

RESEARCH STATEMENT

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1. INTRODUCTION

My research focuses on problems in low-dimensional topology involving invariants of knots, links, and graphs in 3-manifolds. An appealing feature of such problems is the broad array of mathematical techniques that can be brought to bear on them, and my research draws on ideas from gauge theory, symplectic topology, algebraic geometry, and representation theory. These topics are of particular interest to me in part because I have a background in physics, a subject that has strongly influenced the development of modern low-dimensional topology. My background also includes extensive computer programming experience, which I have applied to some of my research in mathematics.

1.1. Knots and links. Knots and links play a key role in the study of 3-manifolds. Indeed, one can show that any 3-manifold can be constructed by performing a simple operation known as Dehn surgery on a suitable link in S^3 . One can consider knots in any 3-manifold, but most work on knot invariants has focused on knots in S^3 . For example, two important knot invariants are Khovanov homology [25] and singular instanton homology [29, 28]. Khovanov homology is easy to calculate (it is the homology of a chain complex constructed from a generic planar projection of a knot), but is defined only for knots in S^3 . Singular instanton homology is difficult to calculate (it is defined in terms of gauge-theoretic nonlinear partial differential equations) and, though it is defined for knots in arbitrary 3-manifolds, the few calculations that have been performed generally rely on techniques applicable only to knots in S^3 . Little is known about invariants of knots in arbitrary 3-manifolds, but such results would be of great interest.

One of my research projects is to generalize Khovanov homology to knots in arbitrary 3-manifolds. Though a number of authors have suggested generalizations for certain specific 3-manifolds [3, 35, 16, 37], this remains an important open problem. Khovanov homology is defined for knots in S^3 in terms of combinatorial rules that can be shown to yield a knot invariant, but do not have a clear geometric meaning and thus do not suggest an obvious generalization. My approach is to reinterpret Khovanov homology in terms of a geometric theory that is more amenable to generalization, attempt to generalize the geometric theory in a natural way, extract from the generalized theory a new set of rules for constructing a chain complex, and finally prove that the homology of the chain complex is a knot invariant. I have considered two approaches along these lines, based on two different geometric interpretations of Khovanov homology.

1.1.1. Khovanov homology for knots in lens spaces. In [12], I described an approach to constructing Khovanov homology for knots in lens spaces inspired by a symplectic interpretation of Khovanov homology due to Hedden, Herald, Hogancamp, and Kirk [21]. This approach relies on a partly conjectural description of the Fukaya category of the traceless $SU(2)$ character variety $R^*(T^2, 2)$ of the 2-torus with two punctures. The space $R^*(T^2, 2)$ is a four-dimensional symplectic manifold whose structure I investigated in [9] from the standpoint of gauge theory, and can also be interpreted as a moduli space of parabolic bundles over an elliptic curve in algebraic geometry, a perspective I considered in [11, 13].

By definition, a *lens space* is a 3-manifold that can be split into two solid tori U_1 and U_2 by cutting along a 2-torus. For a lens space containing a knot K , one can choose the 2-torus so that

it cuts K at two points and splits it into pieces $T_1 \subset U_1$ and $T_2 \subset U_2$ called *1-tangles*. One can project T_i onto the plane to obtain a two-dimensional tangle diagram, from which I construct an object (X_i, δ_i) of the twisted Fukaya category of $R^*(T, 2)$, an A_∞ category that can be thought of as the analog for Fukaya categories of the notion of a chain complex for vector spaces. I show that the homotopy type of the object (X_i, δ_i) is an isotopy invariant of the tangle diagram, which means it is capturing something intrinsic to the three-dimensional topology of T_i . The morphism spaces of the twisted Fukaya category have the structure of chain complexes, and by computing the space of morphisms between the objects (X_1, δ_1) and (X_2, δ_2) corresponding to (U_1, T_1) and (U_2, T_2) , one can construct a chain complex for the knot $K = T_1 \cup T_2$ in the lens space $Y = U_1 \cup U_2$.

Using this approach, I constructed a new chain complex for knots in S^3 , which is the simplest lens space. The homology of the new chain complex reproduces Khovanov homology, as I proved in [10], but the chain complex itself is typically smaller than the Khovanov chain complex. The successful prediction of a new chain complex could be viewed as evidence that this approach will yield invariants analogous to Khovanov homology for knots in other lens spaces. In [12] I used this approach to explicitly construct chain complexes for some knots in $S^2 \times S^1$ and present results that suggest the homology is indeed a knot invariant.

1.1.2. *Singular instanton homology for knots in lens spaces.* In related work, in [9] I generalized ideas of Hedden, Herald, and Kirk [19] to compute generating sets for the singular instanton homology of knots in lens spaces. The generating sets are given by the intersection points of two Lagrangian submanifolds of $R^*(T^2, 2)$; conjecturally, the Lagrangian Floer homology of the pair of submanifolds gives the singular instanton homology itself. I computed generating sets for several example knots, some of which reproduce known results for knots in S^3 and some of which provide original results for knots in lens spaces.

1.1.3. *Hecke modifications.* In [11], I described a second approach to constructing Khovanov homology for knots in lens spaces inspired by Seidel and Smith's interpretation of Khovanov homology as Lagrangian Floer homology in symplectic manifolds known as Seidel-Smith spaces [36]. This interpretation is closely related to a conjectural gauge theory interpretation of Khovanov homology due to Witten [38, 39]. As a first step, I proposed candidate manifolds that generalize the Seidel-Smith spaces. The candidate manifolds are defined as moduli spaces of Hecke modifications of holomorphic vector bundles over an elliptic curve, and the project involved a detailed study of these moduli spaces. A *Hecke modification* is a way of transforming a vector bundle into a new vector bundle, and a *moduli space of Hecke modifications* is a geometric space that parameterizes all the possible ways of doing so. In [13], I used the Hecke-modification techniques I developed in [11] to explicitly describe a moduli space of stable parabolic bundles over an elliptic curve. My work on Hecke modifications was applied to the geometric Langlands program [15], and is related to a quantum field theory interpretation of Langlands duality due to Kapustin and Witten [24].

1.2. **Graphs.** To define invariants of knots, it is often useful to generalize from knots to graphs, also called *webs*, which can be thought of as knots with singular behavior allowed at the vertices. Webs arise naturally in knot invariants defined via representation theory, since they can be viewed as pictorial descriptions of rules for combining representations. To categorify such invariants, it is useful to define a category in which the objects are webs and the morphisms are web cobordisms called *foams*. In fact, Khovanov homology can be viewed as just such a categorification [6].

Webs also play an important role in a new approach to proving the four-color theorem due to Kronheimer and Mrowka [33], which if successful would yield the first proof of this result that does not rely on computers. Kronheimer and Mrowka use a version of singular instanton homology to define a functor J^\sharp from a category of foams to the category of vector spaces over the field of two elements. They show that if the dimension of the vector space $J^\sharp(K)$ associated to a web K is

equal to the number of 3-colorings of K for all webs K , this would imply the four-color theorem. They also show that equality does hold for a special class of *reducible* webs.

In [8] and [14], I investigated a possible combinatorial replacement J^b for J^\sharp that was suggested by Kronheimer and Mrowka [33] and further developed by Khovanov and Robert [26]. I wrote a computer program to calculate lower bounds on $\dim J^b(K)$ and calculated such bounds for a number of example nonreducible webs. In some cases the bounds are sufficiently strong to uniquely determine $\dim J^b(K)$, and these results constitute the first exact calculations of $\dim J^b(K)$ for nonreducible webs.

The relationship between the combinatorial functor J^b and the gauge-theoretic functor J^\sharp is of considerable interest, since results due to Kronheimer, Mrowka, Khovanov, and Robert show that a proof that the two functors are the same would imply the four-color theorem. In [14], I prove that the functors J^b and J^\sharp are in fact *not* the same by exhibiting a specific counterexample.

1.3. Future plans. In later sections I will outline a number of specific ideas for future research, some of which would make good projects for graduate students. In addition, in future work I would like to seek ways to combine my background in physics and mathematics by applying ideas from quantum field theory to problems in topology. Knot and manifold invariants can often be formulated in terms of quantum field theories, and adopting this perspective has proven to be highly fruitful. Many fundamental ideas, such as instantons, pseudo-holomorphic curves, Seiberg-Witten invariants, mirror symmetry, and quantum groups, either originated in or were strongly influenced by quantum field theory. Some of my research ideas do seem to fit into a quantum field theory framework. For example, invariants based on foams are reminiscent of quantum field theories with defects, and attempting to understand the invariants from this standpoint might lead to new insights. Some invariants arising from representation theory, such as the Jones polynomial, can be understood in terms the geometric quantization of certain symplectic manifolds, and it would be interesting to think about whether the categorification of such invariants can be understood in terms of an analogous operation.

2. KHOVANOV HOMOLOGY FOR KNOTS IN LENS SPACES

In [12] I consider an approach to constructing Khovanov homology for knots in lens spaces inspired by a symplectic interpretation of Khovanov homology for knots in S^3 due to Hedden, Herald, Hogancamp, and Kirk [21]. Given a link L in S^3 , they consider a Heegaard splitting

$$(1) \quad (S^3, L) = (B^3, T_1) \cup_{(S^2, 4)} (B^3, T_2)$$

such that the Heegaard surface $(S^2, 4)$ is a 2-sphere that transversely intersects L in four points, the handlebodies (B^3, T_1) and (B^3, T_2) are closed 3-balls containing 2-tangles T_1 and T_2 , and T_1 is trivial. To the Heegaard surface $(S^2, 4)$ they associate the traceless $SU(2)$ character variety $R^*(S^2, 4)$ for the 2-sphere with four punctures, a symplectic manifold known as the *pillowcase*. They project the tangle T_2 onto the plane to obtain a 2-tangle diagram T in the disk. From a cube of resolutions of T , they construct an object (X, δ) in the A_∞ category of twisted complexes over the Fukaya category of $R^*(S^2, 4)$, where X consists of shifted copies of Lagrangians corresponding to planar tangles at the vertices of the cube and $\delta : X \rightarrow X$ consists of maps of Lagrangians corresponding to saddles at the edges of the cube. They prove:

Theorem 2.1. (*Hedden–Herald–Hogancamp–Kirk*) *Given a planar projection of (B^3, T_2) , there is a corresponding object (X, δ) of the twisted Fukaya category of $R^*(S^2, 4)$. The homotopy type of (X, δ) is an isotopy invariant of T_2 rel boundary.*

To the trivial tangle T_1 they assign an object $(W_1, 0)$ of the twisted Fukaya category. The morphism spaces of the twisted Fukaya category have the structure of cochain complexes, so the

space of morphisms from $(W_1, 0)$ to (X, δ) is a cochain complex. They show this cochain complex is the usual cochain complex for the reduced Khovanov homology $\text{Khr}(L)$ of L , thus proving:

Theorem 2.2. (*Hedden–Herald–Hogancamp–Kirk*) *We have an isomorphism of bigraded vector spaces $\text{Khr}(L) \rightarrow H^*(\text{hom}((W_1, 0), (X, \delta)))$.*

Theorem 2.2 shows that the Fukaya category of $R^*(S^2, 4)$ knows about Khovanov homology. My approach to generalizing Khovanov homology to links in lens spaces is based on generalizing this observation. Theorem 2.2 is formulated in terms of $R^*(S^2, 4)$ because this is the character variety of the Heegaard surface $(S^2, 4)$ for the Heegaard splitting (1) of (S^3, L) . More generally, given a link L in a lens space Y , one can consider a Heegaard splitting

$$(2) \quad (Y, L) = (U_1, T_1) \cup_{(T^2, 2)} (U_2, T_2)$$

such that the Heegaard surface $(T^2, 2)$ is a 2-torus that transversely intersects L in two points, the handlebodies (U_1, T_1) and (U_2, T_2) are solid tori containing 1-tangles T_1 and T_2 , and T_1 is trivial. To the Heegaard surface $(T^2, 2)$ I associate the traceless $SU(2)$ character variety $R^*(T^2, 2)$ for the 2-torus with two punctures. I project the tangle T_2 onto the plane to obtain a 1-tangle diagram T in the annulus. Based on a detailed study of the Fukaya category of $R^*(T^2, 2)$, supplemented with some conjectures regarding certain A_∞ operations of this category that I have not calculated explicitly, I generalize Theorem 2.1 for $R^*(S^2, 4)$:

Theorem 2.3. *If the A_∞ operations of $R^*(T^2, 2)$ are as conjectured, then given a planar projection T of (U_2, T_2) there are two corresponding objects (X, δ_+) and (X, δ_-) of the twisted Fukaya category of $R^*(T^2, 2)$. The homotopy type of (X, δ_\pm) is an invariant of T_2 rel boundary.*

Given a 1-tangle diagram T in the annulus, one can close T with an overpass arc A^+ or underpass arc A^- to obtain a diagram of a link T^+ or T^- in S^3 . I use the twisted complex (X, δ_\pm) to explicitly construct a cochain complex (C^\pm, ∂^\pm) for the link T^\pm from a cube of resolutions of T , and in [10] I prove the following generalization of Theorem 2.2:

Theorem 2.4. *There is an isomorphism of bigraded vector spaces $\text{Khr}(T^\pm) \rightarrow H^*(C^\pm, \partial^\pm)$.*

One can also close T to obtain a diagram of a link T^0 in $S^2 \times S^1$. For certain tangle diagrams, I use the twisted complex (X, δ_\pm) to explicitly construct a cochain complex (C^0, ∂^0) for the link T^0 from a cube of resolutions of T . One might hope that the cohomology of (C^0, ∂^0) depends only on the isotopy class of the link T^0 and not on its description as the closure of the particular tangle diagram T . From Theorem 2.3, it follows that the cohomology is an isotopy invariant of T . But it is possible for nonisotopic tangle diagrams T_1 and T_2 to yield isotopic links T_1^0 and T_2^0 in $S^2 \times S^1$, and example calculations show that in this case the cohomology for T_1 and T_2 need not be the same. In all the examples I have checked, however, the dependence on the tangle diagram is reflected only in the bigradings of generators, and the cohomology does agree if the bigradings are collapsed from \mathbb{Z} to \mathbb{Z}_2 . Such examples motivate:

Conjecture 2.1. *The cohomology of the cochain complex (C^0, ∂^0) with bigradings collapsed from \mathbb{Z} to \mathbb{Z}_2 is an isotopy invariant of the link T^0 .*

I have proven a partial invariance result in support of Conjecture 2.1, as described in [12].

2.1. Future plans. These ideas suggest a number of avenues for future research. One project would be to use the techniques I developed to construct and investigate cochain complexes for links in additional lens spaces. I would also like to further investigate the Fukaya category of $R^*(T^2, 2)$ and attempt to prove my conjectures regarding its A_∞ operations. One way to formulate Khovanov homology for links in S^3 is in terms of a cobordism category due to Bar-Natan [5]. My results in [12] suggest that a similar category may be relevant to Khovanov homology for links in lens spaces, and it would be interesting to attempt to construct this category. It would also be interesting to

implement my approach to generalizing Khovanov homology using other character varieties, which would correspond to other Heegaard splittings of 3-manifolds containing a link.

3. SINGULAR INSTANTON HOMOLOGY FOR KNOTS IN LENS SPACES

In related work, in [9] I used the character variety $R^*(T^2, 2)$ to construct generating sets for the singular instanton homology of knots in lens spaces. This work generalizes a scheme due to Hedden, Herald, and Kirk for constructing generating sets for the singular instanton homology of knots in S^3 [19, 20]. Roughly speaking, the singular instanton homology of a knot K in a 3-manifold Y is a kind of Morse homology in which the Chern-Simons functional serves as a Morse function on the space of connections on a trivial $SU(2)$ bundle. The generators of the homology chain complex correspond to gauge orbits of flat connections with prescribed singularities along K , which in turn correspond to conjugacy classes of homomorphisms $\pi_1(Y - K) \rightarrow SU(2)$ that map loops around K to traceless matrices. The set of such conjugacy classes defines a character variety $R(Y, K)$.

The character variety $R(Y, K)$ is not quite the generating set that we seek, since there are two technical complications that must be addressed. First, in order to get a chain complex, with $\partial^2 = 0$, we need to ensure there are no *reducible* connections, which correspond to abelian $SU(2)$ representations. Second, the Chern-Simons functional is generally not Morse, and must be perturbed so as to render it Morse. The modifications needed to address these issues amount to considering homomorphisms $\pi_1(Y - K') \rightarrow SU(2)$ that satisfy certain conditions, where K' is a graph containing K as a subset whose precise form I describe. These homomorphisms define a modified character variety $R_\pi^\natural(Y, K)$ that constitutes the actual generating set for singular instanton homology.

To compute $R_\pi^\natural(Y, K)$, I Heegaard-split (Y, L) as described in equation (2). The solid torus U_1 contains a portion of the knot consisting of a trivial 1-tangle T_1 , together with all of the modifications necessary to eliminate reducible connections and to render the Chern-Simons functional Morse. The solid torus U_2 contains a 1-tangle T_2 that describes the remainder of the knot. In analogy with $R(Y, K)$ and $R_\pi^\natural(Y, K)$, I define character varieties $R^*(U_2, T_2)$ and $R_\pi^\natural(U_1, T_1)$. Pulling back homomorphisms along inclusion maps, we obtain the following commutative diagram:

$$\begin{array}{ccccc} & & R_\pi^\natural(U_1, T_1) & & \\ & \nearrow & & \searrow & \\ R_\pi^\natural(Y, L) & & & & R^*(T^2, 2). \\ & \searrow & & \nearrow & \\ & & R^*(U_2, T_2) & & \end{array}$$

The character variety $R^*(T^2, 2)$ is a symplectic manifold, and the images L_1 and L_2 of $R_\pi^\natural(U_1, T_1)$ and $R^*(U_2, T_2)$ in $R^*(T^2, 2)$ are Lagrangians. I prove theorems that explicitly describe the character variety $R^*(T^2, 2)$ and the Lagrangians L_1 and L_2 . In particular, the Lagrangian L_1 is an immersed 2-sphere with a unique double point. I prove:

Theorem 3.1. *A point $[\rho] \in L_1 \cap L_2 \subset R^*(T^2, 2)$ that is not the double point of L_1 is the image of a unique point $[\rho'] \in R_\pi^\natural(Y, K)$ under the pullback map $R_\pi^\natural(Y, K) \rightarrow R^*(T^2, 2)$. The point $[\rho']$ is nondegenerate if and only if the intersection of L_1 with L_2 at $[\rho]$ is transverse.*

For simplicity I have stated Theorem 3.1 only for the important special case that T_2 is an unknotted arc, so K is a $(1, 1)$ -knot, but I also proved a version of Theorem 3.1 that does not require this assumption. From Theorem 3.1 it follows that if L_2 intersects L_1 transversely and does not contain the double-point of L_1 , then every point in $R_\pi^\natural(Y, K)$ is nondegenerate and the pullback map $R_\pi^\natural(Y, K) \rightarrow R^*(T^2, 2)$ is injective with image $L_1 \cap L_2$. Thus $R_\pi^\natural(Y, K)$ is a generating set for singular instanton homology consisting of $|L_1 \cap L_2|$ generators.

I explicitly calculate generating sets for the singular instanton homology of some example $(1, 1)$ -knots. In particular, one can define the notion of a *simple* knot in a lens space [18], and I prove:

Theorem 3.2. *If K is the unique simple knot in the lens space $L(p, 1)$ representing the homology class $1 \in \mathbb{Z}_p = H_1(L(p, 1); \mathbb{Z})$, then the rank of the singular instanton homology of K is at most p .*

3.1. Future plans. For a simple knot K in the lens space $Y = L(p, q)$, the knot Floer homology $\widehat{HF\widehat{K}}(Y, K)$ has rank p [18]. Thus, Theorem 3.2 is consistent with Kronheimer and Mrowka's conjecture that for a knot K in a 3-manifold Y , the ranks of singular instanton homology and knot Floer homology $\widehat{HF\widehat{K}}(Y, K)$ are the same [27]. In future work, it would be interesting to test Kronheimer and Mrowka's conjecture for additional $(1, 1)$ -knots in S^3 , which should be possible since there is a combinatorial method for computing the knot Floer homology for such knots [17].

4. HECKE MODIFICATIONS

In [11] I considered an approach to constructing Khovanov homology for knots in lens spaces inspired by the work of Seidel and Smith. Seidel and Smith describe a knot in S^3 as the closure of a braid with $2m$ strands. Such a knot can be obtained by gluing together two solid balls, each containing m unknotted arcs, along their common S^2 boundary. To the 2-sphere they associate a symplectic manifold $\mathcal{Y}(S^2, 2m)$ known as a Seidel-Smith space, and to the pair of solid balls they associate a pair of Lagrangians in $\mathcal{Y}(S^2, 2m)$. Seidel and Smith prove that the Lagrangian Floer homology of the pair of Lagrangians is a knot invariant that they conjectured would coincide with the Khovanov homology of the knot [36], and this was later proved by Abouzaid and Smith [1].

It seems reasonable to think that a similar picture might hold for knots in lens spaces. Given a suitable Heegaard splitting of a lens space into two solid tori, perhaps one can associate to their common T^2 boundary a symplectic manifold $\mathcal{Y}(T^2, 2m)$ that generalizes the Seidel-Smith space $\mathcal{Y}(S^2, 2m)$, and to the pair of solid tori a pair of Lagrangians in $\mathcal{Y}(T^2, 2m)$. If the Lagrangian Floer homology of the pair of Lagrangians could be shown to be independent of the Heegaard splitting, it would serve as a natural generalization of Khovanov homology.

The first step in implementing this strategy is to find a suitable symplectic manifold $\mathcal{Y}(T^2, 2m)$, and in [11] I proposed a natural candidate for this space. The space $\mathcal{Y}(S^2, 2m)$ was originally defined in terms of a nilpotent slice in the Lie algebra \mathfrak{sl}_{2m} . This description of $\mathcal{Y}(S^2, 2m)$ does not seem to have a natural generalization, so I first applied a result of Kamnitzer [23] to reinterpret $\mathcal{Y}(S^2, 2m)$ as a moduli space $\mathcal{H}(\mathbb{CP}^1, 2m)$ of Hecke modifications of rank 2 holomorphic vector bundles over a rational curve; roughly, this space parameterizes different ways of transforming a given vector bundle into a new vector bundle. Next, I proposed a notion of a Hecke modification of a parabolic bundle and used this notion to reinterpret the moduli space $\mathcal{H}(\mathbb{CP}^1, n)$ of Hecke modifications of vector bundles as a moduli space $\mathcal{P}(\mathbb{CP}^1, n)$ of Hecke modifications of parabolic bundles. I showed that $\mathcal{P}(\mathbb{CP}^1, n)$ has a natural generalization $\mathcal{P}(E, n)$ to the case of an elliptic curve E , and I proposed the space $\mathcal{P}(E, 2m)$ as a candidate for $\mathcal{Y}(T^2, 2m)$. In summary, the sequence of reinterpretation and generalization is

$$\mathcal{Y}(S^2, 2m) \cong \mathcal{H}(\mathbb{CP}^1, 2m) \cong \mathcal{P}(\mathbb{CP}^1, 2m) \rightsquigarrow \mathcal{P}(E, 2m) =: \mathcal{Y}(T^2, 2m).$$

In fact, I can define the moduli space $\mathcal{P}(C, n)$ for an arbitrary algebraic curve C . I conjecture that $\mathcal{P}(C, n)$ canonically embeds into the moduli space $M^s(C, n+3)$ of stable rank 2 parabolic bundles over C with $n+3$ marked points, and I proved this for $\mathcal{P}(\mathbb{CP}^1, n)$ and $\mathcal{P}(E, n)$:

Theorem 4.1. *The space $\mathcal{P}(\mathbb{CP}^1, n)$ canonically embeds into the moduli space $M^s(\mathbb{CP}^1, n+3)$.*

Theorem 4.2. *The space $\mathcal{P}(E, n)$ canonically embeds into the moduli space $M^s(E, n+3)$.*

Theorem 4.1 appears to be closely related to a theorem due to Woodward [40] that was proven using the nilpotent slice interpretation of $\mathcal{Y}(S^2, 2m)$:

Theorem 4.3 (Woodward). *The space $\mathcal{Y}(S^2, 2m)$ embeds into the moduli space $M^s(\mathbb{CP}^1, 2m+3)$.*

These embedding results suggest that there may be a spectral sequence from symplectic Khovanov homology to symplectic singular instanton homology [22], which is defined in terms of $M^s(C, 2m+3)$, that generalizes a spectral sequence due to Kronheimer and Mrowka from Khovanov homology to singular instanton homology [30]. The embedding results thus provide some additional evidence that $\mathcal{P}(E, 2m)$ is the correct generalization of the Seidel-Smith space $\mathcal{Y}(S^2, 2m)$.

In order to define and characterize the moduli space $\mathcal{P}(E, n)$, I described all possible Hecke modifications of all possible rank 2 vector bundles over an elliptic curve E , using a classification of holomorphic vector bundles on elliptic curves due to Atiyah [4]. As an additional application of these results, in [13] I used Hecke-modification methods to describe the moduli space $M^s(E, 3)$:

Theorem 4.4. *The moduli space $M^s(E, 3)$ is a blow-up of an embedded elliptic curve in $(\mathbb{C}P^1)^3$.*

4.1. Future plans. I plan to continue to pursue this approach to generalizing Khovanov homology to lens spaces, and if possible to connect it with the approach discussed in Section 2 and with a gauge theory interpretation of Khovanov homology due to Witten [38, 39]. It would also be interesting to attempt to use the embedding results to construct a spectral sequence to symplectic singular instanton homology.

5. FOAM EVALUATION

Kronheimer and Mrowka recently described a new approach to proving the four-color theorem [33]. Their approach involves a category Foams whose objects are *webs*, which are unoriented planar trivalent graphs, and whose morphisms are *foams*, which are singular cobordisms between pairs of webs. Using a version of singular instanton homology, they define a functor J^\sharp from the category Foams to the category of vector spaces over \mathbb{F} , the field of two elements. They show that if the dimension of the vector space $J^\sharp(K)$ associated to a web K is equal to the number of 3-colorings $\text{Tait}(K)$ of K for all webs K , this would imply the four-color theorem.

Kronheimer and Mrowka also suggested a possible combinatorial replacement J^b for J^\sharp . They first define $J^b(F) \in \mathbb{F}$ for a *closed foam* $F \subset \mathbb{R}^3$, which is a cobordism from the empty web to itself, via a set of combinatorial rules that they conjectured were well-defined. This was later shown to be the case by Khovanov and Robert [26], who gave an explicit formula for $J^b(F)$. Kronheimer and Mrowka then extend J^b to a functor by using the *universal construction* [7]: for a web K , the corresponding vector space $J^b(K)$ is defined as follows. First, define a vector space $V(K)$ spanned by all *half foams* with top boundary K ; these are cobordisms from the empty web to K . Define a bilinear form $(-, -) : V(K) \otimes V(K) \rightarrow \mathbb{F}$ such that $(H_1, H_2) = J^b(F)$, where $F := H_1 \cup_K \overline{H}_2$ is the closed foam obtained by reflecting the half-foam H_2 top-to-bottom to get \overline{H}_2 and then gluing it to the half-foam H_1 along their common boundary K . Now define $J^b(K)$ to be the quotient of $V(K)$ by the orthogonal complement of $V(K)$ relative to $(-, -)$.

In light of Kronheimer and Mrowka's new approach, it is important to understand the relationships between $\dim J^b(K)$, $\dim J^\sharp(K)$, and $\text{Tait}(K)$. It is known that for any web K we have

$$(3) \quad \dim J^b(K) \leq \text{Tait}(K) \leq \dim J^\sharp(K),$$

and for a special class of *reducible* webs these numbers are all equal [8, 26, 31]. It is thus of interest to compute examples of $\dim J^b(K)$ and $\dim J^\sharp(K)$ for nonreducible webs K . The only results that had previously been obtained were for the dodecahedral web K_1 , for which it was known that $J^b(K_1) \geq 58$, $\text{Tait}(K_1) = 60$, and $J^\sharp(K_1) \leq 68$ [32, 33]. Since $J^b(K)$ is defined in terms of an infinite number of generators mod an infinite number of relations, it is not clear whether $\dim J^b(K)$ can be algorithmically computed. Nevertheless, it is possible to algorithmically compute lower bounds for $\dim J^b(K)$ by computing the rank of the restriction of the bilinear form $(-, -)$ to any finite subspace of $V(K)$.

In [8], I describe a computer program I wrote to determine such lower bounds by constructing a large number of half-foams and computing the rank of the restriction of the bilinear form $(-, -)$

to the vector space that they span, and I compute lower bounds on $\dim \mathcal{J}^b(K)$ for a number of example nonreducible webs. Since $\dim \mathcal{J}^b(K) \leq \text{Tait}(K)$, if the lower bound on $\dim \mathcal{J}^b(K)$ is equal to $\text{Tait}(K)$ then $\dim \mathcal{J}^b(K) = \text{Tait}(K)$. In this way I was able to perform the first calculations for $\dim \mathcal{J}^b(K)$ for K nonreducible.

One way to prove the four-color theorem would be to show that the functors \mathcal{J}^b and \mathcal{J}^\sharp are the same, since the string of inequalities (3) would then imply $\dim \mathcal{J}^\sharp(K) = \text{Tait}(K)$ for all webs K . In [14], I exhibit a specific counterexample to prove:

Theorem 5.1. *The combinatorial functor \mathcal{J}^b and the gauge-theoretic functor \mathcal{J}^\sharp are not the same.*

5.1. Future plans. If \mathcal{J}^b and \mathcal{J}^\sharp agree on certain *closed* foams, which should not be difficult to prove, then the same methods used to establish the counterexample would allow one to calculate upper bounds on $\dim \mathcal{J}^\sharp(K)$, which in some cases would determine this quantity exactly. Such calculations would constitute the first exact calculations of $\dim \mathcal{J}^\sharp(K)$ for nonreducible webs.

Computer methods similar to those I used to study the functor \mathcal{J}^b could also be applied to many other problems involving foams. For example, Robert and Wagner recently constructed a foam-based categorification of the Alexander polynomial [34], and Akhmechet and Khovanov proposed a homology theory for links in the solid torus [2]. Neither of these theories are well-understood, and it would be interesting to investigate them using the computer methods I developed.

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