tait}(K)$ of K; these two numbers are known to be identical for a special class of "reducible" webs, but whether this is the case for nonreducible webs is not known. We describe a computer program that strongly constrains the possibilities for the dimension and graded dimension of $J^\flat(K)$ for a given web K, in some cases determining these quantities uniquely. We present results for a number of nonreducible example webs. For the dodecahedral web W_1 the number of Tait colorings is $\mathrm{Tait}(W_1) = 60$, but our results suggest that $\dim J^\flat(W_1) = 58$.

1. INTRODUCTION

The four-color theorem states that any map in the plane can be colored using no more than four colors in such a way that no two regions that share a common boundary share the same color. The theorem was first proven in 1976 by Appel and Haken via computer calculations [1], and though simplifications to their proof have been made [16, 5], to this day no proof is known that does not rely on computer assistance.

Recently Kronheimer and Mrowka suggested a new approach to the four color theorem that may lead to the first non-computer-assisted proof of this result [9]. Their approach is based on a functor $J^\sharp$, which they define using gauge theory, from the category of webs and foams to the category of $\mathbb{F}$-vector spaces, where $\mathbb{F}$ is the field of two elements. Kronheimer and Mrowka also consider a possible combinatorial replacement $J^\flat$ for $J^\sharp$. The functor $J^\flat$ was originally defined by Kronheimer and Mrowka in terms of a list of combinatorial rules that they conjectured would yield a well-defined result; this was later shown to be the case by Khovanov and Robert [8].

In order to apply the functors $J^\sharp$ and $J^\flat$ to the four-color problem, it is important to understand the relationships between $\dim J^\sharp(K)$, $\dim J^\flat(K)$, and the Tait number $\mathrm{Tait}(K)$ for arbitrary planar webs K. For a special class of "reducible" webs it is known that $J^\flat(K) = \mathrm{Tait}(K)$, but whether this is the case for nonreducible webs is not known. The vector space $J^\flat(K)$ carries a quantum grading, and it is also of interest to compute the quantum dimension $\mathrm{qdim}\, J^\flat(K)$ of this space. We have written a computer program to calculate lower bounds on $\dim J^\flat(K)$ and $\mathrm{qdim}\, J^\flat(K)$, which in some cases are strong enough to determine these quantities uniquely. Our results are summarized in Table 2. In particular, we prove:

Theorem 1.1. *For the nonreducible webs W_2 and W_3 shown in Figure 6 we have $\dim J^\flat(K) = \mathrm{Tait}(K)$.*

So far as we know, Theorem 1.1 constitutes the first exact calculation of $\dim J^\flat(K)$ for K nonreducible. The smallest nonreducible web is the dodecahedral web W_1 shown in Figure 6. For the dodecahedral web the Tait number is $\mathrm{Tait}(W_1) = 60$, but our results show that $\dim J^\flat(W_1)$ must be either 58 and 60, and suggest that it is in fact 58. (We should emphasize that if $\dim J^\flat(W_1) = 58$, this would not invalidate Kronheimer and Mrowka's strategy for proving the four-color theorem using gauge theory; see Remark 2.15.)

Additional insight into the functor $J^\flat$ can be gained by considering a related functor $\langle - \rangle_\phi$ from the category of webs and foams to the category of $\mathbb{F}[E]$-modules. In particular, the $\mathbb{F}[E]$-module $\langle K \rangle_\phi$ associated to a web K carries a quantum grading, which we use to give a simple condition under which $\dim J^\flat(K) = \mathrm{Tait}(K)$:

Theorem 1.2. *If $\mathrm{qrank}\, \langle K \rangle_\phi$ is symmetric under $q \to 1/q$ then $\dim J^\flat(K) = \mathrm{Tait}(K)$ and $\mathrm{qdim}\, J^\flat(K) = \mathrm{qrank}\, \langle K \rangle_\phi$.*
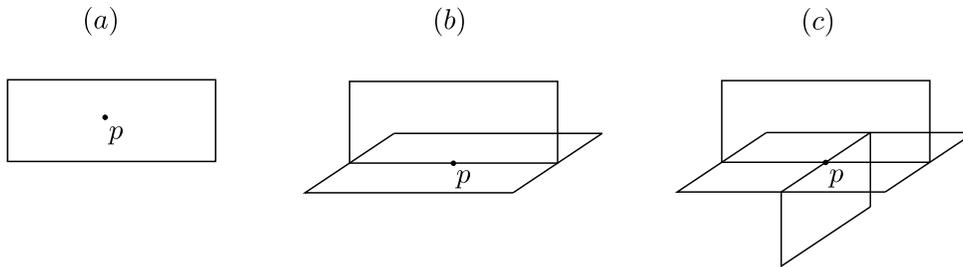
FIGURE 1. Local models for closed foams F. (a) Local model near a regular point $p \in F$. (b) Local model near a seam point $p \in F$. (c) Local model near a tetrahedral point $p \in F$.

The paper is organized as follows. In Section 2 we describe the functors $J^\sharp$ and $J^\flat$ and their relationship to the four-color problem. In Section 3 we prove several results that form the theoretical foundation of our computer calculations. In Section 4 we describe the computer program. In Section 5 we present the computer bounds on $\dim J^\flat(K)$ and $\operatorname{qdim} J^\flat(K)$ for a number of nonreducible example webs K. In Section 6 we discuss several open questions.

2. BACKGROUND

2.1. **The functors $J^\sharp$ and $J^\flat$.** Kronheimer and Mrowka's new approach to the four-color problem relies on concepts involving webs and foams, which we briefly review here. A *web* is an unoriented planar trivalent graph. We allow multiple edges of a web to share the same end nodes (a graph that allows such edges is technically a *multigraph*), we allow an edge to connect a vertex to itself (such edges are called *loops*), and we allow edges to close on themselves to form topological circles. A *foam* is a kind of singular cobordism between two webs. More precisely, a *closed foam* F is a singular 2D surface embedded in $\mathbb{R}^3$ in which every point $p \in F$ has a neighborhood described by one of three local models shown in Figure 1. Points with the local model shown in Figure 1a, Figure 1b, and Figure 1c are called *regular points*, *seam points*, and *tetrahedral points*, respectively. The set of regular points forms a smooth 2D manifold whose connected components are the *facets* of F. Each facet may be decorated with a finite number (possibly zero) of marked points called *dots*. The set of seam points forms a smooth 1D manifold whose connected components are the *seams* of F. In general, we want to consider *foams with boundary* $F \subset \mathbb{R}^2 \times [a, b]$, which have local models $K_- \times [a, a + \epsilon)$ and $K_+ \times (b - \epsilon, b]$ for webs K_- and K_+ near the bottom and top of the foam. We define a *half-foam* H to be a foam with bottom boundary $K_- = \varnothing$. We define a category Foams with webs as objects and foams as morphisms. We will thus sometimes refer to a foam F with bottom boundary K_- and top boundary K_+ as a *cobordism* $F : K_- \to K_+$.

Example 2.1. Any closed surface embedded in $\mathbb{R}^3$ is a closed foam with one facet, no seams, and no tetrahedral points. The union of $S^2 \subset \mathbb{R}^3$ and the equatorial disk is a closed foam known as the *theta-foam*; it has three facets, one seam, and no tetrahedral points. The union of the theta-foam and the upper half of the portion of the $x - z$ plane that is contained within it is a closed foam with six facets, four seams, and two tetrahedral points.

Using a simple argument, one can reformulate the four-color theorem in terms of webs. We first define some additional terminology. An edge e of a web is said to be a *bridge* if removing e increases the number of connected components of the web. A *Tait coloring* of a web K is a 3-coloring of the edges of K such that no two edges incident on any given vertex share the same color. Given a web K, we define the *Tait number* $\operatorname{Tait}(K)$ to be the number of distinct Tait colorings of K. The four-color theorem is then equivalent to:

Theorem 2.2. *(Four-color theorem, reformulated) For any bridgeless web K, we have* $\operatorname{Tait}(K) > 0$.

This reformulation allows Kronheimer and Mrowka to introduce ideas from gauge theory; in essence, they define a version of singular instanton homology in which the gauge fields are required to have prescribed singularities along a given web K. In this manner they define a functor $J^\sharp : \text{Foams} \to \text{Vect}_{\mathbb{F}}$ from the category of foams to the category of $\mathbb{F}$-vector spaces, where $\mathbb{F}$ is the field of two elements. In particular, the functor associates a vector space $J^\sharp(K)$ to each web K.
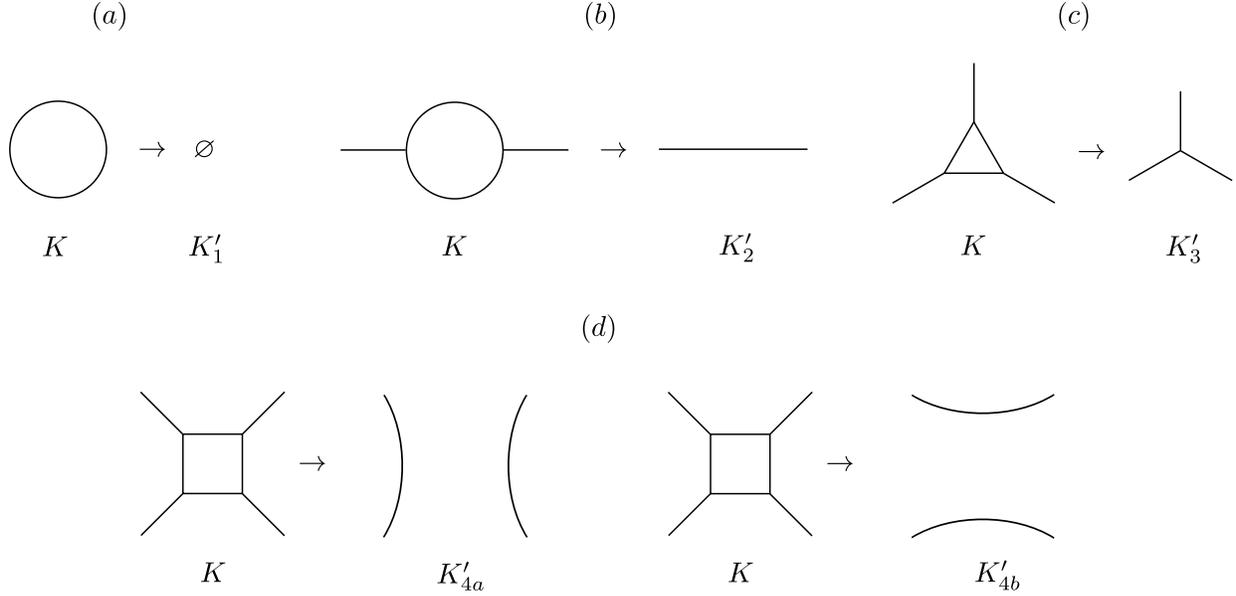
(a)   (b)   (c)

K K_1' K K_2' K K_3'

(d)

K K_{4a}' K K_{4b}'

FIGURE 2. Local replacements $K \to K'$ for reducible webs. The web K' is identical to K outside the depicted region. (a) Disk elimination $K \to K_1'$. (b) Bigon elimination $K \to K_2'$. (c) Triangle elimination $K \to K_3'$. (d) Square elimination $K \to K_{4a}'$ and $K \to K_{4b}'$.

Remark 2.3. The functor $J^\sharp$ can be defined for a more general source category in which the webs are embedded in $\mathbb{R}^3$ and the foams are embedded in $\mathbb{R}^4$. We will not consider these more general notions of webs and foams here.

Kronheimer and Mrowka prove the following theorems:

Theorem 2.4. *(Kronheimer–Mrowka [9, Theorem 1.1]) Given a web K, we have* $\dim J^\sharp(K) = 0$ *if and only if K has a bridge.*

Theorem 2.5. *(Kronheimer–Mrowka [11]) For any web K, we have* $\dim J^\sharp(K) \geq \mathrm{Tait}(K)$.

Based on example calculations, as well as general properties of the functor $J^\sharp$, Kronheimer and Mrowka make the following conjecture:

Conjecture 2.6. *For any web K, we have* $\dim J^\sharp(K) = \mathrm{Tait}(K)$.

Kronheimer and Mrowka show that Conjecture 2.6 is in fact true for a special class of *reducible* webs K (these are called *simple* in [9]), which are defined as follows. Given a web K, we can consider the local replacements shown in Figure 2 in which a *small face* of K, defined to be a disk, bigon, triangle, or square, is eliminated to yield a simpler web K'. We say that a web is *reducible* if there is a series of such local replacements that terminates in the empty web. That is, the empty web is reducible, and a nonempty web K is reducible if K_1' is reducible, K_2' is reducible, K_3' is reducible, or K_{4a}' and K_{4b}' are both reducible, for some choice of local replacements of the form shown in Figure 2.

Remark 2.7. Since $\varnothing$ is initial in sets, it has a unique Tait coloring, thus $\mathrm{Tait}(\varnothing) = 1$. The Tait number for a nonempty web K satisfies the following relations for the local replacements shown in Figure 2:

(1) $\mathrm{Tait}(K) = 3\,\mathrm{Tait}(K_1')$.
(2) $\mathrm{Tait}(K) = 2\,\mathrm{Tait}(K_2')$.
(3) $\mathrm{Tait}(K) = \mathrm{Tait}(K_3')$.
(4) $\mathrm{Tait}(K) = \mathrm{Tait}(K_{4a}') + \mathrm{Tait}(K_{4b}')$.

Given a reducible web K and a choice of reduction, these relations allow for rapid computation of $\mathrm{Tait}(K)$.

The class of reducible webs serves as a useful test case because the recursive definition of reducible webs makes it easy to prove things about them. For example, using the relations described in Remark 2.7, we immediately obtain:

Theorem 2.8. *(Four-color theorem for reducible webs) For any reducible web K, we have $\mathrm{Tait}(K) > 0$.*

Due to Theorem 2.4, Conjecture 2.6 implies Theorem 2.2, the reformulated four-color theorem. It is thus of great interest to determine whether Conjecture 2.6 is in fact true. As a possible route towards that goal, Kronheimer and Mrowka suggested that the gauge-theoretic functor $J^\sharp : \mathrm{Foams} \to \mathrm{Vect}_\mathbb{F}$ might be related to a simpler functor $J^\flat : \mathrm{Foams} \to \mathrm{Vect}_\mathbb{F}$ that could be defined in a purely combinatorial fashion. Kronheimer and Mrowka described a list of combinatorial evaluation rules that they conjectured would assign a well-defined field element $J^\flat(F) \in \mathbb{F}$ to every closed foam F. This conjecture was later shown to be correct by Khovanov and Robert [8], who adapted ideas by Robert and Wagner [15] to describe an explicit formula for $J^\flat(F)$. Kronheimer and Mrowka make the following conjecture:

Conjecture 2.9. *(Kronheimer–Mrowka [9, Conjecture 8.10]) For any closed foam F, we have $J^\flat(F) = J^\sharp(F)$.*

Having defined $J^\flat(F) \in \mathbb{F}$ for closed foams F, one can use the *universal construction* [2] to extend $J^\flat$ to a functor $J^\flat : \mathrm{Foams} \to \mathrm{Vect}_\mathbb{F}$. This is done in [9] as follows.

First we define $J^\flat$ on webs. For the empty web $\varnothing$, define $J^\flat(\varnothing) = \mathbb{F}$. For a nonempty web K, define $S(K)$ to be the set of all half-foams with top boundary K, and define $V(K) := \mathbb{F} \cdot S(K)$ to be the $\mathbb{F}$-vector space freely generated by $S(K)$. Define a bilinear form $(-,-) : V(K) \otimes V(K) \to \mathbb{F}$ such that $(H_1, H_2) = J^\flat(H_1 \cup_K \overline{H}_2)$, where $H_1 \cup_K \overline{H}_2$ is the closed foam obtained by reflecting H_2 top-to-bottom to get $\overline{H}_2$ and then gluing it to H_1 along K. Now define $J^\flat(K)$ to be the quotient of $V(K)$ by the orthogonal complement $V(K)^\perp$ of $V(K)$ relative to $(-,-)$:

$$J^\flat(K) = V(K)/V(K)^\perp.$$

Next we define $J^\flat$ on cobordisms. Given a cobordism $F_{21} : K_1 \to K_2$ from web K_1 to web K_2, define $J^\flat(F_{21}) : J^\flat(K_1) \to J^\flat(K_2)$ such that $J^\flat(F_{21})([H_1]) = [F_{21} \cup_K H_1]$; note that we have defined $J^\flat$ on webs in precisely such a way that $J^\flat$ is well-defined on cobordisms. A closed foam F can be viewed as a cobordism $F : \varnothing \to \varnothing$, so $J^\flat(F)$ is a linear map from $\mathbb{F}$ to $\mathbb{F}$. Identifying this linear map with an element of $\mathbb{F}$, we recover the original closed foam evaluation $J^\flat(F) \in \mathbb{F}$.

Remark 2.10. The bilinear form $(-,-)$ on $V(K)$ induces a nondegenerate bilinear form on $J^\flat(K)$ for any web K. The functor $J^\sharp$ is monoidal, but it is not known if $J^\flat$ is monoidal.

If Conjecture 2.9 is true, then the vector space $J^\flat(K)$ is a subquotient of the vector space $J^\sharp(K)$, as can be seen as follows. Consider the subspace $W(K)$ of $J^\sharp(K)$ spanned by all vectors of the form $J^\sharp(H)$, where $H \in S(K)$ is a half-foam with top boundary K. The bilinear form $(-,-) : V(K) \otimes V(K) \to \mathbb{F}$ restricts to a bilinear form $(-,-)_{W(K)} : W(K) \otimes W(K) \to \mathbb{F}$. Then $J^\flat(K)$ is the quotient of $W(K)$ by the orthogonal complement $W(K)^\perp$ of $W(K)$ relative to $(-,-)_{W(K)}$. In particular, Conjecture 2.9 implies that $\dim J^\flat(K) \leq \dim J^\sharp(K)$, which is in fact the case, as we will see from other considerations.

In Section 3 we derive the following corollary to a result proved by Khovanov and Robert:

Corollary 2.11. *For any web K, we have $\dim J^\flat(K) \leq \mathrm{Tait}(K)$.*

Khovanov and Robert ask the following question, which they answer in the affirmative in the case that K is reducible:

Question 2.12 (Khovanov–Robert [8]). Is $\dim J^\flat(K) = \mathrm{Tait}(K)$ for every web K?

In summary, we can assign three integers to any web K: $\mathrm{Tait}(K)$, $\dim J^\sharp(K)$, and $\dim J^\flat(K)$. Theorem 2.5 and Corollary 2.11 imply that for any web K these three integers are related by

$$(1) \qquad\qquad \dim J^\flat(K) \leq \mathrm{Tait}(K) \leq \dim J^\sharp(K).$$

For a reducible web K these three integers coincide:

$$\dim J^\flat(K) = \mathrm{Tait}(K) = \dim J^\sharp(K).$$

Conjecture 2.6 states that $\dim J^\sharp(K) = \mathrm{Tait}(K)$ for all webs K, and Question 2.12 asks whether $\dim J^\flat(K) = \mathrm{Tait}(K)$ for all webs K. Due to Theorem 2.4, a proof that $\dim J^\sharp(K) = \mathrm{Tait}(K)$ for all webs K would yield a proof of the four-color theorem. Thus, due to the string of inequalities (1), a proof that $J^\flat(K) = J^\sharp(K)$ for all webs K would yield a proof of the four-color theorem.

In light of these interrelated conjectures, it is of interest to compute examples of $\dim J^\flat(K)$ and $\dim J^\sharp(K)$ for nonreducible webs K. The only such results that have yet been obtained are for the dodecahedral web W_1, the smallest nonreducible web, which has Tait number $\mathrm{Tait}(W_1) = 60$:

Theorem 2.13. *(Kronheimer–Mrowka* [9]*) For the dodecahedral web W_1, we have* $\dim J^\flat(W_1) \geq 58$.

Theorem 2.14. *(Kronheimer–Mrowka* [10]*) For the dodecahedral web W_1, we have* $\dim J^\sharp(W_1) \leq 68$.

Due to Theorem 2.5 and Corollary 2.11, it follows that for the dodecahedral web W_1 the current dimension bounds are

$$58 \leq \dim J^\flat(W_1) \leq 60 \leq \dim J^\sharp(W_1) \leq 68.$$

In Section 5 we show that $\dim J^\flat(W_1)$ must be either 58 or 60 (it cannot be 59), and our results suggest that $\dim J^\flat(W_1) = 58$.

Remark 2.15. We should emphasize that if $\dim J^\flat(W_1) = 58$ for the dodecahedral web W_1, as our results suggest, this would not rule out Kronheimer and Mrowka's strategy for proving the four-color theorem, since it leaves open the possibility that $\dim J^\sharp(K) = \mathrm{Tait}(K)$ for all webs K. Rather, if $\dim J^\flat(W_1) = 58$ this would rule out only one possible implementation of their strategy, namely that of showing that $J^\flat(K) = J^\sharp(K)$ for all webs K.

Since $J^\flat(K)$ is defined in terms of an infinite number of generators mod an infinite number of relations, it is not clear whether $\dim J^\flat(K)$ can be algorithmically computed. Nevertheless, it is possible to algorithmically compute lower bounds for $\dim J^\flat(K)$, as can be seen by considering the following two facts. First, given a closed foam F, Khovanov and Robert's foam evaluation formula shows that $J^\flat(F) \in \mathbb{F}$ is algorithmically computable. Second, we have the following observation:

Observation 2.16. Given a finite-dimensional subspace $V_f(K)$ of the vector space $V(K)$ spanned by all half-foams with top boundary K, the rank of the bilinear form $(-,-)$ restricted to $V_f(K)$ bounds $\dim J^\flat(K)$ from below.

Since the restriction of the bilinear form $(-,-)$ to $V_f(K)$ can be determined by evaluating the closed foams resulting from all possible pairings of a basis of half-foams for $V_f(K)$, it follows that lower bounds for $\dim J^\flat(K)$ can be algorithmically computed. Before describing the details of our scheme for calculating lower bounds, we first review some technical material that we will need.

Remark 2.17. Computer programs involving foams have been used before to resolve interesting questions in low-dimensional topology. For example, a computer program due to Lewark [12] for calculating Khovanov's $\mathfrak{sl}_3$-foam knot homology [7] was able to refute Lobb's conjecture [13] that the $\mathfrak{sl}_3$ knot concordance invariant s_3 is equivalent to the Rasmussen invariant s_2.

2.2. **Khovanov–Robert closed foam evaluation formula.** We describe here Khovanov and Robert's closed foam evaluation formula. Given a foam F, define $\mathrm{facets}(F)$ to be its set of facets. A *coloring* of a foam F is a map $c : \mathrm{facets}(F) \to \{1, 2, 3\}$, and we will refer to 1, 2, and 3 as *colors*. A coloring c of a foam F is *admissible* if the three facets incident on any given seam of F are assigned distinct colors. Given a foam F, define $\mathrm{adm}(F)$ to be the (possibly empty) set of admissible colorings of F.

Define a polynomial ring in three variables $R' = \mathbb{F}[X_1, X_2, X_3]$ and a ring $R'' = R'[(X_1 + X_2)^{-1}, (X_1 + X_3)^{-1}, (X_2 + X_3)^{-1}]$ obtained from R' by inverting the elements $X_1 + X_2$, $X_1 + X_3$, and $X_2 + X_3$. Define elementary symmetric polynomials

$$E_1 = X_1 + X_2 + X_3, \qquad E_2 = X_1 X_2 + X_2 X_3 + X_3 X_1, \qquad E_3 = X_1 X_2 X_3,$$

and define a subring $R = \mathbb{F}[E_1, E_2, E_3]$ of R'. Given a closed foam F and a coloring c of F, define $P(F, c) \in R'$ and $Q(F, c) \in R''$ by

$$P(F, c) = \prod_{f \in \mathrm{facets}(F)} X_{c(f)}^{d(f)}, \qquad Q(F, c) = \prod_{1 \leq i < j \leq 3} (X_i + X_j)^{\chi(F_{ij}(c))/2},$$

where $d(f)$ denotes the number of dots decorating the facet f and $\chi(F_{ij}(c))$ denotes the Euler characteristic of the closed surface $F_{ij}(c)$ obtained by taking the closure of the union of all the facets of F colored either

i or j by c. The surface $F_{ij}(c)$ is orientable, since it is embedded in $\mathbb{R}^3$, so its Euler characteristic $\chi(F_{ij}(c))$ is even and thus $Q(F,c)$ is in fact an element of R''. Define a polynomial

$$\langle F \rangle = \sum_{c \in \mathrm{adm}(F)} \frac{P(F,c)}{Q(F,c)} \in R. \tag{2}$$

That $\langle F \rangle$ is a polynomial, rather than just a rational function, is not obvious from equation (2), but it is nonetheless true. Khovanov and Robert's formula for $J^\flat(F)$ is obtained by evaluating $\langle F \rangle$ at $E_1 = E_2 = E_3 = 0$:

$$J^\flat(F) = \langle F \rangle|_{E_1 = E_2 = E_3 = 0} \in \mathbb{F}. \tag{3}$$

From equations (2) and (3) it follows that $J^\flat(F) = 0$ if F has no admissible colorings.

Example 2.18. Let F_n denote the closed foam $S^2 \subset \mathbb{R}^3$ with n dots. We find that

$$J^\flat(F_n) = \begin{cases} 1 & \text{if } n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

This result can be used to compute $J^\flat(K)$ for the circle web $K = S^1$. Let $H_n : \varnothing \to K$ denote the lower hemisphere of $S^2 \subset \mathbb{R}^3$ with n dots. Then $(H_n, H_m) = J^\flat(F_{n+m})$. It follows that $J^\flat(K) = \mathbb{F} \cdot \{[H_0], [H_1], [H_2]\}$, as can be verified by noting that $\dim J^\flat(K) = \mathrm{Tait}(K) = 3$ and the bilinear form induced on $J^\flat(K)$ by $(-,-)$ has full rank.

2.3. **Quantum gradings.** To any foam F, Khovanov and Robert associate a $\mathbb{Z}$-valued degree $\deg F$ that is given by

$$\deg F = 2d(F) - 2\chi(F) - \chi(s(F)), \tag{4}$$

where $d(F)$ is the total number of dots on F, $\chi(F)$ is the Euler characteristic of F, and $\chi(s(F))$ is the Euler characteristic of the union $s(F)$ of the seam points and tetrahedral points of F.

Foam degree is related to a $\mathbb{Z}$-grading on the ring $R' = \mathbb{F}[X_1, X_2, X_3]$ in which the variables X_1, X_2, and X_3 are all assigned degree 2. The grading on R' induces a grading on the subring $R = \mathbb{F}[E_1, E_2, E_3]$ in which E_1, E_2, and E_3 are assigned degrees 2, 4, and 6. The degree of a foam is related to the degree of its evaluation by the following result:

Theorem 2.19 (Khovanov–Robert [8] Theorem 2.17). *Given a closed foam F, the grading of $\langle F \rangle \in R$ is given by $\deg F$ when $\langle F \rangle$ is nonzero.*

Theorem 2.19 gives a useful vanishing condition:

Corollary 2.20. *If the degree of a closed foam F is nonzero, then $J^\flat(F) = 0$.*

Example 2.21. Let F_n denote the closed foam $S^2 \subset \mathbb{R}^3$ with n dots. We find that $\deg F_n = 2(n-2)$. It follows from Corollary 2.20 that $J^\flat(F_n) = 0$ if $n \neq 2$, as is consistent with Example 2.18.

Khovanov and Robert use the notion of foam degree to impose a $\mathbb{Z}$-grading on the vector space $J^\flat(K)$ associated to a web K. Recall that $J^\flat(K)$ is spanned by vectors of the form $J^\flat(H)$, where H is a half-foam with top boundary K. We define the grading of the vector $J^\flat(H) \in J^\flat(K)$ to be $\deg H$. In general, given a graded finite-dimensional vector space V we define V_i to be the subspace of V spanned by vectors of degree i and we define the quantum dimension $\mathrm{qdim}\, V \in \mathbb{Z}[q, q^{-1}]$ of V to be

$$\mathrm{qdim}\, V = \sum_i q^i \dim V_i.$$

2.4. **Foam evaluation in the ring $\mathbb{F}[E]$.** The functor $J^\flat$ is a special case of a more general class of functors that Khovanov and Robert define by evaluating closed foams in various rings. We consider here a functor $\langle - \rangle_\phi$ from the category of foams to the category of $\mathbb{F}[E]$-modules.

For a closed foam F, we define $\langle F \rangle_\phi = \phi(\langle F \rangle) \in \mathbb{F}[E]$, where $\langle F \rangle \in R = \mathbb{F}[E_1, E_2, E_3]$ is given by equation (2) and $\phi : \mathbb{F}[E_1, E_2, E_3] \to \mathbb{F}[E]$ is the ring homomorphism given by $E_1, E_2 \mapsto 0$, $E_3 \mapsto E$. We can extend $\langle - \rangle_\phi$ to a functor from the category of foams to the category of $\mathbb{F}[E]$-modules by using the universal construction. In particular, we define $\langle - \rangle_\phi$ on webs as follows. For the empty web $\varnothing$, define $\langle \varnothing \rangle_\phi = \mathbb{F}[E]$. For a nonempty web K, define $S(K)$ to be the set of all half-foams with top boundary K,

and define $M(K) = \mathbb{F}[E] \cdot S(K)$ to be the $\mathbb{F}[E]$-module freely generated by $S(K)$. Define a bilinear form $(-, -)_\phi : M(K) \otimes M(K) \to \mathbb{F}[E]$ such that $(H_1, H_2)_\phi = \langle H_1 \cup_K \overline{H}_2 \rangle_\phi$. Now define the $\mathbb{F}[E]$-module $\langle K \rangle_\phi$ to be the quotient of $M(K)$ by the orthogonal complement $M(K)^\perp$ of $M(K)$ relative to $(-, -)_\phi$:

$$\langle K \rangle_\phi = M(K)/M(K)^\perp.$$

The functor $\langle - \rangle_\phi$ is useful because, as we will show in Section 3, it is closely related to the functor $J^\flat$, but its properties are better understood. In particular, Khovanov and Robert prove the following result:

Theorem 2.22. *(Khovanov–Robert* [8, Proposition 4.18]*) For any web K, the $\mathbb{F}[E]$-module $\langle K \rangle_\phi$ is free of rank* $\mathrm{Tait}(K)$.

We can view $\mathbb{F}[E]$ as a $\mathbb{Z}$-graded ring with $\deg E = 6$, and the $\mathbb{F}[E]$-module $\langle K \rangle_\phi$ corresponding to a web K carries a $\mathbb{Z}$-grading induced by the degrees of the generating half-foams. In general, we have the following result, which follows directly from the graded Nakayama's lemma:

Theorem 2.23 ([14])**.** *If A is a $\mathbb{Z}$-graded algebra over a field K such that $A = \oplus_{n=0}^\infty A_n$ with $A_0 = K$, and M is a free finite-rank A-module that is $\mathbb{Z}$-graded, then M has a homogeneous basis.*

Remark 2.24. The assumption that $A_0 = K$ in Theorem 2.23 is necessary, as can be seen from the following example taken from [6, Section 1.2.4]. Define a $\mathbb{Z}$-graded K-algebra $A = K[x] \oplus K[x]$ and a $\mathbb{Z}$-graded A-module $M = A \cdot a$, where $a = (1, 0) + (0, x) \in A$. Then M is free of rank 1, but it does not have a homogeneous basis.

Since $\mathbb{F}[E]$ is an $\mathbb{F}$-algebra of the form required by Theorem 2.23, it follows from Theorem 2.22 that $\langle K \rangle_\phi$ has a homogeneous basis. In general, given a free graded $\mathbb{F}[E]$-module M of finite rank r we chose a homogeneous basis $B = \{m_1, \cdots, m_r\}$ of M and define the *quantum rank* $\mathrm{qrank}\, M \in \mathbb{Z}[q, q^{-1}]$ of M to be

$$\mathrm{qrank}\, M = \sum_i q^i \cdot \#\{m_k \in B \mid \deg m_k = i\}.$$

3. Theoretical results

In this section we describe several results that form the theoretical foundation for our computer calculations. Recall that given a nonempty web K we defined $S(K)$ to be the set of all half-foams with top boundary K, and we defined $V(K)$ and $M(K)$ to be the $\mathbb{F}$-vector space and $\mathbb{F}[E]$-module freely generated by $S(K)$:

$$V(K) = \mathbb{F} \cdot S(K), \qquad\qquad M(K) = \mathbb{F}[E] \cdot S(K).$$

We defined the $\mathbb{F}$-vector space $J^\flat(K)$ and free $\mathbb{F}[E]$-module $\langle K \rangle_\phi$ by

$$J^\flat(K) = V(K)/V(K)^\perp, \qquad\qquad \langle K \rangle_\phi = M(K)/M(K)^\perp,$$

where the orthogonal complements $V(K)^\perp$ and $M(K)^\perp$ are taken relative to $(-, -)$ and $(-, -)_\phi$. By comparing the definitions of $J^\flat(K)$ and $\langle K \rangle_\phi$, we obtain the following corollary to Theorem 2.22, which was stated in a weaker form as Corollary 2.11:

Corollary 3.1. *For any web K, we have $\dim J^\flat(K) \le \mathrm{Tait}(K)$ and $\mathrm{qdim}\, J^\flat(K) \le \mathrm{qrank}\, \langle K \rangle_\phi$.*

Proof. This follows from the fact that $(H_1, H_2) \in \mathbb{F}$ can be obtained by evaluating $(H_1, H_2)_\phi \in \mathbb{F}[E]$ at $E = 0$, so $(H_1, H_2) = 0$ whenever $(H_1, H_2)_\phi = 0$. □

We can also use the relationship between the bilinear forms $(-, -)$ and $(-, -)_\phi$ to prove the following result, which gives a condition under which the inequalities in Corollary 3.1 are saturated:

Theorem 3.2. *If $\mathrm{qrank}\, \langle K \rangle_\phi$ is symmetric under $q \to 1/q$ then $\dim J^\flat(K) = \mathrm{Tait}(K)$ and $\mathrm{qdim}\, J^\flat(K) = \mathrm{qrank}\, \langle K \rangle_\phi$.*

Proof. Define $r = \mathrm{Tait}(K)$. As we will show in Section 3.2, we can construct two ordered homogeneous bases $(g_1, \cdots, g_r)$ and $(\tilde{g}_1, \cdots, \tilde{g}_r)$ for $\langle K \rangle_\phi$ such that

$$\deg g_i + \deg \tilde{g}_i = 6d_i$$

for a nonnegative integer d_i, and

$$(g_i, \tilde{g}_j)_\phi = \begin{cases} E^{d_i} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

We will say that g_i and $\tilde{g}_i$ *pair* together. The claim is equivalent to the statement that only generators of opposite degrees pair together. Assume for contradiction that this is not the case, and pick the largest value of n such that a generator in degree $-n$ pairs with a generator in degree $n + 6m$ for $m > 0$. By the symmetry hypothesis, the number of generators in degree $-(n+6m)$ is the same as the number of generators in degree $n+6m$, and by our choice of n the generators in these opposite degrees must mutually pair together; contradiction. $\square$

3.1. **Lower bounds.** Our basic strategy for investigating $J^\flat(K)$ and $\langle K \rangle_\phi$ is to approximate these spaces by considering a finite subset $S_f(K)$ of the (infinite) set $S(K)$ of half-foams with top boundary K. In this section we prove several results that show how information about $J^\flat(K)$ and $\langle K \rangle_\phi$ can be obtained in this manner.

Given a finite subset $S_f(K) \subset S(K)$, we define $V_f(K) \subset V(K)$ and $M_f(K) \subset M(K)$ by

$$V_f(K) := \mathbb{F} \cdot S_f(K), \qquad\qquad M_f(K) := \mathbb{F}[E] \cdot S_f(K).$$

We define

$$\ell(K) := \dim(V_f(K)/V_f(K)^\perp), \qquad\qquad r(K) := \text{rank}(M_f(K)/M_f(K)^\perp),$$
$$\ell_q(K) := \text{qdim}(V_f(K)/V_f(K)^\perp), \qquad\qquad r_q(K) := \text{qrank}(M_f(K)/M_f(K)^\perp).$$

(One can show that the $\mathbb{F}[E]$-module $M_f(K)/M_f(K)^\perp$ is free by using an argument similar that used in [8, Proposition 4.4] and the fact that $\mathbb{F}[E]$ is a PID, so Theorem 2.23 shows that it has a homogeneous basis and thus a well-defined quantum rank.) Using the same reasoning that yields Corollary 3.1, we obtain:

Theorem 3.3. *For any web K, we have $\ell(K) \le r(K)$ and $\ell_q(K) \le r_q(K)$.*

We have the following generalization of Observation 2.16, which we can use to compute lower bounds on the dimension and quantum dimension of $J^\flat(K)$:

Theorem 3.4. *For any web K we have*

$$\ell(K) \le \dim J^\flat(K) \le \text{Tait}(K), \qquad\qquad \ell_q(K) \le \text{qdim} J^\flat(K) \le \text{qrank} \langle K \rangle_\phi.$$

Proof. Define $\overline{V}_f(K) := V_f(K)/V(K)^\perp$ to be the image of $V_f(K)$ in the quotient space $J^\flat(K) = V(K)/V(K)^\perp$. We have

$$(5) \qquad\qquad \dim J^\flat(K) \ge \dim \overline{V}_f(K), \qquad\qquad \text{qdim} J^\flat(K) \ge \text{qdim} \overline{V}_f(K).$$

Since $V_f(K) \cap V(K)^\perp \subseteq V_f(K)^\perp$, we have a surjective linear map $\overline{V}_f(K) \to V_f(K)/V_f(K)^\perp$ that preserves degrees, thus

$$(6) \qquad \dim \overline{V}_f(K) \ge \dim(V_f(K)/V_f(K)^\perp) =: \ell(K), \qquad \text{qdim} \overline{V}_f(K) \ge \text{qdim}(V_f(K)/V_f(K)^\perp) =: \ell_q(K).$$

Equations (5) and (6), together with Corollary 3.1, give the desired result. $\square$

The proof of Theorem 3.4 also shows that the inequalities in Theorem 3.4 are saturated when $\ell(K) = \text{Tait}(K)$:

Corollary 3.5. *If $\ell(K) = \text{Tait}(K)$ then $\dim J^\flat(K) = \text{Tait}(K)$ and $\ell_q(K) = \text{qdim} J^\flat(K) = \text{qrank} \langle K \rangle_\phi$.*

Define $\overline{M}_f(K) := M_f(K)/M(K)^\perp$ to be the image of $M_f(K)$ in the quotient space $\langle K \rangle_\phi = M(K)/M(K)^\perp$. Since $\overline{M}_f(K)$ is a submodule of the free $\mathbb{F}[E]$-module $\langle K \rangle_\phi$ and $\mathbb{F}[E]$ is a PID, it follows that $\overline{M}_f(K)$ is a free $\mathbb{F}[E]$-module. We have the following analog of Theorem 3.4:

Theorem 3.6. *For any web K we have*

$$r(K) \le \text{rank} \langle K \rangle_\phi = \text{Tait}(K), \qquad\qquad r_q(K) \le \text{qrank} \overline{M}_f(K).$$

Proof. We have

$$(7) \qquad \qquad \operatorname{rank} \langle K \rangle_\phi \geq \operatorname{rank} \overline{M}_f(K).$$

Since $M_f(K) \cap M(K)^\perp \subseteq M_f(K)^\perp$, we have a surjective homomorphism $\overline{M}_f(K) \to M_f(K)/M_f(K)^\perp$ that preserves degrees, thus

$$(8) \quad \operatorname{rank} \overline{M}_f(K) \geq \operatorname{rank}(M_f(K)/M_f(K)^\perp) =: r(K), \quad \operatorname{qrank} \overline{M}_f(K) \geq \operatorname{qrank}(M_f(K)/M_f(K)^\perp) =: r_q(K).$$

Equations (7) and (8), together with Theorem 2.22, give the desired result. $\square$

The proof of Theorem 3.6 also shows that inequality in Theorem 3.6 involving the quantum rank is saturated when $r(K) = \operatorname{Tait}(K)$:

Corollary 3.7. *If $r(K) = \operatorname{Tait}(K)$ then* $\operatorname{qrank} \overline{M}_f(K) = r_q(K)$.

In contrast to Corollary 3.5, which uniquely determines the quantum dimension of $J^\flat(K)$ when $\ell(K) = \operatorname{Tait}(K)$, Corollary 3.7 does not uniquely determine the quantum rank of $\langle K \rangle_\phi$ when $r(K) = \operatorname{Tait}(K)$; rather, it shows only that $\langle K \rangle_\phi$ has a submodule $\overline{M}_f(K)$ of full rank and known quantum rank. This information does, however, strongly constrain the possibilities for the quantum rank of $\langle K \rangle_\phi$, since the only possible difference between $\overline{M}_f(K)$ and $\langle K \rangle_\phi$ is that homogeneous generators of $\overline{M}_f(K)$ could be shifted upwards in degree by multiples of $\deg E = 6$ relative to corresponding generators of $\langle K \rangle_\phi$.

Example 3.8. Consider the empty web $K = \varnothing$, for which $\operatorname{Tait}(K) = 1$, $\langle K \rangle_\phi = \mathbb{F}[E]$, and $\operatorname{qrank} \langle K \rangle_\phi = 1$. For each integer $d \geq 0$, we have a submodule $\mathbb{F}[E] \cdot E^d$ of $\langle K \rangle_\phi$ of full rank and quantum rank q^{6d}.

3.2. **Computing the lower bounds.** Given a finite set of half-foams $S_f(K)$, we compute $\ell(K)$, $\ell_q(K)$, $r(K)$, and $r_q(K)$ as follows. We enumerate $S_f(K)$ as an ordered list $S_f(K) = (H_1, \cdots, H_n)$ and define an $n \times n$ matrix A whose (i,j)-matrix element is given by

$$(9) \qquad \qquad A_{ij} = (H_i, H_j)_\phi \in \mathbb{F}[E].$$

Note that A_{ij} is either zero or $E^{a_{ij}}$, where a_{ij} is a nonnegative integer determined by the degrees of the half-foams H_i and H_j:

$$(10) \qquad \qquad \deg H_i + \deg H_j = \deg E^{a_{ij}} = 6 a_{ij}.$$

We perform a Smith decomposition of A to express it in the form

$$(11) \qquad \qquad A = SBT,$$

where S and T are invertible $n \times n$ matrices and B is a diagonal $n \times n$ matrix. From equations (9) and (11), it follows that

$$(12) \qquad \qquad (H_i, H_j)_\phi = \sum_{k=1}^{n} S_{ik} B_{kk} T_{kj}.$$

Define bases $\{f_1, \cdots, f_n\}$ and $\{\tilde{f}_1, \cdots, \tilde{f}_n\}$ of $M_f(K)$ by

$$(13) \qquad f_k = \sum_{i=1}^{n} (S^{-1})_{ki} H_i, \qquad \qquad \tilde{f}_k = \sum_{j=1}^{n} (T^{-1})_{jk} H_j.$$

From equations (12) and (13), and the fact that B is diagonal, it follows that

$$B_{ij} = \begin{cases} (f_i, \tilde{f}_i)_\phi & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

By considering the details of the Smith decomposition procedure, which involves applying a series of row and column operations to A to transform it into B, one can use equation (10) and the fact that $\{H_1, \cdots, H_n\}$ is a homogeneous basis for $M_f(K)$ to show that $\{f_1, \cdots, f_n\}$ and $\{\tilde{f}_1, \cdots, \tilde{f}_n\}$ are also homogeneous bases for $M_f(K)$. After reordering the basis elements if necessary, it follows that B has the form

$$B = \operatorname{diag}(E^{d_1}, \cdots, E^{d_m}, 0, \cdots, 0),$$

where $d_1, \cdots, d_m$ are nonnegative integers defined such that

$$\deg f_k + \deg \tilde{f}_k = \deg E^{d_k} = 6 d_k.$$

We thus obtain free homogeneous bases $\{[f_1], \cdots, [f_m]\}$ and $\{[\tilde{f}_1], \cdots, [\tilde{f}_m]\}$ for $M_f(K)/M_f(K)^\perp$. The degree of $[f_k]$ is given by

$$\deg[f_k] = \deg f_k = \deg(S^{-1})_{ki} + \deg H_i$$

for any value of i for which $(S^{-1})_{ki}$ is nonzero, and the degree of $[\tilde{f}_k]$ is given by

$$\deg[\tilde{f}_k] = \deg \tilde{f}_k = \deg(T^{-1})_{jk} + \deg H_j$$

for any value of j for which $(T^{-1})_{jk}$ is nonzero. The quantities $r(K)$ and $r_q(K)$ are then given by

$$r(K) = \operatorname{rank} M_f(K)/M_f(K)^\perp = m, \qquad r_q(K) = \operatorname{qrank} M_f(K)/M_f(K)^\perp = \sum_{i=1}^m q^{\deg[f_i]}.$$

Bases for the vector space $V_f(K)/V_f(K)^\perp$ can be obtained in a similar manner by evaluating the matrices S, B, and T at $E = 0$, and these bases can be used to compute $\ell(K)$ and $\ell_q(K)$.

4. Computer program

We have written a computer program in Mathematica to determine lower bounds for $\dim J^\flat(K)$ and $\operatorname{qdim} J^\flat(K)$. The program consists of two distinct components. The first component takes as input a web K and produces as output a large set $S_f(K)$ of half-foams with top boundary K. The second component computes the rank and quantum rank of the bilinear form $(-, -)$ restricted to the vector space $V_f(K) = \mathbb{F} \cdot S_f(K)$. This computation is accomplished by applying Khovanov and Robert's formula to the closed foams obtained by taking all possible pairings of half-foams in $S_f(K)$. The program is available from the author's website [3].

Remark 4.1. The program represents foams as *movies*; these are lists $(K_1 = \varnothing, E_1, K_2, E_2, \cdots, E_n, K_{n+1})$ in which webs K_i are interleaved with elementary cobordisms $E_i : K_i \to K_{i+1}$ connecting adjacent pairs of webs. The terminal web K_{n+1} is $K_{n+1} = \varnothing$ for a closed foam and $K_{n+1} = K$ for a half-foam with top boundary K. A movie can be thought of as describing successive horizontal slices through the foam, where the web K_i describes the i-th slice and the cobordism E_i describes the portion of the foam that lies between the i-th and $(i+1)$-th slice. The allowed elementary cobordisms E_i are shown in Figures 3 and 5.

4.1. **Construction of generating set of half-foams.** The first component of the program takes as input a web K and produces as output a set $S_f(K)$ of half-foams with top boundary K. To accomplish this, we rely on the fact that for a reducible web K there is an algorithm for producing a basis $S_f(K)$ of $\operatorname{Tait}(K)$ half-foams for $J^\flat(K)$. The algorithm involves a set of elementary cobordisms shown in Figure 3, which correspond to the local replacements for reducible webs shown in Figure 2.

Given a reducible web K and a reduction of K, we construct the basis $S_f(K)$ by recursively applying the following rules:

  (1) For the empty web $K = \varnothing$, take $S_f(K) = \varnothing$.
  (2) For a local replacement $K \to K_1'$ of the form shown in Figure 2a, construct $S_f(K)$ by applying each of the elementary cobordisms $C_1, \dot{C}_1, \ddot{C}_1 : K_1' \to K$ shown in Figure 3a to each half-foam in $S_f(K_1')$.
  (3) For a local replacement $K \to K_2'$ of the form shown in Figure 2b, construct $S_f(K)$ by applying each of the elementary cobordisms $C_2, \dot{C}_2 : K_2' \to K$ shown in Figure 3b to each half-foam in $S_f(K_2')$.
  (4) For a local replacement $K \to K_3'$ of the form shown in Figure 2c, construct $S_f(K)$ by applying the elementary cobordism $C_3 : K_3' \to K$ shown in Figure 3c to each half-foam in $S_f(K_3')$.
  (5) For local replacements $K \to K_{4a}'$ and $K \to K_{4b}'$ of the form shown in Figure 2d, take $S_f(K)$ to be the union of the two sets of half-foams obtained by applying the elementary cobordism $C_{4a} : K_{4a}' \to K$ to each half-foam in $S_f(K_{4a}')$ and the elementary cobordism $C_{4b} : K_{4b}' \to K$ to each half-foam in $S_f(K_{4b}')$, where C_{4a} and C_{4b} are as shown in Figure 3d.

This algorithm allows us to construct the movie representations of half-foams described in Remark 4.1. The resulting basis $S_f(K)$ is not canonical, since it depends on a choice of reduction of K.

For nonreducible webs K, there is no known algorithm for producing a basis of half-foams for $J^\flat(K)$, and our goal instead is just to produce a large set $S_f(K)$ of half-foams with top boundary K. Ideally, we would like $S_f(K)$ to be large and diverse enough to contain a spanning set for $J^\flat(K)$. To construct $S_f(K)$, we use the fact that a nonreducible web K can often be converted into a reducible web K' by making one of the four
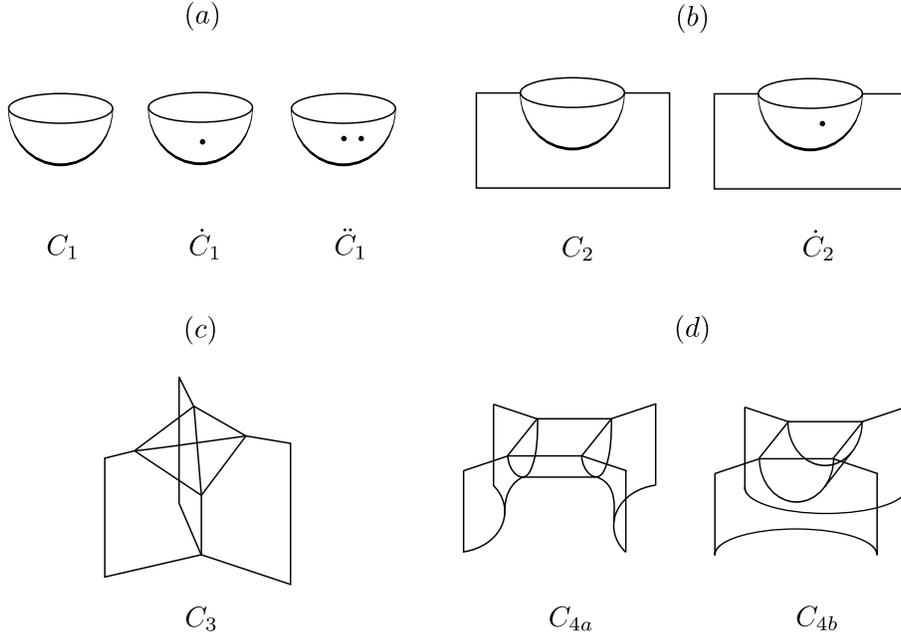
FIGURE 3. Elementary cobordisms for reducible webs. The cobordisms are taken to be the identity outside the depicted region. The cobordisms $\dot{C}_1$, $\ddot{C}_1$, and $\dot{C}_2$ have facets that are decorated with dots.

local replacements $K \to K'$ shown in Figure 4. If such is the case, then we can obtain a set of half-foams with top boundary K by applying the corresponding elementary cobordism $E : K' \to K$ shown in Figure 5 to each of the half-foams in a generating set $S_f(K')$ constructed using the above algorithm for reducible webs. We obtain $S_f(K)$ by taking the union of such sets constructed for all local replacements $K \to K'$ of the form shown in Figure 4 that yield reducible webs K'.
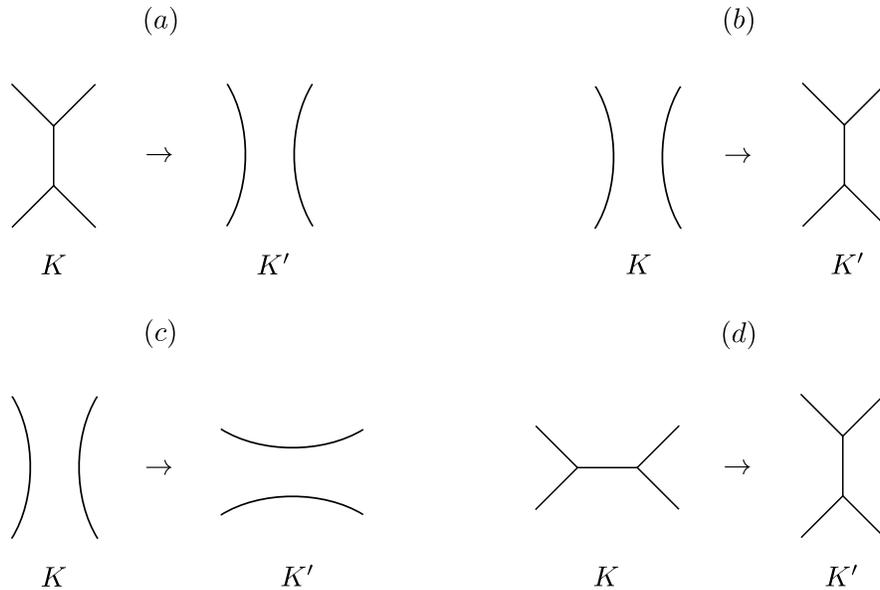

FIGURE 4. Local replacements $K \to K'$ for nonreducible webs. The web K' is identical to K outside the depicted region. (a) Zip. (b) Unzip. (c) Saddle. (d) IH.
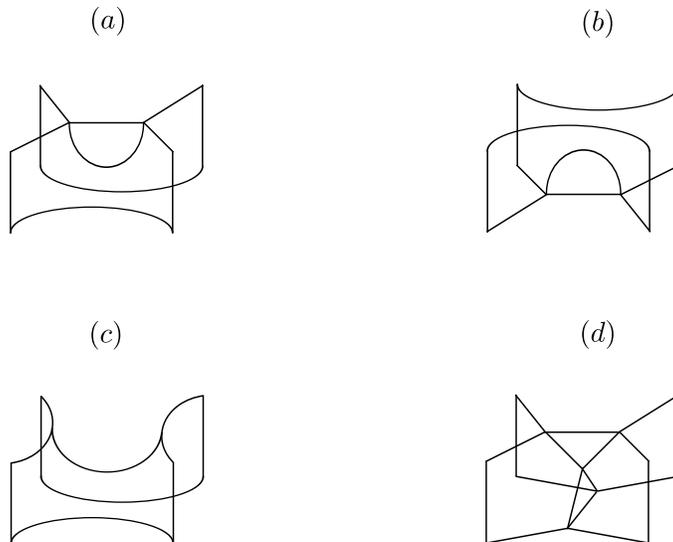
FIGURE 5.   Elementary cobordisms for nonreducible webs. The cobordisms are taken to be the identity outside the depicted region. (a) Zip. (b) Unzip. (c) Saddle. (d) IH.

Remark 4.2. The movie representation, as described in Remark 4.1, of each half-foam in $S_f(K)$ has the form $(K_1 = \varnothing, E_1, \cdots, E_n, K_{n+1} = K)$, where for $i \in \{1, \cdots, n-1\}$ the cobordism E_i is one of the cobordisms shown in Figure 3 and E_n is one of the cobordisms shown in Figure 5.

Remark 4.3. For simplicity, the program does not actually use the above algorithm for reducible webs: instead of systematically searching for a reduction of the web K', we apply the following recursive algorithm. If K' has circles, then randomly select a circle and eliminate it using the local replacement shown in Figure 2a. If K' has no circles, but does have bigons, then randomly select a bigon and eliminate it using the local replacement shown in Figure 2b. If K' has no circles or bigons, but does have triangles, then randomly select a triangle and eliminate it using the local replacement shown in Figure 2c. If K' has no circles, bigons, or triangles, but does have squares, then randomly select a square and eliminate it, using both of the local replacements shown in Figure 2d. If K' has no small faces, then we do not use K' to construct half-foams for our generating set. In practice this algorithm seems quite effective at reducing reducible webs, but we have not proven that it always reduces every reducible web, and consequently the generating sets $S_f(K)$ that we produce for nonreducible webs K might not always be as large as they could be.

In order to compute quantum gradings, we need to compute the degrees of the half-foams in the generating set $S_f(K)$. Degree is additive under composition of foams,

$$\deg(F_1 \circ F_2) = \deg F_1 + \deg F_2,$$

so the degree of any foam built by composing elementary cobordisms can be computed by summing the degrees of each elementary cobordism in the composition. We use equation (4) to compute the degree of each of the elementary cobordisms shown in Figures 3 and 5 and display the results in Table 1.

4.2. **Evaluation of closed foams.** The second component of the computer program takes as input a closed foam F and returns as output $J^\flat(F) \in \mathbb{F}$. This component of the program is essentially a computer implementation of Khovanov and Robert's closed foam evaluation formula, which we described in Section 2.2, using the movie representations of foams described in Remark 4.1. Given a closed foam F, we

   (1) Enumerate the facets of F.
   (2) Compute the adjacency graph for facets(F). The adjacency graph is an unoriented graph with a vertex v_i for each facet $f_i \in$ facets(F) and an edge connecting vertices v_i and v_j if and only if the closures of the corresponding facets f_i and f_j intersect.
   (3) Using the adjacency graph, enumerate the admissible colorings of F.
   (4) Compute $J^\flat(F) \in \mathbb{F}$ using equation (3).

E	C_1	$\dot{C}_1$	$\ddot{C}_1$	C_2	$\dot{C}_2$	C_3	C_{4a}	C_{4b}	Zip	Unzip	Saddle	IH
$\deg E$	-2	0	2	-1	1	0	0	0	1	1	2	1

TABLE 1. Degree $\deg E$ of each elementary cobordism E.

Due to the vanishing result described in Corollary 2.20, which states that $J^\flat(F) = 0$ when $\deg F \neq 0$, we need only apply this algorithm when $\deg F = 0$.

Remark 4.4. The most complicated step of the foam evaluation algorithm is step (1), enumerating the facets of F. To achieve this, we use the fact that the closed foams we consider always have the form $F = H_1 \cup_K \overline{H}_2$ for half-foams H_1 and H_2 with top boundary K. To obtain the facets of F, we enumerate the facets of H_1 and H_2 separately and then combine the two sets of facets using a gluing algorithm. Given a half-foam H with list representation $(K_1 = \varnothing, E_1, \cdots, E_n, K_{n+1} = K)$, we enumerate its facets by using a recursive algorithm to enumerate the facets of the half-foams H_i with list representations $(K_1 = \varnothing, E_1, \cdots, E_{i-1}, K_i)$ for $i \in \{1, \cdots, n+1\}$. It is interesting to note that as we work from $i = 1$ to $i = n$ the number of facets is strictly increasing, and at each step the intersection of the facets with the web K_i is a partition of K_i. The number of facets can remain constant or decrease going from n to $n + 1$ if the last cobordism E_n is a Saddle or an Unzip (for example, two facets that were distinct in the half-foam H_n can describe two pieces of the same facet in the half-foam $H = H_{n+1}$). Also, the half-foams $H_1, H_2, \cdots, H_n$ always have admissible colorings, but for all four possibilities of the last cobordism E_n (Zip, Unzip, Saddle, or IH), it is possible that the half-foam $H = H_{n+1}$ has no admissible colorings.

5. COMPUTER RESULTS

5.1. **Lower bounds.** We use the computer program described in Section 4 to obtain lower bounds $\ell(K)$ on the dimension of $J^\flat(K)$ for the example webs K shown in Figure 6. The results are summarized in Table 2. For each web K, we use the algorithm described in Section 4.1 to construct a list $S_f(K) = (H_1, \cdots, H_N)$ of N half-foams with top boundary K, with a random ordering. For increasing values of n, we compute lower bounds $\ell_n(K)$ on $\dim J^\flat(K)$ by calculating the rank of the bilinear form $(-, -)$ restricted to the vector space spanned by $\{H_1, \cdots, H_n\}$, using the closed-foam evaluation algorithm described in Section 4.2. In order to obtain results in a reasonable amount of time, we compute $\ell_n(K)$ only up to an index $n = N_e$ that is less than the total number of half-foams N that we have generated. We note that $\ell_n(K)$ is a nondecreasing function of n that saturates at a value $\ell(K)$ for some index $n = N_\ell$; that is, $\ell_n(K) = \ell(K)$ for $N_\ell \leq n \leq N_e$, and N_ℓ is the smallest index with this property. The saturation value $\ell(K)$ is the lower bound on $\dim J^\flat(K)$ that is listed in Table 2.

Remark 5.1. A useful class of example webs to consider is the class of *fullerene graphs*; these are planar trivalent graphs with 12 pentagonal faces and an arbitrary number of hexagonal faces. Because they contain no small faces, fullerene graphs are always nonreducible. A computer program for enumerating fullerene graphs is described in [4]. The webs W_1, W_2, and W_5 are the unique fullerene graphs with 20, 24, and 26 vertices, respectively; W_3 and W_4 are the two fullerene graphs with 28 vertices; and W_7 is one of 6 fullerene graphs with 34 vertices.

As an example, consider the dodecahedral web W_1 shown in Figure 6. A graph of $\ell_n(W_1)$ versus n is shown in Figure 7. The Tait number of W_1 is $\mathrm{Tait}(W_1) = 60$, the lower bound on $\dim J^\flat(W_1)$ computed by our program is $\ell(W_1) = 58$, and this bound is attained after examining $N_\ell = 156$ of the $N = 11\,160$ half-foams that we constructed via the algorithm described in Section 4.1. The lower bound remains 58 even after examining $N_e = 6\,727$ of the $N = 11\,160$ half-foams we constructed.

Remark 5.2. Kronheimer and Mrowka also obtain the lower bound $\ell(W_1) = 58$ for the dodecahedral web W_1 [9]. Their lower bound is obtained in a manner similar to ours, but with a generating set of half-foams constructed as follows. Given a 4-coloring c_4 of the faces of a web $K \subset S^2$; that is, a map $c_4 : \{\text{faces of } K\} \to \{1, 2, 3, 4\}$ such that no two adjacent faces share the same color, let T denote the union of the faces of K that are not colored 4 and define an undotted half-foam

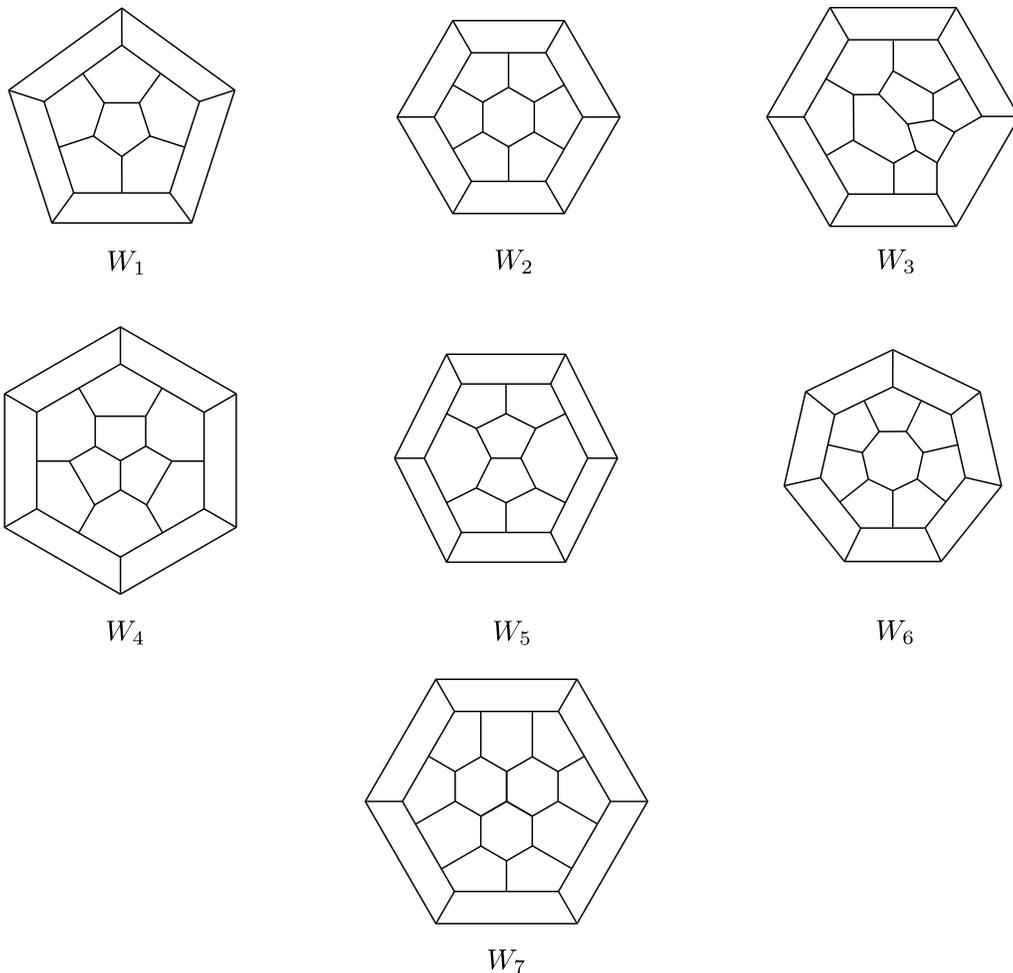$$F(K, c_4) = (T \times \{0\}) \cup (K \times [0, 1]) \subset S^2 \times \mathbb{R}.$$

FIGURE 6. Example webs. The web W_1 is the dodecahedral web.

There are 240 distinct 4-colorings of the faces of the dodecahedral web W_1, corresponding to 20 distinct half-foams $\{F(W_1, c_4)\}$, each of which has degree -3. To each of the half-foams $\{F(W_1, c_4)\}$ one can add 0, 1, 2, or 3 dots to obtain half-foams in degrees -3, -1, 1, or 3. The resulting generating set of dotted half-foams yields the lower bound $\ell(W_1) = 58$. We computed lower bounds $\ell(K)$ for the example webs W_1, W_2, W_3, W_4, W_5, and W_6 shown in Figure 6 using generating sets constructed in a similar manner, but, except for the dodecahedral web W_1, the bounds we obtained by this method are strictly weaker than those shown in Table 2.

It is also of interest to compute lower bounds on $\operatorname{qdim} J^\flat(K)$ and $\operatorname{qrank} \langle K \rangle_\phi$ for each example web K. As described in Section 3.2, for each example web K we use the set of half-foams $S_f(K)$ to compute the quantities $\ell(K)$, $\ell_q(K)$, $r(K)$, and $r_q(K)$. The results are summarized in Table 3. For each example web K, the set of half-foams $S_f(K)$ is sufficiently large that $r(K) = \operatorname{Tait}(K)$. Table 3 gives a proof of Theorem 1.1 from the Introduction, which we restate in a stronger form here:

Theorem 5.3. *For the webs W_2 and W_3 shown in Figure 6 we have* $\dim J^\flat(K) = \operatorname{rank} \langle K \rangle_\phi = \operatorname{Tait}(K)$ *and* $\operatorname{qdim} J^\flat(K) = \operatorname{qrank} \langle K \rangle_\phi = \ell_q(K)$ *for $\ell_q(K)$ as shown in Table 3.*

Proof. Table 3 shows that for the webs W_2 and W_3 we have $\ell(K) = \operatorname{Tait}(K)$. The result now follows from Theorem 2.22 and Corollary 3.5 □

For the remaining webs W_1, W_4, W_5, W_6, and W_7, for which $\ell(K) < \operatorname{Tait}(K)$, we are unable to determine the exact values of $\operatorname{qdim} J^\flat(K)$ and $\operatorname{qrank} \langle K \rangle_\phi$; nevertheless, since $r(K) = \operatorname{Tait}(K)$ we can apply Corollary

Web K	$\ell(K)$	$\mathrm{Tait}(K)$	N_ℓ	N_e	N
W_1	58	60	156	6727	11 160
W_2	120	120	747	5322	27 792
W_3	162	162	822	4902	45 960
W_4	178	180	1193	6351	47 196
W_5	188	192	2447	7153	40 704
W_6	248	252	1726	6331	53 172
W_7	308	312	1190	5458	101 970

TABLE 2.   Lower bounds $\ell(K)$ on $\dim J^\flat(K)$ and Tait number $\mathrm{Tait}(K)$ for example webs K. The numbers N_ℓ, N_e, and N are explained in the main text.

3.7 to strongly constraint these quantities. Recall that Corollary 3.7 states that there is a free $\mathbb{F}[E]$-submodule $\overline{M}_f(K)$ of $\langle K \rangle_\phi$ such that

$$\mathrm{rank}\,\overline{M}_f(K) = \mathrm{rank}\,\langle K \rangle_\phi = \mathrm{Tait}(K), \qquad\qquad \mathrm{qrank}\,\overline{M}_f(K) = r_q(K).$$

As described in Section 3.1, the only possible difference between $\overline{M}_f(K)$ and $\langle K \rangle_\phi$ is that homogeneous generators of $\overline{M}_f(K)$ could be shifted upwards in degree by multiples of $\deg E = 6$ relative to corresponding generators of $\langle K \rangle_\phi$. As an example, consider the dodecahedral web W_1, for which

$$\ell_q(W_1) = 9q^{-3} + 20q^{-1} + 20q + 9q^3, \qquad\qquad r_q(W_1) = 9q^{-3} + 20q^{-1} + 20q + 11q^3.$$

There are only two possible cases. One case is that $\overline{M}_f(W_1) = \langle W_1 \rangle_\phi$. In this case $\dim J^\flat(W_1) = 58$ and

$$\mathrm{qdim}\,J^\flat(W_1) = 9q^{-3} + 20q^{-1} + 20q + 9q^3, \qquad \mathrm{qrank}\,\langle W_1 \rangle_\phi = 9q^{-3} + 20q^{-1} + 20q + 11q^3.$$

The second case is that $\overline{M}_f(W_1)$ is a proper submodule of $\langle W_1 \rangle_\phi$ of full rank. Since $\ell_q(W_1) \leq \mathrm{qdim}\,J^\flat(W_1) \leq \mathrm{qrank}\,\langle W_1 \rangle_\phi$, one of the degree 3 generators of $\overline{M}_f(W_1)$ must be shifted upwards in degree relative to a corresponding generator of $\langle W_1 \rangle_\phi$ in degree -3. In this case $\dim J^\flat(W_1) = 60$ and

$$\mathrm{qdim}\,J^\flat(W_1) = \mathrm{qrank}\,\langle W_1 \rangle_\phi = 10q^{-3} + 20q^{-1} + 20q + 10q^3.$$

Remark 5.4. The data in Tables 2 and 3 was obtained using a desktop computer and took about two weeks of physical running time. The running time for a given web is quadratic in N_e, the number of half-foams examined, since we need to compute the rank of a $N_e \times N_e$ pairing matrix.

5.2. **Empirical evidence that $J^\flat(W_1) = 58$.** Our results suggest that $J^\flat(W_1) = 58$, though we are not able to prove this using our current approach. Here we list several pieces of evidence that point towards this conclusion:
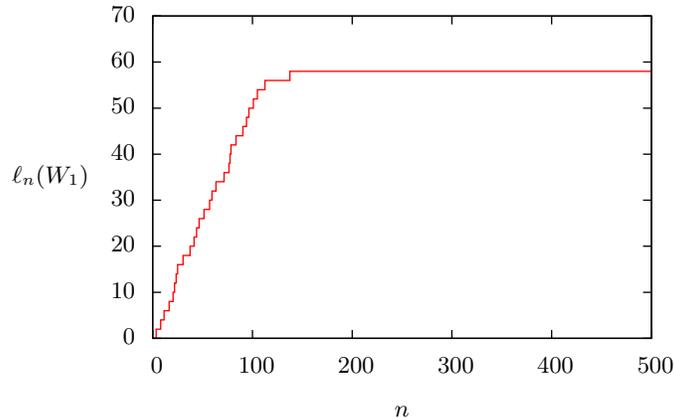

FIGURE 7.   Graph of $\ell_n(W_1)$ versus n for the dodecahedral web W_1.

K	$\ell(K)$	Tait(K)	$\ell_q(K) + (r_q(K) - \ell_q(K))$
W_1	58	60	$9q^{-3} + 20q^{-1} + 20q + 9q^3 + (2q^3)$
W_2	120	120	$3q^{-5} + 2q^{-4} + 16q^{-3} + 6q^{-2} + 29q^{-1} + 8 + 29q + 6q^2 + 16q^3 + 2q^4 + 3q^5$
W_3	162	162	$2q^{-5} + 7q^{-4} + 13q^{-3} + 21q^{-2} + 24q^{-1} + 28 + 24q + 21q^2 + 13q^3 + 7q^4 + 2q^5$
W_4	178	180	$q^{-6} + 11q^{-4} + 10q^{-3} + 29q^{-2} + 19q^{-1} + 38 + 19q + 29q^2 + 10q^3 + 11q^4 + q^6 + (q + q^5)$
W_5	188	192	$4q^{-5} + 31q^{-3} + 59q^{-1} + 59q + 31q^3 + 4q^5 + (q + 2q^3 + q^5)$
W_6	248	252	$20q^{-4} + 62q^{-2} + 84 + 62q^2 + 20q^4 + (2q^2 + 2q^4)$
W_7	308	312	$4q^{-5} + 5q^{-4} + 41q^{-3} + 15q^{-2} + 79q^{-1} + 20 + 79q + 15q^2 + 41q^3 + 5q^4 + 4q^5 + (q + 2q^3 + q^5)$

TABLE 3.   For each web K, we list $\ell(K)$, Tait$(K) = r(K)$, $\ell_q(K)$, and (indicated in parentheses) the difference $r_q(K) - \ell_q(K)$ when this quantity is nonzero.

(1) As shown in Table 2 and Figure 7, for the dodecahedral web W_1 we find 58 generators after examining $N_\ell = 156$ half-foams, but we do not find any additional generators after examining $N_e = 6\,727$ half-foams.

(2) We still obtain the lower bound $\ell(W_1) = 58$ if we use a smaller set $S_f(W_1)$ consisting of half-foams constructed using only one of the four types of cobordisms shown in Figure 5; that is, if we use only Zip cobordisms we obtain $\ell(W_1) = 58$, if we use only Unzip cobordisms we obtain $\ell(W_1) = 58$, if we use only Saddle cobordisms we obtain $\ell(W_1) = 58$, and if use only IH cobordisms we obtain $\ell(W_1) = 58$.

(3) As described in Remark 5.2, Kronheimer and Mrowka also obtain the lower bound $\ell(K) = 58$ using a set of half-foams whose structure is very different from those considered here. This fact reinforces (2), which similarly shows that several very different sets of half-foams contain exactly 58 generators.

(4) For reducible webs we have an algorithm for constructing a basis of half-foams for $J^\flat(K)$, but we can also apply our general method of constructing half-foams to reducible webs. Since we know that $\dim J^\flat(K) = \text{Tait}(K)$ for reducible webs, we can thereby test the effectiveness of our general method at finding generators. We have tested a number of reducible webs of similar complexity to the example nonreducible webs shown in Figure 6, and for all the webs we have tested our method finds all Tait(K) generators.

(5) For any web K we know that rank $\langle K \rangle_\phi = \text{Tait}(K)$, and for all of the example nonreducible webs shown in Figure 6 our method produced enough half-foams to span a submodule $\overline{M}_f(K)$ of $\langle K \rangle_\phi$ of full rank Tait(K). Together with (4), this fact seems to indicate that our method is fairly effective at finding generators, at least for webs at the level of complexity of those shown in Figure 6.

From quantum grading considerations, we have $\dim J^\flat(K)$ is either 58 or 60; it cannot be 59. If $\dim J^\flat(K)$ is in fact 60, our results show that the two missing generators must lie in degrees 3 and -3. Perhaps this information could help to either prove that $\dim J^\flat(K) = 58$, or to find the missing generators.

6. QUESTIONS

We conclude with three open questions. We note that qrank $\langle K \rangle_\phi$ is symmetric under $q \to 1/q$ for all reducible webs K, and also for webs W_2 and W_3 in Table 3 for which $\ell(K) = \text{Tait}(K)$. We can ask if this property holds for all webs:

Question 6.1. Is it the case that qrank $\langle K \rangle_\phi$ is symmetric under $q \to 1/q$ for all webs K?

If Question 6.1 were to be answered in the affirmative, then Theorem 3.2 would imply that $\dim J^\flat(K) = \text{Tait}(K)$ and $\text{qdim}\, J^\flat(K) = \text{qrank}\, \langle K \rangle_\phi$ for all webs K, thus answering Question 2.12 in the affirmative.

For all webs K, except the empty web and circle web, the integer Tait(K) is divisible by $3! = 6$, since we can permute the colors of any Tait coloring of K to obtain another Tait coloring. Since rank $\langle K \rangle_\phi = \text{Tait}(K)$, we can ask whether the quantum analog of this divisibility property holds:

Question 6.2. When is qrank $\langle K \rangle_\phi$ divisible by $[3]! = (q^2 + 1 + q^{-2})(q + q^{-1})$?

Remark 6.3. In general, the quantum analog of a positive integer n is $[n] = q^{n-1} + q^{n-3} + \cdots + q^{-(n-1)}$ and the quantum analog of $n!$ is $[n]! = [n][n-1] \cdots [1]$.

We note that qrank $\langle K \rangle_\phi$ is divisible by [3]! for all reducible webs K, except the empty web and the circle web, and for the webs W_2 and W_3 in Table 3 for which $\ell(K) = \text{Tait}(K)$. For the remaining webs in Table 3, for which $\ell(K) < \text{Tait}(K)$, the computation of $r_q(K)$ shows that if $\dim J^\flat(K) = \text{Tait}(K)$ then qdim $J^\flat(K) = $ qrank $\langle K \rangle_\phi$ is divisible by [3]!. For the dodecahedral web W_1, if qrank $\langle W_1 \rangle_\phi$ is divisible by [3]! then this would force $\dim J^\flat(W_1) = 60$.

From the expressions for $\ell_q(K)$ in Table 3, and the relationship between homogeneous generators of $\overline{M}_f(K)$ and those of $\langle K \rangle_\phi$, it follows that qrank $\langle K \rangle_\phi$ contains only odd powers of q for webs W_1 and W_5; only even powers of q for web W_6; and both even and odd powers of q for webs W_2, W_3, W_4, and W_7. We can ask:

Question 6.4. Under what conditions does qrank $\langle K \rangle_\phi$ contain only even, or only odd, powers of q?

Acknowledgments

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