Research statement

David Boozer

1 Introduction

My research concentrates on invariants of knots, links, and graphs in 3-manifolds. Knots and links play a key role in the study of 3-manifolds. Indeed, one can show that any 3-manifold can be obtained by performing a simple operation known as Dehn surgery on a suitable link in S^3. One can consider knots in any 3-manifold, but so far most work on knot invariants has focused on knots in S^3. For example, two important knot invariants are Khovanov homology [14] and singular instanton homology [16, 17]. Khovanov homology is easy to calculate (it is the homology of a chain complex constructed from a generic planar projection of a knot), but is defined only for knots in S^3. Singular instanton homology is difficult to calculate (it is defined in terms of gauge-theoretic nonlinear PDE's) and, though it is defined for knots in arbitrary 3-manifolds, it has been calculated only for some knots in S^3. Little is known about invariants of knots in arbitrary 3-manifolds, but, partly because of the close connection between knots and 3-manifold topology, such results would be of great interest.

For my first project [5], I considered a possible strategy for generalizing Khovanov homology to knots in lens spaces by exploiting Seidel and Smith's reinterpretation of Khovanov homology as Lagrangian Floer homology in symplectic manifolds known as Seidel-Smith spaces [23]. In particular, I proposed candidate manifolds that generalize the Seidel-Smith spaces to the case of knots in lens spaces. These candidate manifolds are related to moduli spaces of Hecke modifications of holomorphic vector bundles over an elliptic curve, and a major component of the project involved a detailed study of these moduli spaces. Some of my work on these spaces has also been applied to the geometric Langlands program [8].

For my second project [6], I generalized ideas of Hedden, Herald, and Kirk [11] to describe a scheme for computing generating sets for the singular instanton homology of knots in lens spaces. The generating sets are given by the intersection points of two Lagrangians in a symplectic manifold whose structure I describe explicitly (conjecturally, the Lagrangian Floer homology of the Lagrangians gives the singular instanton homology itself). Using this scheme, I computed generating sets for several example knots, some of which reproduce known results for knots in S^3 and some of which provide original results for knots in lens spaces.

To define invariants of knots, it is often useful to generalize from knots to graphs, also called *webs*, which can be thought of as knots with singular behavior allowed at the vertices. For example, webs arise naturally in knot invariants defined via representation theory, since they can be viewed as pictorial descriptions of rules for combining representations. To categorify such invariants, it is useful to define a category in which the objects are webs and the morphisms are web cobordisms called *foams* (in fact, Khovanov homology can be viewed as just such a categorification [3]). Webs also play an important role in a new strategy for proving the four-color theorem due to Kronheimer and Mrowka [19]. Kronheimer and Mrowka use a version of singular instanton homology to define a functor $J^\sharp$ from a suitably defined category of foams to the category of vector spaces over $\mathbb{F}$, the field of two elements. They show that if the dimension of the vector space $J^\sharp(K)$ associated to a web K is equal to a certain number $\mathrm{Tait}(K)$ associated to the web K for all webs K, this would imply the four-color theorem. They also show that $\dim J^\sharp(K)$ and $\mathrm{Tait}(K)$ are in fact equal for a special class of "reducible" webs.

For my third project [7], I investigated a possible combinatorial replacement $J^\flat$ for $J^\sharp$ that was defined by Kronheimer and Mrowka. I wrote a computer program to calculate lower bounds on $\dim J^\flat(K)$ and calculated such bounds for a number of example nonreducible webs. In some cases the bounds are sufficiently strong to uniquely determine $\dim J^\flat(K)$, and these are the first exact calculations of $\dim J^\flat(K)$ for nonreducible webs.

2 Khovanov homology for knots in lens spaces

In [5] I consider the problem of generalizing Khovanov homology to knots in lens spaces. Since Khovanov's original combinatorial formulation of Khovanov homology [14] does not suggest an obvious generalization, my strategy was to first reinterpret Khovanov homology in geometric terms using ideas due to Seidel and Smith. Seidel and Smith describe a knot in S^3 as the closure of a braid with $2m$ strands, which can be viewed as the result of gluing together two solid balls, each containing m unknotted arcs, along their common S^2-boundary with $2m$ marked points. To the 2-sphere they associate a symplectic manifold $\mathcal{Y}(S^2, 2m)$, known as the Seidel-Smith space, and to the pair of solid balls they associate a pair of Lagrangians inside $\mathcal{Y}(S^2, 2m)$. Seidel and Smith prove that the Lagrangian Floer homology of the pair of Lagrangians is a knot invariant that they conjectured would coincide with the Khovanov homology of the knot [23]; this was later shown to be the case by Abouzaid and Smith [1].

One might hope that a similar picture holds for knots in lens spaces. Given a suitable Heegaard splitting of the lens space into two solid tori, perhaps one can associate to their common T^2-boundary a symplectic manifold $\mathcal{Y}(T^2, 2m)$ that generalizes the Seidel-Smith space $\mathcal{Y}(S^2, 2m)$, and to the pair of solid tori a pair of Lagrangians inside $\mathcal{Y}(T^2, 2m)$. If the Lagrangian Floer homology of the pair of Lagrangians could be shown to be invariant under all possible Heegaard splittings, it would serve as a natural generalization of Khovanov homology.

The first step in implementing this strategy is to find a suitable symplectic manifold $\mathcal{Y}(T^2, 2m)$, and in [5] I proposed a natural candidate for this space. The space $\mathcal{Y}(S^2, 2m)$ was originally defined in terms of a nilpotent slice in the Lie algebra $\mathfrak{sl}_{2m}$. This description of $\mathcal{Y}(S^2, 2m)$ does not seem to have a natural generalization, so I first applied a result of Kamnitzer [13] to reinterpret $\mathcal{Y}(S^2, 2m)$ as a moduli space $\mathcal{H}(S^2, 2m)$ of Hecke modifications of rank 2 holomorphic vector bundles over a rational curve; roughly, these spaces parameterize different ways of locally modifying a given vector bundle so as to obtain a new vector bundle. Next, I proposed a notion of a Hecke modification of a parabolic bundle and used this notion to reinterpret the moduli space $\mathcal{H}(S^2, n)$ of Hecke modifications of vector bundles as a moduli space $\mathcal{H}_p(S^2, n)$ of Hecke modifications of parabolic bundles. I showed that the moduli space $\mathcal{H}_p(S^2, n)$ has a natural generalization $\mathcal{H}_p(T^2, n)$ to the case of elliptic curves, and I proposed the space $\mathcal{H}_p(T^2, 2m)$ as a candidate for $\mathcal{Y}(T^2, 2m)$. In summary, the sequence of reinterpretation and generalization is

$$\mathcal{Y}(S^2, 2m) \cong \mathcal{H}(S^2, 2m) \cong \mathcal{H}_p(S^2, 2m) \qquad \rightsquigarrow \qquad \mathcal{H}_p(T^2, 2m) =: \mathcal{Y}(T^2, 2m). \tag{1}$$

As possible evidence that $\mathcal{H}_p(T^2, 2m)$ is the correct generalization, I proved that $\mathcal{H}_p(S^2, 2m)$ and $\mathcal{H}_p(T^2, 2m)$ satisfy parallel embedding theorems:

Theorem 2.1. *The space $\mathcal{H}_p(S^2, n)$ canonically embeds into the moduli space $M^{ss}(S^2, n+3)$ of semistable rank 2 parabolic bundles over $\mathbb{CP}^1$ with $n+3$ marked points, where the underlying vector bundles of the parabolic bundles are required to have trivial determinant bundle.*

Theorem 2.2. *The space $\mathcal{H}_p(T^2, n)$ canonically embeds into the moduli space $M^{ss}(T^2, n+1)$ of semistable rank 2 parabolic bundles over an elliptic curve X with $n+1$ marked points, where the underlying vector bundles of the parabolic bundles are required to have trivial determinant bundle.*

Theorem 2.1 appears to be closely related to a theorem due to Woodward [24] that was proven using the nilpotent slice interpretation of $\mathcal{Y}(S^2, 2m)$:

Theorem 2.3 (Woodward)**.** *The space $\mathcal{Y}(S^2, 2m)$ embeds into the moduli space $M^{ss}(S^2, 2m+3)$ of semistable rank 2 parabolic bundles over $\mathbb{CP}^1$ with $2m+3$ marked points, where the underlying vector bundles of the parabolic bundles are required to have trivial determinant bundle.*

In order to define and characterize the moduli spaces $\mathcal{H}_p(T^2, n)$, I undertook a detailed study of Hecke modifications of rank 2 holomorphic vector bundles over an elliptic curve, and described explicit morphisms that represent all possible Hecke modifications of all possible rank 2 vector bundles. (This project was made possible by the fact that holomorphic vector bundles on elliptic curves were classified by Atiyah [2]. No such classification is available for higher genus, and it is this fact that singles out genus 1 as special and explains why I focused on lens spaces.) Using these results, I explicitly computed the moduli space $\mathcal{H}_p(T^2, n)$ for $n = 0, 1, 2$:

Theorem 2.4. *The moduli space $\mathcal{H}_p(T^2, n)$ for $n = 0, 1, 2$ is given by*

$$\mathcal{H}_p(T^2, 0) = \mathbb{CP}^1, \qquad \mathcal{H}_p(T^2, 1) = (\mathbb{CP}^1)^2, \qquad \mathcal{H}_p(T^2, 2) = (\mathbb{CP}^1)^3 - f(X), \tag{2}$$

where X is the elliptic curve and $f : X \to (\mathbb{CP}^1)^3$ is a holomorphic embedding.

In future work, I would like to continue to pursue this strategy of generalizing Khovanov homology to lens spaces. Specifically, I need to (1) propose a candidate Lagrangian in $\mathcal{Y}(T^2, 2m)$ that would correspond to a solid torus containing m unknotted arcs; (2) determine whether the mapping class group acts on $\mathcal{Y}(T^2, 2m)$ and if so, how; and (3) prove that the Lagrangian Floer homology is invariant under different Heegaard splittings of the lens space. A natural place to start might be with the space $\mathcal{H}_p(T^2, 2)$, since it can be described explicitly.

3 Singular instanton homology for knots in lens spaces

In recent work, Hedden, Herald, and Kirk described a scheme for producing generating sets for the singular instanton homology of a variety of knots in S^3 [11, 12], and in [6] I generalized their scheme to produce generating sets for knots in lens spaces. Roughly speaking, the singular instanton homology of a knot K in a 3-manifold Y is a kind of Morse homology in which the Chern-Simons functional plays the role of a Morse function on the space of connections on a trivial principal $SU(2)$ bundle. The generators of the homology chain complex correspond to gauge orbits of flat connections with prescribed singularities along K, which in turn correspond conjugacy classes of homomorphisms $\pi_1(Y - K) \to SU(2)$ that take loops around K to traceless matrices. The set of such conjugacy classes defines a character variety $R(Y, K)$.

The character variety $R(Y, K)$ is not quite the generating set that we seek, since there are two technical complications that must be addressed. First, in order to get a chain complex, with $\partial^2 = 0$, we need to ensure there are no "reducible" connections, which correspond to abelian $SU(2)$ representations. Second, the Chern-Simons functional is generally not Morse, and must be perturbed so as to render it Morse. The modifications needed to address these issues amount to considering homomorphisms $\pi_1(Y - K') \to SU(2)$ that satisfy certain conditions, where K' is a two-component graph containing K as a subset whose precise form I describe. These homomorphisms define a modified character variety $R^\natural_\pi(Y,K)$ that constitutes the actual generating set for singular instanton homology.

To compute $R^\natural_\pi(Y,K)$, I Heegaard-split a lens space Y containing a knot K into a pair of solid tori U_1 and U_2. The solid torus U_1 contains a portion of the knot consisting of an unknotted arc A_1, together with all of the modifications necessary eliminate reducible connections and to render the Chern-Simons functional Morse. The solid torus U_2 contains a (possibly knotted) arc A_2 that describes the remainder of the knot. In analogy with $R(Y,K)$ and $R^\natural_\pi(Y,K)$, I define character varieties $R(U_2,A_2)$ and $R^\natural_\pi(U_1,A_1)$. I also define a character variety $R(T^2,2)$ for the twice-punctured torus $T^2-\{p_1,p_2\}$ consisting of conjugacy classes of homomorphisms $\pi_1(T^2-\{p_1,p_2\}) \to SU(2)$ that take loops around p_1 and p_2 to traceless matrices. Pulling back homomorphisms along inclusion maps, we obtain the following commutative diagram:

$$\begin{tikzcd} & R^\natural_\pi(U_1,A_1) \arrow[dr] & \\ R^\natural_\pi(Y,K) \arrow[ur] \arrow[dr] & & R(T^2,2) \\ & R(U_2,A_2) \arrow[ur] & \end{tikzcd} \tag{3}$$

The character variety $R(T^2,2)$ carries a symplectic structure, and the images L_1 and L_2 of $R^\natural_\pi(U_1,A_1)$ and $R(U_2,A_2)$ in $R(T^2,2)$ are Lagrangians. I prove theorems that explicitly describe the character variety $R(T^2,2)$, the Lagrangians L_1 and L_2, and their intersection:

Theorem 3.1. *The character variety $R(T^2,2)$ is the union of two pieces P_4 and P_3, where P_4 is homeomorphic to $S^2 \times S^2 - \Delta$ and Δ is the diagonal, and P_3 deformation retracts onto S^2.*

Theorem 3.2. *The space $R^\natural_\pi(U_1,A_1)$ is homeomorphic to S^2, and the map $R^\natural_\pi(U_1,A_1) \to R(T^2,2)$ is injective away from the north and south pole of S^2, which are mapped to the same point. The space $R(U_2,A_2)$ is homeomorphic to the closed disk D^2, and the map $R(U_2,A_2) \to R(T^2,2)$ is injective.*

Theorem 3.3. *A point $[\rho] \in L_1 \cap L_2 \subset R(T^2,2)$ that is not the double-point of L_1 is the image of a unique point $[\rho'] \in R^\natural_\pi(Y,K)$ under the pullback map $R^\natural_\pi(Y,K) \to R(T^2,2)$. The point $[\rho']$ is nondegenerate if and only if the intersection of L_1 with L_2 at $[\rho]$ is transverse.*

For simplicity I have stated Theorems 3.2 and 3.3 only for the important special case that A_2 is an unknotted arc, which means that K is a $(1,1)$-knot, but I also proved a version of Theorem 3.3 that does not require this assumption. From Theorem 3.3 it follows that if L_2 intersects L_1 transversely and does not contain the double-point of L_1, then every point in $R^\natural_\pi(Y,K)$ is nondegenerate and the pullback map $R^\natural_\pi(Y,K) \to R(T^2,2)$ is injective with image $L_1 \cap L_2$. Thus $R^\natural_\pi(Y,K)$ is a generating set for singular instanton homology consisting of $|L_1 \cap L_2|$ generators. In summary, the essential idea of the scheme is to confine all of the perturbation data needed to make the instanton homology well-defined to a Lagrangian L_1 that is independent of the knot K and that can be described explicitly. The problem of computing $R^\natural_\pi(Y,K)$ then reduces to determining the unperturbed Lagrangian L_2 corresponding to the knot K and understanding its intersection with L_1.

Using the scheme, together with an explicit description that I give of the action of the mapping class group on $R(T^2, 2)$, I explicitly calculate generating sets for the singular instanton homology of several example $(1,1)$-knots. In particular, one can define the notion of a "simple" knot in a lens space [10], and I prove the following result:

Theorem 3.4. *If K is the unique simple knot representing the homology class $1 \in \mathbb{Z}_p = H_1(L(p,1);\mathbb{Z})$ of the lens space $L(p,1)$, then the rank of the singular instanton homology of K is at most p.*

For a simple knot K in the lens space $Y = L(p,q)$, the knot Floer homology $\widehat{HFK}(Y,K)$ has rank p [10]. Thus, Theorem 3.4 is consistent with Kronheimer and Mrowka's conjecture that for a knot K in a 3-manifold Y, the ranks of singular instanton homology and knot Floer homology $\widehat{HFK}(Y,K)$ are the same [18].

In future work, it would be interesting to test Kronheimer and Mrowka's conjecture for additional $(1,1)$-knots in S^3 (there is a combinatorial method for computing the knot Floer homology for such knots [9]). It would also be of interest to see if grading information for instanton homology can be recovered from the Lagrangian intersections; if so, perhaps one could use this information to actually compute the instanton homology of some example knots in lens spaces.

4 Computer bounds for Kronheimer-Mrowka foam evaluation

Kronheimer and Mrowka have recently described a new strategy for proving the four-color theorem without the aid of computers [19]. Their strategy involves a category Foams whose objects are *webs*, which are unoriented planar trivalent graphs, and whose morphisms are *foams*, which are singular cobordisms between pairs of webs. Using a version of singular instanton homology, they define a functor $J^\sharp$ from the category Foams to the category $\mathrm{Vect}_{\mathbb{F}}$ of vector spaces over $\mathbb{F}$, the field of two elements. They also consider a possible combinatorial replacement $J^\flat$ for $J^\sharp$. They first define $J^\flat(F) \in \mathbb{F}$ for *closed foams* F, which are cobordisms from the empty web to itself, via a set of combinatorial rules that they conjectured were well-defined. This was later shown to be the case by Khovanov and Robert [15], who gave an explicit formula for $J^\flat(F)$. Kronheimer and Mrowka then extend $J^\flat$ to a functor by using the *universal construction* [4]; in particular, for a web K they define the corresponding vector space $J^\flat(K)$ as follows. First, define a vector space $V(K)$ spanned by all *half foams* with top boundary K; these are cobordisms from the empty web to K. Define a bilinear form $(-,-) : V(K) \otimes V(K) \to \mathbb{F}$ such that $(F_1, F_2) = J^\flat(F_1 \cup_K \bar{F}_2)$, where $F_1 \cup_K \bar{F}_2$ is the closed foam obtained by reflecting F_2 top-to-bottom to get $\bar{F}_2$ and then gluing it to F_1 along K. Now define $J^\flat(K)$ to be the quotient of $V(K)$ by the orthogonal complement of $V(K)$ relative to $(-,-)$.

In light of Kronheimer and Mrowka's new strategy for proving the four-color theorem, it is important to understand the relationships between $\dim J^\flat(K)$, $\dim J^\sharp(K)$, and $\mathrm{Tait}(K)$, the number of 3-colorings of the web K, for arbitrary webs K. It is known that for any web K these three numbers are related by $\dim J^\flat(K) \leq \mathrm{Tait}(K) \leq \dim J^\sharp(K)$, and for a special class of "reducible" webs K these three numbers coincide ($\dim J^\flat(K) = \mathrm{Tait}(K) = \dim J^\sharp(K)$) [7, 15, 20]. It is thus of interest to compute examples of $\dim J^\flat(K)$ and $\dim J^\sharp(K)$ for nonreducible webs K. The only results that have previously been obtained are for the dodecahedral web W_1, for which it is known that $J^\flat(W_1) \geq 58$, $\mathrm{Tait}(W_1) = 60$, and $J^\sharp(W_1) \leq 68$ [19, 21]. Since $J^\flat(K)$ is defined in terms of an infinite number of generators mod an infinite number of relations, it is not clear whether $\dim J^\flat(K)$ can be algorithmically computed. Nevertheless, it is possible to algorithmically compute lower bounds for $\dim J^\flat(K)$ by computing the rank of the restriction of the bilinear form $(-,-)$ to any finite subspace of $V(K)$.

W_2 W_3

Figure 1: Nonreducible webs W_2 and W_3.

In [7], I describe a computer program I wrote to determine such lower bounds by enumerating a large number of half-foams and computing the rank of the restriction of the bilinear form $(-,-)$ to the vector space that they span. I compute lower bounds on $\dim J^\flat(K)$ for a number of example nonreducible webs. Since $\dim J^\flat(K) \leq \mathrm{Tait}(K)$, if the lower bound on $\dim J^\flat(K)$ is equal to $\mathrm{Tait}(K)$ then $\dim J^\flat(K) = \mathrm{Tait}(K)$. In this way I was able to perform the first calculations for $\dim J^\flat(K)$ for K nonreducible:

Theorem 4.1. *For the nonreducible webs W_2 and W_3 shown in Figure 1, we have that* $\dim J^\flat(W_2) = \mathrm{Tait}(W_2) = 120$ *and* $\dim J^\flat(W_3) = \mathrm{Tait}(W_3) = 162$.

The vector spaces $J^\flat(K)$ carry a quantum grading, which is induced from a grading on foams, and the computer program can also find lower bounds on the quantum dimension $\mathrm{qdim}\, J^\flat(K)$, which is some cases yield exact results. In particular, I prove

Theorem 4.2. *The quantum dimensions of the nonreducible webs W_2 and W_3 are*

$$\mathrm{qdim}\, J^\flat(W_2) = 3q^{-5} + 2q^{-4} + 16q^{-3} + 6q^{-2} + 29q^{-1} + 8 + 29q + 6q^2 + 16q^3 + 2q^4 + 3q^5, \tag{4}$$

$$\mathrm{qdim}\, J^\flat(W_3) = 2q^{-5} + 7q^{-4} + 13q^{-3} + 21q^{-2} + 24q^{-1} + 28 + 24q + 21q^2 + 13q^3 + 7q^4 + 2q^5. \tag{5}$$

In future work, the computer program could be adapted to other projects involving categorification in terms of foams. For example, I could write a program to compute a knot homology theory recently proposed by Robert and Wagner [22] that categorifies the Alexander polynomial, and which they conjecture coincides with knot Floer homology. Also, it would be of interest to attempt to answer:

Question 4.1. *Can one compute the dimension of $J^\sharp(K)$ for some nonreducible webs K?*

References

[1] M. Abouzaid and I. Smith, *Khovanov homology from Floer cohomology*, preprint, **arXiv:1504.01230**.

[2] M. F. Atiyah, *Vector bundles over an elliptic curve*, Proc. London Math. Soc., **7**(1957), 414–452.

[3] C. Blanchet, *An oriented model for Khovanov homology*, J. Knot Theor. Ramif., **19**(2010), no. 2, 291–312.

[4] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel, *Topological quantum field theories derived from the Kauffman bracket*, Topology, **34**(1995), 883–927.

[5] D. Boozer, *Moduli spaces of Hecke modifications for rational and elliptic curves*, preprint, **arXiv:1805.11184**.

[6] D. Boozer. *Holonomy perturbations of the Chern-Simons functional for lens spaces*, preprint, **arXiv:1811.01536**.

[7] D. Boozer, *Computer Bounds for Kronheimer-Mrowka Foam Evaluation*, preprint, **arXiv:1908.07133**.

[8] E. Frenkel, *Is there an analytic theory of automorphic functions for complex algebraic curves?* preprint, **arXiv:1812.08160**.

[9] H. Goda, H. Matsuda, T. Morifuji, *Knot Floer homology of* $(1,1)$*-knots*, Geom. Dedic., **112**(2005), 197-214.

[10] M. Hedden, *On Floer homology and the Berge conjecture on knots admitting lens space surgeries*, Tran.s Amer. Math. Soc., **363**(2011), 949-968.

[11] M. Hedden, C. Herald, and P. Kirk, *The pillowcase and perturbations of traceless representations of knot groups*, Geom. Topol., **18**(2014) 211-287.

[12] M. Hedden, C. Herald, and P. Kirk, *The pillowcase and traceless representations of knot groups II: a Lagrangian-Floer theory in the pillowcase*, preprint, **arXiv:1501.00028**.

[13] J. Kamnitzer, *The Beilinson-Drinfeld Grassmannian and symplectic knot homology*, in *Grassmannians, moduli spaces and vector bundles*, Amer. Math. Soc., Providence, RI, volume 14 of *Clay Math. Proc.*, pp. 81–94, 2011.

[14] M. Khovanov, *A categorification of the Jones polynomial*, Duke Math. J., **101**(2000), no. 3, 359–426.

[15] M. Khovanov and L.-H. Robert, *Foam evaluation and Kronheimer–Mrowka theories*, **arXiv:1808.09662**.

[16] P. Kronheimer and T. Mrowka, *Knot homology groups from instantons*, J. Topol. **4** (2011), no. 4, 835–918.

[17] P. Kronheimer and T. Mrowka, *Khovanov homology is an unknot-detector*, Publ. Math. Inst. Hautes Etudes Sci. No. 113 (2011), 97–208.

[18] P. Kronheimer and T. Mrowka, *Knots, sutures, and excision*, J. Differential Geom., **84**(2010), 301–364.

[19] P. Kronheimer and T. Mrowka, *Tait colorings, and an instanton homology for webs and foams*, preprint, **arXiv:1508.07205**.

[20] P. Kronheimer and T. Mrowka, *Exact triangles for* $SO(3)$ *instanton homology of webs*, preprint, **arXiv:1508.07207**.

[21] P. Kronheimer and T. Mrowka, *A deformation of instanton homology for webs*, preprint, **arXiv:1710.05002**.

[22] L.-H. Robert and E. Wagner, *A quantum categorification of the Alexander Polynomial*, preprint, **arXiv:1902.05648**.

[23] P. Seidel and I. Smith, *A link invariant from the symplectic geometry of nilpotent slices*, Duke Math. J., **134**(2006), no. 3, 453–514.

[24] C. Woodward, unpublished notes.