THE MODULI SPACE OF STABLE RANK 2 PARABOLIC BUNDLES OVER AN ELLIPTIC CURVE WITH 3 MARKED POINTS

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ABSTRACT. We explicitly describe the moduli space $M^s(X, 3)$ of stable rank 2 parabolic bundles over an elliptic curve $X$ with trivial determinant bundle and 3 marked points. Specifically, we exhibit $M^s(X, 3)$ as a blow-up of an embedded elliptic curve in $(\mathbb{CP}^1)^3$. The moduli space $M^s(X, 3)$ can also be interpreted as the $SU(2)$ character variety of the 3-punctured torus. Our description of $M^s(X, 3)$ reproduces the known Poincaré polynomial for this space.

1. Introduction

Given a curve $C$, one can define a moduli space $M^s(C, n)$ of stable rank 2 parabolic bundles over $C$ with trivial determinant bundle and $n$ marked points. The space $M^s(C, n)$ has the structure of a smooth complex manifold of dimension $3(g-1)+n$, where $g$ is the genus of the curve $C$. In general, the space $M^s(C, n)$ depends on a positive real parameter $\mu$ known as the weight. For $\mu$ sufficiently small ($\mu < 1/n$ will suffice), the space $M^s(C, n)$ is independent of $\mu$, but as $\mu$ increases it may cross critical values at which $M^s(C, n)$ undergoes certain birational transformations [2, 13]. The moduli space $M^s(C, n)$ can also be interpreted as an $SU(2)$-character variety, which is defined as the space of conjugacy classes of $SU(2)$-representations of the fundamental group of $C$ with $n$ punctures, where loops around the punctures are required to correspond to $SU(2)$-matrices conjugate to $\text{diag}(e^{2\pi i \mu}, e^{-2\pi i \mu})$. Moduli spaces of parabolic bundles on curves are natural objects of study in algebraic geometry, and also play an important role in low-dimensional topology. In particular, these spaces have a canonical symplectic structure and can be used to define Floer homology theories of links [4, 5, 6, 7].

Explicit descriptions of $M^s(C, n)$ are known for small values of $n$ and $C$ a rational or elliptic curve. For rational curves, it is well-known that for small weight we have

$$M^s(\mathbb{CP}^1, 0) = M^s(\mathbb{CP}^1, 1) = M^s(\mathbb{CP}^1, 2) = \emptyset,$$

$$M^s(\mathbb{CP}^1, 3) = \{pt\},$$

$$M^s(\mathbb{CP}^1, 4) = \mathbb{CP}^1 - \{3 \text{ points}\},$$

$$M^s(\mathbb{CP}^1, 5) = \mathbb{CP}^2 \# 4\mathbb{CP}^2.$$  

The structure of $M^s(\mathbb{CP}^1, 6)$ for $\mu = 1/4$, corresponding to the traceless character variety, was recently described by Kirk [9]. For an elliptic curve $X$, it is straightforward to show that for small weight we have

$$M^s(X, 0) = \emptyset, \quad M^s(X, 1) = \mathbb{CP}^1.$$  

It was recently shown by Vargas [15] that $M^s(X, 2)$ is the complement of an embedded elliptic curve in $(\mathbb{CP}^1)^2$.

Our goal in this paper is to explicitly describe the structure of $M^s(X, 3)$. We prove the following result:

**Theorem 1.1.** For small weight, the moduli space $M^s(X, 3)$ for an elliptic curve $X$ is a blow-up of an embedded elliptic curve in $(\mathbb{CP}^1)^3$.

To prove Theorem 1.1, we make use of an explicit description of Hecke modifications of rank 2 holomorphic vector bundles on elliptic curves that is derived in [3]. Roughly speaking, a Hecke modification is a way of locally modifying a vector bundle near a point to obtain a new vector bundle. The moduli space $M^s(X, 3)$ plays an important role in a conjectural Floer homology theory for links in lens spaces discussed in [3], and Theorem 1.1 was motivated by this application.

The paper is organized as follows. In Sections 2 and 3, we review the background material we will need on parabolic bundles and vector bundles on elliptic curves. In Section 4, we use Hecke-modification methods to explicitly describe $M^s(X, 3)$. In Section 5, we relate $M^s(X, 3)$ to $M^s(\mu, X, 2)$, the moduli space of $S$-equivalence classes of rank 2 semistable parabolic bundles with trivial determinant bundle and 2 marked points. In Section 6, we use our description of $M^s(X, 3)$ to reproduce the known Poincaré polynomial for this space.
2. PARABOLIC BUNDLES

The concept of a parabolic bundle was introduced in [10]. We will not need this concept in its full generality; rather, we will consider only parabolic bundles of a certain restricted form, which is discussed at greater length in [3, Appendix B]. For our purposes here, a rank 2 parabolic bundle over a curve $C$ consists of a rank 2 holomorphic vector bundle $E$ over $C$ with trivial determinant bundle, distinct marked points $p_1, \cdots, p_n \in C$, a line $\ell_{p_i} \in \mathbb{P}(E_{p_i})$ in the fiber $E_{p_i}$ over each marked point $p_i$, and a positive real parameter $\mu$ known as the "weight." For simplicity, we will suppress the curve $C$ and weight $\mu$ in the notation and denote a parabolic bundle as $(E, \ell_{p_1}, \cdots, \ell_{p_n})$.

In order to describe the stability properties of parabolic bundles, it is helpful to introduce some additional terminology. Recall that the degree of the proper subbundles of a vector bundle $E$ on a curve $C$ is bounded above. Given a rank 2 holomorphic vector bundle $E$, we say that a line $\ell_p \in \mathbb{P}(E_p)$ is bad if there is a line subbundle $L$ of $E$ of maximal degree such that $\ell_p = L_p$, and good otherwise. We say that lines $\ell_{p_1}, \ell_{p_2}, \ell_{p_3} \in \mathbb{P}(E_{p_1}), \mathbb{P}(E_{p_2})$ are bad in the same direction if there is a line subbundle $L$ of $E$ of maximal degree such that $\ell_{p_i} = L_p$ for $i = 1, \ldots, n$.

Consider a parabolic bundle $E = (E, \ell_{p_1}, \cdots, \ell_{p_n})$. Let $m$ denote the maximum number of lines of $E$ that are bad in the same direction. For sufficiently small weight ($\mu < 1/n$ will suffice), we can characterize the stability and semistability of $E$ as follows. If $E$ is unstable, then $E$ is unstable. If $E$ is semistable, then $E$ is stable if $m < n/2$, semistable if $m \leq n/2$, and unstable if $m > n/2$. Note that if $n$ is odd then stability and semistability are equivalent.

We define a moduli space $M^s(C,n)$ of isomorphism classes of stable parabolic bundles with $n$ marked points. As with vector bundles, one can define a notion of $S$-equivalent semistable parabolic bundles, and we define a moduli space $M^{ss}(C,n)$ of $S$-equivalence classes of semistable parabolic bundles. For odd $n$ we have that $M^s(C,n) = M^{ss}(C,n)$. For even $n$ we have that $M^s(C,n)$ is an open subset of $M^{ss}(C,n)$.

3. VECTOR BUNDLES ON ELLIPTIC CURVES

Vector bundles on elliptic curves were classified by Atiyah [1], and are well understood. Here we briefly summarize the results regarding vector bundles on elliptic curves that we will need; these results are either well-known (see for example [1, 8, 12, 14]) or derived in [3, Section 5].

3.1. Line bundles. Isomorphism classes of degree 0 line bundles on an elliptic curve $X$ are parameterized by the Jacobian $\text{Jac}(X)$. Given a basepoint $e \in X$, we define the Abel-Jacobi isomorphism $X \to \text{Jac}(X)$, $p \mapsto [\mathcal{O}(p-e)]$. Given points $p, e \in X$, we define a translation map $\tau_p \colon \text{Jac}(X) \to \text{Jac}(X)$, $[L] \mapsto [L \otimes \mathcal{O}(p-e)]$. We say that a line bundle $L$ is 2-torsion if $L^2 = \mathcal{O}$. There are four 2-torsion line bundles, which we denote $L_i$ for $i = 1, 2, 3, 4$.

3.2. Semistable rank 2 vector bundles. For our purposes here, we need only consider semistable rank 2 vector bundles on an elliptic curve $X$ with trivial determinant bundle. There are three classes of such bundles.

First, we have vector bundles of the form $E = L \oplus L^{-1}$, where $L$ is a degree 0 line bundle such that $L^2 \neq \mathcal{O}$. There are two bad lines $L_p, (L^{-1})_p \in \mathbb{P}(E_p)$ in the fiber $E_p$ over a point $p \in X$, and all other lines in $\mathbb{P}(E_p)$ are good. The automorphism group $\text{Aut}(E)$ of $E$ consists of $GL(2, \mathbb{C})$ matrices of the form

$$
\begin{pmatrix}
A & 0 \\
0 & D
\end{pmatrix}.
$$

Each bad line $L_p, (L^{-1})_p \in \mathbb{P}(E_p)$ is fixed by the automorphisms of $E$, and there is a unique (up to rescaling by a constant) automorphism carrying any good line $\ell_p \in \mathbb{P}(E_p)$ to any other good line $\ell'_p \in \mathbb{P}(E_p)$.

Second, we have four vector bundles of the form $E = L_i \oplus L_i$, where $L_i$ is a 2-torsion line bundle. All lines $\ell_p \in \mathbb{P}(E_p)$ in the fiber $E_p$ over a point $p \in X$ are bad. The automorphism group $\text{Aut}(E)$ of $E$ is $GL(2, \mathbb{C})$, and there is a unique (up to rescaling by a constant) automorphism carrying any triple of lines $(\ell_{p_1}, \ell_{p_2}, \ell_{p_3}) \in \mathbb{P}(E_{p_1}) \times \mathbb{P}(E_{p_2}) \times \mathbb{P}(E_{p_3})$ such that no two lines are bad in the same direction to any other triple of lines $(\ell'_{p_1}, \ell'_{p_2}, \ell'_{p_3}) \in \mathbb{P}(E_{p_1}) \times \mathbb{P}(E_{p_2}) \times \mathbb{P}(E_{p_3})$ such that no two lines are bad in the same direction.

Third, we have four vector bundles of the form $E = F_2 \otimes L_i$, where $L_i$ is a 2-torsion line bundle and $F_2$ is the unique non-split extension of $\mathcal{O}$ by $\mathcal{O}$:

$$
0 \longrightarrow \mathcal{O} \longrightarrow F_2 \longrightarrow \mathcal{O} \longrightarrow 0.
$$

There is a unique bad $(L_i)_p \in \mathbb{P}(E_p)$ in the fiber $E_p$ over a point $p \in X$, and all other lines in $\mathbb{P}(E_p)$ are good. The automorphism group $\text{Aut}(E)$ of $E$ consists of $GL(2, \mathbb{C})$ matrices of the form

$$
\begin{pmatrix}
A & B \\
0 & A
\end{pmatrix}.
$$

The bad line $(L_i)_p \in \mathbb{P}(E_p)$ is fixed by the automorphisms of $E$, and there is a unique (up to rescaling by a constant) automorphism carrying any good line $\ell_p \in \mathbb{P}(E_p)$ to any other good line $\ell'_p \in \mathbb{P}(E_p)$.
We define $M^{ss}(X)$ to be the moduli space of $S$-equivalence classes of semistable rank 2 vector bundles on $X$ with trivial determinant bundle. In [14] it is shown that $M^{ss}(X)$ is isomorphic to $\mathbb{CP}^1$, as can be understood as follows. The above classification shows that we can parameterize semistable rank 2 vector bundles with trivial determinant bundle as $L \oplus L^{-1}$ for $[L] \in \text{Jac}(X)$, together with the four bundles $F_2 \otimes L_i$. The bundles $L \oplus L^{-1}$ and $L^{-1} \oplus L$ are isomorphic, hence $S$-equivalent, and are thus identified in $M^{ss}(X)$. One can show that the bundles $F_2 \otimes L_i$ and $L_i \oplus L_i$ are $S$-equivalent, and are thus identified in $M^{ss}(X)$. It follows that $M^{s}(X)$ is the quotient of $\text{Jac}(X)$ by the involution $[L] \mapsto [L^{-1}]$, which yields a space known as the pillowcase, that is isomorphic to $\mathbb{CP}^1$. We define a map $p : \text{Jac}(X) \to M^{ss}(X)$, $[L] \mapsto [L \oplus L^{-1}]$, which is a branched double-cover with four branch points $p([L_i]) \in M^{ss}(X)$ corresponding to four ramification points $[L_i] \in \text{Jac}(X)$ that are fixed by the involution $[L] \mapsto [L^{-1}]$.

### 3.3. Hecke modifications of rank 2 vector bundles

Given a rank 2 vector bundle $E$ over a curve $C$, distinct points $p_1, \ldots, p_n \in C$, and lines $\ell_i \in \mathbb{P}(E_{p_i})$ for each point $p_i$, one can perform a Hecke modification of $E$ at each point $p_i$ using data provided by the line $\ell_i$, so as to obtain a new vector bundle that we will denote $H(E, \ell_{p_1}, \ldots, \ell_{p_n})$. One way to describe $H(E, \ell_{p_1}, \ldots, \ell_{p_n})$ is as follows. Let $\mathcal{E}$ denote the sheaf of sections of $E$, and define a subsheaf $\mathcal{F}$ of $\mathcal{E}$ whose set of sections over an open subset $U$ of $X$ is given by

$$\mathcal{F}(U) = \{ s \in \mathcal{E}(U) \mid p_i \in U \implies s(p_i) \in \ell_i, \text{ for } i = 1, \ldots, n \}.$$ 

We define $H(E, \ell_{p_1}, \ldots, \ell_{p_n})$ to be the vector bundle whose sheaf of sections is $\mathcal{F}$.

Hecke modifications of rank 2 vector bundles on elliptic curves are described explicitly in [3]. For our purposes here, all the results we will need can be described in terms of a few properties of a certain map $h_e$, which we define as follows. Given a semistable rank 2 vector bundle $E$ with trivial determinant bundle over an elliptic curve $X$, distinct points $p, q, e \in X$ such that $p + q = 2e$, and lines $\ell_p \in \mathbb{P}(E_p)$ and $\ell_q \in \mathbb{P}(E_q)$ that are not bad in the same direction, we define $h_e(E, \ell_p, \ell_q) \in M^{ss}(X)$ as

$$h_e(E, \ell_p, \ell_q) = [H(E, \ell_p, \ell_q) \otimes \mathcal{O}(e)].$$

One can show that if the lines $\ell_p$ and $\ell_q$ are not bad in the same direction, then $H(E, \ell_p, \ell_q) \otimes \mathcal{O}(e)$ is in fact a semistable vector bundle with trivial determinant bundle and thus represents a point in $M^{ss}(X)$. It is clear that the order of the lines doesn’t matter in the definition of $H(E, \ell_p, \ell_q)$, so

$$h_e(E, \ell_p, \ell_q) = h_e(E, \ell_q, \ell_p).$$

If $(E, \ell_p, \ell_q)$ and $(E', \ell'_p, \ell'_q)$ are isomorphic parabolic bundles, one can show that

$$h_e(E, \ell_p, \ell_q) = h_e(E', \ell'_p, \ell'_q).$$

In particular, for $\phi \in \text{Aut}(E)$ we have

$$h_e(E, \ell_p, \ell_q) = h_e(E, \phi(\ell_p), \phi(\ell_q)).$$

If the line $\ell_p$ is good and $E \neq L_i \oplus L_i$, then we have the following result:

**Theorem 3.1** ([3, Lemma 5.26]). If $E = L \oplus L^{-1}$ for $L^2 \neq \mathcal{O}$ or $E = F_2 \otimes L_i$, and $\ell_p \in \mathbb{P}(E_p)$ is a good line, then the map $\mathbb{P}(E_q) \to M^{ss}(X), \ell_q \mapsto h_e(E, \ell_p, \ell_q)$ is an isomorphism.

If the line $\ell_p$ is bad, then $h_e(E, \ell_p, \ell_q)$ is uniquely determined by $E$ and the points $p$ and $e$:

**Theorem 3.2** ([3, Lemma 5.27]). If $\ell_p \in \mathbb{P}(E_p)$ is a bad line and $\ell_q \in \mathbb{P}(E_q)$ is not bad in the same direction as $\ell_p$, then $h_e(E, \ell_p, \ell_q)$ is given by

$$h_e(L \oplus L^{-1}, \ell_p, \ell_q) = (p \circ \tau_{p-e})([L]),$$

$$h_e(F_2 \otimes L_i, \ell_p, \ell_q) = (p \circ \tau_{p-e})([L_i]),$$

$$h_e(L_i \oplus L_i, \ell_p, \ell_q) = (p \circ \tau_{p-e})([L_i]).$$

Note that $(p \circ \tau_{p-e})([L_i]) = (p \circ \tau_{e-p})([L_i])$. Note also that in Theorem 3.2 the line $\ell_q \in \mathbb{P}(E_q)$ is allowed to be bad, just not bad in the same direction as $\ell_p \in \mathbb{P}(E_p)$. For example, we have

$$h_e(L \oplus L^{-1}, \ell_p, \ell_q) = (p \circ \tau_{p-e})([L]) = (p \circ \tau_{e-q})([L]).$$

### 4. Description of $M^s(X, 3)$

We consider here the moduli space $M^s(X, 3)$ of stable rank 2 parabolic bundles over an elliptic curve $X$ with trivial determinant bundle and 3 marked points $p_1, p_2, p_3 \in X$. If $[E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}] \in M^s(X, 3)$, then $E$ is semistable and no two of the lines $\ell_{p_1}, \ell_{p_2}, \ell_{p_3}$ are bad in the same direction. It follows that $E$ has one of the three forms described in Section 3.2; that is, $E = L \oplus L^{-1}$ for $L^2 \neq \mathcal{O}$, $E = L_i \oplus L_i$, or $E = F_2 \otimes L_i$. Choose points $e_1, e_2, e_3 \in X$ such that

$$p_1 + p_2 = 2e_3,$$

$$p_3 + p_1 = 2e_2,$$

$$p_2 + p_3 = 2e_1.$$
Define a map $\pi = (\pi_1, \pi_2, \pi_3) : M^s(X, 3) \to (M^{ss}(X))^3$ by

$$\pi([E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}]) = ([E], h_{e_2}(E, \ell_{p_1}, \ell_{p_3}), h_{e_3}(E, \ell_{p_2}, \ell_{p_3})).$$

Note that since $E$ is semistable and no two of the lines $\ell_{p_1}, \ell_{p_2}, \ell_{p_3}$ are bad in the same direction, we can in fact define $h_{e_2}(E, \ell_{p_1}, \ell_{p_3})$ and $h_{e_3}(E, \ell_{p_2}, \ell_{p_3})$. It is useful to decompose $M^s(X, 3)$ as

$$M^s(X, 3) = \{\ell_{p_3} \text{ good}\} \cup \{\ell_{p_3} \text{ bad}\},$$

where the open submanifold $\{\ell_{p_3} \text{ good}\}$ and the closed submanifold $\{\ell_{p_3} \text{ bad}\}$ consist of points $[E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}] \in M^s(X, 3)$ for which $\ell_{p_3}$ is a good and bad line, respectively. The open submanifold $\{\ell_{p_3} \text{ good}\}$ is described by the following result:

**Theorem 4.1.** The restriction of $\pi : M^s(X, 3) \to (M^{ss}(X))^3$ to $\{\ell_{p_3} \text{ good}\} \to \pi(\{\ell_{p_3} \text{ good}\})$ is an isomorphism.

**Proof.** If $[E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}] \in \{\ell_{p_3} \text{ good}\}$, then $E$ must have good lines, hence $E = L \oplus L^{-1}$ for $L^2 \neq \mathcal{O}$ or $E = F_2 \otimes L$. For each point $[E] \in M^{ss}(X)$, choose a representative $E$ of $[E]$ and a good line $\ell'_{p_3} \in \mathbb{P}(E_{p_3})$. We can define a map $\pi^{-1}_1([E]) \cap \{\ell_{p_3} \text{ good}\} \to \mathbb{P}(E_{p_3}) \times \mathbb{P}(E_{p_2}),$

$$[E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}] \mapsto (\phi(\ell_{p_1}), \phi(\ell_{p_2})), $$

where the unique (up to rescaling by a constant) automorphism of $E$ such that $\phi(\ell_{p_3}) = \ell'_{p_3}$. This map is an isomorphism onto its image, hence by Theorem 3.1 the map $(\pi_2, \pi_3) : \pi^{-1}_1([E]) \cap \{\ell_{p_3} \text{ good}\} \to (M^{ss}(X))^2$ is an isomorphism onto its image.

Next we consider the closed submanifold $\{\ell_{p_3} \text{ bad}\}$. Elements of $\{\ell_{p_3} \text{ bad}\}$ have one of three forms:

$$[L \oplus L^{-1}, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}],$$

$$[F_2 \otimes L, \ell_{p_1}, \ell_{p_2}, (L_i)_{p_3}],$$

$$[L_i \oplus L_i, \ell_{p_1}, \ell_{p_2}, (L_i)_{p_3}],$$

where $L^2 \neq \mathcal{O}$ and $L_i$ is a 2-torsion line bundle. Note that elements of the form $[L \oplus L^{-1}, \ell_{p_1}, \ell_{p_2}, (L_i)_{p_3}]$ can be converted into the first of the three listed forms by applying the isomorphism $\phi : L \oplus L^{-1} \to L^{-1} \oplus L$:

$$[L \oplus L^{-1}, \ell_{p_1}, \ell_{p_2}, (L_i)_{p_3}] = [L^{-1} \oplus L, \phi(\ell_{p_1}), \phi(\ell_{p_2}), (L_i)_{p_3}] = [M \oplus M^{-1}, \phi(\ell_{p_1}), \phi(\ell_{p_2}), M_{p_3}],$$

where we have defined $M = L^{-1}$ and used the fact that $\phi((L_i)_{p_3}) = (L_i)_{p_3}$.

Recall that we defined a map $\pi_1 : M^s(X, 3) \to M^{ss}(X)$, $[E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}] \to [E]$. We can lift $\pi_1 : \{\ell_{p_3} \text{ bad}\} \to M^{ss}(X)$ to the branched double-cover $p : \text{Jac}(X) \to M^{ss}(X)$ by using the bad line $\ell_{p_3}$ to distinguish between distinct vector bundles $L \oplus L^{-1}$ and $L^{-1} \oplus L$ that are identified in $M^{ss}(X)$:

$$\begin{array}{ccc}
\text{Jac}(X) & \xrightarrow{\widetilde{\pi}_1} & \{\ell_{p_3} \text{ bad}\} \\
\downarrow & & \downarrow \pi_1 \\
M^{ss}(X) & \xrightarrow{\pi} & (M^{ss}(X))^3
\end{array}$$

where $\pi_1 : \{\ell_{p_3} \text{ bad}\} \to \text{Jac}(X)$ is defined such that

$$\widetilde{\pi}_1([L \oplus L^{-1}, \ell_{p_1}, \ell_{p_2}, L_{p_3}]) = [L],$$

$$\widetilde{\pi}_1([F_2 \otimes L, \ell_{p_1}, \ell_{p_2}, (L_i)_{p_3}]) = \pi_1([L_i \oplus L_i, \ell_{p_1}, \ell_{p_2}, (L_i)_{p_3}]) = [L_i].$$

Define a map $f : \text{Jac}(X) \to (M^{ss}(X))^3$, $f = (p, p \circ \tau_{p_3-e_2}, p \circ \tau_{p_3-e_1})$.

**Theorem 4.2.** We have a commutative diagram

$$\begin{array}{ccc}
\{\ell_{p_3} \text{ bad}\} & \xrightarrow{\pi_1} & \text{Jac}(X) \\
\downarrow & & \downarrow f \\
& & (M^{ss}(X))^3
\end{array}$$

**Proof.** The fiber of $\pi_1$ over a point $[L] \in \text{Jac}(X)$ such that $L^2 \neq \mathcal{O}$ is

$$\pi_1^{-1}([L]) = \{[L \oplus L^{-1}, \ell_{p_1}, \ell_{p_2}, L_{p_3}]\}.$$ 

From Theorem 3.2 it follows that $\pi(\pi_1^{-1}([L])) = f([L])$. The fiber of $\pi_1$ over the point $[L_i] \in \text{Jac}(X)$ is

$$\pi_1^{-1}([L_i]) = \{[F_2 \otimes L, \ell_{p_1}, \ell_{p_2}, (L_i)_{p_3}] \cup \{[L_i \oplus L_i, \ell_{p_1}, \ell_{p_2}, (L_i)_{p_3}]\}.$$

From Theorem 3.2 it follows that

$$\pi([F_2 \otimes L, \ell_{p_1}, \ell_{p_2}, (L_i)_{p_3}]) = \pi([L_i \oplus L_i, \ell_{p_1}, \ell_{p_2}, (L_i)_{p_3}]) = f([L_i]).$$

Thus $\pi(\pi_1^{-1}([L_i])) = f([L_i])$. 

\[ \square \]
Theorem 4.3. The map \( f : \text{Jac}(X) \to (M^{ss}(X))^3 \) is injective.

Proof. Take \([L], [M] \in \text{Jac}(X)\) such that \( f([L]) = f([M])\). Projecting onto the first factor of \((M^{ss}(X))^3\), we find that either \([M] = [L]\) or \([M] = [L^{-1}]\). If \([M] = [L^{-1}]\), then projecting onto the second factor of \((M^{ss}(X))^3\) gives either \([L \otimes \mathcal{O}(p_3 - e_2)] = [L^{-1} \otimes \mathcal{O}(p_3 - e_2)]\) or \([L \otimes \mathcal{O}(p_3 - e_2)] = [L \otimes \mathcal{O}(e_2 - p_3)]\). In the first case \([L] = [L^{-1}]\). The second case cannot actually occur, since otherwise \(2p_3 = 2e_2 = p_3 + p_1\) and thus \(p_1 = p_3\), contradiction. \(\square\)

Given a point \([E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}] \in \{\ell_{p_3} \text{ bad}\}\), we have that \(E\) is semistable and \(\ell_{p_1}\) and \(\ell_{p_2}\) cannot be bad in the same direction, hence we can define a map \(h : \{\ell_{p_3} \text{ bad}\} \to (M^{ss}(X))\),

\[
h([E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}]) = h_{e_3}(E, \ell_{p_1}, \ell_{p_2}).
\]

Theorem 4.4. The map \((\tilde{\pi}_1, h) : \{\ell_{p_3} \text{ bad}\} \to \text{Jac}(X) \times M^{ss}(X)\) is an isomorphism.

Proof. The fiber of \(\tilde{\pi}_1\) over a point \([L] \in \text{Jac}(X)\) such that \(L^{2} \neq \mathcal{O}\) is

\[
\tilde{\pi}_1^{-1}([L]) = \{([L] \oplus [L^{-1}], \ell_{p_1}, \ell_{p_2}, L_{p_3}]\} \to \mathbb{CP}^3.
\]

A canonical version of this statement is that the restriction of \(h : \{\ell_{p_3} \text{ bad}\} \to M^{ss}(X)\) to \(\tilde{\pi}_1^{-1}([L])\) gives an isomorphism \(\tilde{\pi}_1^{-1}([L]) \to M^{ss}(X)\). In particular, from Theorems 3.1 and 3.2 it follows that

\[
h([L] \oplus [L^{-1}], \ell_{p_1}, \ell_{p_2}, L_{p_3}] | \ell_{p_1} \neq (L^{-1})_{p_1}, \ell_{p_2} \neq (L^{-1})_{p_2}) = M^{ss}(X) - \{(p \circ \tau_{e_3 - p_1})([L]), (p \circ \tau_{e_3 - p_2})([L])\},
\]

\[
h([L] \oplus [L^{-1}], \ell_{p_1}, \ell_{p_2}, L_{p_3}] | \ell_{p_1} = (L^{-1})_{p_1}, \ell_{p_2} = (L^{-1})_{p_2}) = \{(p \circ \tau_{e_3 - p_1})([L]), (p \circ \tau_{e_3 - p_2})([L])\},
\]

\[
h([L] \oplus [L^{-1}], \ell_{p_1}, \ell_{p_2}, L_{p_3}] | \ell_{p_1} \neq (L^{-1})_{p_1}, \ell_{p_2} = (L^{-1})_{p_2}) = \{(p \circ \tau_{e_3 - p_1})([L]), (p \circ \tau_{e_3 - p_2})([L])\}.
\]

The fiber of \(\tilde{\pi}_1\) over the point \([L_i] \in \text{Jac}(X)\) is

\[
\tilde{\pi}_1^{-1}([L_i]) = \{[F_2 \oplus L_i, \ell_{p_1}, \ell_{p_2}, (L_{i}p_3)]\} \cup \{[L_i] \oplus [L_i], \ell_{p_1}, \ell_{p_2}, L_{p_3}]\}.
\]

Note that \(\{[L_i] \oplus [L_i], \ell_{p_1}, \ell_{p_2}, L_{p_3}\}\) consists of a single point, since there is a unique (up to rescaling by a constant) automorphism of \(L_i \oplus L_i\) that induces an isomorphism of any pair of stable parabolic bundles \((L_i \oplus L_i, \ell_{p_1}, \ell_{p_2}, L_{p_3})\) and \((L_i \oplus L_i, \ell'_{p_1}, \ell'_{p_2}, L_{p_3})\). We will argue that \(\{[F_2 \oplus L_i, \ell_{p_1}, \ell_{p_2}, (L_{i}p_3)]\}\) is isomorphic to \(\mathbb{C}\). Choose a local trivialization of \(E := F_2 \oplus L_i\) over an open set containing \(p_1\) and \(p_2\) so as to obtain identifications \(\psi_i : \mathbb{P}(E_{p_i}) \to \mathbb{CP}^3\) for \(i = 1, 2\). We can choose the local trivialization such that \(\psi_i(L_{p_i}) = E\) and \(\psi_i((L^{-1})_{p_i}) = 0\). Define \(z_i = \psi_i(\ell_{p_i})\) and note that \((z_1, z_2) \in \mathbb{C}^2 - \{(0, 0)\}\).

An automorphism of \(E\) induces the transformation \((z_1, z_2) \to a(z_1, z_2)\) for \(a \in \mathbb{C}^*\), hence we have an isomorphism

\[
\tilde{\pi}_1^{-1}([L_i]) = \{[F_2 \oplus L_i, \ell_{p_1}, \ell_{p_2}, (L_{i}p_3)] \to \mathbb{C}, \quad [F_2 \oplus L_i, \ell_{p_1}, \ell_{p_2}, L_{p_3}] \to z_2 - z_1\}.
\]

A canonical version of these results is that the restriction of \(h : \{\ell_{p_3} \text{ bad}\} \to M^{ss}(X)\) to \(\tilde{\pi}_1^{-1}([L_i])\) gives an isomorphism \(\tilde{\pi}_1^{-1}([L_i]) \to M^{ss}(X)\). In particular, from Theorems 3.1 and 3.2 it follows that

\[
h([F_2 \oplus L_i, \ell_{p_1}, \ell_{p_2}, L_{p_3}]) = M^{ss}(X) - \{(p \circ \tau_{p_1 - e_3})([L_i]), (p \circ \tau_{p_2 - e_3})([L_i])\},
\]

\[
h([L_i] \oplus [L_i], \ell_{p_1}, \ell_{p_2}, L_{p_3}) = \{(p \circ \tau_{p_1 - e_3})([L_i]), (p \circ \tau_{p_2 - e_3})([L_i])\}.
\]

Note that \((p \circ \tau_{p_1 - e_3})([L_i]) = (p \circ \tau_{p_2 - e_3})([L_i]) = (p \circ \tau_{e_3 - p_i})([L_i])\). \(\square\)

Theorems 4.1–4.4 prove Theorem 1.1 from the Introduction.

5. Relationship between \(M^s(X, 3)\) and \(M^{ss}(X, 2)\)

In [15] it is shown that \(M^{ss}(X, 2)\) is isomorphic to \((\mathbb{CP}^1)^2\). From our perspective, we can describe this result by defining a map \(M^{ss}(X, 2) \to (M^{ss}(X))^2\),

\[
[E, \ell_{p_1}, \ell_{p_2}] \mapsto ([E], h_{e_3}(E, \ell_{p_1}, \ell_{p_2})),
\]

One can show that this map is an isomorphism.

We can relate the closed subset \(\{\ell_{p_3} \text{ bad}\}\) of \(M^s(X, 3)\) to the moduli space \(M^{ss}(X, 2)\) as follows. Define a map \(\{\ell_{p_3} \text{ bad}\} \to M^{ss}(X, 2)\),

\[
[E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}] \mapsto [E, \ell_{p_1}, \ell_{p_2}],
\]

\[\text{THE MODULI SPACE OF STABLE RANK 2 PARABOLIC BUNDLES OVER AN ELLIPTIC CURVE WITH 3 MARKED POINTS 5}\]
We have a commutative diagram
\[
\begin{array}{ccc}
\{\ell_{p_3} \text{ bad}\} & \longrightarrow & M^{ss}(X, 2) \\
\mathbb{Z}_2 & \downarrow & \downarrow \\
\text{Jac}(X) & \overset{p}{\longrightarrow} & M^{ss}(X),
\end{array}
\]
where we have defined a map $M^{ss}(X, 2) \rightarrow M^{ss}(X)$, $[E, \ell_{p_1}, \ell_{p_2}] \mapsto [E]$.

6. Poincaré Polynomial of $M^s(X, 3)$

The Poincaré polynomial of $M^s(C, n)$ is given in [11, Theorem 3.8] for the case $\mu = 1/4$, corresponding to the traceless character variety, and $n$ odd:
\begin{equation}
P_t(M^s(C, n)) = \frac{(1 + t^2)^n (1 + t^3)^2g - 2^{n-1}t^{2g+n-1}(1+t)^2g(1+t^2)}{(1-t^2)(1-t^4)},
\end{equation}
where $g$ is the genus of $C$. In fact, the results of [11] are stated for parabolic bundles with fixed determinant bundle of odd degree, but since $\mu = 1/4$, corresponding to a traceless character variety, the results also hold for the moduli space $M^s(C, n)$ for which the determinant bundle of the parabolic bundles is trivial. For an elliptic curve $X$ with 3 marked points, equation (1) gives
\begin{equation}
P_t(M^s(X, 3)) = 1 + 4t^2 + 2t^3 + 4t^4 + t^6.
\end{equation}

We can reproduce equation (2) using our explicit description of $M^s(X, 3)$. Since $\pi : M^s(X, 3) \rightarrow (M^{ss}(X))^3$ restricts to an isomorphism $M^s(X, 3) - \{\ell_{p_3} \text{ bad}\} \rightarrow (M^{ss}(X))^3 - \pi(\{\ell_{p_3} \text{ bad}\})$, we obtain the following equation for the Poincaré polynomials for cohomology with compact supports:
\begin{equation}
P_t(M^s(X, 3) - \{\ell_{p_3} \text{ bad}\}) = P_t((M^{ss}(X))^3 - \pi(\{\ell_{p_3} \text{ bad}\})).
\end{equation}
From the long exact sequence for cohomology with compact supports, we have
\begin{equation}
P_t(M^s(X, 3) - \{\ell_{p_3} \text{ bad}\}) = P_t(M^s(X, 3)) - P_t(\{\ell_{p_3} \text{ bad}\}),
\end{equation}
\begin{equation}
P_t((M^{ss}(X))^3 - \pi(\{\ell_{p_3} \text{ bad}\})) = P_t((M^{ss}(X))^3) - P_t(\pi(\{\ell_{p_3} \text{ bad}\})).
\end{equation}
We have that $M^{ss}(X)$ is isomorphic to $\mathbb{C}P^1$, $\pi(\{\ell_{p_3} \text{ bad}\})$ isomorphic to Jac$(X)$, and $\{\ell_{p_3} \text{ bad}\}$ is isomorphic to Jac$(X) \times M^{ss}(X)$, so
\begin{equation}
P_t(M^{ss}(X))^3 = (1+t^2)^3, \quad P_t(\pi(\{\ell_{p_3} \text{ bad}\})) = 1+2t+t^2, \quad P_t(\{\ell_{p_3} \text{ bad}\}) = (1+2t+t^2)(1+t^2).
\end{equation}
Combining equations (3)–(6), we reproduce equation (2) for $P_t(M^s(X, 3))$.

References