THE MODULI SPACE OF STABLE RANK 2 PARABOLIC BUNDLES OVER AN ELLIPTIC CURVE WITH 3 MARKED POINTS

DAVID BOOZER

ABSTRACT. We explicitly describe the moduli space $M^s(X,3)$ of stable rank 2 parabolic bundles over an elliptic curve X with trivial determinant bundle and 3 marked points. Specifically, we exhibit $M^s(X,3)$ as a blow-up of an embedded elliptic curve in $(\mathbb{CP}^1)^3$. The moduli space $M^s(X,3)$ can also be interpreted as the $SU(2)$ character variety of the 3-punctured torus. Our description of $M^s(X,3)$ reproduces the known Poincaré polynomial for this space.

1. INTRODUCTION

Given a curve C, one can define a moduli space $M^s(C,n)$ of stable rank 2 parabolic bundles over C with trivial determinant bundle and n marked points. The space $M^s(C,n)$ has the structure of a smooth complex manifold of dimension $3(g-1)+n$, where g is the genus of the curve C. In general, the space $M^s(C,n)$ depends on a positive real parameter μ known as the *weight*. For μ sufficiently small ($\mu < 1/n$ will suffice), the space $M^s(C,n)$ is independent of μ, but as μ increases it may cross critical values at which $M^s(C,n)$ undergoes certain birational transformations [2, 13]. The moduli space $M^s(C,n)$ can also be interpreted as an $SU(2)$-character variety, which is defined as the space of conjugacy classes of $SU(2)$-representations of the fundamental group of C with n punctures, where loops around the punctures are required to correspond to $SU(2)$-matrices conjugate to $\operatorname{diag}(e^{2\pi i\mu}, e^{-2\pi i\mu})$. Moduli spaces of parabolic bundles on curves are natural objects of study in algebraic geometry, and also play an important role in low-dimensional topology. In particular, these spaces have a canonical symplectic structure and can be used to define Floer homology theories of links [4, 5, 6, 7].

Explicit descriptions of $M^s(C,n)$ are known for small values of n and C a rational or elliptic curve. For rational curves, it is well-known that for small weight we have

$$\begin{aligned}
&M^s(\mathbb{CP}^1,0) = M^s(\mathbb{CP}^1,1) = M^s(\mathbb{CP}^1,2) = \varnothing,\\
&M^s(\mathbb{CP}^1,3) = \{pt\},\\
&M^s(\mathbb{CP}^1,4) = \mathbb{CP}^1 - \{3 \text{ points}\},\\
&M^s(\mathbb{CP}^1,5) = \mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2.
\end{aligned}$$

The structure of $M^s(\mathbb{CP}^1,6)$ for $\mu = 1/4$, corresponding to the traceless character variety, was recently described by Kirk [9]. For an elliptic curve X, it is straightforward to show that for small weight we have

$$M^s(X,0) = \varnothing, \qquad\qquad M^s(X,1) = \mathbb{CP}^1.$$

It was recently shown by Vargas [15] that $M^s(X,2)$ is the complement of an embedded elliptic curve in $(\mathbb{CP}^1)^2$.

Our goal in this paper is to explicitly describe the structure of $M^s(X,3)$. We prove the following result:

Theorem 1.1. *For small weight, the moduli space $M^s(X,3)$ for an elliptic curve X is a blow-up of an embedded elliptic curve in $(\mathbb{CP}^1)^3$.*

To prove Theorem 1.1, we make use of an explicit description of Hecke modifications of rank 2 holomorphic vector bundles on elliptic curves that is derived in [3]. Roughly speaking, a Hecke modification is a way of locally modifying a vector bundle near a point to obtain a new vector bundle. The moduli space $M^s(X,3)$ plays an important role in a conjectural Floer homology theory for links in lens spaces discussed in [3], and Theorem 1.1 was motivated by this application.

The paper is organized as follows. In Sections 2 and 3, we review the background material we will need on parabolic bundles and vector bundles on elliptic curves. In Section 4, we use Hecke-modification methods to explicitly describe $M^s(X,3)$. In Section 5, we relate $M^s(X,3)$ to $M^{ss}(X,2)$, the moduli space of S-equivalence classes of rank 2 semistable parabolic bundles with trivial determinant bundle and 2 marked points. In Section 6, we use our description of $M^s(X,3)$ to reproduce the known Poincaré polynomial for this space.

Date: July 5, 2020.

2. Parabolic bundles

The concept of a parabolic bundle was introduced in [10]. We will not need this concept in its full generality; rather, we will consider only parabolic bundles of a certain restricted form, which is discussed at greater length in [3, Appendix B]. For our purposes here, a rank 2 parabolic bundle over a curve C consists of a rank 2 holomorphic vector bundle E over C with trivial determinant bundle, distinct marked points $p_1, \cdots, p_n \in C$, a line $\ell_{p_i} \in \mathbb{P}(E_{p_i})$ in the fiber E_{p_i} over each marked point p_i, and a positive real parameter μ known as the *weight*. For simplicity, we will suppress the curve C and weight μ in the notation and denote a parabolic bundle as $(E, \ell_{p_1}, \cdots, \ell_{p_n})$.

In order to describe the stability properties of parabolic bundles, it is helpful to introduce some additional terminology. Recall that the degree of the proper subbundles of a vector bundle E on a curve C is bounded above. Given a rank 2 holomorphic vector bundle E, we say that a line $\ell_p \in \mathbb{P}(E_p)$ is *bad* if there is a line subbundle L of E of maximal degree such that $\ell_p = L_p$, and *good* otherwise. We say that lines $\ell_{p_1} \in \mathbb{P}(E_{p_1}), \cdots, \ell_{p_n} \in \mathbb{P}(E_{p_n})$ are *bad in the same direction* if there is a line subbundle L of E of maximal degree such that $\ell_{p_i} = L_{p_i}$ for $i = 1, \cdots, n$.

Consider a parabolic bundle $\mathcal{E} = (E, \ell_{p_1}, \cdots, \ell_{p_n})$. Let m denote the maximum number of lines of $\mathcal{E}$ that are bad in the same direction. For sufficiently small weight ($\mu < 1/n$ will suffice), we can characterize the stability and semistability of $\mathcal{E}$ as follows. If E is unstable, then $\mathcal{E}$ is unstable. If E is semistable, then $\mathcal{E}$ is stable if $m < n/2$, semistable if $m \leq n/2$, and unstable if $m > n/2$. Note that if n is odd then stability and semistability are equivalent.

We define a moduli space $M^s(C, n)$ of isomorphism classes of stable parabolic bundles with n marked points. As with vector bundles, one can define a notion of S-equivalent semistable parabolic bundles, and we define a moduli space $M^{ss}(C, n)$ of S-equivalence classes of semistable parabolic bundles. For odd n we have that $M^s(C, n) = M^{ss}(C, n)$. For even n we have that $M^s(C, n)$ is an open subset of $M^{ss}(C, n)$.

3. Vector bundles on elliptic curves

Vector bundles on elliptic curves were classified by Atiyah [1], and are well understood. Here we briefly summarize the results regarding vector bundles on elliptic curves that we will need; these results are either well-known (see for example [1, 8, 12, 14]) or derived in [3, Section 5].

3.1. **Line bundles.**

Isomorphism classes of degree 0 line bundles on an elliptic curve X are parameterized by the Jacobian $\mathrm{Jac}(X)$. Given a basepoint $e \in X$, we define the *Abel-Jacobi* isomorphism $X \to \mathrm{Jac}(X)$, $p \mapsto [\mathcal{O}(p - e)]$. Given points $p, e \in X$, we define a *translation map* $\tau_{p-e} : \mathrm{Jac}(X) \to \mathrm{Jac}(X)$, $[L] \mapsto [L \otimes \mathcal{O}(p - e)]$. We say that a line bundle L is *2-torsion* if $L^2 = \mathcal{O}$. There are four 2-torsion line bundles, which we denote L_i for $i = 1, 2, 3, 4$.

3.2. **Semistable rank 2 vector bundles.**

For our purposes here, we need only consider semistable rank 2 vector bundles on an elliptic curve X with trivial determinant bundle. There are three classes of such bundles.

First, we have vector bundles of the form $E = L \oplus L^{-1}$, where L is a degree 0 line bundle such that $L^2 \neq \mathcal{O}$. There are two bad lines $L_p, (L^{-1})_p \in \mathbb{P}(E_p)$ in the fiber E_p over a point $p \in X$, and all other lines in $\mathbb{P}(E_p)$ are good. The automorphism group $\mathrm{Aut}(E)$ of E consists of $GL(2, \mathbb{C})$ matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}.$$

Each bad line $L_p, (L^{-1})_p \in \mathbb{P}(E_p)$ is fixed by the automorphisms of E, and there is a unique (up to rescaling by a constant) automorphism carrying any good line $\ell_p \in \mathbb{P}(E_p)$ to any other good line $\ell'_p \in \mathbb{P}(E_p)$.

Second, we have four vector bundles of the form $E = L_i \oplus L_i$, where L_i is a 2-torsion line bundle. All lines $\ell_p \in \mathbb{P}(E_p)$ in the fiber E_p over a point $p \in X$ are bad. The automorphism group $\mathrm{Aut}(E)$ of E is $GL(2, \mathbb{C})$, and there is a unique (up to rescaling by a constant) automorphism carrying any triple of lines $(\ell_{p_1}, \ell_{p_2}, \ell_{p_3}) \in \mathbb{P}(E_{p_1}) \times \mathbb{P}(E_{p_2}) \times \mathbb{P}(E_{p_3})$ such that no two lines are bad in the same direction to any other triple of lines $(\ell'_{p_1}, \ell'_{p_2}, \ell'_{p_3}) \in \mathbb{P}(E_{p_1}) \times \mathbb{P}(E_{p_2}) \times \mathbb{P}(E_{p_3})$ such that no two lines are bad in the same direction.

Third, we have four vector bundles of the form $E = F_2 \otimes L_i$, where L_i is a 2-torsion line bundle and F_2 is the unique non-split extension of $\mathcal{O}$ by $\mathcal{O}$:

$$0 \longrightarrow \mathcal{O} \longrightarrow F_2 \longrightarrow \mathcal{O} \longrightarrow 0.$$

There is a unique bad $(L_i)_p \in \mathbb{P}(E_p)$ in the fiber E_p over a point $p \in X$, and all other lines in $\mathbb{P}(E_p)$ are good. The automorphism group $\mathrm{Aut}(E)$ of E consists of $GL(2, \mathbb{C})$ matrices of the form

$$\begin{pmatrix} A & B \\ 0 & A \end{pmatrix}.$$

The bad line $(L_i)_p \in \mathbb{P}(E_p)$ is fixed by the automorphisms of E, and there is a unique (up to rescaling by a constant) automorphism carrying any good line $\ell_p \in \mathbb{P}(E_p)$ to any other good line $\ell'_p \in \mathbb{P}(E_p)$.

We define $M^{ss}(X)$ to be the moduli space of S-equivalence classes of semistable rank 2 vector bundles on X with trivial determinant bundle. In [14] it is shown that $M^{ss}(X)$ is isomorphic to $\mathbb{CP}^1$, as can be understood as follows. The above classification shows that we can parameterize semistable rank 2 vector bundles with trivial determinant bundle as $L \oplus L^{-1}$ for $[L] \in \mathrm{Jac}(X)$, together with the four bundles $F_2 \otimes L_i$. The bundles $L \oplus L^{-1}$ and $L^{-1} \oplus L$ are isomorphic, hence S-equivalent, and are thus identified in $M^{ss}(X)$. One can show that the bundles $F_2 \otimes L_i$ and $L_i \oplus L_i$ are S-equivalent, and are thus identified in $M^{ss}(X)$. It follows that $M^s(X)$ is the quotient of $\mathrm{Jac}(X)$ by the involution $[L] \mapsto [L^{-1}]$, which yields a space known as the *pillowcase* that is isomorphic to $\mathbb{CP}^1$. We define a map $p : \mathrm{Jac}(X) \to M^{ss}(X)$, $[L] \mapsto [L \oplus L^{-1}]$, which is a branched double-cover with four branch points $p([L_i]) \in M^{ss}(X)$ corresponding to four ramification points $[L_i] \in \mathrm{Jac}(X)$ that are fixed by the involution $[L] \mapsto [L^{-1}]$.

3.3. **Hecke modifications of rank 2 vector bundles.** Given a rank 2 vector bundle E over a curve C, distinct points $p_1, \cdots, p_n \in C$, and lines $\ell_{p_i} \in \mathbb{P}(E_{p_i})$ for each point p_i, one can perform a *Hecke modification* of E at each point p_i using data provided by the line ℓ_{p_i} so as to obtain a new vector bundle that we will denote $H(E, \ell_{p_1}, \cdots, \ell_{p_n})$. One way to describe $H(E, \ell_{p_1}, \cdots, \ell_{p_n})$ is as follows. Let $\mathcal{E}$ denote the sheaf of sections of E, and define a subsheaf $\mathcal{F}$ of $\mathcal{E}$ whose set of sections over an open subset U of X is given by

$$\mathcal{F}(U) = \{s \in \mathcal{E}(U) \mid p_i \in U \implies s(p_i) \in \ell_{p_i} \text{ for } i = 1, \cdots, n\}.$$

We define $H(E, \ell_{p_1}, \cdots, \ell_{p_n})$ to be the vector bundle whose sheaf of sections is $\mathcal{F}$.

Hecke modifications of rank 2 vector bundles on elliptic curves are described explicitly in [3]. For our purposes here, all the results we will need can be described in terms of a few properties of a certain map h_e, which we define as follows. Given a semistable rank 2 vector bundle E with trivial determinant bundle over an elliptic curve X, distinct points $p, q, e \in X$ such that $p + q = 2e$, and lines $\ell_p \in \mathbb{P}(E_p)$ and $\ell_q \in \mathbb{P}(E_q)$ that are not bad in the same direction, we define $h_e(E, \ell_p, \ell_q) \in M^{ss}(X)$ as

$$h_e(E, \ell_p, \ell_q) = [H(E, \ell_p, \ell_q) \otimes \mathcal{O}(e)].$$

One can show that if the lines ℓ_p and ℓ_q are not bad in the same direction, then $H(E, \ell_p, \ell_q) \otimes \mathcal{O}(e)$ is in fact a semistable vector bundle with trivial determinant bundle and thus represents a point in $M^{ss}(X)$. It is clear that the order of the lines doesn't matter in the definition of $H(E, \ell_p, \ell_q)$, so

$$h_e(E, \ell_p, \ell_q) = h_e(E, \ell_q, \ell_p).$$

If (E, ℓ_p, ℓ_q) and (E', ℓ'_p, ℓ'_q) are isomorphic parabolic bundles, one can show that

$$h_e(E, \ell_p, \ell_q) = h_e(E', \ell'_p, \ell'_q).$$

In particular, for $\phi \in \mathrm{Aut}(E)$ we have

$$h_e(E, \ell_p, \ell_q) = h_e(E, \phi(\ell_p), \phi(\ell_q)).$$

If the line ℓ_p is good and $E \neq L_i \oplus L_i$, then we have the following result:

Theorem 3.1 ([3, Lemma 5.26])**.** *If $E = L \oplus L^{-1}$ for $L^2 \neq \mathcal{O}$ or $E = F_2 \otimes L_i$, and $\ell_p \in \mathbb{P}(E_p)$ is a good line, then the map $\mathbb{P}(E_q) \to M^{ss}(X)$, $\ell_q \mapsto h_e(E, \ell_p, \ell_q)$ is an isomorphism.*

If the line ℓ_p is bad, then $h_e(E, \ell_p, \ell_q)$ is uniquely determined by E and the points p and e:

Theorem 3.2 ([3, Lemma 5.27])**.** *If $\ell_p \in \mathbb{P}(E_p)$ is a bad line and $\ell_q \in \mathbb{P}(E_q)$ is not bad in the same direction as ℓ_p, then $h_e(E, \ell_p, \ell_q)$ is given by*

$$h_e(L \oplus L^{-1}, L_p, \ell_q) = (p \circ \tau_{p-e})([L]), \qquad h_e(L \oplus L^{-1}, (L^{-1})_p, \ell_q) = (p \circ \tau_{e-p})([L]),$$
$$h_e(F_2 \otimes L_i, (L_i)_p, \ell_q) = (p \circ \tau_{p-e})([L_i]), \qquad h_e(L_i \oplus L_i, \ell_p, \ell_q) = (p \circ \tau_{p-e})([L_i]).$$

Note that $(p \circ \tau_{p-e})([L_i]) = (p \circ \tau_{e-p})([L_i])$. Note also that in Theorem 3.2 the line $\ell_q \in \mathbb{P}(E_q)$ is allowed to be bad, just not bad in the same direction as $\ell_p \in \mathbb{P}(E_p)$. For example, we have

$$h_e(L \oplus L^{-1}, L_p, L_q^{-1}) = (p \circ \tau_{p-e})([L]) = (p \circ \tau_{e-q})([L]).$$

4. Description of $M^s(X, 3)$

We consider here the moduli space $M^s(X, 3)$ of stable rank 2 parabolic bundles over an elliptic curve X with trivial determinant bundle and 3 marked points $p_1, p_2, p_3 \in X$. If $[E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}] \in M^s(X, 3)$, then E is semistable and no two of the lines $\ell_{p_1}, \ell_{p_2}, \ell_{p_3}$ are bad in the same direction. It follows that E has one of the three forms described in Section 3.2; that is, $E = L \oplus L^{-1}$ for $L^2 \neq \mathcal{O}$, $E = L_i \oplus L_i$, or $E = F_2 \otimes L_i$. Choose points $e_1, e_2, e_3 \in X$ such that

$$p_1 + p_2 = 2e_3, \qquad p_3 + p_1 = 2e_2, \qquad p_2 + p_3 = 2e_1.$$

Define a map $\pi = (\pi_1, \pi_2, \pi_3) : M^s(X, 3) \to (M^{ss}(X))^3$ by

$$\pi([E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}]) = ([E],\, h_{e_2}(E, \ell_{p_1}, \ell_{p_3}),\, h_{e_1}(E, \ell_{p_2}, \ell_{p_3})).$$

Note that since E is semistable and no two of the lines ℓ_{p_1}, ℓ_{p_2}, ℓ_{p_3} are bad in the same direction, we can in fact define $h_{e_2}(E, \ell_{p_1}, \ell_{p_3})$ and $h_{e_1}(E, \ell_{p_2}, \ell_{p_3})$. It is useful to decompose $M^s(X, 3)$ as

$$M^s(X, 3) = \{\ell_{p_3} \text{ good}\} \cup \{\ell_{p_3} \text{ bad}\},$$

where the open submanifold $\{\ell_{p_3} \text{ good}\}$ and the closed submanifold $\{\ell_{p_3} \text{ bad}\}$ consist of points $[E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}] \in M^s(X, 3)$ for which ℓ_{p_3} is a good and bad line, respectively. The open submanifold $\{\ell_{p_3} \text{ good}\}$ is described by the following result:

Theorem 4.1. *The restriction of* $\pi : M^s(X, 3) \to (M^{ss}(X))^3$ *to* $\{\ell_{p_3} \text{ good}\} \to \pi(\{\ell_{p_3} \text{ good}\})$ *is an isomorphism.*

Proof. If $[E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}] \in \{\ell_{p_3} \text{ good}\}$, then E must have good lines, hence $E = L \oplus L^{-1}$ for $L^2 \neq \mathcal{O}$ or $E = F_2 \otimes L_i$. For each point $[E] \in M^{ss}(X)$, choose a representative E of $[E]$ and a good line $\ell'_{p_3} \in \mathbb{P}(E_{p_3})$. We can define a map $\pi_1^{-1}([E]) \cap \{\ell_{p_3} \text{ good}\} \to \mathbb{P}(E_{p_1}) \times \mathbb{P}(E_{p_2})$,

$$[E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}] \mapsto (\phi(\ell_{p_1}), \phi(\ell_{p_2})),$$

where ϕ is the unique (up to rescaling by a constant) automorphism of E such that $\phi(\ell_{p_3}) = \ell'_{p_3}$. This map is an isomorphism onto its image, hence by Theorem 3.1 the map $(\pi_2, \pi_3) : \pi_1^{-1}([E]) \cap \{\ell_{p_3} \text{ good}\} \to (M^{ss}(X))^2$ is an isomorphism onto its image. □

Next we consider the closed submanifold $\{\ell_{p_3} \text{ bad}\}$. Elements of $\{\ell_{p_3} \text{ bad}\}$ have one of three forms:

$$[L \oplus L^{-1}, \ell_{p_1}, \ell_{p_2}, L_{p_3}], \qquad [F_2 \otimes L_i, \ell_{p_1}, \ell_{p_2}, (L_i)_{p_3}], \qquad [L_i \oplus L_i, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}],$$

where $L^2 \neq \mathcal{O}$ and L_i is a 2-torsion line bundle. Note that elements of the form $[L \oplus L^{-1}, \ell_{p_1}, \ell_{p_2}, (L^{-1})_{p_3}]$ can be converted into the first of the three listed forms by applying the isomorphism $\phi : L \oplus L^{-1} \to L^{-1} \oplus L$:

$$[L \oplus L^{-1}, \ell_{p_1}, \ell_{p_2}, (L^{-1})_{p_3}] = [L^{-1} \oplus L, \phi(\ell_{p_1}), \phi(\ell_{p_2}), (L^{-1})_{p_3}] = [M \oplus M^{-1}, \phi(\ell_{p_1}), \phi(\ell_{p_2}), M_{p_3}],$$

where we have defined $M = L^{-1}$ and used the fact that $\phi((L^{-1})_{p_3}) = (L^{-1})_{p_3}$.

Recall that we defined a map $\pi_1 : M^s(X, 3) \to M^{ss}(X)$, $[E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}] \mapsto [E]$. We can lift $\pi_1 : \{\ell_{p_3} \text{ bad}\} \to M^{ss}(X)$ to the branched double-cover $p : \mathrm{Jac}(X) \to M^{ss}(X)$ by using the bad line ℓ_{p_3} to distinguish between distinct vector bundles $L \oplus L^{-1}$ and $L^{-1} \oplus L$ that are identified in $M^{ss}(X)$:

$$\begin{array}{ccc} & & \mathrm{Jac}(X) \\ & \overset{\tilde{\pi}_1}{\nearrow} & \big\downarrow p \\ \{\ell_{p_3} \text{ bad}\} & \xrightarrow{\pi_1} & M^{ss}(X), \end{array}$$

where $\tilde{\pi}_1 : \{\ell_{p_3} \text{ bad}\} \to \mathrm{Jac}(X)$ is defined such that

$$\tilde{\pi}_1([L \oplus L^{-1}, \ell_{p_1}, \ell_{p_2}, L_{p_3}]) = [L], \qquad \tilde{\pi}_1([F_2 \otimes L_i, \ell_{p_1}, \ell_{p_2}, (L_i)_{p_3}]) = \tilde{\pi}_1([L_i \oplus L_i, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}]) = [L_i].$$

Define a map $f : \mathrm{Jac}(X) \to (M^{ss}(X))^3$, $f = (p,\, p \circ \tau_{p_3 - e_2},\, p \circ \tau_{p_3 - e_1})$.

Theorem 4.2. *We have a commutative diagram*

$$\begin{array}{ccc} \{\ell_{p_3} \text{ bad}\} & & \\ \tilde{\pi}_1 \big\downarrow & \overset{\pi}{\searrow} & \\ \mathrm{Jac}(X) & \xrightarrow{f} & (M^{ss}(X))^3. \end{array}$$

Proof. The fiber of $\tilde{\pi}_1$ over a point $[L] \in \mathrm{Jac}(X)$ such that $L^2 \neq \mathcal{O}$ is

$$\tilde{\pi}_1^{-1}([L]) = \{[L \oplus L^{-1}, \ell_{p_1}, \ell_{p_2}, L_{p_3}]\}.$$

From Theorem 3.2 it follows that $\pi(\tilde{\pi}_1^{-1}([L])) = f([L])$. The fiber of $\tilde{\pi}_1$ over the point $[L_i] \in \mathrm{Jac}(X)$ is

$$\tilde{\pi}_1^{-1}([L_i]) = \{[F_2 \otimes L_i, \ell_{p_1}, \ell_{p_2}, (L_i)_{p_3}]\} \cup \{[L_i \oplus L_i, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}]\}.$$

From Theorem 3.2 it follows that

$$\pi([F_2 \otimes L_i, \ell_{p_1}, \ell_{p_2}, (L_i)_{p_3}]) = \pi([L_i \oplus L_i, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}]) = f([L_i]).$$

Thus $\pi(\tilde{\pi}_1^{-1}([L_i])) = f([L_i])$. □

Theorem 4.3. *The map* $f : \mathrm{Jac}(X) \to (M^{ss}(X))^3$ *is injective.*

Proof. Take $[L], [M] \in \mathrm{Jac}(X)$ such that $f([L]) = f([M])$. Projecting onto the first factor of $(M^{ss}(X))^3$, we find that either $[M] = [L]$ or $[M] = [L^{-1}]$. If $[M] = [L^{-1}]$, then projecting onto the second factor of $(M^{ss}(X))^3$ gives either $[L \otimes \mathcal{O}(p_3 - e_2)] = [L^{-1} \otimes \mathcal{O}(p_3 - e_2)]$ or $[L \otimes \mathcal{O}(p_3 - e_2)] = [L \otimes \mathcal{O}(e_2 - p_3)]$. In the first case $[L] = [L^{-1}]$. The second case cannot actually occur, since otherwise $2p_3 = 2e_2 = p_3 + p_1$ and thus $p_1 = p_3$, contradiction. □

Given a point $[E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}] \in \{\ell_{p_3} \text{ bad}\}$, we have that E is semistable and ℓ_{p_1} and ℓ_{p_2} cannot be bad in the same direction, hence we can define a map $h : \{\ell_{p_3} \text{ bad}\} \to M^{ss}(X)$,

$$h([E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}]) = h_{e_3}(E, \ell_{p_1}, \ell_{p_2}).$$

Theorem 4.4. *The map* $(\tilde{\pi}_1, h) : \{\ell_{p_3} \text{ bad}\} \to \mathrm{Jac}(X) \times M^{ss}(X)$ *is an isomorphism.*

Proof. The fiber of $\tilde{\pi}_1$ over a point $[L] \in \mathrm{Jac}(X)$ such that $L^2 \neq \mathcal{O}$ is

$$\tilde{\pi}_1^{-1}([L]) = \{[L \oplus L^{-1}, \ell_{p_1}, \ell_{p_2}, L_{p_3}]\}.$$

We will argue that $\tilde{\pi}_1^{-1}([L])$ is isomorphic to $\mathbb{CP}^1$. Choose a local trivialization of $E := L \oplus L^{-1}$ over an open set containing p_1 and p_2 so as to obtain identifications $\psi_i : \mathbb{P}(E_{p_i}) \to \mathbb{CP}^1$ for $i = 1, 2$. We can choose the local trivialization such that $\psi_i(L_{p_i}) = \infty$ and $\psi_i((L^{-1})_{p_i}) = 0$. Define $z_i = \psi_i(\ell_{p_i})$ and note that $(z_1, z_2) \in \mathbb{C}^2 - \{(0, 0)\}$. An automorphism of E induces the transformation $(z_1, z_2) \mapsto a(z_1, z_2)$ for $a \in \mathbb{C}^\times$, hence we have an isomorphism

$$\tilde{\pi}_1^{-1}([L]) = \{[L \oplus L^{-1}, \ell_{p_1}, \ell_{p_2}, L_{p_3}]\} \to \mathbb{CP}^1, \qquad [L \oplus L^{-1}, \ell_{p_1}, \ell_{p_2}, L_{p_3}] \mapsto [z_1 : z_2].$$

A canonical version of this statement is that the restriction of $h : \{\ell_{p_3} \text{ bad}\} \to M^{ss}(X)$ to $\tilde{\pi}_1^{-1}([L])$ gives an isomorphism $\tilde{\pi}_1^{-1}([L]) \to M^{ss}(X)$. In particular, from Theorems 3.1 and 3.2 it follows that

$$h(\{[L \oplus L^{-1}, \ell_{p_1}, \ell_{p_2}, L_{p_3}] \mid \ell_{p_1} \neq (L^{-1})_{p_1}, \ \ell_{p_2} \neq (L^{-1})_{p_2}\}) = M^{ss}(X) - \{(p \circ \tau_{e_3 - p_1})([L]), \ (p \circ \tau_{e_3 - p_2})([L])\},$$
$$h(\{[L \oplus L^{-1}, \ell_{p_1}, \ell_{p_2}, L_{p_3}] \mid \ell_{p_1} = (L^{-1})_{p_1}, \ \ell_{p_2} \neq (L^{-1})_{p_2}\}) = \{(p \circ \tau_{e_3 - p_1})([L])\},$$
$$h(\{[L \oplus L^{-1}, \ell_{p_1}, \ell_{p_2}, L_{p_3}] \mid \ell_{p_1} \neq (L^{-1})_{p_1}, \ \ell_{p_2} = (L^{-1})_{p_2}\}) = \{(p \circ \tau_{e_3 - p_2})([L])\}.$$

The fiber of $\tilde{\pi}_1$ over the point $[L_i] \in \mathrm{Jac}(X)$ is

$$\tilde{\pi}_1^{-1}([L_i]) = \{[F_2 \otimes L_i, \ell_{p_1}, \ell_{p_2}, (L_i)_{p_3}]\} \cup \{[L_i \oplus L_i, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}]\}.$$

Note that $\{[L_i \oplus L_i, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}]\}$ consists of a single point, since there is a unique (up to rescaling by a constant) automorphism of $L_i \oplus L_i$ that induces an isomorphism of any pair of stable parabolic bundles $(L_i \oplus L_i, \ell_{p_1}, \ell_{p_2}, \ell_{p_3})$ and $(L_i \oplus L_i, \ell'_{p_1}, \ell'_{p_2}, \ell'_{p_3})$. We will argue that $\{[F_2 \otimes L_i, \ell_{p_1}, \ell_{p_2}, (L_i)_{p_3}]\}$ is isomorphic to $\mathbb{C}$. Choose a local trivialization of $E := F_2 \otimes L_i$ over an open set containing p_1 and p_2 so as to obtain identifications $\psi_i : \mathbb{P}(E_{p_i}) \to \mathbb{CP}^1$ for $i = 1, 2$. We can choose the local trivialization such that $\psi_i((L_i)_{p_i}) = \infty$. Define $z_i = \psi_i(\ell_{p_i})$ and note that $(z_1, z_2) \in \mathbb{C}^2$. An automorphism of E induces the transformation $(z_1, z_2) \mapsto (z_1 + b, z_2 + b)$ for $b \in \mathbb{C}$, hence we have an isomorphism

$$\{[F_2 \otimes L_i, \ell_{p_1}, \ell_{p_2}, L_{p_3}]\} \to \mathbb{C}, \qquad [F_2 \otimes L_i, \ell_{p_1}, \ell_{p_2}, L_{p_3}] \mapsto z_2 - z_1.$$

A canonical version of these results is that the restriction of $h : \{\ell_{p_3} \text{ bad}\} \to M^{ss}(X)$ to $\tilde{\pi}_1^{-1}([L_i])$ gives an isomorphism $\tilde{\pi}_1^{-1}([L_i]) \to M^{ss}(X)$. In particular, from Theorems 3.1 and 3.2 it follows that

$$h(\{[F_2 \otimes L_i, \ell_{p_1}, \ell_{p_2}, L_{p_3}]\}) = M^{ss}(X) - \{(p \circ \tau_{p_1 - e_3})([L_i])\},$$
$$h(\{[L_i \oplus L_i, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}]\}) = \{(p \circ \tau_{p_1 - e_3})([L_i])\}.$$

Note that $(p \circ \tau_{p_1 - e_3})([L_i]) = (p \circ \tau_{e_3 - p_1})([L_i]) = (p \circ \tau_{p_2 - e_3})([L_i])$. □

Theorems 4.1–4.4 prove Theorem 1.1 from the Introduction.

5. Relationship between $M^s(X, 3)$ and $M^{ss}(X, 2)$

In [15] it is shown that $M^{ss}(X, 2)$ is isomorphic to $(\mathbb{CP}^1)^2$. From our perspective, we can describe this result by defining a map $M^{ss}(X, 2) \to (M(X)^{ss})^2$,

$$[E, \ell_{p_1}, \ell_{p_2}] \mapsto ([E], \ h_{e_3}(E, \ell_{p_1}, \ell_{p_2})).$$

One can show that this map is an isomorphism.

We can relate the closed subset $\{\ell_{p_3} \text{ bad}\}$ of $M^s(X, 3)$ to the moduli space $M^{ss}(X, 2)$ as follows. Define a map $\{\ell_{p_3} \text{ bad}\} \to M^{ss}(X, 2)$,

$$[E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}] \mapsto [E, \ell_{p_1}, \ell_{p_2}].$$

We have a commutative diagram

$$\begin{array}{ccc} \{\ell_{p_3} \text{ bad}\} & \longrightarrow & M^{ss}(X,2) \\ {\scriptstyle \tilde{\pi}_1}\downarrow & & \downarrow \\ \mathrm{Jac}(X) & \xrightarrow{p} & M^{ss}(X), \end{array}$$

where we have defined a map $M^{ss}(X,2) \to M^{ss}(X)$, $[E, \ell_{p_1}, \ell_{p_2}] \mapsto [E]$.

6. Poincaré polynomial of $M^s(X,3)$

The Poincaré polynomial of $M^s(C,n)$ is given in [11, Theorem 3.8] for the case $\mu = 1/4$, corresponding to the traceless character variety, and n odd:

$$P_t(M^s(C,n)) = \frac{(1+t^2)^n(1+t^3)^{2g} - 2^{n-1}t^{2g+n-1}(1+t)^{2g}(1+t^2)}{(1-t^2)(1-t^4)}, \tag{1}$$

where g is the genus of C. In fact, the results of [11] are stated for parabolic bundles with fixed determinant bundle of odd degree, but since $\mu = 1/4$, corresponding to a traceless character variety, the results also hold for the moduli space $M^s(C,n)$ for which the determinant bundle of the parabolic bundles is trivial. For an elliptic curve X with 3 marked points, equation (1) gives

$$P_t(M^s(X,3)) = 1 + 4t^2 + 2t^3 + 4t^4 + t^6. \tag{2}$$

We can reproduce equation (2) using our explicit description of $M^s(X,3)$. Since $\pi : M^s(X,3) \to (M^{ss}(X))^3$ restricts to an isomorphism $M^s(X,3) - \{\ell_{p_3} \text{ bad}\} \to (M^{ss}(X))^3 - \pi(\{\ell_{p_3} \text{ bad}\})$, we obtain the following equation for the Poincaré polynomials for cohomology with compact supports:

$$P_t(M^s(X,3) - \{\ell_{p_3} \text{ bad}\}) = P_t((M^{ss}(X))^3 - \pi(\{\ell_{p_3} \text{ bad}\})). \tag{3}$$

From the long exact sequence for cohomology with compact supports, we have

$$P_t(M^s(X,3) - \{\ell_{p_3} \text{ bad}\}) = P_t(M^s(X,3)) - P_t(\{\ell_{p_3} \text{ bad}\}), \tag{4}$$

$$P_t((M^{ss}(X))^3 - \pi(\{\ell_{p_3} \text{ bad}\})) = P_t((M^{ss}(X))^3) - P_t(\pi(\{\ell_{p_3} \text{ bad}\})). \tag{5}$$

We have that $M^{ss}(X)$ is isomorphic to $\mathbb{CP}^1$, $\pi(\{\ell_{p_3} \text{ bad}\})$ isomorphic to $\mathrm{Jac}(X)$, and $\{\ell_{p_3} \text{ bad}\}$ is isomorphic to $\mathrm{Jac}(X) \times M^{ss}(X)$, so

$$P_t(M^{ss}(X)^3) = (1+t^2)^3, \qquad P_t(\pi(\{\ell_{p_3} \text{ bad}\})) = 1 + 2t + t^2, \qquad P_t(\{\ell_{p_3} \text{ bad}\}) = (1+2t+t^2)(1+t^2). \tag{6}$$

Combining equations (3)–(6), we reproduce equation (2) for $P_t(M^s(X,3))$.

References

[1] M. F. Atiyah. Vector bundles over an elliptic curve. *Proc. London Math. Soc.*, 7:414–452, 1957.

[2] H. U. Boden and Y. Hu. Variations of moduli of parabolic bundles. *Math. Ann.*, 301:539–559, 1995.

[3] D. Boozer. Hecke modifications for rational and elliptic curves. *Algeb. Geom. Topol., to appear; arXiv preprint arXiv:1805.11184v2*, 2018.

[4] D. Boozer. Holonomy perturbations of the Chern–Simons functional for lens spaces. *arXiv preprint arXiv:1811.01536*, 2018.

[5] M. Hedden, C. Herald, and P. Kirk. The pillowcase and perturbations of traceless representations of knot groups. *Geom. Topol.*, 18(1):211–287, 2014.

[6] M. Hedden, C. Herald, and P. Kirk. The pillowcase and traceless representations of knot groups II: a Lagrangian-Floer theory in the pillowcase. *J. Symplect. Geom.*, 18(3):721–815, 2018.

[7] H. T. Horton. A symplectic instanton homology via traceless character varieties. *arXiv preprint arXiv:1611.09927v2*, 2016.

[8] O. Iena. Vector bundles on elliptic curves and factors of automorphy. *Rend. Istit. Mat. Univ. Trieste*, 43:61–94, 2011.

[9] P. Kirk. On the traceless $SU(2)$ character variety of the 6-punctured 2-sphere. *J. Knot Theory Ram.*, 26(2):1740009, 2017.

[10] V. B. Mehta and C. S. Seshadri. Moduli of vector bundles on curves with parabolic structures. *Math. Ann.*, 248:205–239, 1980.

[11] E. J. Street. Recursive relations in the cohomology ring of moduli spaces of rank 2 parabolic bundles on the Riemann sphere. *arXiv preprint arXiv:1205.1730*, 2012.

[12] M. Teixidor i Bigas. Vector bundles on curves. `http://emerald.tufts.edu/~mteixido/files/vectbund.pdf`. Accessed 2018-05-23.

[13] M. Thaddeus. Geometric invariant theory and flips. *J. Amer. Math. Soc.*, 9:691–723, 1996.

[14] L. W. Tu. Semistable bundles over an elliptic curve. *Adv. Math.*, 98:1–26, 1993.

[15] N. F. Vargas. Geometry of the moduli of parabolic bundles on elliptic curves. *Trans. Am. Math., to appear; arXiv preprint arXiv:1611.05417*, 2016.