KHOVANOV HOMOLOGY VIA 1-TANGLE DIAGRAMS IN THE ANNULUS

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ABSTRACT. We show that the reduced Khovanov homology of a link L in S^3 can be expressed as the homology of a chain complex constructed from a description of L as the closure of a 1-tangle diagram in the annulus. The chain complex is constructed using a resolution of the 1-tangle diagram into planar tangles in a manner analogous to ordinary reduced Khovanov homology, but with some novel features. In particular, unlike for ordinary Khovanov homology, terms appear in the differential that are products of linear maps corresponding to pairs of saddles obtained from the resolution. We also use the chain complex to construct a spectral sequence that converges to reduced Khovanov homology.

1. INTRODUCTION

Khovanov homology is a powerful invariant defined for oriented links in S^3 [8]. The Khovanov homology of an oriented link is a finitely-generated bigraded abelian group that categorifies its Jones polynomial [7]; roughly speaking, the relationship between the Khovanov homology of a link and its Jones polynomial is analogous to the relationship between the singular homology of a topological space and its Euler characteristic. The Jones polynomial of a link can be recovered from its Khovanov homology, but the Khovanov homology generally contains more information: it can sometimes distinguish between links with the same Jones polynomial, and Khovanov homology detects the unknot [9], but it is not known whether the same is true of the Jones polynomial.

Here we formulate reduced Khovanov homology in terms of descriptions of links as closures of 1-tangle diagrams in the annulus. A *1-tangle* is a 1-dimensional submanifold of the thickened annulus $A \times I$ with boundary $\{(p, 1/2), (q, 1/2)\}$, where p and q are fixed points on the inner and outer boundary circles of the annulus $A = S^1 \times I$. A 1-tangle can be projected onto the annulus to yield a *1-tangle diagram*. For a generic 1-tangle, the associated 1-tangle diagram is an immersed 1-manifold with finitely many transverse double-points with crossing data.

Given an oriented 1-tangle diagram T, we can close T using an unknotted overpass or underpass arc to obtain oriented link diagrams T^+ and T^-. These link diagrams determine links in S^3 that are defined up to isotopy, which for simplicity we also denote by T^+ and T^-. We construct a bigraded chain complex $(C_{T^\pm}, \partial_{T^\pm})$ by resolving the tangle diagram T into planar tangles in the annulus. We prove:

Theorem 1.1. *The chain complex $(C_{T^\pm}, \partial_{T^\pm})$ is chain-homotopy equivalent to the chain complex for the reduced Khovanov homology of the link $T^\pm$.*

In particular, the homology of the chain complex $(C_{T^\pm}, \partial_{T^\pm})$ is the reduced Khovanov homology of the link $T^\pm$.

We also use the chain complex $(C_{T^\pm}, \partial_{T^\pm})$ to construct a spectral sequence to reduced Khovanov homology. We can express the differential $\partial_{T^\pm}$ as $\partial_{T^\pm} = \partial^0_{T^\pm} + \partial^\pm_{T^\pm}$, where $(\partial^0_{T^\pm})^2 = 0$. The term $\partial^0_{T^\pm}$ is a sum of linear maps corresponding to saddles obtained from the resolution of T, and the term $\partial^\pm_{T^\pm}$ is a sum of products of linear maps corresponding to pairs of saddles obtained from the resolution of T. We prove:

Theorem 1.2. *There is a spectral sequence with E_2 page given by the homology of $(C_{T^\pm}, \partial^0_{T^\pm})$ that converges to the reduced Khovanov homology of the link $T^\pm$.*

If we forget the bigrading of the chain complex $(C_{T^\pm}, \partial_{T^\pm})$, we have that $C_{T^+} = C_{T^-}$ as ungraded vector spaces and $\partial^0_{T^+} = \partial^0_{T^-}$ as linear maps of ungraded vector spaces. So the E_2 pages of the spectral sequences for the links T^+ and T^- are the same up to grading shifts. It is interesting to note that the E_2 page of our spectral sequence is an invariant of the 1-tangle in $A \times I$ corresponding to the tangle diagram T, as follows from a more general result regarding tangles in thickened surfaces due to Asaeda, Przytycki, and Sikora [2].

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Khovanov homology was originally defined only for links in S^3, and an important open problem is to generalize Khovanov homology to links in arbitrary 3-manifolds. Khovanov homology has been extended to links in I-bundles over arbitrary surfaces by Asaeda, Przytycki, and Sikora [1], to links in $S^2 \times S^1$ by Rozansky [10], to links in $\mathbb{RP}^3$ by Gabrovšek [5], and recently to links in all connected sums of $S^2 \times S^1$ by Willis [11]. The results presented here are part of an attempt to construct Khovanov homology for links in arbitrary lens spaces. Our strategy for constructing Khovanov homology for such links relies on first generalizing a result due to Hedden, Herald, Hogancamp, and Kirk. In [6], they show that Bar-Natan's functor from the tangle cobordism category for 2-tangles in the 3-ball to chain complexes, which Bar-Natan defined in [3], can be factored through the twisted Fukaya category for a symplectic manifold $R(S^2, 4)$ known as the pillowcase. Their factorization result involves splitting a link in S^3 along a 2-sphere that transversely intersects the link in four points. To the 4-punctured 2-sphere they associate its traceless $SU(2)$ character variety, which is the pillowcase $R(S^2, 4)$.

We conjecture that an analogous factorization result may hold for the twisted Fukaya category of $R(T^2, 2)$, a symplectic manifold that can be interpreted as the traceless $SU(2)$ character variety for the 2-punctured 2-torus. If so, the corresponding functor would provide a natural way of generalizing Khovanov homology to links in lens spaces, since a lens space containing a link can always be Heegaard-split into a pair of solid tori glued along a 2-torus that transversely intersects the link in two points. Using results from [4], we were able to describe some of the structure of the Fukaya category of $R(T^2, 2)$ that is relevant to generalizing Hedden, Herald, Hogancamp, and Kirk's factorization result, assuming such a generalization is possible. This information, together with the results of [6], provided the clues needed to construct the chain complex for annular 1-tangle diagrams described here. The information we have obtained regarding the Fukaya category of $R(T^2, 2)$ suggests that the factorization result of [6] does indeed generalize, with the resulting chain complex as described by Theorem 1.1, but we have not proven this. Our hope is that Theorem 1.1 is an indication that the Fukaya category of $R(T^2, 2)$ "knows" about Khovanov homology, and that further investigation of this Fukaya category may yield a candidate construction of reduced Khovanov homology for links in lens spaces.

The paper is organized as follows. In Section 2, we define several vector spaces and linear maps that we use throughout the paper. In Section 3, we show how to construct the chain complex $(C_{T^\pm}, \partial_{T^\pm})$ from an oriented annular 1-tangle diagram T, and we sketch the proof of Theorem 1.1. Our proof of Theorem 1.1 involves first reformulating the chain complex $(C_{T^\pm}, \partial_{T^\pm})$ in terms of planar m-tangles in the disk. In Section 4, we discuss the relevant concepts involving such planar disk tangles. In Section 5, we reformulate the chain complex $(C_{T^\pm}, \partial_{T^\pm})$ in terms of planar disk tangles. In Sections 6, 7, and 8, we describe the planar disk tangles and saddles that are obtained from resolving the (nonplanar) annular 1-tangle diagram T. In Sections 9 and 10 we combine the results of Sections 4–8 to complete the proof of Theorem 1.1. In Section 11, we prove Theorem 1.2. In Section 12, we illustrate our results with an example.

2. Vector spaces and linear maps

In this section we define several vector spaces and linear maps that we will use to construct the chain complex $(C_{T^\pm}, \partial_{T^\pm})$. Our notation is generally consistent with that of [6].

We define $\mathbb{F}$ to be the field of two elements. We also view $\mathbb{F}$ as a $\mathbb{Z}$-graded $\mathbb{F}$-vector space lying entirely in grading 0. We define a $\mathbb{Z}$-graded $\mathbb{F}$-vector space $A = \mathbb{F} \cdot \{1_A^{(1)}, x^{(-1)}\}$, where the superscripts indicate that the vectors 1_A and x are homogeneous with gradings 1 and -1. We refer to the $\mathbb{Z}$-grading on $\mathbb{F}$ and A as a *quantum grading*. We define the following $\mathbb{F}$-linear maps:

$$
\begin{aligned}
&\eta : \mathbb{F} \to A, && \eta(1) = 1_A, \\
&\dot{\eta} : \mathbb{F} \to A, && \dot{\eta}(1) = x, \\
&\epsilon : A \to \mathbb{F}, && \epsilon(1_A) = 0, \qquad \epsilon(x) = 1, \\
&\dot{\epsilon} : A \to \mathbb{F}, && \dot{\epsilon}(1_A) = 1, \qquad \dot{\epsilon}(x) = 0, \\
&1_{x1} : A \to A, && 1_{x1}(1_A) = x, \qquad 1_{x1}(x) = 0, \\
&\Delta : A \to A \otimes A, && \Delta(1_A) = 1_A \otimes x + x \otimes 1_A, \qquad \Delta(x) = x \otimes x, \\
&m : A \otimes A \to A, && m(1_A \otimes 1_A) = 1_A, \qquad m(1_A \otimes x) = m(x \otimes 1_A) = x, \qquad m(x \otimes x) = 0.
\end{aligned}
$$

The quantum gradings of these maps are

$$\eta^{(1)}, \qquad \dot\eta^{(-1)}, \qquad \epsilon^{(1)}, \qquad \dot\epsilon^{(-1)}, \qquad (1_{x1})^{(-2)}, \qquad \Delta^{(-1)}, \qquad m^{(-1)}.$$

The graded vector space A, together with the multiplication m, comultiplication Δ, unit η, and counit ϵ, gives Khovanov's Frobenius algebra. For notational simplicity, given an $\mathbb{F}$-vector space V we often identify V with $\mathbb{F} \otimes V$. For example, we have that

$$m = \dot\epsilon \otimes 1_A + \epsilon \otimes 1_{x1}, \qquad \Delta = \dot\eta \otimes 1_A + \eta \otimes 1_{x1}.$$

The chain complex $(C_{T^\pm}, \partial_{T^\pm})$ is built out of $\mathbb{F}$-vector spaces that carry both a homological $\mathbb{Z}$-grading h and a quantum $\mathbb{Z}$-grading q. We will view the vector spaces $\mathbb{F}$ and A as lying entirely in homological grading 0. Given a bigraded vector space V, we indicate the bigrading of a homogeneous vector $v \in V$ with superscripts as $v^{(h,q)}$. We define the vector space $V[h, q]$ to be V with gradings shifted upwards by (h, q). That is, if $v \in V$ is homogeneous with bigrading (h_v, q_v), then the corresponding vector $v \in V[h, q]$ is homogeneous with bigrading $(h + h_v, q + q_v)$.

3. Chain complex for tangles in the annulus

Given an oriented 1-tangle diagram T in the annulus, we close T above or below with an unknotted arc to obtain oriented link diagrams T^+ and T^-. For each link diagram $T^\pm$, we define a chain complex $(C_{T^\pm}, \partial_{T^\pm})$ as follows.

First we describe the bigraded vector space $C_{T^\pm}$. Let c denote the total number of crossings of T. Choose an arbitrary ordering of the crossings. Define the 0-resolution (1-resolution) of a crossing such that the overpass turns left (right). Define $I = \{0, 1\}^c$ to be the set of binary strings of length c. Given a binary string $i \in I$, define T_i to be the planar 1-tangle in the annulus obtained by resolving the crossings of T in the manner indicated by i; specifically, resolve crossing number k as indicated by the k-th bit of i.

A planar 1-tangle in the annulus consists of a single arc connecting the marked points on the inner and outer boundary circles, together with a finite number of circle components. We orient the arc of each planar 1-tangle in a direction from the inner boundary circle to the outer boundary circle; note that this orientation is unrelated to the orientation of the (nonplanar) tangle T. We leave the circle components unoriented.

For each integer n, define a bigraded vector space

$$\tilde C_n = \bigoplus \{A^{\otimes c(T_i)}[r(i), r(i)] \mid i \in I \text{ such that } w(T_i) = n\},$$

where $r(i)$ is the number of 1's in the binary string $i \in I$, $c(T_i)$ is the number of circle components of the planar tangle T_i, and $w(T_i) \in \mathbb{Z}$ is the number of times the (oriented) arc component of T_i winds counterclockwise around the annulus. Recall that we define the link diagram T^+, respectively T^-, by closing the tangle diagram T with an arc a that passes over, respectively under, the tangle diagram T. Define the *loop number* ℓ of the pair (T, a) to be the total number of times a crosses T. We can depict the pair (T, a) as shown in Figure 1; note that we assume all the crossings of T take place in a disk T_D that does not intersect the overpass/underpass arc a. Define $n_+(T^\pm)$ and $n_-(T^\pm)$ to be the number of positive and negative crossings for the link diagram $T^\pm$. For each integer n, define a grading-shifted vector space C_n by

$$C_n = \tilde C_n[h_\pm(T) + h_\pm(A_n), q_\pm(T) + q_\pm(A_n)],$$

where

$$h_\pm(T) = -n_-(T^\pm), \qquad q_\pm(T) = n_+(T^\pm) - 2n_-(T^\pm),$$
$$h_\pm(A_n) = (1/2)(\ell \pm n), \qquad q_\pm(A_n) = (1/2)(\ell \pm 3n).$$

Define bigraded vector spaces C_{T^+} and C_{T^-} by

$$C_{T^\pm} = \bigoplus_n C_n.$$

We note that the grading shift $(h_\pm(T), q_\pm(T))$ of C_n is familiar from the definition of ordinary reduced Khovanov homology, with the convention that the reduced Khovanov homology of the unknot is $\mathbb{F}[0, 0]$. The additional grading shift $(h_\pm(A_n), q_\pm(A_n))$ will be explained in Lemma 6.7.

Next we describe the differential $\partial_{T^\pm}$. We first define several linear maps by summing over saddles $p \to p'$ obtained from the resolution of T. Define linear maps $\partial_n : C_n \to C_n$ by summing the following terms:

(1) For a saddle $p \to p'$ with $w(p) = w(p') = n$ splitting a circle from the arc:

$$\dot{\eta} \otimes 1_{A^{\otimes r}} : A^{\otimes r} \to A \otimes A^{\otimes r}, \qquad c(p) = r, \qquad c(p') = r + 1.$$

(2) For a saddle $p \to p'$ with $w(p) = w(p') = n$ merging a circle with the arc:

$$\dot{\epsilon} \otimes 1_{A^{\otimes r}} : A \otimes A^{\otimes r} \to A^{\otimes r}, \qquad c(p) = r + 1, \qquad c(p') = r.$$

(3) For a saddle $p \to p'$ with $w(p) = w(p') = n$ splitting a circle from a circle:

$$\Delta \otimes 1_{A^{\otimes r}} : A \otimes A^{\otimes r} \to A \otimes A \otimes A^{\otimes r}, \qquad c(p) = r + 1, \qquad c(p') = r + 2.$$

(4) For a saddle $p \to p$ with $w(p) = w(p') = n$ merging two circles:

$$m \otimes 1_{A^{\otimes r}} : A \otimes A \otimes A^{\otimes r} \to A \otimes A^{\otimes r}, \qquad c(p) = r + 2, \qquad c(p') = r + 1.$$

Define linear maps $\tilde{\partial}^L_n : C_n \to C_n$ (respectively $\tilde{\partial}^R_n : C_n \to C_n$) by summing the following terms:

(1) For a saddle $p \to p'$ with $w(p) = w(p') = n$ splitting a circle from the arc that connects to the arc on the left (respectively right) side:

$$\eta \otimes 1_{A^{\otimes r}} : A^{\otimes r} \to A \otimes A^{\otimes r}, \qquad c(p) = r, \qquad c(p') = r + 1.$$

(2) For a saddle $p \to p'$ with $w(p) = w(p') = n$ merging a circle with the arc that connects to the arc on the left (respectively right) side:

$$\epsilon \otimes 1_{A^{\otimes r}} : A \otimes A^{\otimes r} \to A^{\otimes r}, \qquad c(p) = r + 1, \qquad c(p') = r.$$

Define linear maps $Q_n : C_n \to C_{n-2}$ by summing the following terms:

(1) For a saddle $p \to p'$ with $w(p) = n$ and $w(p') = n - 2$,

$$1_{A^{\otimes r}} : A^{\otimes r} \to A^{\otimes r}, \qquad c(p) = r, \qquad c(p') = r.$$

Define linear maps $P_n : C_n \to C_{n+2}$ by summing the following terms:

(1) For a saddle $p \to p'$ with $w(p) = n$ and $w(p') = n + 2$,

$$1_{A^{\otimes r}} : A^{\otimes r} \to A^{\otimes r}, \qquad c(p) = r, \qquad c(p') = r.$$

The grading shifts in the definition of the vector spaces C_n imply that the bigradings of the linear maps ∂_n, $\tilde{\partial}^L_n$, and $\tilde{\partial}^R_n$ are

$$(\partial_n)^{(1,0)}, \qquad (\tilde{\partial}^L_n)^{(1,2)}, \qquad (\tilde{\partial}^R_n)^{(1,2)}.$$

The bigradings of the linear maps Q_n and P_n are different for T^+ and T^-. For T^+, the bigradings are

$$(Q_n)^{(0,-2)}, \qquad (P_n)^{(2,4)}.$$

For T^-, the bigradings are

$$(Q_n)^{(2,4)}, \qquad (P_n)^{(0,-2)}.$$

Define a linear map $\partial^0_{T^\pm} : C_{T^\pm} \to C_{T^\pm}$ with bigrading $(\partial^0_{T^\pm})^{(1,0)}$ by

$$\partial^0_{T^\pm} = \sum_n \partial_n.$$

Define linear maps $\partial^\pm_{T^\pm} : C_{T^\pm} \to C_{T^\pm}$ with bigrading $(\partial^\pm_{T^\pm})^{(1,0)}$ by

$$\partial^+_{T^+} = \sum_n (\tilde{\partial}^L_{n-2} Q_n + Q_n \tilde{\partial}^L_n), \qquad \partial^-_{T^-} = \sum_n (\tilde{\partial}^L_{n+2} P_n + P_n \tilde{\partial}^L_n).$$

Define linear maps $\partial_{T^\pm} : C_{T^\pm} \to C_{T^\pm}$ by $\partial_{T^\pm} = \partial^0_{T^\pm} + \partial^\pm_{T^\pm}$.

Having defined the bigraded vector space $C_{T^\pm}$ and differential $\partial_{T^\pm}$, we are now ready to restate Theorem 1.1 from the Introduction:

Theorem 3.1. *Given an oriented 1-tangle diagram T in the annulus, the pair $(C_{T^\pm}, \partial_{T^\pm})$ is a chain complex that is chain-homotopy equivalent to the chain complex for the reduced Khovanov homology of the link $T^\pm$, where the marked point for the reduced Khovanov homology is taken to be the endpoint of T that lies on the inner boundary circle of the annulus.*

Our basic tool for constructing the chain-homotopy equivalence is the following lemma:

Lemma 3.2 (Reduction Lemma)**.** *Consider a chain complex (C, ∂) such that the vector space C has the form $C = A \oplus B \oplus B$ for vector spaces A and B and the differential $\partial : C \to C$ has the form*

$$\partial = \begin{pmatrix} \partial_A & \beta_1 & \beta_2 \\ \alpha_1 & \partial_B & 0 \\ \alpha_2 & 1_B & \partial_B \end{pmatrix}.$$

Then $(A, \partial_A + \beta_1\alpha_2)$ is a chain complex that is chain-homotopy equivalent to (C, ∂).

Proof. The fact that $\partial^2 = 0$ implies that $(\partial_A + \beta_1\alpha_2)^2 = 0$, so $(A, \partial_A + \beta_1\alpha_2)$ is a chain complex. Define linear maps $F : C \to A$ and $G : A \to C$ by

$$F = \begin{pmatrix} 1_A & 0 & \beta_1 \end{pmatrix}, \qquad G = \begin{pmatrix} 1_A \\ \alpha_2 \\ 0 \end{pmatrix}.$$

The fact that $\partial^2 = 0$ implies that F and G are chain maps. Define a linear map $H : C \to C$ by

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1_B \\ 0 & 0 & 0 \end{pmatrix}.$$

We have that

$$GF = 1_C + \partial H + H\partial, \qquad FG = 1_A,$$

so F and G are chain-homotopy equivalences. □

We refer to the chain complex $(A, \partial_A + \beta_1\alpha_2)$ as the *reduced chain complex* corresponding to (C, ∂), and say that it is obtained by *reducing* on the identity map $1_B : B \to B$ in ∂. In the differential $\partial_A + \beta_1\alpha_2$ for the reduced chain complex, we refer to terms of ∂_A as *residual terms* and terms of $\beta_1\alpha_2$ as *reduction terms.* We refer to the maps α_2 and β_1 as *reduction factors.*

We can now sketch the outline of the proof of Theorem 3.1:

Proof of Theorem 3.1. We will prove the claim for the case T^+; the case T^- is similar. We prove the claim by induction on the loop number ℓ of the annular 1-tangle diagram T. For the base case $\ell = 0$, the chain complex $(C_{T^+}, \partial_{T^+})$ is the chain complex for the reduced Khovanov homology of T^+, so the claim is trivially true.

For the induction step, assume the claim is true for all annular 1-tangle diagrams of loop number $\ell - 1$, and consider an annular 1-tangle diagram T with loop number ℓ. As shown in Figure 1, we can flip the outermost loop of T under the annulus to obtain an annular 1-tangle diagram $\overline{T}$ with loop number $\ell - 1$ and one additional crossing. Since T^+ and $\overline{T}^+$ describe isotopic links in S^3, the chain complexes for the reduced Khovanov homology of T^+ and $\overline{T}^+$ are chain-homotopy equivalent. The induction hypothesis implies that $(C_{\overline{T}^+}, \partial_{\overline{T}^+})$ is chain-homotopy equivalent to the reduced Khovanov homology of $\overline{T}^+$, and is hence chain-homotopy equivalent to the reduced Khovanov homology of T^+. As we will describe in Section 6, we can apply the Reduction Lemma 3.2 to $(C_{\overline{T}^+}, \partial_{\overline{T}^+})$ to obtain a reduced chain complex $(C_{red}, \partial_{red})$, whose precise definition is given in Definition 6.4. To complete the proof, we will show that $(C_{red}, \partial_{red}) = (C_{T^+}, \partial_{T^+})$. In Lemma 6.8 we show that $C_{red} = C_{T^+}$ as bigraded vector spaces, and in Lemmas 9.3 and 10.4 we show that $\partial_{red} = \partial_{T^+}$. □

Remark 3.3. A link diagram L in $\mathbb{R}^2$ with at least one crossing can be decomposed into two disk 2-tangle diagrams D_1 and D_2 or into two annular 1-tangle diagrams A_1 and A_2, as shown in Figure 2. We have shown that the factorization result of [6] can be generalized to the case of such disk 2-tangle diagram decompositions, and the resulting chain complex is as described by Theorem 3.1 for the case of annular 1-tangle diagrams with loop number $\ell = 1$.

4. Disk tangles and saddles

In order to prove Theorem 3.1, we will first reformulate the chain complex $(C_{T^\pm}, \partial_{T^\pm})$ in terms of a resolution of the oriented annular 1-tangle diagram T into planar tangles in the disk. In this section we discuss the relevant concepts involving the planar disk tangles and saddles obtained from the resolution.

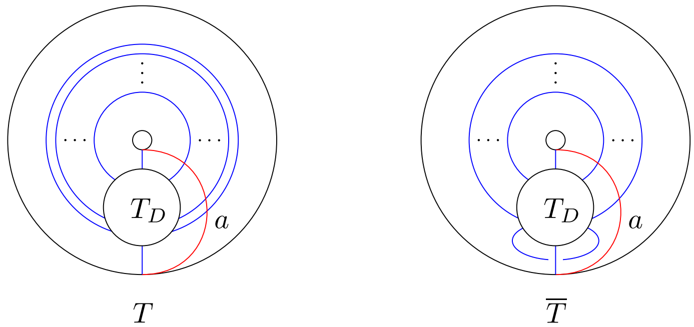

FIGURE 1. An annular 1-tangle diagram T with loop number ℓ determines an annular 1-tangle diagram $\overline{T}$ with loop number $\ell - 1$. The circle labeled T_D is the disk m-tangle corresponding to T, where $m = \ell + 1$, as explained in Section 4.1. The red curve labeled a is the overpass/underpass arc used to define the link diagrams $T^{\pm}$ and $\overline{T}^{\pm}$.

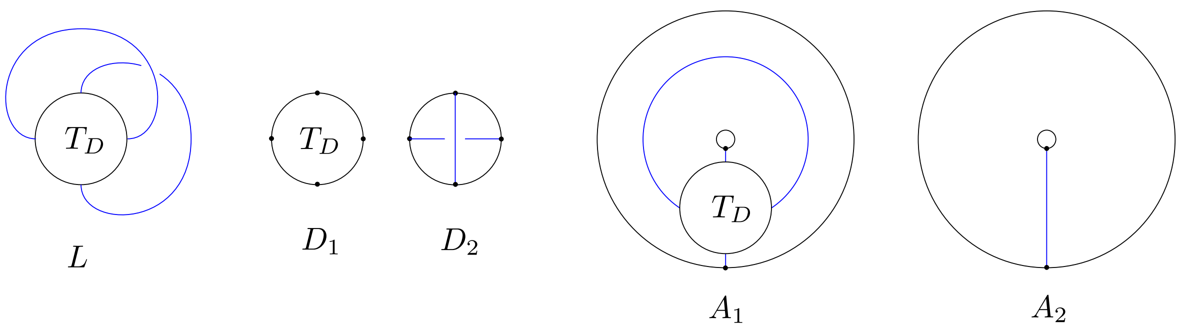

FIGURE 2. A link diagram L with at least one crossing can be decomposed into two disk 2-tangle diagrams D_1 and D_2, or into two annular 1-tangle diagrams A_1 and A_2.

4.1. **Tangle diagrams in the disk.** Consider an oriented 1-tangle diagram T in the annulus and a choice of overpass/underpass arc a used to define the link diagrams T^+ and T^-. Recall that we defined the loop number ℓ of the pair (T, a) to be the number of times a crosses T. Given a pair (T, a), we can cut T along the arc a to obtain an oriented m-tangle diagram T_D in a topological disk, where $m = \ell + 1$. The boundary of T_D defines $2m$ marked points on the boundary circle of the disk. Conversely, an m-tangle diagram T_D in the disk determines a 1-tangle diagram T in the annulus and an overpass/underpass arc a, as shown in Figure 1, such that the pair (T, a) has loop number $\ell = m - 1$. We enumerate the $2m$ marked points on the boundary circle of the disk by counting counterclockwise around the circle, where points 1 and $m + 1$ are located at the top and bottom.

4.2. **Planar tangles in the disk.** We define d_{2m} to be the set of planar m-tangles in the disk. A planar tangle $p \in d_{2m}$ consists of m arcs that connect the $2m$ marked points in pairs, together with some number $c_d(p)$ of circle components that we refer to as *disk circles*. A planar m-tangle $p \in d_{2m}$ in the disk uniquely determines a planar 1-tangle p_A in the annulus, which consists of a single arc together with some number of circle components. We refer to an arc a of $p \in d_{2m}$ as a *strand arc* if it lies on the single arc of p_A and a *circle arc* if it lies on a circle component of p_A. We refer to the set of circle arcs of p that lie on a given circle component of p_A as an *annular circle*, and we define $c_a(p)$ to be the number of annular circles of p. We define the *circle number* $c(p) = c_d(p) + c_a(p)$ of p to be the number of circle components of p_A. We refer to the set of strand arcs of p as the *strand component* of p. We refer to the annular circles and disk circles of p as *circle components* of p. Given a planar tangle $p \in d_{2m}$ with $m \geq 2$, we define arcs A, B, D, and C that contain the marked points 1, m, $m + 1$, and $m + 2$, respectively, as shown in Figure 3. Note that $B \neq C$, but otherwise the arcs A, B, C, and D need not be distinct.

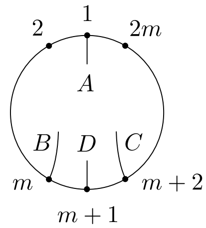

FIGURE 3. Arcs A, B, C, and D of a planar disk m-tangle $p \in d_{2m}$ (we depict only a portion of each arc).

We orient the strand arcs of a planar disk tangle $p \in d_{2m}$ by orienting the single arc of the corresponding annular 1-tangle p_A in a direction from the inner boundary circle of the annulus to the outer boundary circle. We leave the circle arcs of p unoriented. Our choice of orientation of the single arc of p_A induces an ordering of the strand arcs of p, and we will use notation such as $a_1 < a_2$ and $a_1 \leq a_2$ to indicate how strand arcs a_1 and a_2 of p are related under this ordering. To indicate that a sequence of circle arcs $(a_1, \cdots, a_n)$ belong to the same annular circle and that the arcs are encountered in sequence as we move either clockwise or counterclockwise starting from a_1, we use the notation $a_1 - a_2 - \cdots - a_n$; equivalently $a_n - \cdots - a_2 - a_1$. To indicate that a sequence of arcs $(a_1, \cdots, a_n)$ belong to different annular circles, we use the notation $a_1, a_2, \cdots, a_n$.

We define the *winding number* $w(p) \in \{-\ell, \cdots, \ell\}$ of p to be the number of times the single arc of p_A winds counterclockwise around the annulus. We will sometimes indicate the winding number and circle number of a planar tangle p by using subscripts and parentheses: $p_n(r)$ indicates that p has winding number $w(p) = n$ and circle number $c(p) = r$.

Given a planar tangle $p \in d_{2m}$, we assign a *type* $\tau(a) \in \{e, u, c\}$ to each arc a in p as follows. Starting at marked point 1, we move along the single arc of p_A in the direction of its orientation (from the inner boundary circle of the annulus to the outer boundary circle) and alternately label each arc a of p that we encounter as *earringed* ($\tau(a) = e$) or *unearringed* ($\tau(a) = u$); this terminology is adapted from [6]. Note in particular that $\tau(A) = e$, $\tau(D) = e$ for m odd, and $\tau(D) = u$ for m even. This procedure assigns types to all of the strand arcs. Each remaining arc a is a circle arc, and we define its type to be $\tau(a) = c$. We define the type of a disk circle d to be $\tau(d) = c$. We define $\overline{e} = u$ and $\overline{u} = e$.

The arc components of a planar tangle $p \in d_{2m}$ describe a crossingless matching of the $2m$ marked points on the boundary of the disk. We define a set D_{2m} of such crossingless matchings; note that $|D_{2m}| = C_m$, where m is the m-th Catalan number. Given a planar tangle $p \in d_{2m}$, we define the corresponding crossingless matching $\tau(p) \in D_{2m}$ to be the *type* of the planar tangle. We can also think of $\tau(p)$ as the planar tangle obtained from p by removing all of the disk circles, and under this interpretation many of our notions involving planar tangles carry over to tangle types. For example, we define the winding number $w(P)$ of a tangle type P to be the winding number of any planar tangle p of type $\tau(p) = P$. We can also speak of the arcs and types of arcs of a tangle type. Figure 4 depicts the possible tangle types for loop number $\ell = 0, 1, 2$. Note that for loop number $\ell = 2$ we have three distinct tangle types P_0^L, P_0^C, and P_0^R that all have winding number 0.

4.3. **Saddles.** We now consider saddles $S : p \leftrightarrow p'$ that relate two planar disk tangles $p, p' \in d_{2m}$. We define a pair of *saddle insertion points* (p_1, p_2) for p and (p'_1, p'_2) for p' as shown in Figure 5. If both saddle insertion points in a pair (q_1, q_2) lie on the strand component, we define q_1 and q_2 such that $q_1 < q_2$. If one saddle insertion point in a pair (q_1, q_2) lies on the strand component and the other lies on a circle component, we define q_1 to be the point on the strand component and q_2 to be the point on the circle component. For each saddle insertion point q, we define a corresponding *saddle arc* or *saddle disk circle* s that contains q. We

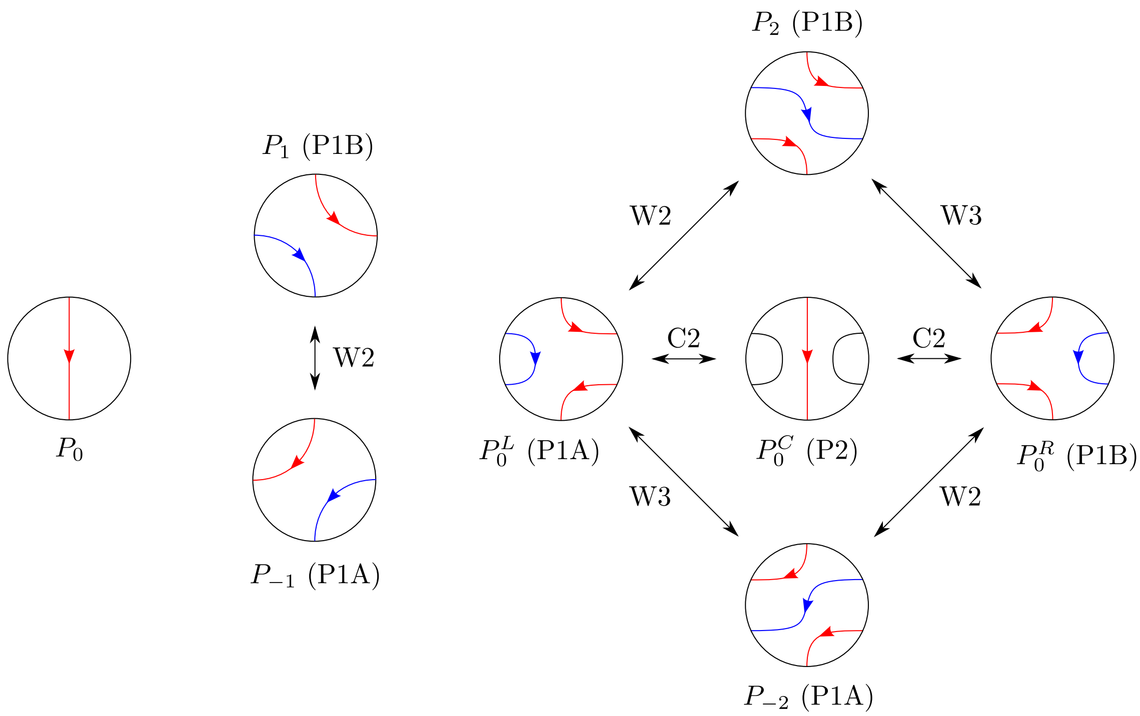

FIGURE 4. Tangle types for loop number $\ell = 0$ (left), $\ell = 1$ (center), and $\ell = 2$ (right). Earringed arcs are red, unearringed arcs are blue, and circle arcs are black. The subscripts indicate the winding number for each tangle type. The label in parentheses indicates the type of each tangle type, as explained in Section 6. Arrows connecting a pair of tangle types indicate that they are related by saddles. Each arrow is labeled by its corresponding saddle type, as explained in Sections 7 and 8.

assign an *orientation* $\sigma(s) \in \{L, R, U\}$ to each saddle arc s as follows.

$$\sigma(s) = \begin{cases} L & \text{if } \tau(s) \in \{e, u\} \text{ and the saddle attaches to the left side of } s, \\ R & \text{if } \tau(s) \in \{e, u\} \text{ and the saddle attaches to the right side of } s, \\ U & \text{if } \tau(s) = c. \end{cases}$$

For a saddle disk circle s we define $\sigma(s) = U$. We define $\overline{L} = R$ and $\overline{R} = L$. We will sometimes denote the type and orientation of a saddle arc s by using superscripts and subscripts: $s_{\sigma(s)}^{\tau(s)}$. For the saddle $S : p \to p'$ shown in Figure 5, the types and orientations of the saddle arcs are

$$(s_1)_R^e, \qquad (s_2)_R^e, \qquad (s_1')_L^e, \qquad (s_2')_U^c.$$

We now consider saddles $S : p \leftrightarrow p'$ for which the saddle insertion points (p_1, p_2) lie on distinct saddle arcs (s_1, s_2), which implies the saddle insertion points (p_1', p_2') must lie on distinct saddle arcs (s_1', s_2'). For such saddles we enumerate the possibilities for the types and orientations of the saddle arcs, and we indicate the relationship between the winding number and annular circle number of p and p':

Lemma 4.1. *Consider a saddle $S : p \leftrightarrow p'$ between planar tangles $p, p' \in d_{2m}$ such that the saddle insertion points (p_1, p_2) lie on distinct saddle arcs (s_1, s_2) of p and the saddle insertion points (p_1', p_2') lie on distinct saddle arcs (s_1', s_2') of p'. Without loss of generality we will assume that $w(p') \geq w(p)$ and if $w(p) = w(p')$*

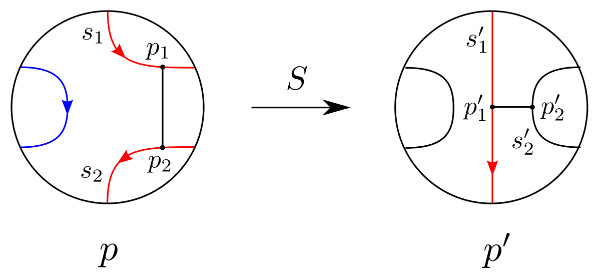

FIGURE 5. Saddle insertion points and saddle arcs for a saddle $S : p \to p'$. Shown are the saddle insertion points (p_1, p_2) and saddle arcs (s_1, s_2) for S in p, and the saddle insertion points (p'_1, p'_2) and saddle arcs (s'_1, s'_2) for S in p'.

then $c_a(p') \geq c_a(p)$. *The possible such saddles are*

$w(p') - w(p)$	$c_a(p') - c_a(p)$	(s_1, s_2)	(s'_1, s'_2)
2	0	$(s_1)^\tau_L < (s_2)^{\overline{\tau}}_R$	$(s'_1)^\tau_R < (s'_2)^{\overline{\tau}}_L$
0	1	$(s_1)^\tau_\sigma < (s_2)^\tau_\sigma$	$(s'_1)^\tau_{\overline{\sigma}}$, $(s'_2)^c_U$
0	1	$(s_1)^c_U - (s_2)^c_U$	$(s'_1)^c_U$, $(s'_2)^c_U$,

where $\tau \in \{e, u\}$ *and* $\sigma \in \{L, R\}$.

Proof. First consider the case where s_1 and s_2 are strand arcs; that is, $\tau(s_1) = \tau(s_2) \in \{e, u\}$. Let p_A and p'_A denote the annular 1-tangles corresponding to p and p', and let a_{12} denote the portion of the single arc of p_A that lies between the saddle insertion points p_1 and p_2. Let s_{12} denote the arc in p connecting p_1 to p_2 that corresponds to the saddle S. Then $\sigma := a_{12} \cup s_{12}$ is a circle in the annulus. If σ is essential, then the saddle S flips the orientation of a_{12} from clockwise in p_A to counterclockwise in p'_A, so $w(p') - w(p) = 2$ and $c_a(p') - c_a(p) = 0$. If σ is inessential, then the saddle S splits a circle from p_A, hence $w(p') - w(p) = 0$ and $c_a(p') - c_a(p) = 1$. The statements regarding the orientations of the saddle arcs are clear, and the statements regarding the types of the saddle arcs follow from the observation that s_{12} must cross the overpass/underpass arc a shown in Figure 1 an odd number of times if σ is essential and an even number of times if σ is inessential. The case where s_1 and s_2 are circle arcs; that is, $\tau(s_1) = \tau(s_2) = c$, is clear. □

4.4. **Interleaved, nested, and disjoint saddles.** Consider a pair of saddles $(S : p \leftrightarrow p', T : p \leftrightarrow \tilde{p})$ such that the saddle insertion points (p_1, p_2) for S in p lie on strand arcs (s_1, s_2) of p, the saddle insertion points (q_1, q_2) for T in p lie on strand arcs (t_1, t_2) of p, $\tau(s_1) = \overline{\tau(s_2)} \in \{e, u\}$, and $\tau(t_1), \tau(t_2) \in \{e, u\}$. The saddles S and T induce saddles $\overline{S} : \tilde{p} \leftrightarrow p''$ and $\overline{T} : p' \leftrightarrow p''$ for some planar tangle p'', resulting in a commuting square of saddles:

$$\begin{array}{ccc} p & \overset{S}{\longleftrightarrow} & p' \\ \big\updownarrow{\scriptstyle T} & & \big\updownarrow{\scriptstyle \overline{T}} \\ \tilde{p} & \overset{\overline{S}}{\longleftrightarrow} & p''. \end{array}$$

There are three types of such pairs (S, T): interleaved, nested, and disjoint.

Definition 4.2. We say that saddles $(S : p \leftrightarrow p', T : p \leftrightarrow \tilde{p})$ are *interleaved* if $p_1 < q_1 < p_2 < q_2$, where (p_1, p_2) are the saddle insertion points for S in p and (q_1, q_2) are the saddle insertion points for T in p, $\tau(s_1) = \overline{\tau(s_2)} \in \{e, u\}$, and $\tau(t_1) = \overline{\tau(t_2)} \in \{e, u\}$.

The commuting square for interleaved saddles $(S : p \leftrightarrow p', T : p \leftrightarrow \tilde{p})$ is depicted in Figure 6. The following lemma describes the relationships among the planar tangles in this commuting square:

Lemma 4.3. *Consider interleaved saddles* $(S : p \leftrightarrow p', T : p \leftrightarrow \tilde{p})$ *with induced saddles* $\overline{T} : p' \leftrightarrow p''$ *and* $\overline{S} : \tilde{p} \leftrightarrow p''$. *The winding numbers and circle numbers of the planar tangles are related by the following*

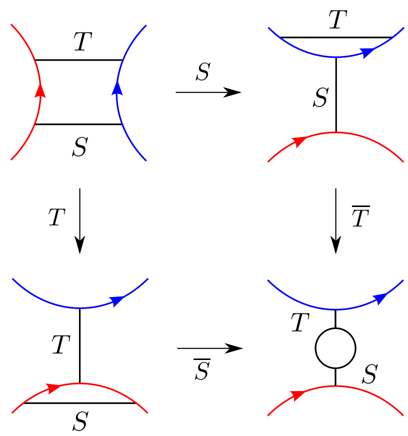

FIGURE 6. Interleaved saddles (S, T) and induced saddles $\overline{S}$ and $\overline{T}$.

commutative diagram:

$$\begin{array}{ccc} p_{n\pm 2}(r) & \xleftrightarrow{\;S\;} & p'_n(r) \\ \big\updownarrow{\scriptstyle T} & & \big\updownarrow{\scriptstyle \overline{T}} \\ \tilde{p}_n(r) & \xleftrightarrow{\;\overline{S}\;} & p''_n(r+1), \end{array}$$

where the plus sign in $n \pm 2$ is for $\sigma(s_1) = R$ and the minus sign is for $\sigma(s_1) = L$. The saddles $\overline{S}$ and $\overline{T}$ connect the additional circle component of p'' to opposite sides of the strand (i.e., $\sigma(s''_1) = \overline{\sigma(t''_1)} \in \{L, R\}$). For $s_1 = t_1$ and $s_2 = t_2$ we have that $\tau(p'_n) = \tau(\tilde{p}_n) = \tau(p''_n)$, otherwise $\tau(p'_n)$, $\tau(\tilde{p}_n)$, and $\tau(p''_n)$ are all distinct.

Proof. The types and orientations of the saddle arcs are as shown in the following diagram:

$$\begin{array}{ccc} (s_1)^{\tau(s_1)}_{\sigma(s_1)} \leq (t_1)^{\tau(t_1)}_{\sigma(s_1)} \leq (s_2)^{\overline{\tau(s_1)}}_{\overline{\sigma(s_1)}} \leq (t_2)^{\overline{\tau(t_1)}}_{\overline{\sigma(s_1)}} & \xleftrightarrow{\;S\;} & (s'_1)^{\tau(s_1)}_{\overline{\sigma(s_1)}} \leq (t'_1)^{\overline{\tau(t_1)}}_{\overline{\sigma(s_1)}} \leq (s'_2)^{\overline{\tau(s_1)}}_{\sigma(s_1)} \leq (t'_2)^{\overline{\tau(t_1)}}_{\overline{\sigma(s_1)}} \\ \big\updownarrow{\scriptstyle T} & & \big\updownarrow{\scriptstyle \overline{T}} \\ (\tilde{s}_1)^{\tau(s_1)}_{\sigma(s_1)} \leq (\tilde{t}_1)^{\tau(t_1)}_{\overline{\sigma(s_1)}} \leq (\tilde{s}_2)^{\tau(s_1)}_{\sigma(s_1)} \leq (\tilde{t}_2)^{\overline{\tau(t_1)}}_{\sigma(s_1)} & \xleftrightarrow{\;\overline{S}\;} & (s''_1)^{\tau(s_1)}_{\overline{\sigma(s_1)}} \leq (t''_1)^{\overline{\tau(t_1)}}_{\sigma(s_1)},\ (s''_2)^c_U - (t''_2)^c_U, \end{array}$$

where $\sigma(s_1) \in \{L, R\}$. The claim regarding the tangle types is clear. □

Definition 4.4. We say that saddles $(S : p \leftrightarrow p', T : p \leftrightarrow \tilde{p})$ are *nested* if $p_1 < q_1 < q_2 < p_2$, where (p_1, p_2) are the saddle insertion points for S in p and (q_1, q_2) are the saddle insertion points for T in p, $\tau(s_1) = \overline{\tau(s_2)} \in \{e, u\}$, and $\tau(t_1) = \tau(t_2) \in \{e, u\}$.

Remark 4.5. It is not possible to have saddles $(S : p \leftrightarrow p', T : p \leftrightarrow \tilde{p})$ such that $q_1 \leq p_1 < p_2 \leq q_2$, where (p_1, p_2) are the saddle insertion points for S in p, (q_1, q_2) are the saddle insertion points for T in p, $\tau(s_1) = \overline{\tau(s_2)} \in \{e, u\}$, and $\tau(t_1) = \tau(t_2) \in \{e, u\}$.

The commuting square for nested saddles $(S : p \leftrightarrow p', T : p \leftrightarrow \tilde{p})$ is depicted in Figure 7. The following lemma describes the relationships among the planar tangles in this commuting square:

Lemma 4.6. *Consider nested saddles $(S : p \leftrightarrow p', T : p \leftrightarrow \tilde{p})$ with induced saddles $\overline{T} : p' \leftrightarrow p''$ and $\overline{S} : \tilde{p} \leftrightarrow p''$. The winding numbers and circle numbers of the planar tangles are related by the following*

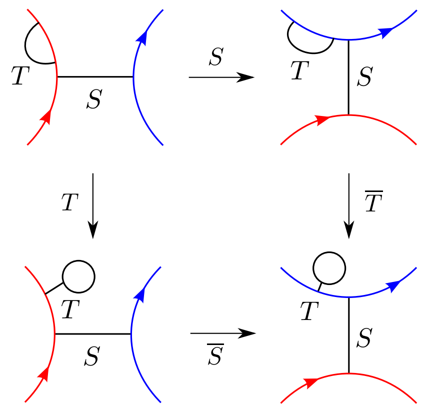

FIGURE 7. Nested saddles (S, T) and induced saddles $\overline{S}$ and $\overline{T}$.

commutative diagram:

$$\begin{array}{ccc} p_{n\pm 2}(r) & \xleftrightarrow{\;S\;} & p'_n(r) \\ \updownarrow T & & \updownarrow \overline{T} \\ \tilde{p}_{n\pm 2}(r+1) & \xleftrightarrow{\;\overline{S}\;} & p''_n(r+1), \end{array}$$

where the plus sign in $n\pm 2$ *is for* $\sigma(s_1) = R$ *and the minus sign is for* $\sigma(s_1) = L$. *The saddles* T *and* $\overline{T}$ *connect the additional circle components of* $\tilde{p}$ *and* p'' *to opposite sides of the strand (i.e.,* $\sigma(\tilde{t}_1) = \overline{\sigma(t''_1)} \in \{L, R\}$*). For* $t_1 = t_2$ *we have that* $\tau(p_{n\pm 2}) = \tau(\tilde{p}_{n\pm 2})$ *and* $\tau(p'_n) = \tau(p''_n)$*, otherwise* $\tau(p_{n\pm 2}) \neq \tau(\tilde{p}_{n\pm 2})$ *and* $\tau(p'_n) \neq \tau(p''_n)$.

Proof. The types and orientations of the saddle arcs are as shown in the following diagram:

$$\begin{array}{ccc} (s_1)^{\tau(s_1)}_{\sigma(s_1)} \leq (t_1)^{\tau(t_1)}_{\sigma(t_1)} < (t_2)^{\tau(t_1)}_{\sigma(t_1)} \leq (s_2)^{\overline{\tau(s_1)}}_{\overline{\sigma(s_1)}} & \xleftrightarrow{\;S\;} & (s'_1)^{\tau(s_1)}_{\overline{\sigma(s_1)}} \leq (t'_1)^{\overline{\tau(t_1)}}_{\overline{\sigma(t_1)}} < (t'_2)^{\overline{\tau(t_1)}}_{\overline{\sigma(t_1)}} \leq (s'_2)^{\overline{\tau(s_1)}}_{\sigma(s_1)} \\ \updownarrow T & & \updownarrow \overline{T} \\ (\tilde{s}_1)^{\tau(s_1)}_{\sigma(s_1)} \leq (\tilde{t}_1)^{\tau(t_1)}_{\overline{\sigma(t_1)}} \leq (\tilde{s}_2)^{\overline{\tau(s_1)}}_{\overline{\sigma(s_1)}},\ (\tilde{t}_2)^c_U & \xleftrightarrow{\;\overline{S}\;} & (s''_1)^{\tau(s_1)}_{\overline{\sigma(s_1)}} < (t''_1)^{\overline{\tau(t_1)}}_{\sigma(t_1)} \leq (s''_2)^{\overline{\tau(s_1)}}_{\sigma(s_1)},\ (t''_2)^c_U. \end{array}$$

The claim regarding the tangle types is clear. ☐

Definition 4.7. We say that saddles $(S : p \leftrightarrow p', T : p \leftrightarrow \tilde{p})$ are *disjoint* if $p_1 < p_2 < q_1 < q_2$ or $q_1 < q_2 < p_1 < p_2$, where (p_1, p_2) are the saddle insertion points for S in p and (q_1, q_2) are the saddle insertion points for T in p, $\tau(s_1) = \overline{\tau(s_2)} \in \{e, u\}$, and $\tau(t_1) = \tau(t_2) \in \{e, u\}$.

The commuting square for disjoint saddles $(S : p \leftrightarrow p', T : p \leftrightarrow \tilde{p})$ is depicted in Figure 8. The following lemma describes the relationships among the planar tangles in this commuting square:

Lemma 4.8. *Consider disjoint saddles* $(S : p \leftrightarrow p', T : p \leftrightarrow \tilde{p})$ *with induced saddles* $\overline{T} : p' \leftrightarrow p''$ *and* $\overline{S} : \tilde{p} \leftrightarrow p''$. *The winding numbers and circle numbers of the planar tangles are related by the following commutative diagram:*

$$\begin{array}{ccc} p_{n\pm 2}(r) & \xleftrightarrow{\;S\;} & p'_n(r) \\ \updownarrow T & & \updownarrow \overline{T} \\ \tilde{p}_{n\pm 2}(r+1) & \xleftrightarrow{\;\overline{S}\;} & p''_n(r+1), \end{array}$$

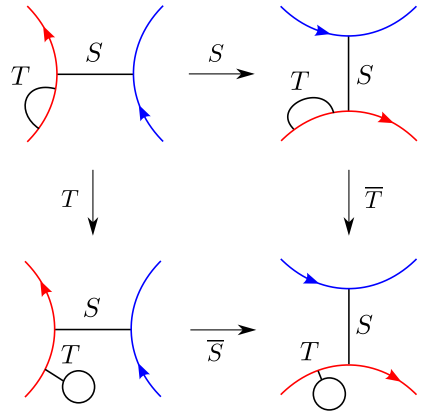

FIGURE 8. Disjoint saddles (S, T) and induced saddles $\overline{S}$ and $\overline{T}$.

where the plus sign in $n \pm 2$ *is for* $\sigma(s_1) = R$ *and the minus sign is for* $\sigma(s_1) = L$. *The saddles* T *and* $\overline{T}$ *connect the additional circle components of* $\tilde{p}$ *and* p'' *to the same side of the strand (i.e.,* $\sigma(\tilde{t}_1) = \sigma(t''_1) \in \{L, R\}$*). For* $t_1 = t_2$ *we have that* $\tau(p_{n\pm 2}) = \tau(\tilde{p}_{n\pm 2})$ *and* $\tau(p'_n) = \tau(p''_n)$*, otherwise* $\tau(p_{n\pm 2}) \neq \tau(\tilde{p}_{n\pm 2})$ *and* $\tau(p'_n) \neq \tau(p''_n)$*.*

Proof. For the case $p_1 < p_2 < q_1 < q_2$, the types and orientations of the saddle arcs are as shown in the following diagram:

$$\begin{array}{ccc}
(s_1)^{\tau(s_1)}_{\sigma(s_1)} < (s_2)^{\overline{\tau(s_1)}}_{\sigma(s_1)} \leq (t_1)^{\tau(t_1)}_{\sigma(t_1)} < (t_2)^{\tau(t_1)}_{\sigma(t_1)} & \xleftrightarrow{\;S\;} & (s'_1)^{\tau(s_1)}_{\sigma(s_1)} < (s'_2)^{\overline{\tau(s_1)}}_{\sigma(s_1)} \leq (t'_1)^{\tau(t_1)}_{\sigma(t_1)} < (t'_2)^{\tau(t_1)}_{\sigma(t_1)} \\
\updownarrow T & & \updownarrow \overline{T} \\
(\tilde{s}_1)^{\tau(s_1)}_{\sigma(s_1)} < (\tilde{s}_2)^{\overline{\tau(s_1)}}_{\sigma(s_1)} \leq (\tilde{t}'_1)^{\tau(t_1)}_{\sigma(t_1)},\ (\tilde{t}_2)^{c}_{U} & \xleftrightarrow{\;\overline{S}\;} & (s''_1)^{\tau(s_1)}_{\sigma(s_1)} < (s''_2)^{\overline{\tau(s_1)}}_{\sigma(s_1)} \leq (t''_1)^{\tau(t_1)}_{\sigma(t_1)},\ (t''_2)^{c}_{U}.
\end{array}$$

The case $q_1 < q_2 < p_1 < p_2$ is similar. The claim regarding the tangle types is clear. □

5. CHAIN COMPLEX FOR TANGLES IN THE DISK

We will now express the chain complex $(C_{T^+}, \partial_{T^+})$ in terms of planar tangles in the disk. Given an oriented 1-tangle diagram T in the annulus and an arc a such that (T, a) has loop number ℓ, let T_D denote the corresponding m-tangle diagram in the disk, where $m = \ell + 1$, as shown in Figure 1.

First we express the bigraded vector space C_{T^+} in terms of planar tangles in the disk. Recall that we defined $I = \{0, 1\}^c$ to be the set of binary strings of length c, where c is the number of crossings of T. Given a binary string $i \in I$, define $T_i \in d_{2m}$ to be the planar tangle obtained by resolving the crossings of T_D in the manner described by i. For each tangle type $P \in D_{2m}$, define a bigraded vector space

$$\tilde{C}_P = \bigoplus \{A^{\otimes c(T_i)}[r(i), r(i)] \mid i \in I \text{ such that } \tau(T_i) = P\}, \tag{1}$$

where $c(T_i)$ is the circle number of the planar tangle T_i, and a grading-shifted vector space C_P by

$$C_P = \tilde{C}_P[h_+(T) + h_+(P), q_+(T) + q_+(P)],$$

where

$$h_+(T) = -n_-(T^+), \qquad q_+(T) = n_+(T^+) - 2n_-(T^+), \tag{2}$$

$$h_+(P) = (1/2)(\ell + w(P)), \qquad q_+(P) = (1/2)(\ell + 3w(P)). \tag{3}$$

The bigraded vector space C_{T^+} can then be expressed as

$$C_{T^+} = \bigoplus_n C_n = \bigoplus_P C_P.$$

Next we express the differential ∂_{T+} in terms of planar tangles in the disk. We first define several linear maps by summing over saddles $p \to p'$ obtained from the resolution of T. For each pair of tangle types $P, P' \in D_{2m}$, define a linear map $T_{P'P} : C_P \to C_{P'}$ by summing the following terms:

(1) For a saddle $p \to p'$ with $\tau(p) = P$ and $\tau(p') = P'$ splitting a disk circle or circle arc from a strand arc:

$$\dot\eta \otimes 1_{A^{\otimes r}} : A^{\otimes r} \to A \otimes A^{\otimes r}, \qquad c(p) = r, \qquad c(p') = r + 1.$$

(2) For a saddle $p \to p'$ with $\tau(p) = P$ and $\tau(p') = P'$ merging a disk circle or circle arc with a strand arc:

$$\dot\epsilon \otimes 1_{A^{\otimes r}} : A \otimes A^{\otimes r} \to A^{\otimes r}, \qquad c(p) = r + 1, \qquad c(p') = r.$$

(3) For a saddle $p \to p'$ with $\tau(p) = P$ and $\tau(p') = P'$ splitting a disk circle or circle arc from a disk circle or circle arc:

$$\Delta \otimes 1_{A^{\otimes r}} : A \otimes A^{\otimes r} \to A \otimes A \otimes A^{\otimes r}, \qquad c(p) = r + 1, \qquad c(p') = r + 2.$$

(4) For a saddle $p \to p'$ with $\tau(p) = P$ and $\tau(p') = P'$ merging a disk circle or circle arc with a disk circle or circle arc:

$$m \otimes 1_{A^{\otimes r}} : A \otimes A \otimes A^{\otimes r} \to A \otimes A^{\otimes r}, \qquad c(p) = r + 2, \qquad c(p') = r + 1.$$

For each pair of tangle types $P, P' \in D_{2m}$, define a linear map $T^L_{P'P} : C_P \to C_{P'}$ (respectively $T^R_{P'P} : C_P \to C_{P'}$) by summing the following terms:

(1) For a saddle $p \to p'$ with $\tau(p) = P$ and $\tau(p') = P'$ splitting a disk circle or circle arc from a strand arc that connects to the strand on the left (respectively right) side:

$$\dot\eta \otimes 1_{A^{\otimes r}} : A^{\otimes r} \to A \otimes A^{\otimes r}, \qquad c(p) = r, \qquad c(p') = r + 1.$$

(2) For a saddle $p \to p'$ with $\tau(p) = P$ and $\tau(p') = P'$ merging a disk circle or circle arc with a strand arc that connects to the strand on the left (respectively right) side:

$$\dot\epsilon \otimes 1_{A^{\otimes r}} : A \otimes A^{\otimes r} \to A^{\otimes r}, \qquad c(p) = r + 1, \qquad c(p') = r.$$

For each pair of tangle types $P, P' \in D_{2m}$, define a linear map $\tilde T^L_{P'P} : C_P \to C_{P'}$ (respectively $\tilde T^R_{P'P} : C_P \to C_{P'}$) by summing the following terms:

(1) For a saddle $p \to p'$ with $\tau(p) = P$ and $\tau(p') = P'$ splitting a disk circle or circle arc from a strand arc that connects to the strand on the left (respectively right) side:

$$\eta \otimes 1_{A^{\otimes r}} : A^{\otimes r} \to A \otimes A^{\otimes r}, \qquad c(p) = r, \qquad c(p') = r + 1.$$

(2) For a saddle $p \to p'$ with $\tau(p) = P$ and $\tau(p') = P'$ merging a disk circle or circle arc with a strand arc that connects to the strand on the left (respectively right) side:

$$\epsilon \otimes 1_{A^{\otimes r}} : A \otimes A^{\otimes r} \to A^{\otimes r}, \qquad c(p) = r + 1, \qquad c(p') = r.$$

For each pair of tangle types $P, P' \in D_{2m}$, define a linear map $Q_{P'P} : C_P \to C_{P'}$ by summing the following terms:

(1) If $w(P') = w(P) - 2$, then for a saddle $p \to p'$:

$$1_{A^{\otimes r}} : A^{\otimes r} \to A^{\otimes r}, \qquad c(p) = r, \qquad c(p') = r.$$

Define

$$\partial_P = T_{PP}, \qquad \tilde\partial^L_P = \tilde T^L_{PP}, \qquad \tilde\partial^R_P = \tilde T^R_{PP}.$$

Given a saddle $S : p \to p'$ with $\tau(p) = P$ and $\tau(p') = P'$, we define, for example, $T_{P'P}(S)$ to be the term in $T_{P'P}$ corresponding to this saddle, and similarly for the other linear maps. Note that $T_{P'P}$, $\tilde T^R_{P'P}$, and $\tilde T^L_{P'P}$ correspond to saddles $p \to p'$ that preserve the winding number and change the circle number by one, and $Q_{P'P}$ corresponds to saddles that lower the winding number by two and preserve the circle number. The linear map $\partial^0_{T+} : C_{T+} \to C_{T+}$ can be expressed as

$$\partial^0_{T+} = \sum_n C_n = \sum_P \sum_{P'} T_{P'P} = \sum_P \partial_P + \sum_P \sum_{P' \neq P} T_{P'P}, \tag{4}$$

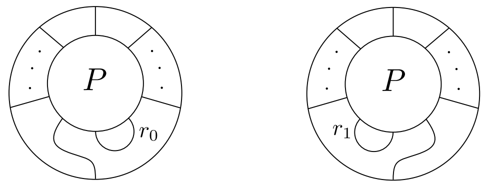

FIGURE 9. The 0-resolution $[P]_0$ and 1-resolution $[P]_1$ of a tangle type P.

and the linear map $\partial^+_{T^+} : C_{T^+} \to C_{T^+}$ can be expressed as

$$\partial^+_{T^+} = \sum_n (\tilde{\partial}^L_{n-2} Q_n + Q_n \tilde{\partial}^L_n) = \sum_P \sum_{P'} \sum_{P''} (\tilde{T}^L_{P''P'} Q_{P'P} + Q_{P''P'} \tilde{T}^L_{P'P}). \tag{5}$$

Recall that the differential $\partial_{T^+} : C_{T^+} \to C_{T^+}$ is given by $\partial_{T^+} = \partial^0_{T^+} + \partial^+_{T^+}$.

Having reformulated the chain complex $(C_{T^+}, \partial_{T^+})$ in terms of planar tangles in the disk, we can now return to the task of completing the proof of Theorem 3.1. Recall that we started with an oriented annular 1-tangle diagram T with loop number ℓ. We flipped the outermost loop of T under the annulus to obtain an annular 1-tangle diagram $\overline{T}$ with loop number $\ell - 1$ and one additional crossing, as shown in Figure 1, and we considered the chain complex $(C_{\overline{T}^+}, \partial_{\overline{T}^+})$ for the link diagram $\overline{T}^+$. Our goal is to use the Reduction Lemma 3.2 to show that $(C_{\overline{T}^+}, \partial_{\overline{T}^+})$ is chain-homotopy equivalent to $(C_{T^+}, \partial_{T^+})$. The first step is to understand the structure of $(C_{\overline{T}^+}, \partial_{\overline{T}^+})$ as expressed in terms of planar tangles in the disk.

We first consider the vector space $C_{\overline{T}^+}$. For each tangle type $P \in D_{2m}$, where $m = \ell + 1$, we define tangle types $[P]_0, [P]_1 \in D_{2m-2}$ as shown in Figure 9 by taking the 0-resolution and 1-resolution of the one additional crossing in $\overline{T}$. We refer to $[P]_0$ and $[P]_1$ as the *0-resolution* and *1-resolution* of P. We refer to the arcs r_0 and r_1 shown in Figure 9 as *resolution arcs*. We can then express $C_{\overline{T}^+}$ as

$$C_{\overline{T}^+} = \bigoplus_P \overline{C}_P, \tag{6}$$

where

$$\overline{C}_P = C_{[P]_0} \oplus C_{[P]_1}.$$

Next we consider the differential $\partial_{\overline{T}^+}$. We first want to identify the various linear maps corresponding to the saddles obtained from the resolution of $\overline{T}$. We classify these saddles into three types:

(1) Saddles $[p]_0 \to [p]_1$ obtained by resolving the one additional crossing of $\overline{T}$.
(2) Saddles $[p]_0 \to [p']_0$ and $[p]_1 \to [p']_1$ that are induced by a type-preserving (i.e. $\tau(p) = \tau(p')$) saddle $S : p \to p'$ obtained from the resolution of T.
(3) Saddles $[p]_0 \to [p']_0$ and $[p]_1 \to [p']_1$ that are induced by a type-changing (i.e. $\tau(p) \neq \tau(p')$) saddle $S : p \to p'$ obtained from the resolution of T.

For each tangle type P, we define a linear map $\overline{\partial}_P : \overline{C}_P \to \overline{C}_P$ by keeping only those terms of $\partial_{\overline{T}^+}$ that map $\overline{C}_P$ to $\overline{C}_P$; such terms are maps or products of pairs of maps corresponding to saddles of types (1) and (2). The map $\overline{\partial}_P$ has the form

$$\overline{\partial}_P = \begin{pmatrix} \partial_{[P]_0} & 0 \\ \partial_{[P]_1[P]_0} & \partial_{[P]_1} \end{pmatrix},$$

where $\partial_{[P]_1[P]_0} : C_{[P]_0} \to C_{[P]_1}$ consists of those terms of $\overline{\partial}_P$ that map $C_{[P]_0}$ to $C_{[P]_1}$. In Section 6, we describe $(\overline{C}_P, \overline{\partial}_P)$ for each tangle type $P \in D_{2m}$.

Remark 5.1. We show in Lemma 11.1 below that $(\partial^0_{T^+})^2 = 0$, but it is generally not the case that $(\overline{\partial}_P)^2 = 0$, so $(\overline{C}_P, \overline{\partial}_P)$ is not a chain complex.

The remaining saddles obtained from the resolution of $\overline{T}$ are of type (3), and are induced by saddles $S : p \to p'$ obtained from the resolution of T that change the tangle type (i.e., $\tau(p) = P$ and $\tau(p') = P' \neq P$). For each type-changing saddle $S : p \to p'$ we define a vector space

$$\overline{C}_S = \overline{C}_P \oplus \overline{C}_{P'}.$$

We define a linear map $\overline{\partial}_S : \overline{C}_S \to \overline{C}_S$ by keeping only those terms of $\partial_{\overline{T}^+}$ that are maps or products of maps corresponding to saddles used to define ∂_P or $\partial_{P'}$, or to saddles $[p]_0 \to [p']_0$ or $[p]_1 \to [p']_1$ induced by $S : p \to p'$. We describe $(\overline{C}_S, \overline{\partial}_S)$ for each type-changing saddle $S : p \to p'$. In Section 7 we describe saddles that change the winding number by two and preserve the circle number, and in Section 8 we describe saddles that preserve the winding number and change the annular circle number by one; according to Lemma 4.1, these are the only two possibilities. In Sections 9 and 10 we use the results of Sections 6, 7, and 8 to reduce the chain complex $(\overline{C}_{T^+}, \overline{\partial}_{T^+})$ and thereby complete our proof of Theorem 3.1.

We end this section with a few observations regarding the linear maps $\partial^0_{T^+}$ and $\partial^+_{T^+}$ defined in equations (4) and (5) that will be useful in what follows:

Remark 5.2. The linear map $\partial^0_{T^+}$ is a sum of maps corresponding to saddles obtained in the resolution of T that preserve the winding number.

Remark 5.3. The linear map $\partial^+_{T^+}$ is a sum of products of pairs of maps corresponding to pairs of saddles obtained from the resolution of T. The pairs of saddles whose corresponding maps are included in the sum comprise two adjacent sides of a commuting square of saddles corresponding to interleaved saddles (S, T) of the following forms:

$$(7)\qquad \begin{array}{ccc} p_n(r) & \xrightarrow{S} & p'_{n-2}(r) \\ \downarrow T & & \downarrow \overline{T} \\ \tilde{p}_{n-2}(r) & \xrightarrow{\overline{S}} & p''_{n-2}(r+1), \end{array} \qquad \begin{array}{ccc} p''_{n-2}(r) & \xleftarrow{S} & p'_n(r) \\ \uparrow T & & \uparrow \overline{T} \\ \tilde{p}_n(r) & \xleftarrow{\overline{S}} & p_n(r+1), \end{array}$$

and nested and disjoint saddles (S, T) of the following forms:

$$(8)\qquad \begin{array}{ccc} p_n(r) & \xrightarrow{S} & p'_{n-2}(r) \\ \downarrow T & & \downarrow \overline{T} \\ \tilde{p}_n(r+1) & \xrightarrow{\overline{S}} & p''_{n-2}(r+1), \end{array} \qquad \begin{array}{ccc} p''_{n-2}(r) & \xleftarrow{S} & p'_n(r) \\ \uparrow T & & \uparrow \overline{T} \\ \tilde{p}_{n-2}(r+1) & \xleftarrow{\overline{S}} & p_n(r+1), \end{array}$$

$$(9)\qquad \begin{array}{ccc} p'_n(r) & \xrightarrow{S} & p''_{n-2}(r) \\ \uparrow T & & \uparrow \overline{T} \\ p_n(r+1) & \xrightarrow{\overline{S}} & \tilde{p}_{n-2}(r+1), \end{array} \qquad \begin{array}{ccc} p'_{n-2}(r) & \xleftarrow{S} & p_n(r) \\ \downarrow T & & \downarrow \overline{T} \\ p''_{n-2}(r+1) & \xleftarrow{\overline{S}} & \tilde{p}_n(r+1). \end{array}$$

We can thus express $\partial^+_{T^+}$ in terms of a sum over all interleaved, nested, and disjoint pairs of saddles (S, T) obtained from the resolution of T. In fact, as we show in Corollary 5.5 below, disjoint saddles do not contribute, so it suffices to sum over interleaved and nested saddles.

As an application of Remark 5.3, we prove the following lemma, which shows we could just as well use right-handed maps, as opposed to left-handed maps, in the definition of $\partial^+_{T^+}$:

Lemma 5.4. *We have*

$$(10)\qquad \sum_{\overline{P}} (\tilde{T}^L_{P''\overline{P}} Q_{\overline{P}P} + Q_{P''\overline{P}} \tilde{T}^L_{\overline{P}P}) = \sum_{\overline{P}} (\tilde{T}^R_{P''\overline{P}} Q_{\overline{P}P} + Q_{P''\overline{P}} \tilde{T}^R_{\overline{P}P}).$$

Proof. The left and right sides of equation (10) are given by summing products of maps in the commuting square corresponding to interleaved saddles (S, T) of the forms shown in diagram (7) and nested and disjoint saddles (S, T) of the forms shown in diagrams (8) and (9), where $\tau(p) = P$ and $\tau(p'') = P''$. We claim that in each case the contribution of the pair (S, T) to the left and right sides of equation (10) is the same. We will show this for a few representative examples.

For the first form of interleaved saddles (S,T) shown in diagram (7), the contributions of the corresponding commuting square of saddles to the left and right sides of equation (10) are

$$\tilde{T}^L_{P''P'}(\overline{T})Q_{P'P}(S) + \tilde{T}^L_{P''\tilde{P}}(\overline{S})Q_{\tilde{P}P}(T), \qquad \tilde{T}^R_{P''P'}(\overline{T})Q_{P'P}(S) + \tilde{T}^R_{P''\tilde{P}}(\overline{S})Q_{\tilde{P}P}(T).$$

From Lemma 4.3 for interleaved saddles, it follows that

$$\tilde{T}^L_{P''\tilde{P}}(\overline{S})Q_{\tilde{P}P}(T) = \tilde{T}^R_{P''P'}(\overline{T})Q_{P'P}(S), \qquad \tilde{T}^L_{P''P'}(\overline{T})Q_{P'P}(S) = \tilde{T}^R_{P''\tilde{P}}(\overline{S})Q_{\tilde{P}P}(T) = 0.$$

So the contribution of (S,T) to the left and right sides of equation (10) is the same.

For the first form of nested saddles (S,T) shown in diagram (8), the contributions of the corresponding commuting square of saddles to the left and right sides of equation (10) are

$$\tilde{T}^L_{P''P'}(\overline{T})Q_{P'P}(S) + Q_{P''\tilde{P}}(\overline{S})\tilde{T}^L_{\tilde{P}P}(T), \qquad \tilde{T}^R_{P''P'}(\overline{T})Q_{P'P}(S) + Q_{P''\tilde{P}}(\overline{S})\tilde{T}^R_{\tilde{P}P}(T).$$

From Lemma 4.6 for nested saddles, it follows that there is an orientation $\sigma \in \{L, R\}$ such that

$$\tilde{T}^\sigma_{P''P'}(\overline{T})Q_{P'P}(S) = Q_{P''\tilde{P}}(\overline{S})\tilde{T}^{\overline{\sigma}}_{\tilde{P}P}(T), \qquad Q_{P''\tilde{P}}(\overline{S})\tilde{T}^\sigma_{\tilde{P}P}(T) = \tilde{T}^{\overline{\sigma}}_{P''P'}(\overline{T})Q_{P'P}(S) = 0.$$

So the contribution of (S,T) to the left and right sides of equation (10) is the same.

For the first form of disjoint saddles (S,T) shown in diagram (8), the contributions of the corresponding commuting square of saddles to the left and right sides of equation (10) are

$$\tilde{T}^L_{P''P'}(\overline{T})Q_{P'P}(S) + Q_{P''\tilde{P}}(\overline{S})\tilde{T}^L_{\tilde{P}P}(T), \qquad \tilde{T}^R_{P''P'}(\overline{T})Q_{P'P}(S) + Q_{P''\tilde{P}}(\overline{S})\tilde{T}^R_{\tilde{P}P}(T).$$

From Lemma 4.8 for disjoint saddles, it follows that there is an orientation $\sigma \in \{L, R\}$ such that

$$\tilde{T}^\sigma_{P''P'}(\overline{T})Q_{P'P}(S) = Q_{P''\tilde{P}}(\overline{S})\tilde{T}^{\overline{\sigma}}_{\tilde{P}P}(T), \qquad Q_{P''\tilde{P}}(\overline{S})\tilde{T}^\sigma_{\tilde{P}P}(T) = \tilde{T}^{\overline{\sigma}}_{P''P'}(\overline{T})Q_{P'P}(S) = 0.$$

So the contribution of (S,T) to the left and right sides of equation (10) is zero. □

The proof of Lemma 5.4 also proves:

Corollary 5.5. *Pairs of disjoint saddles do not contribute to $\partial^+_{T^+}$.*

6. Reduction of planar tangles

We classify each planar tangle p and tangle type P according to the types and relative positions of the arcs B and C:

$$\begin{aligned} &\text{Type P1A}: && \tau(B) = \overline{\tau(C)} \in \{e, u\} \text{ and } B < C, \\ &\text{Type P1B}: && \tau(B) = \overline{\tau(C)} \in \{e, u\} \text{ and } C < B, \\ &\text{Type P2}: && \tau(B) = \tau(C) = c. \end{aligned}$$

We say a planar tangle is type P1 if it is either type P1A or type P1B. The following lemma describes the relationship between a planar tangle p and its 0-resolution and 1-resolution $[p]_0$ and $[p]_1$:

Lemma 6.1. *For a planar tangle p with winding number $w(p) = n$ and circle number $c(p) = r$, the winding number and circle number of the planar tangles $[p]_0$ and $[p]_1$ are given by*

p	$w([p]_0)$	$w([p]_1)$	$c([p]_0)$	$c([p]_1)$
P1A	$n+1$	$n+1$	$r+1$	r
P1B	$n-1$	$n-1$	r	$r+1$
P2	$n+1$	$n-1$	$r-1$	$r-1$

Proof. First consider a type P1A planar tangle p. The union of the reduction arc r_0 and the segment of the strand starting at point $m+2$ on the disk and ending at the point $m+1$ on the disk constitutes a new circle component of $[p]_0$, hence $c([p]_0) = c(p) + 1$. The saddle $[p]_0 \to [p]_1$ connects this circle component to the strand component, hence $c([p]_1) = c(p)$ and $w([p]_1) = w([p]_0)$. Arc C is oriented inwards at point $m+1$, so in going from p to $[p]_0$ and $[p]_1$ we are removing from p a single loop that winds clockwise, hence $w([p]_0) = w([p]_1) = w(p) + 1$. The case of a type P1B planar tangle is similar.

Next consider a type P2 planar tangle p. Since the reduction arc r_0 connects the circle component of p containing B and C to the strand, we have that $c_a([p]_0) = c_a(p) - 1$. The reduction arc r_0 connects points $m+1$ and $m+2$ on the disk, and the point $m+1$ on the disk is oriented outwards, so the addition of the reduction arc r_0 to p extends the strand component by adding loops with net winding number 1. Thus $w([p]_0) = w(p) + 1$. Similarly $c_a([p]_1) = c_a(p) - 1$ and $w([p]_1) = w(p) - 1$. □

We can use Lemma 6.1 to relate the vector spaces C_P, $C_{[P]_0}$, and $C_{[P]_1}$ for the tangle types P, $[P]_0$, and $[P]_1$. We will initially view these as ungraded vector spaces, then in Lemma 6.7 we consider the bigradings. If P is of type P1A, then $[P]_0$ has one more circle component than P and $[P]_1$, so as ungraded vector spaces

$$C_{[P]_0} = C_P \otimes A, \qquad\qquad C_{[P]_1} = C_P.$$

If P is of type P1B, then $[P]_1$ has one more circle component than P and $[P]_0$, so as ungraded vector spaces

$$C_{[P]_0} = C_P, \qquad\qquad C_{[P]_1} = C_P \otimes A.$$

If P is of type P2, then P has one more circle component than $[P]_0$ and $[P]_1$, so can define a vector space W such that as ungraded vector spaces $C_P = W \otimes A$ and

$$C_{[P]_0} = W, \qquad\qquad C_{[P]_1} = W.$$

We can also relate the linear maps ∂_P, $\partial_{[P]_0}$, and $\partial_{[P]_1}$ for the tangle types P, $[P]_0$, and $[P]_1$. Recall from Lemma 6.1 that if P is of type P1A then $[P]_0$ has one more circle component than P and $[P]_1$, if P is of type P1B then $[P]_1$ has one more circle component than P and $[P]_0$, and if P is of type P2 then P has one more circle component than $[P]_0$ and $[P]_1$. Given a tangle type P of type P1A, P1B, or P2, define linear maps $\tilde{\partial}_P^{c_0} : C_P \to C_P$, $\tilde{\partial}_P^{c_1} : C_P \to C_P$, or $\tilde{\partial}_P^{c} : W \to W$ by summing the following terms over saddles $p \to p'$ obtained from the resolution of T:

(1) For a saddle $p \to p'$ with $\tau(p) = \tau(p') = P$ splitting a disk circle from a circle arc in the additional circle component of $[P]_0$, $[P]_1$, or P:

$$\eta \otimes 1_{A^{\otimes r}} : A^{\otimes r} \to A \otimes A^{\otimes r}, \qquad c(p) = r, \qquad c(p') = r + 1.$$

(2) For a saddle $p \to p'$ with $\tau(p) = \tau(p') = P$ merging a disk circle and a circle arc in the additional circle component of $[P]_0$, $[P]_1$, or P:

$$\epsilon \otimes 1_{A^{\otimes r}} : A \otimes A^{\otimes r} \to A^{\otimes r}, \qquad c(p) = r + 1, \qquad c(p') = r.$$

For a tangle type P of type P1, we can use the linear maps $\tilde{\partial}_P^{c_0}$ and $\tilde{\partial}_P^{c_1}$, together with the fact that $m = \dot{\epsilon} \otimes 1_A + \epsilon \otimes 1_{x1}$ and $\Delta = \dot{\eta} \otimes 1_A + \eta \otimes 1_{x1}$, to express the linear maps $\partial_{[P]_0}$ and $\partial_{[P]_1}$ in terms of ∂_P. If P is of type P1A, then

$$(C_{[P]_0}, \partial_{[P]_0}) = (C_P \otimes A, \partial_P \otimes 1_A + \tilde{\partial}_P^{c_0} \otimes 1_{x1}), \qquad (C_{[P]_1}, \partial_{[P]_1}) = (C_P, \partial_P).$$

If P is of type P1B, then

$$(C_{[P]_0}, \partial_{[P]_0}) = (C_P, \partial_P), \qquad (C_{[P]_1}, \partial_{[P]_1}) = (C_P \otimes A, \partial_P \otimes 1_A + \tilde{\partial}_P^{c_1} \otimes 1_{x1}).$$

The relationship between ∂_P, $\partial_{[P]_0}$, and $\partial_{[P]_1}$ for a tangle type P of type P2 is described by Lemma 6.3 below.

Recall from Section 5 that for each tangle type P we defined a vector space $\overline{C}_P$ and a linear map $\overline{\partial}_P : \overline{C}_P \to \overline{C}_P$ by

$$\overline{C}_P = C_{[P]_0} \oplus C_{[P]_1}, \qquad \overline{\partial}_P = \begin{pmatrix} \partial_{[P]_0} & 0 \\ \partial_{[P]_1[P]_0} & \partial_{[P]_1} \end{pmatrix},$$

where $\partial_{[P]_1[P]_0} : C_{[P]_0} \to C_{[P]_1}$ consists of those terms of $\overline{\partial}_P$ that map $C_{[P]_0}$ to $C_{[P]_1}$. The relationship between $(\overline{C}_P, \overline{\partial}_P)$ and (C_P, ∂_P) is described by the following results:

Lemma 6.2. *For a type P1A tangle type P, we have*

$$\overline{C}_P = (C_P \otimes A) \oplus C_P, \qquad \overline{\partial}_P = \begin{pmatrix} \partial_P \otimes 1_A + \tilde{\partial}_P^{c_0} \otimes 1_{x1} & 0 \\ 1_{C_P} \otimes \dot{\epsilon} & \partial_P \end{pmatrix},$$

where $\tilde{\partial}_P^{c_0} : C_P \to C_P$ is the linear map corresponding to saddles connecting disk circles to arcs in P that lie on the additional circle component of $[P]_0$. For a type P1B tangle type P, we have

$$\overline{C}_P = C_P \oplus (C_P \otimes A), \qquad \overline{\partial}_P = \begin{pmatrix} \partial_P & 0 \\ 1_{C_P} \otimes \dot{\eta} & \partial_P \otimes 1_A + \tilde{\partial}_P^{c_1} \otimes 1_{x1} \end{pmatrix},$$

where $\tilde{\partial}_P^{c_1} : C_P \to C_P$ is the linear map corresponding to saddles connecting disk circles to arcs in P that lie on the additional circle component of $[P]_1$.

Proof. We will prove the claim for P of type P1A; the case of type P1B is similar. We have that

$$\overline{C}_P = C_{[P]_0} \oplus C_{[P]_1} = (C_P \otimes A) \oplus C_P.$$

The linear maps $C_{[P]_0} \to C_{[P]_1}$ corresponding to saddles $[p]_0 \to [p]_1$ obtained by resolving the one additional crossing of $\overline{T}$ are described by the diagram

$$\begin{array}{c} C_{[P]_0} = (C_P \otimes A)_{n+1} \\ {\scriptstyle T_{[P]_1[P]_0} = 1_{C_P} \otimes \dot{\epsilon}} \\ {\scriptstyle \tilde{T}^L_{[P]_1[P]_0} = 1_{C_P} \otimes \epsilon} \Big\downarrow \\ C_{[P]_1} = (C_P)_{n+1}, \end{array}$$

where $n = w(P)$ is the winding number of P. Note that such saddles merge the additional circle component of $[p]_0$ with the strand. The claim now follows from the definition of $\partial_{\overline{T}^+}$. □

Lemma 6.3. *For each type P2 tangle type P, we have $(\overline{C}_P, \overline{\partial}_P) = (C_P, \partial_P)$.*

Proof. We define

$$(W, \partial_W) := (C_{[P]_0}, \partial_{[P]_0}) = (C_{[P]_1}, \partial_{[P]_1}).$$

Then

$$\overline{C}_P = C_{[P]_0} \oplus C_{[P]_1} = (W)_{n+1} \oplus (W)_{n-1},$$

where $n = w(P)$ is the winding number of P. The linear map $C_{[P]_0} \to C_{[P]_1}$ corresponding to saddles $[p]_0 \to [p]_1$ obtained by resolving the one additional crossing of $\overline{T}$ is described by the diagram

$$\begin{array}{c} C_{[P]_0} = (W)_{n+1} \\ {\scriptstyle Q_{[P]_1[P]_0} = 1_W} \Big\downarrow \\ C_{[P]_1} = (W)_{n-1}. \end{array}$$

From the definition of $\partial_{\overline{T}^+}$, it follows that

$$\overline{\partial}_P = \begin{pmatrix} \partial_W & 0 \\ \tilde{\partial}^L_{[P]_1} Q_{[P]_1[P]_0} + Q_{[P]_1[P]_0} \tilde{\partial}^L_{[P]_0} & \partial_W \end{pmatrix}.$$

Substituting $Q_{[P]_1[P]_0} = 1_W$, we obtain

$$\tilde{\partial}^L_{[P]_1} Q_{[P]_1[P]_0} + Q_{[P]_1[P]_0} \tilde{\partial}^L_{[P]_0} = \tilde{\partial}^L_{[P]_1} + \tilde{\partial}^L_{[P]_0}.$$

We claim that

$$\tilde{\partial}^L_{[P]_1} + \tilde{\partial}^L_{[P]_0} = \tilde{\partial}^c_W, \tag{11}$$

where $\tilde{\partial}^c_W : W \to W$ is the linear map corresponding to saddles connecting disk circles to circle arcs in planar tangles p of type $\tau(p) = P$ that lie on the additional circle component of p that is not present in $[p]_0$ or $[p]_1$. We can see this as follows. Consider a saddle connecting a disk circle to an arc a of a planar tangle p with tangle type $\tau(p) = P$. The arc a of p lies on a unique arc $[a]_0$ of $[p]_0$ and $[a]_1$ of $[p]_1$. If a lies in the strand component of p we have

$$\sigma(a) = \sigma([a]_0) = \sigma([a]_1) \in \{L, R\},$$

so such saddles do not contribute to either side of equation (11). If a lies in the additional annular circle of p, which contains the arcs B and C, we have

$$\sigma(a) = U, \qquad \sigma([a]_0) = \overline{\sigma([a]_1)} \in \{L, R\},$$

so such saddles contribute equally to the left and right sides of equation (11). If a lies in an annular circle of p that does not contain the arcs B and C, and hence is also present in $[p]_0$ and $[p]_1$, we have

$$\sigma(a) = \sigma([a]_0) = \sigma([a]_1) = U,$$

so such saddles do not contribute to either side of equation (11). So in each case the contribution to both sides of equation (11) is the same.

We can identify $W \otimes 1_A := (W)_{n+1}$ and $W \otimes x := (W)_{n-1}$ to obtain

$$\overline{C}_P = (W)_{n+1} \oplus (W)_{n-1} = W \otimes A = C_P, \qquad \overline{\partial}_P = \begin{pmatrix} \partial_W & 0 \\ \tilde{\partial}^c_W & \partial_W \end{pmatrix} = \partial_W \otimes 1_A + \tilde{\partial}^c_W \otimes 1_{x1} = \partial_P.$$

□

Recall from equation (6) that the vector space $C_{\overline{T}^+}$ for the chain complex $(C_{\overline{T}^+}, \partial_{\overline{T}^+})$ is the direct sum of the spaces $\overline{C}_P$ over all tangle types P. The differential $\partial_{\overline{T}^+}$ contains all of the terms of $\overline{\partial}_P$ for each tangle type P. Lemma 6.2 shows that the chain complex $(C_{\overline{T}^+}, \overline{\partial}_{\overline{T}^+})$ is such that for each tangle type P of type type P1 we can apply the Reduction Lemma 3.2. In particular, for P of type P1A we can reduce on the map $1_{C_P} \otimes \dot{\epsilon} : C_P \otimes 1_A \to C_P$ in $\overline{\partial}_P$, and for P of type P1B we can reduce on the map $1_{C_P} \otimes \dot{\eta} : C_P \to C_P \otimes x$ in $\overline{\partial}_P$. We will refer to such a reduction as a *reduction on* P. We can thus make the following definition:

Definition 6.4. We define $(C_{red}, \partial_{red})$ to be the chain complex that results from reducing $(C_{\overline{T}^+}, \partial_{\overline{T}^+})$ on P for each type P1 planar tangle P (i.e, reducing on the map $1_{C_P} \otimes \dot{\epsilon} : C_P \otimes 1_A \to C_P$ for P of type P1A and reducing on the map $1_{C_P} \otimes \dot{\eta} : C_P \to C_P \otimes x$ for P of type P1B).

For a type P1A tangle type P, we identify C_P with the subspace $C_P \otimes x$ of $\overline{C}_P = (C_P \otimes A) \oplus C_P$. Define a subspace V_P of $C_{\overline{T}^+}$ by

$$V_P = \bigoplus_{P' \neq P} \overline{C}_{P'} \oplus (C_P \otimes x).$$

Linear maps

$$\alpha_1 : V_P \to C_P \otimes 1_A, \qquad \beta_2 : C_P \to V_P$$

that occur in $\partial_{\overline{T}^+}$ are eliminated when we reduce on P, and linear maps

$$\alpha_2 : V_P \to C_P, \qquad \beta_1 : C_P \otimes 1_A \to V_P$$

that occur in $\partial_{\overline{T}^+}$ yield reduction factors when we reduce on P.

For a type P1B tangle type P, we identify C_P with the subspace $C_P \otimes 1_A$ of $\overline{C}_P = C_P \oplus (C_P \otimes A)$. Define a subspace V_P of $C_{\overline{T}^+}$ by

$$V_P = \bigoplus_{P' \neq P} \overline{C}_{P'} \oplus (C_P \otimes 1_A).$$

Linear maps

$$\alpha_1 : V_P \to C_P, \qquad \beta_2 : C_P \otimes x \to V_P$$

that occur in $\partial_{\overline{T}^+}$ are eliminated when we reduce on P, and linear maps

$$\alpha_2 : V_P \to C_P \otimes x, \qquad \beta_1 : C_P \to V_P$$

that occur in $\partial_{\overline{T}^+}$ yield reduction factors when we reduce on P.

Thus, from Lemmas 6.2 and 6.3 we obtain the following lemma:

Lemma 6.5. *For each type P1A tangle type P, we obtain a residual term ∂_P and a reduction factor*

$$\beta_1(P) = \tilde{\partial}^{c_0}_P : C_P \to C_P.$$

For each type P1B tangle type P, we obtain a residual term ∂_P and reduction factor

$$\alpha_2(P) = \tilde{\partial}^{c_1}_P : C_P \to C_P.$$

For each type P2 tangle P, we obtain a residual term ∂_P and no reduction factors.

To complete the proof of Theorem 3.1, we want to show that $(C_{red}, \partial_{red}) = (C_{T^+}, \partial_{T^+})$. Lemmas 6.2 and 6.3 show that

Lemma 6.6. *We have $C_{red} = C_{T^+}$ as ungraded vector spaces.*

We now discuss the bigrading of the vector space C_P. Recall that in equation (1) we defined a bigraded vector space $\tilde{C}_P$. The following lemma explains the grading shifts described in equations (2) and (3):

Lemma 6.7. *For a tangle type P of loop number ℓ, the bigraded vector space C_P is given by*

$$C_P = \tilde{C}_P[h_+(T,P), q_+(T,P)],$$

where

$$h_+(T,P) = h_+(T) + h_+(P), \qquad h_+(T) = -n_-(T^+), \qquad h_+(P) = (1/2)(\ell + w(P)), \tag{12}$$

$$q_+(T,P) = q_+(T) + q_+(P), \qquad q_+(T) = n_+(T^+) - 2n_-(T^+), \qquad q_+(P) = (1/2)(\ell + 3w(P)). \tag{13}$$

Proof. We will prove the claim by induction on the loop number ℓ. It is clearly true for the base case $\ell = 0$, since there is a unique tangle type P_0, as shown in Figure 4, and

$$C_{P_0} = \tilde{C}_{P_0}[h_+(T,P_0),\, q_+(T,P_0)] = \tilde{C}_{P_0}[h_+(T),\, q_+(T)]$$

is the underlying bigraded vector space of the chain complex for the reduced Khovanov homology of the link $T^+ = T^-$.

For the induction step, assume the claim is true for loop number $\ell - 1$, and let P be a tangle type with loop number ℓ. By the induction hypothesis, the bigrading of $\overline{C}_P$ is given by

$$\overline{C}_P = C_{[P]_0} \oplus C_{[P]_1}[1,1] = \tilde{C}_{[P]_0}[h_+(T,[P]_0), q_+(T,[P]_0)] \oplus \tilde{C}_{[P]_1}[h_+(T,[P]_1)+1, q_+(T,[P]_1)+1]. \tag{14}$$

We will use equation (14) to determine the bigrading of C_P for each tangle type P.

If P is of type P1A, then

$$\tilde{C}_{[P]_0} = \tilde{C}_P \otimes A, \qquad \tilde{C}_{[P]_1} = \tilde{C}_P.$$

When we reduce on P, we identify C_P as a grading-shifted version of the subspace $\tilde{C}_P \otimes x$ of $\tilde{C}_{[P]_0}$, and x has quantum grading -1, so from equation (14) it follows that

$$C_P = \tilde{C}_P[h_+(T,[P]_0),\, q_+(T,[P]_0) - 1].$$

The loop number of $[P]_0$ is $\ell - 1$, and by Lemma 6.1 the winding number of $[P]_0$ is $w([P]_0) = w(P) + 1$, so equations (12) and (13) imply that

$$(h_+(T,[P]_0),\, q_+(T,[P]_0) - 1) = (h_+(T,P),\, q_+(T,P)). \tag{15}$$

Thus the claim holds if P is of type P1A.

If P is of type P1B, then

$$\tilde{C}_{[P]_0} = \tilde{C}_P, \qquad \tilde{C}_{[P]_1} = \tilde{C}_P \otimes A.$$

When we reduce on P, we identify C_P as a grading-shifted version of the subspace $\tilde{C}_P \otimes 1_A$ of $\tilde{C}_{[P]_1}$, and 1_A has quantum grading 1, so from equation (14) it follows that

$$C_P = \tilde{C}_P[h_+(T,[P]_1) + 1,\, q_+(T,[P]_0) + 2].$$

The loop number of $[P]_1$ is $\ell - 1$, and by Lemma 6.1 the winding number of $[P]_1$ is $w([P]_0) = w(P) - 1$, so equations (12) and (13) imply that

$$(h_+(T,[P]_1) + 1,\, q_+(T,[P]_1) + 2) = (h_+(T,P),\, q_+(T,P)). \tag{16}$$

Thus the claim holds if P is of type P1B.

If P is of type P2, then we can define a bigraded vector space W such that

$$\tilde{C}_P = W \otimes A = (W \otimes 1_A) \oplus (W \otimes x), \qquad \tilde{C}_{[P]_0} = W, \qquad \tilde{C}_{[P]_1} = W. \tag{17}$$

The loop numbers of $[P]_0$ and $[P]_1$ are $\ell - 1$, and by Lemma 6.1 the winding numbers of $[P]_0$ and $[P]_1$ are $w([P]_0) = w(P) + 1$ and $w([P]_1) = w(P) - 1$, so equations (15) and (16) hold for $[P]_0$ and $[P]_1$. We have $C_P = \overline{C}_P$ as bigraded vector spaces, so from equations (14), (15), (16), and (17), it follows that

$$C_P = \overline{C}_P = W[h_+(T,P),\, q_+(T,P) + 1] \oplus W[h_+(T,P),\, q_+(T,P) - 1] = \tilde{C}_P[h_+(T,P),\, q_+(T,P)],$$

where we have used the fact that the quantum gradings of 1_A and x are 1 and -1. Thus the claim holds if P is of type P2. □

Thus we obtain:

Lemma 6.8. *We have $C_{red} = C_{T^+}$ as bigraded vector spaces.*

7. Reduction of type-changing saddles $p_{n+1}(r) \leftrightarrow p'_{n-1}(r)$

Recall from Section 5 that for each type-changing saddle $S : p \to p'$ obtained from the resolution of T we defined a vector space $\overline{C}_S$ and a linear map $\overline{\partial}_S : \overline{C}_S \to \overline{C}_S$. The vector space $\overline{C}_S$ is given by

$$\overline{C}_S = \overline{C}_P \oplus \overline{C}_{P'}$$

for $P = \tau(p)$ and $P' = \tau(p')$. The linear map $\overline{\partial}_S$ is obtained by keeping only those terms of $\partial_{\overline{T}^+}$ that are linear maps or products of linear maps corresponding to saddles used to define ∂_P or $\partial_{P'}$, or to saddles $[p]_0 \to [p']_0$ or $[p]_1 \to [p']_1$ induced by $S : p \to p'$.

For a type-changing saddle $p_{n+1}(r) \leftrightarrow p'_{n-1}(r)$, Lemma 4.1 implies that the saddle arcs (s_1, s_2) in $p_{n+1}(r)$ are such that $\tau(s_1) = \overline{\tau(s_2)} \in \{e, u\}$. We classify saddles $p_{n+1}(r) \leftrightarrow p'_{n-1}(r)$ into the following types, depending on the types and relative positions of the arcs s_1, s_2, B, and C of $p_{n+1}(r)$:

Type W1:	$\tau(B) = \overline{\tau(C)} \in \{e, u\}$ and either $B < C \le s_1 < s_2$ or $C < B \le s_1 < s_2$,
Type W2:	$\tau(B) = \overline{\tau(C)} \in \{e, u\}$ and either $s_1 \le B < C \le s_2$ or $s_1 \le C < B \le s_2$,
Type W3:	$\tau(B) = \overline{\tau(C)} \in \{e, u\}$ and either $s_1 < s_2 \le B < C$ or $s_1 < s_2 \le C < B$,
Type W4:	$\tau(B) = \tau(C) = c$.

For a saddle $S : p \to p'$, the linear map $\overline{\partial}_S$ has the form

$$\overline{\partial}_S = \begin{pmatrix} \overline{\partial}_P & 0 \\ \overline{\partial}_{P'P}(S) & \overline{\partial}_{P'} \end{pmatrix}$$

for some linear map $\overline{\partial}_{P'P}(S) : \overline{C}_P \to \overline{C}_{P'}$. For each type of saddle, we describe the residual terms and reduction factors obtained from $\overline{\partial}_{P'P}(S)$. (Recall that in Lemma 6.5 we described the residual terms and reduction factors obtained from $\overline{\partial}_P$ and $\overline{\partial}_{P'}$.)

Lemma 7.1. *Type W1 saddles $p_{n+1}(r) \leftrightarrow p'_{n-1}(r)$ do not actually occur.*

Proof. We will prove the claim for the case $B < C$; the case $C < B$ is similar. We observe that the union of the resolution arc r_0 and the segment of the strand starting at point $m+2$ on arc C and ending at point $m+1$ on arc D is an inessential circle in the annulus, so a saddle connecting two arcs in this segment cannot change the winding number. □

Lemma 7.2. *Type W2 saddles $S : p_{n+1}(r) \to p'_{n-1}(r)$ must have $C < B$.*

Proof. For a type W2 saddle the arcs B and C lie between the saddle arcs s_1 and s_2, hence the order of the arcs B and C in P must be opposite to the order of the arcs B' and C' in P'. The saddle $p_{n+1}(r) \leftrightarrow p'_{n-1}(r)$ induces saddles $[p_{n+1}(r)]_k \leftrightarrow [p'_{n-1}(r)]_k$ for $k = 0, 1$. By Lemma 6.1, if $C < B$ and $B' < C'$ then $w([p'_{n-1}(r)]_k) - w([p_{n+1}(r)]_k) = 0$, whereas if $B < C$ and $C' < B'$ then $w([p'_{n-1}(r)]_k) - w([p_{n+1}(r)]_k) = -4$, which is not possible. It also follows from Lemma 6.1 that the induced saddles $[p_{n+1}(r)]_k \leftrightarrow [p'_{n-1}(r)]_k$ connect circles components to the strand. □

Lemma 7.3. *For a type W2 saddle $S : p_{n+1}(r) \to p'_{n-1}(r)$ we have the following terms. There are no residual terms. Reduction on P_{n+1} yields the reduction factor*

$$\beta_1(P, S) = Q_{P'P}(S) : C_P \to C_{P'}.$$

Reduction on P'_{n-1} yields the reduction factor

$$\alpha_2(P', S) = Q_{P'P}(S) : C_P \to C_{P'}.$$

For a type W2 saddle $S : p'_{n-1}(r) \to p_{n+1}(r)$, there are no reduction factors or residual terms.

Proof. We have

$$\overline{C}_S = \overline{C}_P \oplus \overline{C}_{P'}, \qquad \overline{C}_P = (C_P)_n \oplus (C_P \otimes A)_n, \qquad \overline{C}_{P'} = (C_{P'} \otimes A)_n \oplus (C_{P'})_n.$$

We have linear maps $C_{[P]_0} \to C_{[P]_1}$ and $C_{[P']_0} \to C_{[P']_1}$ corresponding to saddles obtained by resolving the one additional crossing of $\overline{T}$, and linear maps $C_{[P]_0} \leftrightarrow C_{[P']_0}$ and $C_{[P]_1} \leftrightarrow C_{[P']_1}$ corresponding to saddles $[p_{n+1}(r)]_0 \leftrightarrow [p'_{n-1}(r)]_0$ and $[p_{n+1}(r)]_1 \leftrightarrow [p'_{n-1}(r)]_1$ induced by $S : p_{n+1}(r) \leftrightarrow p'_{n-1}(r)$:

$$\begin{array}{ccc} C_{[P]_0} = (C_P)_n & \overset{\substack{T_{[P']_0[P]_0}=Q_{P'P}(S)\otimes\dot{\eta} \\ \tilde{T}^L_{[P']_0[P]_0}=Q_{P'P}(S)\otimes\eta}}{\underset{\substack{T_{[P]_0[P']_0}=P_{PP'}(S)\otimes\dot{\epsilon} \\ \tilde{T}^L_{[P]_0[P']_0}=P_{PP'}(S)\otimes\epsilon}}{\rightleftarrows}} & C_{[P']_0} = (C_{P'} \otimes A)_n \\ {\scriptstyle T_{[P]_1[P]_0}=1_{C_P}\otimes\dot{\eta}}\big\downarrow & & \big\downarrow{\scriptstyle \substack{T_{[P']_1[P']_0}=1_{C_{P'}}\otimes\dot{\epsilon} \\ \tilde{T}^L_{[P']_1[P']_0}=1_{C_{P'}}\otimes\epsilon}} \\ C_{[P]_0} = (C_P \otimes A)_n & \overset{T_{[P']_1[P]_1}=Q_{P'P}(S)\otimes\dot{\epsilon}}{\underset{T_{[P]_1[P']_1}=P_{PP'}(S)\otimes\dot{\eta}}{\rightleftarrows}} & C_{[P']_1} = (C_{P'})_n. \end{array} \tag{18}$$

For a saddle $S : p_{n+1}(r) \to p'_{n-1}(r)$, the map $\overline{\partial}_{P'P}(S) : \overline{C}_P \to \overline{C}_{P'}$ is given by

$$\overline{\partial}_{P'P}(S) = \begin{pmatrix} Q_{P'P}(S) \otimes \dot{\eta} & 0 \\ 0 & Q_{P'P}(S) \otimes \dot{\epsilon} \end{pmatrix}.$$

The claim for a saddle $S : p_{n+1}(r) \to p'_{n-1}(r)$ now follows from reading off terms in $\overline{\partial}_{P'P}(S)$. For a saddle $S : p'_{n-1}(r) \to p_{n+1}(r)$, the map $\overline{\partial}_{PP'}(S) : \overline{C}_{P'} \to \overline{C}_P$ is given by

$$\overline{\partial}_{PP'}(S) = \begin{pmatrix} P_{PP'}(S) \otimes \dot{\epsilon} & 0 \\ 0 & P_{PP'}(S) \otimes \dot{\eta} \end{pmatrix}.$$

The claim for a saddle $S : p'_{n-1}(r) \to p_{n+1}(r)$ now follows from reading off terms in $\overline{\partial}_{PP'}(S)$. Note, for example, that the map $\alpha_1(P, S) := P_{PP'}(S) \otimes \dot{\epsilon} : C_{P'} \otimes 1_A \to C_P$ is eliminated when we reduce on P, and the map $\beta_2(P', S) : P_{PP'}(S) \otimes \dot{\eta} : C_{P'} \to C_P \otimes x$ is eliminated when we reduce on P'. ☐

Lemma 7.4. *For a type W3 or W4 saddle $S : p_{n+1}(r) \to p'_{n-1}(r)$, we have the following terms. There are no reduction factors. We have the residual term*

$$\tilde{\partial}^L_{P'} Q_{P'P}(S) + Q_{P'P}(S) \tilde{\partial}^L_P : C_P \to C_{P'}.$$

For a type W3 or W4 saddle $S : p'_{n-1}(r) \to p_{n+1}(r)$, there are no reduction factors or residual terms.

Proof. First consider a type W3 saddle. We will prove the claim for the case $B < C$; the case $C < B$ is similar. We have

$$\overline{C}_S = \overline{C}_P \oplus \overline{C}_{P'}, \qquad \overline{C}_P = (C_P \otimes A)_{n+2} \oplus (C_P)_{n+2}, \qquad \overline{C}_{P'} = (C_{P'} \otimes A)_n \oplus (C_{P'})_n.$$

We have linear maps $C_{[P]_0} \to C_{[P]_1}$ and $C_{[P']_0} \to C_{[P']_1}$ corresponding to saddles obtained by resolving the one additional crossing of $\overline{T}$, and linear maps $C_{[P]_0} \leftrightarrow C_{[P']_0}$ and $C_{[P]_1} \leftrightarrow C_{[P']_1}$ corresponding to saddles $[p_{n+1}(r)]_0 \leftrightarrow [p'_{n-1}(r)]_0$ and $[p_{n+1}(r)]_1 \leftrightarrow [p'_{n-1}(r)]_1$ induced by $S : p_{n+1}(r) \leftrightarrow p'_{n-1}(r)$:

$$\begin{array}{ccc} C_{[P]_0} = (C_P \otimes A)_{n+2} & \xrightarrow{Q_{[P']_0[P]_0}=Q_{P'P}(S)\otimes 1_A} & C_{[P']_0} = (C_{P'} \otimes A)_n \\ {\scriptstyle \substack{T_{[P']_0[P]_0}=1_{C_P}\otimes\dot{\epsilon} \\ \tilde{T}^L_{[P']_0[P]_0}=1_{C_P}\otimes\epsilon}}\big\downarrow & & \big\downarrow{\scriptstyle \substack{T_{[P']_1[P]_1}=1_{C_{P'}}\otimes\dot{\epsilon} \\ \tilde{T}^L_{[P']_1[P]_1}=1_{C_{P'}}\otimes\epsilon}} \\ C_{[P]_1} = (C_P)_{n+2} & \xrightarrow{Q_{[P']_1[P]_1}=Q_{P'P}(S)} & C_{[P']_1} = (C_{P'})_n. \end{array} \tag{19}$$

For a type W3 saddle $S : p_{n+1}(r) \to p'_{n-1}(r)$, the map $\overline{\partial}_{P'P}(S) : \overline{C}_P \to \overline{C}_{P'}$ is

$$\overline{\partial}_{P'P}(S) = \begin{pmatrix} \tilde{\partial}^L_{[P']_0} Q_{[P']_0[P]_0} + Q_{[P']_0[P]_0} \tilde{\partial}^L_{[P]_0} & 0 \\ 0 & \tilde{\partial}^L_{[P']_1} Q_{[P']_1[P]_1} + Q_{[P']_1[P]_1} \tilde{\partial}^L_{[P]_1} \end{pmatrix}.$$

The fact that the matrix element $(C_P \otimes A)_{n+2} \to (C_{P'})_n$ is zero follows from the fact that the saddles $(S : [p_{n+1}(r)]_1 \to [p'_{n-1}(r)]_1,\, T : [p_{n+1}(r)]_0 \to [p_{n+1}(r)]_1)$ are disjoint.

We claim that

$$\tilde{\partial}^L_{[P']_0} Q_{[P']_0[P]_0} + Q_{[P']_0[P]_0} \tilde{\partial}^L_{[P]_0} = (\tilde{\partial}^L_{P'} Q_{P'P}(S) + Q_{P'P}(S) \tilde{\partial}^L_P) \otimes 1_A. \tag{20}$$

We can see this as follows. Let (p_1, p_2) denote the saddle insertion points of S in p_{n+1}. Consider a saddle connecting a disk circle to an arc a of p_{n+1} with saddle insertion point $q \in a$. There is a corresponding saddle connecting the disk circle to an arc a' of p'_{n-1} with saddle insertion point $q' \in a'$. The arc a of p_{n+1} lies on a unique arc or disk circle $[a]_0$ of $[p_{n+1}]_0$. The arc a' of p'_{n-1} lies on a unique arc or disk circle $[a']_0$ of $[p'_{n-1}]_0$. If the arc a is a strand arc, the orientations of these arcs are as follows:

$$\begin{aligned}
&q < p_1 : && \sigma(a) = \sigma([a]_0) = \sigma(a') = \sigma([a']_0) \in \{L, R\},\\
&p_1 < q < p_2 : && \sigma(a) = \sigma([a]_0) = \overline{\sigma(a')} = \overline{\sigma([a']_0)} \in \{L, R\},\\
&p_2 < q,\, a \leq B : && \sigma(a) = \sigma([a]_0) = \sigma(a') = \sigma([a']_0) \in \{L, R\},\\
&p_2 < q,\, C \leq a : && \sigma(a) = \sigma(a') \in \{L, R\},\ \sigma([a]_0) = \sigma([a']_0) = U.
\end{aligned}$$

If the arc a is a circle arc, we have

$$\sigma(a) = \sigma([a]_0) = \sigma(a') = \sigma([a']_0) = U.$$

In each case, the contribution to both sides of equation (20) is the same. The claim for a type W3 saddle $S : p_{n+1}(r) \to p'_{n-1}(r)$ now follows from reading off terms in $\overline{\partial}_{P'P}(S)$. For a type W3 saddle $S : p'_{n-1}(r) \to p_{n+1}(r)$, the map $\overline{\partial}_{PP'}(S) : \overline{C}_{P'} \to \overline{C}_P$ is zero, thus proving the claim for this type of saddle.

Next consider a type W4 saddle. We have

$$\overline{C}_S = \overline{C}_P \oplus \overline{C}_{P'}, \qquad \overline{C}_P = C_P = (W)_{n+2} \oplus (W)_n, \qquad \overline{C}_{P'} = C_{P'} = (W')_n \oplus (W')_{n-2}.$$

We can define vector spaces W and W' such that $C_P = W \otimes A$, $C_{P'} = W' \otimes A$, and $Q_{P'P}(S) : C_P \to C_{P'}$ is given by $Q_{P'P}(S) = [Q_{P'P}] \otimes 1_A$ for a linear map $[Q_{P'P}] : W \to W'$. We have linear maps $C_{[P]_0} \to C_{[P]_1}$ and $C_{[P']_0} \to C_{[P']_1}$ corresponding to saddles obtained by resolving the one additional crossing of $\overline{T}$, and linear maps $C_{[P]_0} \leftrightarrow C_{[P']_0}$ and $C_{[P]_1} \leftrightarrow C_{[P']_1}$ corresponding to saddles $[p_{n+1}(r)]_0 \leftrightarrow [p'_{n-1}(r)]_0$ and $[p_{n+1}(r)]_1 \leftrightarrow [p'_{n-1}(r)]_1$ induced by $S : p_{n+1}(r) \leftrightarrow p'_{n-1}(r)$:

$$\begin{array}{ccc}
C_{[P]_0} = (W)_{n+2} & \xrightarrow{Q_{[P']_0[P]_0} = [Q_{P'P}]} & C_{[P']_0} = (W')_n \\
{\scriptstyle Q_{[P]_1[P]_0} = 1_W}\big\downarrow & & \big\downarrow{\scriptstyle Q_{[P']_1[P']_0} = 1_{W'}} \\
C_{[P]_1} = (W)_n & \xrightarrow{Q_{[P']_1[P]_1} = [Q_{P'P}]} & C_{[P']_1} = (W')_{n-2}.
\end{array} \tag{21}$$

For a type W4 saddle $S : p_{n+1}(r) \to p'_{n-1}(r)$, the map $\overline{\partial}_{P'P}(S) : \overline{C}_P \to \overline{C}_{P'}$ is

$$\overline{\partial}_{P'P}(S) = \begin{pmatrix} \tilde{\partial}^L_{[P']_0} Q_{[P']_0[P]_0} + Q_{[P']_0[P]_0} \tilde{\partial}^L_{[P]_0} & 0 \\ 0 & \tilde{\partial}^L_{[P']_1} Q_{[P']_1[P]_1} + Q_{[P']_1[P]_1} \tilde{\partial}^L_{[P]_1} \end{pmatrix}.$$

An argument similar to that used for type W3 saddles shows that

$$\overline{\partial}_{P'P}(S) = (\tilde{\partial}^L_{W'}[Q_{P'P}] + [Q_{P'P}]\tilde{\partial}^L_W) \otimes 1_A = \tilde{\partial}^L_{P'} Q_{P'P}(S) + Q_{P'P}(S) \tilde{\partial}^L_P,$$

thus proving the claim for this type of saddle. For a type W4 saddle $S : p'_{n-1}(r) \to p_{n+1}(r)$, the map $\overline{\partial}_{PP'}(S) : \overline{C}_{P'} \to \overline{C}_P$ is zero, thus proving the claim for this type of saddle. □

8. Reduction of type-changing saddles $p_n(r) \leftrightarrow p'_n(r+1)$

Recall from Section 5 that for each type-changing saddle $S : p \to p'$ obtained from the resolution of T we defined a vector space $\overline{C}_S$ and a linear map $\overline{\partial}_S : \overline{C}_S \to \overline{C}_S$. The vector space $\overline{C}_S$ is given by

$$\overline{C}_S = \overline{C}_P \oplus \overline{C}_{P'}$$

for $P = \tau(p)$ and $P' = \tau(p')$. The linear map $\overline{\partial}_S$ is obtained by keeping only those terms of $\partial_{\overline{T}^+}$ that are linear maps or products of linear maps corresponding to saddles used to define ∂_P or $\partial_{P'}$, or to saddles $[p]_0 \to [p']_0$ or $[p]_1 \to [p']_1$ induced by $S : p \to p'$.

For a type-changing saddle $p_n(r) \leftrightarrow p'_n(r+1)$, Lemma 4.1 implies that the saddle arcs (s_1, s_2) in $p_n(r)$ are such that $\tau(s_1) = \tau(s_2) \in \{e, u\}$ or $\tau(s_1) = \tau(s_2) = c$. We classify saddles $p_n(r) \leftrightarrow p'_n(r+1)$ into the following types, depending on the types and relative positions of the arcs s_1, s_2, B, and C of $p_n(r)$:

Type C1: $\tau(B) = \overline{\tau(C)} \in \{e, u\}$, $\tau(s_1) = \tau(s_2) \in \{e, u\}$, and either $B < C \leq s_1 < s_2$ or $C < B \leq s_1 < s_2$,

Type C2: $\tau(B) = \overline{\tau(C)} \in \{e, u\}$, $\tau(s_1) = \tau(s_2) \in \{e, u\}$, and either $s_1 \leq B < C \leq s_2$ or $s_1 \leq C < B \leq s_2$,

Type C3: $\tau(B) = \overline{\tau(C)} \in \{e, u\}$, $\tau(s_1) = \tau(s_2) \in \{e, u\}$, and either $s_1 < s_2 \leq B < C$ or $s_1 < s_2 \leq C < B$,

Type C4: $\tau(B) = \overline{\tau(C)} \in \{e, u\}$ and $\tau(s_1) = \tau(s_2) = c$,

Type C5: $\tau(B) = \tau(C) = c$, and (s_1, s_2) do not lie on the circle component containing arcs B and C,

Type C6: $\tau(B) = \tau(C) = c$, and (s_1, s_2) lie on the circle component containing arcs B and C.

For a saddle $S : p \to p'$, the linear map $\overline{\partial}_S$ has the form

$$\overline{\partial}_S = \begin{pmatrix} \overline{\partial}_P & 0 \\ \overline{\partial}_{P'P}(S) & \overline{\partial}_{P'} \end{pmatrix}$$

for some linear map $\overline{\partial}_{P'P}(S) : \overline{C}_P \to \overline{C}_{P'}$. For each type of saddle, we describe the residual terms and reduction factors obtained from $\overline{\partial}_{P'P}(S)$. (Recall that in Lemma 6.5 we described the residual terms and reduction factors obtained from $\overline{\partial}_P$ and $\overline{\partial}_{P'}$.)

For saddles $p(r) \leftrightarrow p'_n(r+1)$ connecting a strand arc and a circle arc, we can define a vector space W' such that

$$(C_{P'}, \partial_{P'}) = (W' \otimes A, \partial_{W'} \otimes 1_A + \tilde{\partial}^c_{W'} \otimes 1_{x1}),$$

where $\tilde{\partial}^c_{W'} : W' \to W'$ is the linear map corresponding to saddles connecting disk circles to circle arcs in P' that lie in the additional circle component of P'. We can express $T_{P'P}(S) : C_P \to C_{P'}$ as $T_{P'P}(S) = [T_{P'P}] \otimes \dot{\eta}$ for a linear map $[T_{P'P}] : C_P \to W'$, and we can express $T_{PP'}(S) : C_{P'} \to C_P$ as $T_{PP'}(S) = [T_{PP'}] \otimes \dot{\epsilon}$ for a linear map $[T_{PP'}] : W' \to C_P$.

Lemma 8.1. *For a type C1 saddle $S : p_n(r) \to p'_n(r+1)$ we have the following terms. We have the residual term*

$$T_{P'P}(S) : C_P \to C_{P'}.$$

For $B < C$, reduction on P_n yields the reduction factor

$$\beta_1(P, S) = \tilde{T}^L_{P'P}(S) + \tilde{T}^R_{P'P}(S) : C_P \to C_{P'}.$$

For $C < B$, reduction on P'_n yields the reduction factor

$$\alpha_2(P', S) = \tilde{T}^L_{P'P}(S) + \tilde{T}^R_{P'P}(S) : C_P \to C_{P'}.$$

For a type C1 saddle $S : p'_n(r+1) \to p_n(r)$ we have the following terms. We have the residual term

$$T_{PP'}(S) : C_{P'} \to C_P.$$

For $B < C$, reduction on P'_n yields the reduction factor

$$\beta_1(P', S) = \tilde{T}^L_{PP'}(S) + \tilde{T}^R_{PP'}(S) : C_{P'} \to C_P.$$

For $C < B$, reduction on P_n yields the reduction factor

$$\alpha_2(P, S) = \tilde{T}^L_{PP'}(S) + \tilde{T}^R_{PP'}(S) : C_{P'} \to C_P.$$

Proof. We will prove the claim for the case $B < C$; the case $C < B$ is similar. We have

$$\overline{C}_S = \overline{C}_P \oplus \overline{C}_{P'}, \qquad \overline{C}_P = (C_P \otimes A)_{n+1} \oplus (C_P)_{n+1}, \qquad \overline{C}_{P'} = (C_{P'} \otimes A)_{n+1} \oplus (C_{P'})_{n+1}.$$

We have linear maps $C_{[P]_0} \to C_{[P]_1}$ and $C_{[P']_0} \to C_{[P']_1}$ corresponding to saddles obtained by resolving the one additional crossing of $\overline{T}$, and linear maps $C_{[P]_0} \leftrightarrow C_{[P']_0}$ and $C_{[P]_1} \leftrightarrow C_{[P']_1}$ corresponding to saddles

$[p_n(r)]_0 \leftrightarrow [p'_n(r+1)]_0$ and $[p_n(r)]_1 \leftrightarrow [p'_n(r+1)]_1$ induced by $S : p_n(r) \leftrightarrow p'_n(r+1)$:

(22)
$$\begin{array}{ccc}
C_{[P]_0} = (C_P \otimes A)_{n+1} & \overset{T_{[P']_0[P]_0}=[T_{P'P}]\otimes\Delta}{\underset{T_{[P]_0[P']_0}=[T_{PP'}]\otimes m}{\rightleftarrows}} & C_{[P']_0} = (C_{P'} \otimes A)_{n+1} \\
\Big\downarrow {\scriptstyle T_{[P]_1[P]_0}=1_{C_P}\otimes\dot\epsilon,\ \tilde{T}^L_{[P]_1[P]_0}=1_{C_P}\otimes\epsilon} & & \Big\downarrow {\scriptstyle T_{[P']_1[P']_0}=1_{C_{P'}}\otimes\dot\epsilon,\ \tilde{T}^L_{[P']_1[P']_0}=1_{C_{P'}}\otimes\epsilon} \\
C_{[P]_1} = (C_P)_{n+1} & \overset{T_{[P']_1[P]_1}=[T_{P'P}]\otimes\dot\eta,\ \tilde{T}^L_{[P']_1[P]_1}=[T^R_{P'P}]\otimes\eta}{\underset{T_{[P]_1[P']_1}=[T_{PP'}]\otimes\dot\epsilon,\ \tilde{T}^L_{[P]_1[P']_1}=[T^R_{PP'}]\otimes\epsilon}{\rightleftarrows}} & C_{[P']_1} = (C_{P'})_{n+1}.
\end{array}$$

For a saddle $S : p_n(r) \to p'_n(r+1)$, the map $\overline{\partial}_{P'P}(S) : \overline{C}_P \to \overline{C}_{P'}$ is given by

$$\overline{\partial}_{P'P}(S) = \begin{pmatrix} [T_{P'P}] \otimes \Delta & 0 \\ 0 & [T_{P'P}] \otimes \dot\eta \end{pmatrix}.$$

The claim for a saddle $S : p_n(r) \to p'_n(r+1)$ now follows from reading off terms in $\overline{\partial}_{P'P}(S)$ and using the fact that $\Delta = \dot\eta \otimes 1_A + \eta \otimes 1_{x1}$. The case of a saddle $S : p'_n(r+1) \to p_n(r)$ is similar. ☐

Lemma 8.2. *For a type C2 saddle $S : p_n(r) \to p'_n(r+1)$ we have the following terms. We have the residual term*

$$T_{P'P}(S) : C_P \to C_{P'}.$$

For $B < C$, reduction on P_n yields the reduction factor

$$\beta_1(P, S) = \tilde{T}^L_{P'P}(S) + \tilde{T}^R_{P'P}(S) : C_P \to C_{P'}.$$

For a type C2 saddle $S : p'_n(r+1) \to p_n(r)$ we have the following terms. We have the residual term

$$T_{PP'}(S) : C_{P'} \to C_P.$$

For $C < B$, reduction on P_n yields the reduction factor

$$\alpha_2(P, S) = \tilde{T}^L_{PP'}(S) + \tilde{T}^R_{PP'}(S) : C_{P'} \to C_P.$$

Proof. We will prove the claim for the case $B < C$; the case $C < B$ is similar. We have

$$\overline{C}_S = \overline{C}_P \oplus \overline{C}_{P'}, \qquad \overline{C}_P = (C_P \otimes A)_{n+1} \oplus (C_P)_{n+1}, \qquad \overline{C}_{P'} = (W')_{n+1} \oplus (W')_{n-1}.$$

We have linear maps $C_{[P]_0} \to C_{[P]_1}$ and $C_{[P']_0} \to C_{[P']_1}$ corresponding to saddles obtained by resolving the one additional crossing of $\overline{T}$, and linear maps $C_{[P]_0} \leftrightarrow C_{[P']_0}$ and $C_{[P]_1} \leftrightarrow C_{[P']_1}$ corresponding to saddles $[p_n(r)]_0 \leftrightarrow [p'_n(r+1)]_0$ and $[p_n(r)]_1 \leftrightarrow [p'_n(r+1)]_1$ induced by $S : p_n(r) \leftrightarrow p'_n(r+1)$:

(23)
$$\begin{array}{ccc}
C_{[P]_0} = (C_P \otimes A)_{n+1} & \overset{T_{[P']_0[P]_0}=[T_{P'P}]\otimes\dot\epsilon,\ \tilde{T}^L_{[P']_0[P]_0}=[T^R_{P'P}]\otimes\epsilon}{\underset{T_{[P]_0[P']_0}=[T_{PP'}]\otimes\dot\eta,\ \tilde{T}^L_{[P]_0[P']_0}=[T^R_{PP'}]\otimes\eta}{\rightleftarrows}} & C_{[P']_0} = (W')_{n+1} \\
\Big\downarrow {\scriptstyle T_{[P]_1[P]_0}=1_{C_P}\otimes\dot\epsilon,\ \tilde{T}^L_{[P]_1[P]_0}=1_{C_P}\otimes\epsilon} & & \Big\downarrow {\scriptstyle Q_{[P']_1[P']_0}=1_{W'}} \\
C_{[P]_1} = (C_P)_{n+1} & \xrightarrow{Q_{[P']_1[P]_1}=[T_{P'P}]} & C_{[P']_1} = (W')_{n-1}.
\end{array}$$

For a saddle $S : p_n(r) \to p'_n(r+1)$, the map $\overline{\partial}_{P'P}(S) : \overline{C}_P \to \overline{C}_{P'}$ is given by

$$\overline{\partial}_{P'P}(S) = \begin{pmatrix} [T_{P'P}] \otimes \dot\epsilon & 0 \\ [T_{P'P}] \otimes \epsilon & \tilde{\partial}^L_{[P']_1}[T_{P'P}] + [T_{P'P}]\tilde{\partial}^L_{[P]_1} \end{pmatrix}.$$

The matrix element $[T_{P'P}] \otimes \epsilon : (C_P \otimes A)_{n+1} \to (W')_{n-1}$ is obtained from the interleaved saddles ($S : [p_n(r)]_1 \to [p'_n(r+1)]_1$, $T : [p'_n(r+1)]_0 \to [p'_n(r+1)]_1$). The claim for a saddle $S : p_n(r) \to p'_n(r+1)$ now follows from reading off terms in $\overline{\partial}_{P'P}(S)$. For a saddle $S : p'_n(r+1) \to p_n(r)$, the map $\overline{\partial}_{PP'}(S) : \overline{C}_{P'} \to \overline{C}_P$ is given by

$$\overline{\partial}_{PP'}(S) = \begin{pmatrix} [T_{PP'}] \otimes \dot\eta & 0 \\ 0 & 0 \end{pmatrix}.$$

The claim for a saddle $S : p'_n(r+1) \to p_n(r)$ now follows from reading off terms in $\overline{\partial}_{PP'}(S)$.

□

Lemma 8.3. *For a type C3, C4, C5, or C6 saddle $S : p_n(r) \to p'_n(r+1)$, we have the following terms. There are no reduction factors. We have the residual term*

$$T_{P'P}(S) : C_P \to C_{P'}.$$

For a type C3, C4, C5, or C6 saddle $S : p'_n(r+1) \to p_n(r)$, we have the following terms. There are no reduction factors. We have the residual term

$$T_{PP'}(S) : C_{P'} \to C_P.$$

Proof. First consider a type C3 saddle. We will prove the claim for the case $B < C$; the case $C < B$ is similar. We have

$$\overline{C}_S = \overline{C}_P \oplus \overline{C}_{P'}, \qquad \overline{C}_P = (C_P \otimes A)_{n+1} \oplus (C_P)_{n+1}, \qquad \overline{C}_{P'} = (C_{P'} \otimes A)_{n+1} \oplus (C_{P'})_{n+1}.$$

We have linear maps $C_{[P]_0} \to C_{[P]_1}$ and $C_{[P']_0} \to C_{[P']_1}$ corresponding to saddles obtained by resolving the one additional crossing of $\overline{T}$, and linear maps $C_{[P]_0} \leftrightarrow C_{[P']_0}$ and $C_{[P]_1} \leftrightarrow C_{[P']_1}$ corresponding to saddles $[p_n(r)]_0 \leftrightarrow [p'_n(r+1)]_0$ and $[p_n(r)]_1 \leftrightarrow [p'_n(r+1)]_1$ induced by $S : p_n(r) \leftrightarrow p'_n(r+1)$:

$$\begin{array}{ccc} C_{[P]_0} = (C_P \otimes A)_{n+1} & \underset{\substack{T_{[P]_0[P']_0} = [T_{PP'}]\otimes\dot{\epsilon}\otimes 1_A \\ \tilde{T}^L_{[P]_0[P']_0} = [T^L_{PP'}]\otimes\epsilon\otimes 1_A}}{\overset{\substack{T_{[P']_0[P]_0} = [T_{P'P}]\otimes\dot{\eta}\otimes 1_A \\ \tilde{T}^L_{[P']_0[P]_0} = [T^L_{P'P}]\otimes\eta\otimes 1_A}}{\rightleftarrows}} & C_{[P']_0} = (C_{P'} \otimes A)_{n+1} \\ {\scriptstyle T_{[P]_1[P]_0} = 1_{C_P}\otimes\dot{\epsilon},\ \tilde{T}^L_{[P]_1[P]_0} = 1_{C_P}\otimes\epsilon}\Big\downarrow & & \Big\downarrow{\scriptstyle T_{[P']_1[P']_0} = 1_{C_{P'}}\otimes\dot{\epsilon},\ \tilde{T}^L_{[P']_1[P']_0} = 1_{C_{P'}}\otimes\epsilon} \\ C_{[P]_1} = (C_P)_{n+1} & \underset{\substack{T_{[P]_1[P']_1} = [T_{PP'}]\otimes\dot{\epsilon} \\ \tilde{T}^L_{[P]_1[P']_1} = [T^L_{PP'}]\otimes\epsilon}}{\overset{\substack{T_{[P']_1[P]_1} = [T_{P'P}]\otimes\dot{\eta} \\ \tilde{T}^L_{[P']_1[P]_1} = [T^L_{P'P}]\otimes\eta}}{\rightleftarrows}} & C_{[P']_1} = (C_{P'})_{n+1}. \end{array} \tag{24}$$

For a type C3 saddle $S : p_n(r) \to p'_n(r+1)$, the map $\overline{\partial}_{P'P}(S) : \overline{C}_P \to \overline{C}_{P'}$ is given by

$$\overline{\partial}_{P'P}(S) = \begin{pmatrix} [T_{P'P}] \otimes \dot{\eta} \otimes 1_A & 0 \\ 0 & [T_{P'P}] \otimes \dot{\eta} \end{pmatrix}.$$

The claim for type C3 saddles now follows from reading off terms in $\overline{\partial}_{P'P}(S)$. The case of a type C3 saddle $S : p'_n(r+1) \to p_n(r)$ is similar.

Next consider a type C4 saddle. We will prove the claim for the case $B < C$; the case $C < B$ is similar. A circle component in P that does not contain the arcs B and C is split into two circle components in P', so we can define vector spaces W and W' such that $C_P = W \otimes A$ and $C_{P'} = W' \otimes A \otimes A$. We can express $T_{P'P}(S)$ as $T_{P'P}(S) = [T_{P'P}] \otimes \Delta$ for a linear map $[T_{P'P}] : W \to W'$, and we can express $T_{PP'}(S)$ as $T_{PP'}(S) = [T_{PP'}] \otimes m$ for a linear map $[T_{PP'}] : W' \to W$. We have

$$\overline{C}_S = \overline{C}_P \oplus \overline{C}_{P'}, \qquad \overline{C}_P = (C_P \otimes A)_{n+1} \oplus (C_P)_{n+1}, \qquad \overline{C}_{P'} = (C_{P'} \otimes A)_{n+1} \oplus (C_{P'})_{n+1}.$$

We have linear maps $C_{[P]_0} \to C_{[P]_1}$ and $C_{[P']_0} \to C_{[P']_1}$ corresponding to saddles obtained by resolving the one additional crossing of $\overline{T}$, and linear maps $C_{[P]_0} \leftrightarrow C_{[P']_0}$ and $C_{[P]_1} \leftrightarrow C_{[P']_1}$ corresponding to saddles $[p_n(r)]_0 \leftrightarrow [p'_n(r+1)]_0$ and $[p_n(r)]_1 \leftrightarrow [p'_n(r+1)]_1$ induced by $S : p_n(r) \leftrightarrow p'_n(r+1)$:

$$\begin{array}{ccc} C_{[P]_0} = (C_P \otimes A)_{n+1} & \underset{T_{[P]_0[P']_0} = [T_{PP'}]\otimes m\otimes 1_A}{\overset{T_{[P']_0[P]_0} = [T_{P'P}]\otimes\Delta\otimes 1_A}{\rightleftarrows}} & C_{[P']_0} = (C_{P'} \otimes A)_{n+1} \\ {\scriptstyle T_{[P]_1[P]_0} = 1_{C_P}\otimes\dot{\epsilon},\ \tilde{T}^L_{[P]_1[P]_0} = 1_{C_P}\otimes\epsilon}\Big\downarrow & & \Big\downarrow{\scriptstyle T_{[P']_1[P']_0} = 1_{C_{P'}}\otimes\dot{\epsilon},\ \tilde{T}^L_{[P']_1[P']_0} = 1_{C_{P'}}\otimes\epsilon} \\ C_{[P]_1} = (C_P)_{n+1} & \underset{T_{[P]_1[P']_1} = [T_{PP'}]\otimes m}{\overset{T_{[P']_1[P]_1} = [T_{P'P}]\otimes\Delta}{\rightleftarrows}} & C_{[P']_1} = (C_{P'})_{n+1}. \end{array} \tag{25}$$

For a type C4 saddle $S : p_n(r) \to p'_n(r+1)$, the map $\overline{\partial}_{P'P}(S) : \overline{C}_P \to \overline{C}_{P'}$ is given by

$$\overline{\partial}_{P'P}(S) \begin{pmatrix} [T_{P'P}] \otimes \Delta \otimes 1_A & 0 \\ 0 & [T_{P'P}] \otimes \Delta \end{pmatrix}.$$

The claim for type C4 saddles now follows from reading off terms in $\overline{\partial}_{P'P}(S)$. The case of a type C4 saddle $S : p'_n(r+1) \to p_n(r)$ is similar.

We claim that for a type C5 or C6 saddle $S : p(r) \to p'(r+1)$ we have

$$\overline{C}_S = \overline{C}_P \oplus \overline{C}_{P'}, \qquad \overline{C}_P = C_P, \qquad \overline{C}_{P'} = C_{P'}, \qquad \overline{\partial}_{P'P}(S) = T_{P'P}(S).$$

Consider a type C5 saddle. We can define vector spaces W and W' such that $C_P = W \otimes A$ and $C_{P'} = W' \otimes A$. We can express $T_{P'P}(S)$ as $T_{P'P}(S) = [T_{P'P}] \otimes 1_A$ for a linear map $[T_{P'P}] : W \to W'$, and we can express $T_{PP'}(S)$ as $T_{PP'}(S) = [T_{PP'}] \otimes 1_A$ for a linear map $[T_{PP'}] : W' \to W$. We have

$$\overline{C}_P = C_P = (W)_{n+1} \oplus (W)_{n-1}, \qquad \overline{C}_{P'} = C_{P'} = (W')_{n+1} \oplus (W')_{n-1}.$$

We have linear maps $C_{[P]_0} \to C_{[P]_1}$ and $C_{[P']_0} \to C_{[P']_1}$ corresponding to saddles obtained by resolving the one additional crossing of $\overline{T}$, and linear maps $C_{[P]_0} \leftrightarrow C_{[P']_0}$ and $C_{[P]_1} \leftrightarrow C_{[P']_1}$ corresponding to saddles $[p_n(r)]_0 \leftrightarrow [p'_n(r+1)]_0$ and $[p_n(r)]_1 \leftrightarrow [p'_n(r+1)]_1$ induced by $S : p_n(r) \leftrightarrow p'_n(r+1)$:

$$\begin{array}{ccc}
C_{[P]_0} = (W)_{n+1} & \underset{\substack{T_{[P]_0[P']_0}=[T_{PP'}] \\ \tilde{T}^L_{[P]_0[P']_0}=[\tilde{T}^L_{PP'}]}}{\overset{\substack{T_{[P']_0[P]_0}=[T_{P'P}] \\ \tilde{T}^L_{[P']_0[P]_0}=[\tilde{T}^L_{P'P}]}}{\rightleftarrows}} & C_{[P']_0} = (W')_{n+1} \\
\Big\downarrow{\scriptstyle Q_{[P]_1[P]_0}=1_W} & & \Big\downarrow{\scriptstyle Q_{[P']_1[P']_0}=1_{W'}} \\
C_{[P]_1} = (W)_{n-1} & \underset{\substack{T_{[P]_1[P']_1}=[T_{PP'}] \\ \tilde{T}^L_{[P]_1[P']_1}=[\tilde{T}^L_{PP'}]}}{\overset{\substack{T_{[P']_1[P]_1}=[T_{P'P}] \\ \tilde{T}^L_{[P']_1[P]_1}=[\tilde{T}^L_{P'P}]}}{\rightleftarrows}} & C_{[P']_1} = (W')_{n-1}.
\end{array} \tag{26}$$

For a type C5 saddle $S : p_n(r) \to p'_n(r+1)$, the map $\overline{\partial}_{P'P}(S) : \overline{C}_P \to \overline{C}_{P'}$ is given by

$$\overline{\partial}_{P'P}(S) = \begin{pmatrix} [T_{P'P}] & 0 \\ 0 & [T_{P'P}] \end{pmatrix} = [T_{P'P}] \otimes 1_A = T_{P'P}(S).$$

The case of a type C5 saddle $S : p'_n(r+1) \to p_n(r)$ is similar.

Consider a type C6 saddle. A circle component in P that contains the arcs B and C is split into two circle components in P', so we can define vector spaces W and W' such that $C_P = W \otimes A$ and $C_{P'} = W' \otimes A \otimes A$. We can express $T_{P'P}(S)$ as $T_{P'P}(S) = [T_{P'P}] \otimes \Delta$ for a linear map $[T_{P'P}] : W \to W'$, and we can express $T_{PP'}(S)$ as $T_{PP'}(S) = [T_{PP'}] \otimes m$ for a linear map $[T_{PP'}] : W' \to W$. We have

$$\overline{C}_S = \overline{C}_P \oplus \overline{C}_{P'}, \qquad \overline{C}_P = C_P = (W)_{n+1} \oplus (W)_{n-1}, \qquad \overline{C}_{P'} = C_{P'} = (W' \otimes A)_{n+1} \oplus (W' \otimes A)_{n-1}.$$

We have linear maps $C_{[P]_0} \to C_{[P]_1}$ and $C_{[P']_0} \to C_{[P']_1}$ corresponding to saddles obtained by resolving the one additional crossing of $\overline{T}$, and linear maps $C_{[P]_0} \leftrightarrow C_{[P']_0}$ and $C_{[P]_1} \leftrightarrow C_{[P']_1}$ corresponding to saddles $[p_n(r)]_0 \leftrightarrow [p'_n(r+1)]_0$ and $[p_n(r)]_1 \leftrightarrow [p'_n(r+1)]_1$ induced by $S : p_n(r) \leftrightarrow p'_n(r+1)$:

$$\begin{array}{ccc}
C_{[P]_0} = (W)_{n+1} & \underset{\substack{T_{[P]_0[P']_0}=[T_{PP'}]\otimes\dot{\epsilon} \\ \tilde{T}^L_{[P]_0[P']_0}=[T^{\sigma}_{PP'}]\otimes\epsilon}}{\overset{\substack{T_{[P']_0[P]_0}=[T_{P'P}]\otimes\dot{\eta} \\ \tilde{T}^L_{[P']_0[P]_0}=[T^{\sigma}_{P'P}]\otimes\eta}}{\rightleftarrows}} & C_{[P']_0} = (W' \otimes A)_{n+1} \\
\Big\downarrow{\scriptstyle Q_{[P]_1[P]_0}=1_W} & & \Big\downarrow{\scriptstyle Q_{[P']_1[P']_0}=1_{W'}\otimes 1_A} \\
C_{[P]_1} = (W)_{n-1} & \underset{\substack{T_{[P]_1[P']_1}=[T_{PP'}]\otimes\dot{\epsilon} \\ \tilde{T}^L_{[P]_1[P']_1}=[T^{\overline{\sigma}}_{PP'}]\otimes\epsilon}}{\overset{\substack{T_{[P']_1[P]_1}=[T_{P'P}]\otimes\dot{\eta} \\ \tilde{T}^L_{[P']_1[P]_1}=[T^{\overline{\sigma}}_{P'P}]\otimes\eta}}{\rightleftarrows}} & C_{[P']_1} = (W' \otimes A)_{n-1}.
\end{array} \tag{27}$$

The orientation $\sigma \in \{L, R\}$ depends on the particular saddle. For a type C6 saddle $S : p(r) \to p'(r+1)$, the map $\overline{\partial}_{P'P}(S) : \overline{C}_P \to \overline{C}_{P'}$ is given by

$$\overline{\partial}_{P'P}(S) = \begin{pmatrix} [T_{P'P}] \otimes \dot{\eta} & 0 \\ [T_{P'P}] \otimes \eta & [T_{P'P}] \otimes \dot{\eta} \end{pmatrix} = [T_{P'P}] \otimes \dot{\eta} \otimes 1_A + [T_{P'P}] \otimes \eta \otimes 1_{x1} = [T_{P'P}] \otimes \Delta = T_{P'P}(S).$$

The matrix element $[T_{P'P}] \otimes \eta : (W)_{n+1} \to (W' \otimes A)_{n-1}$ is obtained from the nested saddles $(S : [p_n(r)]_0 \to [p_n(r)]_1,\ T : [p_n(r)]_0 \to [p'_n(r+1)]_0)$. The case of a type C6 saddle $S : p'(r+1) \to p(r)$ is similar. $\square$

9. All terms of $\partial^0_{T^+}$ occur in ∂_{red}

We are now ready to complete the proof of Theorem 3.1. Recall that in Definition 6.4 we defined the reduction $(C_{red}, \partial_{red})$ of the chain complex $(C_{\overline{T}^+}, \partial_{\overline{T}^+})$. In Lemma 6.8 we showed that $C_{red} = C_{T^+}$ as bigraded vector spaces. It remains to show that $\partial_{red} = \partial_{T^+}$. In this section we show that all the terms of ∂_{T^+} occur in ∂_{red}, and in Section 10 we show that all the terms of ∂_{red} occur in ∂_{T^+}. Recall that $\partial_{T^+} = \partial^0_{T^+} + \partial^+_{T^+}$, where

$$\partial^0_{T^+} = \sum_P \partial_P + \sum_P \sum_{P' \neq P} T_{P'P}, \qquad \partial^+_{T^+} = \sum_P \sum_{P'} \sum_{P''} (\tilde{T}^L_{P''P'} Q_{P'P} + Q_{P''P'} \tilde{T}^L_{P'P}).$$

Lemma 9.1. *All the terms of $\partial^0_{T^+}$ occur as residual terms of ∂_{red}.*

Proof. In Lemma 6.5 we showed that for each tangle type P we obtain ∂_P as a residual term. For $P' \neq P$, we have

$$T_{P'P} = \sum_S T_{P'P}(S),$$

where the sum is taken over all type-changing saddles $S : p_n(r) \to p'_n(r \pm 1)$ with $\tau(p) = P$ and $\tau(p') = P'$. In Section 8 we showed that for each type-changing saddle $S : p_n(r) \to p'_n(r \pm 1)$ we obtain the residual term $T_{P'P}(S) : C_P \to C_{P'}$. □

Lemma 9.2. *All the terms of $\partial^+_{T^+}$ occur as residual or reduction terms of ∂_{red}.*

Proof. We can express $\partial^+_{T^+}$ as

$$\partial^+_{T^+} = \sum_P \sum_{\overline{P}} \sum_{P''} (\tilde{T}^L_{P''\overline{P}} Q_{\overline{P}P} + Q_{P''\overline{P}} \tilde{T}^L_{\overline{P}P}) = \sum_P \sum_{P''} \partial_{P''P},$$

where

$$\partial_{P''P} = \sum_{\overline{P}} (\tilde{T}^L_{P''\overline{P}} Q_{\overline{P}P} + Q_{P''\overline{P}} \tilde{T}^L_{\overline{P}P}).$$

We can express $\partial_{P''P}$ as $\partial_{P''P} = \partial^{=}_{P''P} + \partial^{\neq}_{P''P}$, where

$$\partial^{=}_{P''P} = \tilde{\partial}^L_{P''} Q_{P''P} + Q_{P''P} \tilde{\partial}^L_P, \qquad \partial^{\neq}_{P''P} = \sum_{\overline{P} \neq P''} \tilde{T}^L_{P''\overline{P}} Q_{\overline{P}P} + \sum_{\overline{P} \neq P} Q_{P''\overline{P}} \tilde{T}^L_{\overline{P}P}.$$

Lemma 9.4 below shows that each term of $\partial^{=}_{P''P}$ is a term of ∂_{red}.

As discussed in Remark 5.3, the map $\partial^{\neq}_{P''P} : C_P \to C_{P''}$ is a sum of products of pairs of maps corresponding to pairs of saddles obtained from the resolution of T. The pairs of saddles whose corresponding maps are included in the sum comprise two adjacent sides of a commuting square of saddles corresponding to interleaved saddles (S, T) of the forms shown in diagram (7) and nested saddles (S, T) of the forms shown in diagrams (8) and (9), where $\tau(p) = P$, $\tau(p'') = P''$, and all saddles in the commuting square change the tangle type. We claim that the contribution of each such pair occurs in ∂_{red}. We will show this for a few representative examples.

For the first form of interleaved saddles (S, T) shown in diagram (7), the contribution to $\partial^{\neq}_{P''P}$ is

$$\tilde{T}^L_{P''P'}(\overline{T}) Q_{P'P}(S) + \tilde{T}^L_{P''\tilde{P}}(\overline{S}) Q_{\tilde{P}P}(T).$$

Lemma 9.5 below shows that this contribution occurs in ∂_{red}. For the first form of nested saddles (S, T) shown in diagram (8), the contribution to $\partial^{\neq}_{P''P}$ is

$$\tilde{T}^L_{P''P'}(\overline{T}) Q_{P'P}(S) + Q_{P''\tilde{P}}(\overline{S}) \tilde{T}^L_{\tilde{P}P}(T).$$

Lemma 9.6 below shows that this contribution occurs in ∂_{red}. □

Lemmas 9.1 and 9.2 give

Lemma 9.3. *All the terms of ∂_{T^+} occur as residual or reduction terms of ∂_{red}.*

We now complete the proof of Lemma 9.2:

Lemma 9.4. *For each saddle $S : p_{n+1}(r) \to p'_{n-1}(r)$ with $\tau(p) = P$ and $\tau(p') = P'$, we obtain a reduction or residual term*

$$\tilde{\partial}^L_{P'} Q_{P'P}(S) + Q_{P'P}(S)\tilde{\partial}^L_P : C_P \to C_{P'}.$$

Proof. For saddles $S : p_{n+1}(r) \to p'_{n-1}(r)$ of type W2, Lemmas 6.5 and 7.3 show that we obtain the term $\tilde{\partial}^{c_0}_{P'} Q_{P'P}(S)$ from the reduction on P' and the term $Q_{P'P}(S)\tilde{\partial}^{c_1}_P$ from the reduction on P. We claim

$$\tilde{\partial}^{c_0}_{P'} Q_{P'P}(S) + Q_{P'P}(S)\tilde{\partial}^{c_1}_P = \tilde{\partial}^L_{P'} Q_{P'P}(S) + Q_{P'P}(S)\tilde{\partial}^L_P. \tag{28}$$

We can see this as follows. Let (p_1, p_2) denote the saddle insertion points of S in p_{n+1}. Consider a saddle connecting a disk circle to an arc a of p_{n+1} with saddle insertion point $q \in a$. There is a corresponding saddle connecting the disk circle to an arc a' of p'_{n-1} with saddle insertion point $q' \in a'$. The arc a of p_{n+1} lies on a unique arc or disk circle $[a]_0$ of $[p_{n+1}]_0$ and $[a]_1$ of $[p_{n+1}]_1$. The arc a' of p'_{n-1} lies on a unique arc or disk circle $[a']_0$ of $[p'_{n-1}]_0$ and $[a']_1$ of $[p'_{n-1}]_1$. If a is a strand arc, the orientations and types of these arcs are as follows:

$q < p_1$:	$\sigma(a) = \sigma(a') \in \{L, R\}$,	$\tau([a]_1) \in \{e, u\}$,	$\tau([a']_0) \in \{e, u\}$,
$p_1 < q < p_2,\ a \le C$:	$\sigma(a) = \overline{\sigma(a')} \in \{L, R\}$,	$\tau([a]_1) \in \{e, u\}$,	$\tau([a']_0) = c_0$,
$p_1 < q < p_2,\ B \le a$:	$\sigma(a) = \overline{\sigma(a')} \in \{L, R\}$,	$\tau([a]_1) = c_1$,	$\tau([a']_0) \in \{e, u\}$,
$p_2 < q$:	$\sigma(a) = \sigma(a') \in \{L, R\}$,	$\tau([a]_1) = c_1$,	$\tau([a']_0) = c_0$,

where c_0 and c_1 indicate circle arcs on the additional circle components of $[p']_0$ and $[p]_1$, respectively. If a is a circle arc, we have

$$\tau(a) = \tau([a]_1) = \tau(a') = \tau([a']_0) = c;$$

note that the circle arc $[a]_1$ in $[p]_1$ does not lie on additional circle component of $[p]_1$, and the circle arc $[a']_0$ in $[p']_0$ does not lie on the additional circle component of $[p']_0$. In each case, the contribution to both sides of equation (28) is the same.

For saddles $S : p_{n+1}(r) \to p'_{n-1}(r)$ of type W3 or W4, Lemma 7.4 shows that we obtain $\tilde{\partial}^L_{P'} Q_{P'P}(S) + Q_{P'P}(S)\tilde{\partial}^L_P$ as a residual term. Thus the claim holds for saddles $S : p_{n+1}(r) \to p'_{n-1}(r)$ of all possible types. □

We classify interleavings $(S : p_n(r) \to p'_{n-2}(r), T : p_n(r) \to \tilde{p}_{n-2}(r))$ into the following types, depending on the types and relative positions of the arcs B and C, the saddle arcs (s_1, s_2) of S, and the saddle arcs (t_1, t_2) of T in $p_n(r)$:

- Type I1: $\tau(B) = \overline{\tau(C)} \in \{e, u\}$ and $s_1 \le t_1 \le C < B \le s_2 \le t_2$,
- Type I2: $\tau(B) = \overline{\tau(C)} \in \{e, u\}$ and $s_1 \le t_1 \le s_2 \le C < B \le t_2$,
- Type I3: $\tau(B) = \overline{\tau(C)} \in \{e, u\}$ and either $s_1 \le t_1 \le s_2 \le t_2 \le B < C$ or $s_1 \le t_1 \le s_2 \le t_2 \le C < B$,
- Type I4: $\tau(B) = \tau(C) = c$.

In the classification we have made use of Lemma 7.1, which states that type W1 saddles do not actually occur, and Lemma 7.2, which states that type W2 saddles must have $C < B$.

Lemma 9.5. *Consider interleaved saddles $(S : p_n(r) \to p'_{n-2}(r), T : p_n(r) \to \tilde{p}_{n-2}(r))$ and induced saddles $\overline{T} : p'_{n-2}(r) \to p''_{n-2}(r+1)$ and $\overline{S} : \tilde{p}_{n-2}(r) \to p''_{n-2}(r+1)$. We obtain the term*

$$\tilde{T}^L_{P''\tilde{P}}(\overline{S}) Q_{\tilde{P}P}(T) = \tilde{T}^R_{P''P'}(\overline{T}) Q_{P'P}(S) : C_P \to C_{P''},$$

and we have

$$\tilde{T}^L_{P''P'}(\overline{T}) Q_{P'P}(S) = \tilde{T}^R_{P''\tilde{P}}(\overline{S}) Q_{\tilde{P}P}(T) = 0.$$

Proof. The interleaved saddles (S,T) are described by the commutative diagram

$$\begin{array}{ccc} p_n(r) & \xrightarrow{\quad S \quad} & p'_{n-2}(r) \\ {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle \overline{T}} \\ \tilde{p}_{n-2}(r) & \xrightarrow{\quad \overline{S} \quad} & p''_{n-2}(r+1). \end{array}$$

We have the following cases:

I	S	$\overline{T}$	T	$\overline{S}$	reduction term $\beta_1(\overline{T})\alpha_2(S)$?	reduction term $\beta_1(\overline{S})\alpha_2(T)$?	residual term?
I1	W2	C1	W2	C3	y	n	n
I2	W3	C2	W2	C2	n	y	n
I3	W3	C3	W3	C3	n	n	y
I4	W4	C5	W4	C5	n	n	y

The column marked I indicates the type of interleaving of the pair of saddles (S,T), the columns marked S, $\overline{T}$, T, and $\overline{S}$ indicate the corresponding types of each of these saddles, and the remaining three columns indicate whether we obtain $\tilde{T}^L_{P''\tilde{P}}(\overline{S})Q_{\tilde{P}P}(T) = \tilde{T}^R_{P''P'}(\overline{T})Q_{P'P}(S)$ as a reduction term $\beta_1(\overline{T})\alpha_2(S)$, as a reduction term $\beta_1(\overline{S})\alpha_2(T)$, or as a residual term. The entries for reduction terms follow from the enumeration of such terms in the proof of Lemma 10.3, and the entries for residual terms follow from the enumeration of such terms in the proof of Lemma 10.1. The fact that $\tilde{T}^L_{P''P'}(\overline{T})Q_{P'P}(S) = \tilde{T}^R_{P''\tilde{P}}(\overline{S})Q_{\tilde{P}P}(T) = 0$ was shown in the proof of Lemma 5.4. □

We classify nestings $(S : p_n(r) \to p'_{n-2}(r), T : p_n(r) \to \tilde{p}_n(r+1))$ into the following types, depending on the types and relative positions of the arcs B and C, the saddle arcs (s_1, s_2) of S, and the saddle arcs (t_1, t_2) of T in $p_n(r)$:

Type N1: $\tau(B) = \overline{\tau(C)} \in \{e, u\}$ and $s_1 \leq C < B \leq t_1 < t_2 \leq s_2$,

Type N2: $\tau(B) = \overline{\tau(C)} \in \{e, u\}$ and $s_1 \leq t_1 \leq C < B \leq t_2 \leq s_2$,

Type N3: $\tau(B) = \overline{\tau(C)} \in \{e, u\}$ and $s_1 \leq t_1 < t_2 \leq C < B \leq s_2$,

Type N4: $\tau(B) = \overline{\tau(C)} \in \{e, u\}$ and either $s_1 \leq t_1 < t_2 \leq s_2 \leq B < C$ or $s_1 \leq t_1 < t_2 \leq s_2 \leq C < B$,

Type N5: $\tau(B) = \tau(C) = c$.

In the classification we have made use of Lemma 7.1, which states that type W1 saddles do not actually occur, and Lemma 7.2, which states that type W2 saddles must have $C < B$.

Lemma 9.6. *Consider nested saddles $(S : p_n(r) \to p'_{n-2}(r), T : p_n(r) \to \tilde{p}_n(r+1))$ with induced saddles $\overline{T} : p'_{n-2}(r) \to p''_{n-2}(r+1)$ and $\overline{S} : \tilde{p}_n(r+1) \to p''_{n-2}(r+1)$. For each such pair of saddles, we obtain a term*

$$\tilde{T}^{\sigma}_{P''P'}(\overline{T})Q_{P'P}(S) = Q_{P''\tilde{P}}(\overline{S})\tilde{T}^{\overline{\sigma}}_{\tilde{P}P}(T) : C_P \to C_{P''}$$

for some orientation $\overline{\sigma} \in \{L, R\}$, and we have

$$Q_{P''\tilde{P}}(\overline{S})\tilde{T}^{\sigma}_{\tilde{P}P}(T) = \tilde{T}^{\overline{\sigma}}_{P''P'}(\overline{T})Q_{P'P}(S) = 0.$$

Proof. The nested saddles (S,T) are described by the commutative diagram

$$\begin{array}{ccc} p_n(r) & \xrightarrow{\quad S \quad} & p'_{n-2}(r) \\ {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle \overline{T}} \\ \tilde{p}_n(r+1) & \xrightarrow{\quad \overline{S} \quad} & p''_{n-2}(r+1). \end{array}$$

We have the following cases:

N	S	$\overline{T}$	T	$\overline{S}$	reduction term $\beta_1(\overline{T})\alpha_2(S)$?	reduction term $\beta_1(\overline{S})\alpha_2(T)$?	residual term?
N1	W2	C3	C1	W2	n	y	n
N2	W2	C2	C2	W4	y	n	n
N3	W2	C1	C3	W2	y	n	n
N4	W3	C3	C3	W3	n	n	y
N5	W4	C5	C5	W4	n	n	y

The column marked N indicates the type of nesting of the pair of saddles (S, T), the columns marked S, $\overline{T}$, T, and $\overline{S}$ indicate the corresponding types of each of these saddles, and the remaining three columns indicate whether we obtain $\tilde{T}^{\sigma}_{P''P'}(\overline{T})Q_{P'P}(S) = Q_{P''P}(\overline{S})\tilde{T}^{\sigma}_{\overline{P}P}(T)$ as a reduction term $\beta_1(\overline{T})\alpha_2(S)$, as a reduction term $\beta_1(\overline{S})\alpha_2(T)$, or as a residual term. The entries for reduction terms follow from the enumeration of such terms in the proof of Lemma 10.3, and the entries for residual terms follow from the enumeration of such terms in the proof of Lemma 10.1. The fact that $Q_{P''\tilde{P}}(\overline{S})\tilde{T}^{\sigma}_{\tilde{P}P}(T) = \tilde{T}^{\sigma}_{P''P'}(\overline{T})Q_{P'P}(S) = 0$ was shown in the proof of Lemma 5.4. □

10. All terms of ∂_{red} occur in $\partial^0_{T^+}$

Recall that in Definition 6.4 we defined the reduction $(C_{red}, \partial_{red})$ of the chain complex $(C_{\overline{T}^+}, \partial_{\overline{T}^+})$. In Section 9 we showed that all terms of ∂_{T^+} occur in ∂_{red}. In this section we show that all terms of ∂_{red} occur in ∂_{T^+}, thus completing the proof of Theorem 3.1.

Recall that $\partial_{\overline{T}^+} = \partial^0_{\overline{T}^+} + \partial^+_{\overline{T}^+}$. The map $\partial^0_{\overline{T}^+}$ is a sum of maps corresponding to saddles obtained from the resolution of $\overline{T}$. The map $\partial^+_{\overline{T}^+}$ is a sum of products of maps corresponding to pairs of saddles obtained from the resolution of $\overline{T}$. As explained in Section 5, we classify these saddles into three types:

(1) Saddles $[p]_0 \to [p]_1$ obtained by resolving the one additional crossing of $\overline{T}$.
(2) Saddles $[p]_0 \to [p']_0$ and $[p]_1 \to [p']_1$ that are induced by a type-preserving (i.e. $\tau(p) = \tau(p')$) saddle $S : p \to p'$ obtained from the resolution of T.
(3) Saddles $[p]_0 \to [p']_0$ and $[p]_1 \to [p']_1$ that are induced by a type-changing (i.e. $\tau(p) \neq \tau(p')$) saddle $S : p \to p'$ obtained from the resolution of T.

In Sections 6, 7, and 8, we described the residual terms and reduction factors that are obtained from terms of $\partial^0_{\overline{T}^+}$ that correspond to saddles of all three types (1), (2), and (3), and from terms of $\partial^+_{\overline{T}^+}$ that correspond to pairs of saddles that are not both of type (3). It remains to describe the additional residual terms and reduction factors obtained from terms of $\partial^+_{\overline{T}^+}$ that correspond to pairs of saddles that are both of type (3).

Lemma 10.1. *All the residual terms obtained from $\partial_{\overline{T}^+}$ occur in ∂_{T^+}.*

Proof. In Section 6 we showed that terms of $\partial_{\overline{T}^+}$ corresponding to saddles of types (1) and (2) yield the residual term

$$\sum_P \partial_P.$$

In Sections 7 and 8 we showed that if we now include saddles of type (3), but exclude terms of $\partial^+_{\overline{T}^+}$ corresponding to pairs of saddles that are both of type (3), we obtain the additional residual term

$$\sum_P \sum_{P' \neq P} T_{P'P} + \sum_S \sum_P \sum_{P'} (\tilde{\partial}^L_{P'} Q_{P'P}(S) + Q_{P'P}(S)\tilde{\partial}^L_P),$$

where the sum on S is taken over all saddles $S : p_{n+1}(r) \to p'_{n-1}(r)$ of type W3 or W4. All of these residual terms occur in $\partial^+_{T^+}$.

It remains to consider terms of $\partial^+_{\overline{T}^+}$ that correspond to pairs of saddles that are both of type (3). Such terms take one of the following forms:

$$\tilde{T}^L_{[P'']_0[P']_0} Q_{[P']_0[P]_0}, \qquad \tilde{T}^L_{[P'']_1[P']_1} Q_{[P']_1[P]_1}, \qquad Q_{[P'']_0[P']_0} \tilde{T}^L_{[P']_0[P]_0}, \qquad Q_{[P'']_1[P']_1} \tilde{T}^L_{[P']_1[P]_1},$$

where, for example, $Q_{[P']_0[P]_0}$ and $\tilde{T}^L_{[P'']_0[P']_0}$ are maps corresponding to type (3) saddles $[p]_0 \to [p']_0$ and $[p']_0 \to [p'']_0$ that occur in the resolution of $\overline{T}$, which are induced by type-changing saddles $S_1 : p \to p'$ and $S_2 : p' \to p''$ that occur in the resolution of T. In general, such a residual term has the form

$$\partial_{P''P}(S_1, S_2) = \partial^t(S_2)\partial^i(S_1) : C_P \to C_{P''},$$

where $\partial^i(S_1) : C_P \to C_{\overline{T}^+}$ and $\partial^t(S_2) : C_{\overline{T}^+} \to C_{P''}$ are linear maps corresponding to saddles induced by S_1 and S_2. We will refer to the maps $\partial^i(S_1)$ and $\partial^t(S_2)$ as the *initial factor* and *terminal factor* of the residual term $\partial_{P''P}(S_1, S_2)$. For tangle types P of type P1A, we identify C_P with the subspace $C_P \otimes x$ of $\overline{C}_P = (C_P \otimes A) \oplus C_P$, so we have initial and terminal factors

$$\partial^i(S) : C_P \otimes x \to C_{\overline{T}^+}, \qquad \partial^t(S) : C_{\overline{T}^+} \to C_P \otimes x.$$

For tangle types P of type P1B, we identify C_P with the subspace $C_P \otimes 1_A$ of $\overline{C}_P = C_P \oplus (C_P \otimes A)$, so we have initial and terminal factors

$$\partial^i(S) : C_P \otimes 1_A \to C_{\overline{T}^+}, \qquad \partial^t(S) : C_{\overline{T}^+} \to C_P \otimes 1_A.$$

For tangle types P of type P2, we have $\overline{C}_P = C_P = W \otimes A$ for a vector space W, so we have initial and terminal factors

$$\partial^i(S) : W \to C_{\overline{T}^+}, \qquad \partial^t(S) : C_{\overline{T}^+} \to W.$$

Using the results of Sections 7 and 8, for each type-changing saddle $S : p \leftrightarrow p'$ obtained from the resolution of T we indicate the possible initial and terminal factors that could be obtained:

For a type W2 saddle $S : p_{n+1}(r) \leftrightarrow p'_{n-1}(r)$:

$$\begin{array}{ccc} \text{P1B} & \xleftarrow{\text{W2}} & \text{P1A}'. \\ 0 & & \partial^i(S) = \tilde{T}^L_{[P]_0[P']_0} \\ 0 & & 0 \end{array} \tag{29}$$

For a type W3 saddle $S : p_{n+1}(r) \leftrightarrow p'_{n-1}(r)$:

$$\begin{array}{ccc} \text{P1A} & \xrightarrow{\text{W3}} & \text{P1A}', \\ \partial^i(S) = Q_{[P']_0[P]_0} & & \partial^t(S) = Q_{[P']_0[P]_0} \\ 0 & & 0 \end{array} \qquad \begin{array}{ccc} \text{P1B} & \xrightarrow{\text{W3}} & \text{P1B}'. \\ 0 & & 0 \\ \partial^i(S) = Q_{[P']_1[P]_1} & & \partial^t(S) = Q_{[P']_1[P]_1} \end{array}$$

For a type W4 saddle $S : p_{n+1}(r) \leftrightarrow p'_{n-1}(r)$:

$$\begin{array}{ccc} \text{P2} & \xrightarrow{\text{W4}} & \text{P2}'. \\ \partial^i(S) = Q_{[P']_0[P]_0} & & \partial^t(S) = Q_{[P']_0[P]_0} \\ \partial^i(S) = Q_{[P']_1[P]_1} & & \partial^t(S) = Q_{[P']_1[P]_1} \end{array}$$

For a type C2 saddle $S : p_n(r) \leftrightarrow p'_n(r+1)$

$$\begin{array}{ccc} \text{P1A} & \xrightarrow{\text{C2}} & \text{P2}', \\ \partial^i(S) = \tilde{T}^L_{[P']_0[P]_0} & & \partial^t(S) = \tilde{T}^L_{[P']_0[P]_0} \\ 0 & & \partial^t(S) = Q_{[P']_1[P]_1} \end{array} \qquad \begin{array}{ccc} \text{P1A} & \xleftarrow{\text{C2}} & \text{P2}', \\ 0 & & \partial^i(S) = \tilde{T}^L_{[P]_0[P']_0} \\ 0 & & 0 \end{array}$$

$$\begin{array}{ccc} \text{P1B} & \xrightarrow{\text{C2}} & \text{P2}', \\ 0 & & 0 \\ 0 & & \partial^t(S) = \tilde{T}^L_{[P']_1[P]_1} \end{array} \qquad \begin{array}{ccc} \text{P1B} & \xleftarrow{\text{C2}} & \text{P2}'. \\ 0 & & \partial^i(S) = Q_{[P]_0[P']_0} \\ \partial^t(S) = \tilde{T}^L_{[P]_1[P']_1} & & \partial^i(S) = \tilde{T}^L_{[P]_1[P']_1} \end{array}$$

For a type C3 saddle $S : p_n(r) \leftrightarrow p'_n(r+1)$:

$$\begin{array}{ccc} \mathrm{P1A} & \xrightarrow{\mathrm{C3}} & \mathrm{P1A}', \\ \partial^i(S) = \tilde{T}^L_{[P']_0[P]_0} & & \partial^t(S) = \tilde{T}^L_{[P']_0[P]_0} \\ 0 & & 0 \end{array} \qquad \begin{array}{ccc} \mathrm{P1A} & \xleftarrow{\mathrm{C3}} & \mathrm{P1A}', \\ \partial^t(S) = \tilde{T}^L_{[P]_0[P']_0} & & \partial^i(S) = \tilde{T}^L_{[P]_0[P']_0} \\ 0 & & 0 \end{array}$$

$$\begin{array}{ccc} \mathrm{P1B} & \xrightarrow{\mathrm{C3}} & \mathrm{P1B}', \\ 0 & & 0 \\ \partial^i(S) = \tilde{T}^L_{[P']_1[P]_1} & & \partial^t(S) = \tilde{T}^L_{[P']_1[P]_1} \end{array} \qquad \begin{array}{ccc} \mathrm{P1B} & \xleftarrow{\mathrm{C3}} & \mathrm{P1B}'. \\ 0 & & 0 \\ \partial^t(S) = \tilde{T}^L_{[P]_1[P']_1} & & \partial^i(S) = \tilde{T}^L_{[P]_1[P']_1} \end{array}$$

For a type C5 or C6 saddle $S : p_n(r) \leftrightarrow p'_n(r+1)$:

$$\begin{array}{ccc} \mathrm{P2} & \xrightarrow{\mathrm{C5,C6}} & \mathrm{P2}', \\ \partial^i(S) = \tilde{T}^L_{[P']_0[P]_0} & & \partial^t(S) = \tilde{T}^L_{[P']_0[P]_0} \\ \partial^i(S) = \tilde{T}^L_{[P']_1[P]_1} & & \partial^t(S) = \tilde{T}^L_{[P']_1[P]_1} \end{array} \qquad \begin{array}{ccc} \mathrm{P2} & \xleftarrow{\mathrm{C5,C6}} & \mathrm{P2}'. \\ \partial^t(S) = \tilde{T}^L_{[P]_0[P']_0} & & \partial^i(S) = \tilde{T}^L_{[P]_0[P']_0} \\ \partial^t(S) = \tilde{T}^L_{[P]_1[P']_1} & & \partial^i(S) = \tilde{T}^L_{[P]_1[P']_1} \end{array}$$

At the head and tail of each arrow we indicate the type of the corresponding planar tangle. Below the head (tail) of each arrow we indicate the corresponding map that could constitute the terminal (initial) factor of a residual term, where the first row corresponds to $[p]_0 \leftrightarrow [p']_0$ and the second row corresponds to $[p]_1 \leftrightarrow [p']_1$. Saddles that do not yield initial or terminal factors are not included in the list.

As an example, consider a type W2 saddle $S : p'_{n-1}(r) \to p_{n+1}(r)$, as described by diagram (18). First consider maps out of $\overline{C}_{P'} = C_{[P']_0} \oplus C_{[P']_1}$ that could yield initial factors. When we reduce on P', we identify $C_{P'}$ with the subspace $C_{P'} \otimes x$ of $C_{[P']_0} = C_{P'} \otimes A$. So maps out of $C_{[P']_1} = C_{P'}$ do not yield initial factors, as we indicate by putting a zero in the lower-right corner of table (29), and we need only consider maps out of $C_{P'} \otimes x$. In fact we do have a nonzero map $\tilde{T}^L_{[P]_0[P']_0} = P_{PP'}(S) \otimes \epsilon : C_{P'} \otimes x \to C_P$, as we indicate by putting $\partial^i(S) = \tilde{T}^L_{[P]_0[P']_0}$ in the upper-right corner of table (29). Next consider maps into $\overline{C}_P = C_{[P]_0} \oplus C_{[P]_1}$ that could yield terminal factors. When we reduce on P, we identify C_P with the subspace $C_P \otimes 1_A$ of $C_{[P]_1} = C_P \otimes A$. So maps into $C_{[P]_0} = C_P$ do not yield terminal factors, as we indicate by putting a zero in the upper-left corner of table (29), and we need only consider maps into $C_P \otimes 1_A$. There are no such maps that could yield a terminal factor, so we put a zero in the lower-left corner of table (29). The remaining tables in our list are constructed similarly.

From the above list, we find that the terms of $\partial^+_{\overline{T}^+}$ that could yield residual terms are of the following forms. We have pairs of type (W3,C2):

(30)
$$\begin{array}{ccccc} \mathrm{P1A} & \xrightarrow{\mathrm{W3}} & \mathrm{P1A}' & \xrightarrow{\mathrm{C2}} & \mathrm{P2}'', \\ Q_{[P']_0[P]_0} & & & & \tilde{T}^L_{[P'']_0[P']_0} \end{array} \qquad \begin{array}{ccccc} \mathrm{P2} & \xrightarrow{\mathrm{C2}} & \mathrm{P1B}' & \xrightarrow{\mathrm{W3}} & \mathrm{P1B}'', \\ \tilde{T}^L_{[P']_1[P]_1} & & & & Q_{[P'']_1[P']_1} \end{array}$$

(31)
$$\begin{array}{ccccc} \mathrm{P2} & \xrightarrow{\mathrm{C2}} & \mathrm{P1A}' & \xrightarrow{\mathrm{W3}} & \mathrm{P1A}'', \\ \tilde{T}^L_{[P']_0[P]_0} & & & & Q_{[P'']_0[P']_0} \end{array} \qquad \begin{array}{ccccc} \mathrm{P1B} & \xrightarrow{\mathrm{W3}} & \mathrm{P1B}' & \xrightarrow{\mathrm{C2}} & \mathrm{P2}''. \\ Q_{[P']_1[P]_1} & & & & \tilde{T}^L_{[P'']_1[P']_1} \end{array}$$

We have pairs of type (W3,C3):

(32)
$$\begin{array}{ccccc} \mathrm{P1A} & \xrightarrow{\mathrm{W3}} & \mathrm{P1A}' & \xrightarrow{\mathrm{C3}} & \mathrm{P1A}'', \\ Q_{[P']_0[P]_0} & & & & \tilde{T}^L_{[P'']_0[P']_0} \end{array} \qquad \begin{array}{ccccc} \mathrm{P1B} & \xrightarrow{\mathrm{W3}} & \mathrm{P1B}' & \xrightarrow{\mathrm{C3}} & \mathrm{P1B}'', \\ Q_{[P']_1[P]_1} & & & & \tilde{T}^L_{[P'']_1[P']_1} \end{array}$$

(33)
$$\begin{array}{ccccc} \mathrm{P1A} & \xrightarrow{\mathrm{C3}} & \mathrm{P1A}' & \xrightarrow{\mathrm{W3}} & \mathrm{P1A}'', \\ \tilde{T}^L_{[P']_0[P]_0} & & & & Q_{[P'']_0[P']_0} \end{array} \qquad \begin{array}{ccccc} \mathrm{P1B} & \xrightarrow{\mathrm{C3}} & \mathrm{P1B}' & \xrightarrow{\mathrm{W3}} & \mathrm{P1B}''. \\ \tilde{T}^L_{[P']_1[P]_1} & & & & Q_{[P'']_1[P']_1} \end{array}$$

We have pairs of type (W4,C2):

$$\begin{array}{llllll} & \mathrm{P1A} \xrightarrow{\mathrm{C2}} \mathrm{P2}' \xrightarrow{\mathrm{W4}} \mathrm{P2}'', & & \mathrm{P2} \xrightarrow{\mathrm{W4}} \mathrm{P2}' \xrightarrow{\mathrm{C2}} \mathrm{P1B}''. & \\ (34) & \tilde{T}^L_{[P']_0[P]_0} & Q_{[P'']_0[P']_0} & Q_{[P']_1[P]_1} & \tilde{T}^L_{[P'']_1[P']_1} \end{array}$$

We have pairs of type (W4,C5):

$$\begin{array}{llllll} & \mathrm{P2} \xrightarrow{\mathrm{W4}} \mathrm{P2}' \xrightarrow{\mathrm{C5}} \mathrm{P2}'', & & \mathrm{P2} \xrightarrow{\mathrm{C5}} \mathrm{P2}' \xrightarrow{\mathrm{W4}} \mathrm{P2}''. & \\ (35) & Q_{[P']_0[P]_0} & \tilde{T}^L_{[P'']_0[P']_0} & \tilde{T}^L_{[P']_0[P]_0} & Q_{[P'']_0[P']_0} \\ & Q_{[P']_1[P]_1} & \tilde{T}^L_{[P'']_1[P']_1} & \tilde{T}^L_{[P']_1[P]_1} & Q_{[P'']_1[P']_1} \end{array}$$

We have pairs of type (W4,C6):

$$\begin{array}{llllll} & \mathrm{P2} \xrightarrow{\mathrm{W4}} \mathrm{P2}' \xrightarrow{\mathrm{C6}} \mathrm{P2}'', & & \mathrm{P2} \xrightarrow{\mathrm{C6}} \mathrm{P2}' \xrightarrow{\mathrm{W4}} \mathrm{P2}''. & \\ (36) & Q_{[P']_0[P]_0} & \tilde{T}^L_{[P'']_0[P']_0} & \tilde{T}^L_{[P']_0[P]_0} & Q_{[P'']_0[P']_0} \\ & Q_{[P']_1[P]_1} & \tilde{T}^L_{[P'']_1[P']_1} & \tilde{T}^L_{[P']_1[P]_1} & Q_{[P'']_1[P']_1} \end{array}$$

We claim that pairs of saddles $S_1 : p \to p'$ and $S_2 : p' \to p''$ of the forms shown in tables (30), (34), and (36) are always disjoint. It follows that the corresponding induced saddles $[p]_k \to [p']_k$ and $[p']_k \to [p'']_k$ are disjoint, and thus do not yield residual terms. We will show this for the first form in table (30). Let (s'_1, s'_2) denote the saddle arcs of S_1 in p', and let (t'_1, t'_2) denote the saddle arcs of S_2 in p'. Since S_1 is of type W3 and S_2 is of type C2, we have

$$s'_1 < s'_2 \leq B' < C', \qquad t'_1 \leq B' < C' \leq t'_2.$$

From Remark 4.5 and Lemma 7.2, it follows that

$$s'_1 < s'_2 \leq t'_1 \leq B' < C' \leq t'_2,$$

so the pair (S_1, S_2) is disjoint.

We claim that pairs of saddles $S_1 : p \to p'$ and $S_2 : p' \to p''$ of the forms shown in table (31) yield residual terms that are zero. We will show this for the first form in table (31). When we reduce on P'', we identify $C_{P''}$ with the subspace $C_{P''} \otimes x$ of $C_{[P'']_0} = C_{P''} \otimes A$. From diagrams (23) and (19) for type C2 and W3 saddles, we have

$$\tilde{T}^L_{[P']_0[P]_0} = [T^R_{P'P}(S_1)] \otimes \eta, \qquad Q_{[P'']_0[P']_0} = Q_{P''P'}(S_2) \otimes 1_A.$$

Since the image of $Q_{[P'']_0[P']_0}\tilde{T}^L_{[P']_0[P]_0}$ lies in $C_{P''} \otimes 1_A$, the residual term is zero.

We claim that pairs of saddles $S_1 : p \to p'$ and $S_2 : p' \to p''$ of the forms shown in tables (32), (33), and (35) yield residual terms of the form $\tilde{T}^L_{P''P'}(S_2)Q_{P'P}(S_1)$ or $Q_{P''P'}(S_1)\tilde{T}^L_{P'P}(S_1)$, both of which occur in $\partial^+_{T^+}$. We will show this for the first form in table (32). When we reduce on P, we identify C_P with the subspace $C_P \otimes x$ of $C_{[P]_0} = C_P \otimes A$. When we reduce on P'', we identify $C_{P''}$ with the subspace $C_{P''} \otimes x$ of $C_{[P'']_0} = C_{P''} \otimes A$. From diagrams (19) and (24) for type W3 and C3 saddles, we have

$$Q_{[P']_0[P]_0}(S_1) = Q_{P'P}(S_1) \otimes 1_A, \qquad \tilde{T}^L_{[P'']_0[P']_0}(S_2) = \tilde{T}^L_{P''P'}(S_2) \otimes 1_A.$$

So we obtain the residual term $\tilde{T}^L_{P''P'}(S_2)Q_{P'P}(S_1) : C_P \to C_{P''}$. □

Lemma 10.2. *Terms of $\partial^+_{\overline{T}^+}$ that correspond to pairs of saddles that are both of type (3) do not yield reduction factors.*

Proof. Terms of $\partial_{\overline{T}^+}$ that correspond to pairs of saddles that are both of type (3) take one of the following forms:

$$\tilde{T}^L_{[P'']_0[P']_0}Q_{[P']_0[P]_0}, \qquad \tilde{T}^L_{[P'']_1[P']_1}Q_{[P']_1[P]_1}, \qquad Q_{[P'']_0[P']_0}\tilde{T}^L_{[P']_0[P]_0}, \qquad Q_{[P'']_1[P']_1}\tilde{T}^L_{[P']_1[P]_1},$$

where, for example, $Q_{[P']_0[P]_0}$ and $\tilde{T}^L_{[P'']_0[P']_0}$ are maps corresponding to type (3) saddles $[p]_0 \to [p']_0$ and $[p']_0 \to [p'']_0$ that occur in the resolution of $\overline{T}$, which are induced by type-changing saddles $p \to p'$ and $p' \to p''$ that occur in the resolution of T. For tangle types P of type P1A, we could obtain reduction factors

$$\alpha_2(P, S_1, S_2) = \alpha_2^m(S_2)\partial^i(S_1) : C_{P''} \to C_P, \qquad \beta_1(P, S_1, S_2) = \partial^t(S_2)\beta_1^m(S_1) : C_P \otimes 1_A \to C_{P''},$$

where $\partial^i(S_1) : C_{P''} \to C_{\overline{T}^+}$ and $\alpha_2^m(S_2) : C_{\overline{T}^+} \to C_P$ are maps corresponding to saddles induced by type-changing saddles $S_1 : p'' \to p'$ and $S_2 : p' \to p$, and $\beta_1^m(S_1) : C_P \otimes 1_A \to C_{\overline{T}^+}$ and $\partial^t(S_2) : C_{\overline{T}^+} \to C_{P''}$ are maps corresponding to saddles induced by type-changing saddles $S_1 : p \to p'$ and $S_2 : p' \to p''$. For tangle types P of type P1B, we could obtain reduction factors

$$\alpha_2(P, S_1, S_2) = \alpha_2^m(S_2)\partial^i(S_1) : C_{P''} \to C_P \otimes x, \qquad \beta_1(P, S_1, S_2) = \partial^t(S_2)\beta_1^m(S_1) : C_P \to C_{P''},$$

where $\partial^i(S_1) : C_{P''} \to C_{\overline{T}^+}$ and $\alpha_2^m(S_2) : C_{\overline{T}^+} \to C_P \otimes x$ are maps corresponding to saddles induced by type-changing saddles $S_1 : p'' \to p'$ and $S_2 : p' \to p$, and $\beta_1^m(S_1) : C_P \to C_{\overline{T}^+}$ and $\partial^t(S_2) : C_{\overline{T}^+} \to C_{P''}$ are maps corresponding to saddles induced by type-changing saddles $S_1 : p \to p'$ and $S_2 : p' \to p''$.

The maps $\partial^i(S)$ and $\partial^t(S)$ that appear in the reduction factors $\alpha_2(P, S_1, S_2)$ and $\beta_1(P, S_1, S_2)$ are the initial and terminal factors that we described in the proof of Lemma 10.1. We will refer to the maps $\alpha_2^m(S)$ and $\beta_1^m(S)$ as the *middle factors* of $\alpha_2(P, S_1, S_2)$ and $\beta_1(P, S_1, S_2)$. For tangle types P of type P1A, we have middle factors

$$\alpha_2^m(S) : C_{\overline{T}^+} \to C_P, \qquad \beta_1^m(S) : C_P \otimes 1_A \to C_{\overline{T}^+}.$$

For tangle types P of type P1B, we have middle factors

$$\alpha_2^m(S) : C_{\overline{T}^+} \to C_P \otimes x, \qquad \beta_1^m(S) : C_P \to C_{\overline{T}^+}.$$

Using the results of Sections 7 and 8, for each type-changing saddle $S : p \leftrightarrow p'$ that occurs in the resolution of T we indicate the possible middle factors that could be obtained:

For a type W2 saddle $S : p_{n+1}(r) \leftrightarrow p'_{n-1}(r)$:

$$\begin{array}{ccc} \text{P1B} & \xrightarrow{\text{W2}} & \text{P1A}'. \\ \beta_1^m(S) = \tilde{T}^L_{[P']_0[P]_0} & & 0 \\ 0 & & 0 \end{array} \tag{37}$$

For a type W3 saddle $S : p_{n+1}(r) \leftrightarrow p'_{n-1}(r)$:

$$\begin{array}{ccc} \text{P1A} & \xrightarrow{\text{W3}} & \text{P1A}', \\ \beta_1^m(S) = Q_{[P']_0[P]_0} & & 0 \\ 0 & & \alpha_2^m(S) = Q_{[P']_1[P]_1} \end{array} \qquad \begin{array}{ccc} \text{P1B} & \xrightarrow{\text{W3}} & \text{P1B}'. \\ \beta_1^m(S) = Q_{[P']_0[P]_0} & & 0 \\ 0 & & \alpha_2^m(S) = Q_{[P']_1[P]_1} \end{array}$$

For a type C1 saddle $S : p_n(r) \leftrightarrow p'_n(r+1)$:

$$\begin{array}{ccc} \text{P1A} & \xrightarrow{\text{C1}} & \text{P1A}', \\ 0 & & 0 \\ 0 & & \alpha_2^m(S) = \tilde{T}^L_{[P']_1[P]_1} \end{array} \qquad \begin{array}{ccc} \text{P1A} & \xleftarrow{\text{C1}} & \text{P1A}', \\ 0 & & 0 \\ \alpha_2^m(S) = \tilde{T}^L_{[P]_1[P']_1} & & 0 \end{array}$$

$$\begin{array}{ccc} \text{P1B} & \xrightarrow{\text{C1}} & \text{P1B}', \\ \beta_1^m(S) = \tilde{T}^L_{[P']_1[P]_1} & & 0 \\ 0 & & 0 \end{array} \qquad \begin{array}{ccc} \text{P1B} & \xleftarrow{\text{C1}} & \text{P1B}'. \\ 0 & & \beta_1^m(S) = \tilde{T}^L_{[P]_1[P']_1} \\ 0 & & 0 \end{array}$$

For a type C3 saddle $S : p_n(r) \leftrightarrow p'_n(r+1)$:

$$\begin{array}{ccc} \text{P1A} & \xrightarrow{\quad \text{C3} \quad} & \text{P1A}', \\ \beta_1^m(S) = \tilde{T}^L_{[P']_0[P]_0} & & 0 \\ 0 & & \alpha_2^m(S) = \tilde{T}^L_{[P']_1[P]_1} \end{array} \qquad \begin{array}{ccc} \text{P1A} & \xleftarrow{\quad \text{C3} \quad} & \text{P1A}', \\ 0 & & \beta_1^m(S) = \tilde{T}^L_{[P]_0[P']_0} \\ \alpha_2^m(S) = \tilde{T}^L_{[P]_1[P']_1} & & 0 \end{array}$$

$$\begin{array}{ccc} \text{P1B} & \xrightarrow{\quad \text{C3} \quad} & \text{P1B}', \\ \beta_1^m(S) = \tilde{T}^L_{[P']_0[P]_0} & & 0 \\ 0 & & \alpha_2^m(S) = \tilde{T}^L_{[P']_1[P]_1} \end{array} \qquad \begin{array}{ccc} \text{P1B} & \xleftarrow{\quad \text{C3} \quad} & \text{P1B}'. \\ 0 & & \beta_1^m(S) = \tilde{T}^L_{[P]_0[P']_0} \\ \alpha_2^m(S) = \tilde{T}^L_{[P]_1[P']_1} & & 0 \end{array}$$

At the head and tail of each arrow we indicate the type of the corresponding planar tangle. Below the head (tail) of each arrow we indicate the corresponding map that could contribute to a middle factor α_2^m (β_1^m), where the first row corresponds to $[p]_0 \leftrightarrow [p']_0$ and the second row corresponds to $[p]_1 \leftrightarrow [p']_1$. Type-changing saddles saddles $S : p \leftrightarrow p'$ that do not yield middle factors are not included in the list.

As an example, consider the middle factors that can be obtained from a type W2 saddle $S : p_{n+1}(r) \to p'_{n-1}(r)$, as described by diagram (18). Since $p_{n+1}(r)$ is of type P1B and $p_{n-1}(r)$ is of type P1A, it is not possible to obtain a middle factor $\beta_1^m(S)$ mapping out of $C_P \otimes A$ or $\alpha_2^m(S)$ mapping into $C_{P'} \otimes A$, so we put zeros in the lower-left and upper-right corners of table (37). There are potential middle factors of the form

$$\beta_1^m(S) : C_P \to C_{\overline{T}^+}, \qquad \alpha_2^m(S) : C_{\overline{T}^+} \to C_{P'}.$$

From diagram (18), we see that we have a nonzero map

$$\tilde{T}^L_{[P']_0[P]_0} = Q_{P'P}(S) \otimes \eta : C_P \to C_{P'} \otimes A,$$

so we obtain the middle factor $\beta_1^m(S) = \tilde{T}^L_{[P']_0[P]_0}$ indicated in the upper-left corner of table (37). Diagram (18) shows there is no nonzero map into $C_{P'}$, so we set the lower-right corner of table (37) to zero. The remaining tables in our list are constructed similarly.

From the above list, we find that the terms of $\partial^+_{\overline{T}^+}$ that could yield reduction factors are of the following forms. We have pairs of type (W2,W3):

$$\begin{array}{ccccc} & \text{P1B} & \xrightarrow{\quad \text{W2} \quad} & \text{P1A}' & \xrightarrow{\quad \text{W3} \quad} & \text{P1A}''. \\ \beta_1^m(S_1) = \tilde{T}^L_{[P']_0[P]_0} & & & & & \partial^t(S_2) = Q_{[P'']_0[P']_0} \end{array} \tag{38}$$

We have pairs of type (W3,C2):

$$\begin{array}{cccccc} & \text{P1A} & \xrightarrow{\quad \text{W3} \quad} & \text{P1A}' & \xrightarrow{\quad \text{C2} \quad} & \text{P2}'', \\ \beta_1^m(S_1) = Q_{[P']_0[P]_0} & & & & & \partial^t(S_2) = \tilde{T}^L_{[P'']_0[P']_0} \end{array} \tag{39}$$

$$\begin{array}{cccccc} & \text{P2}'' & \xrightarrow{\quad \text{C2} \quad} & \text{P1B}' & \xrightarrow{\quad \text{W3} \quad} & \text{P1B}. \\ \partial^i(S_1) = \tilde{T}^L_{[P']_1[P'']_1} & & & & & \alpha_2^m(S_2) = Q_{[P]_1[P']_1} \end{array} \tag{40}$$

We have pairs of type (W3,C3):

$$\text{P1A} \xrightarrow{\text{W3}} \text{P1A}' \xrightarrow{\text{C3}} \text{P1A}'',$$
$$\beta_1^m(S_1) = Q_{[P']_0[P]_0} \qquad \partial^t(S_2) = \tilde{T}^L_{[P'']_0[P']_0}$$

$$\text{P1A} \xrightarrow{\text{C3}} \text{P1A}' \xrightarrow{\text{W3}} \text{P1A}'',$$
$$\beta_1^m(S_1) = \tilde{T}^L_{[P']_0[P]_0} \qquad \partial^t(S_2) = Q_{[P'']_0[P']_0}$$

$$\text{P1B}'' \xrightarrow{\text{C3}} \text{P1B}' \xrightarrow{\text{W3}} \text{P1B},$$
$$\partial^i(S_1) = \tilde{T}^L_{[P']_1[P'']_1} \qquad \alpha_2^m(S_2) = Q_{[P]_1[P']_1}$$

$$\text{P1B}'' \xrightarrow{\text{W3}} \text{P1B}' \xrightarrow{\text{C3}} \text{P1B}.$$
$$\partial^i(S_1) = Q_{[P']_1[P]_1} \qquad \alpha_2^m(S_2) = \tilde{T}^L_{[P'']_1[P']_1}$$

We claim that in each case the reduction factor is zero. First consider the form shown in table (38). From diagrams (18) and (19) for type W2 and W3 saddles, we have that

$$\tilde{T}^L_{[P']_0[P]_0} = Q_{P'P}(S_1) \otimes \eta : C_P \to C_{P'} \otimes A, \qquad Q_{[P'']_0[P']_0} = Q_{P''P'}(S_2) \otimes 1_A : C_{P'} \otimes A \to C_{P''} \otimes A.$$

Since the image of $Q_{[P'']_0[P']_0}\tilde{T}^L_{[P']_0[P]_0}$ lies in $C_{P''} \otimes 1_A$, it follows that $\beta_1(P, S_1, S_2) = Q_{[P'']_0[P']_0}\tilde{T}^L_{[P']_0[P]_0} = 0$. Next consider the form shown in table (39). From diagrams (19) and (23) for type W3 and C2 saddles, we have that

$$Q_{[P']_0[P]_0} = Q_{P'P}(S_1) \otimes 1_A : C_P \otimes A \to C_{P'} \otimes A, \qquad \tilde{T}^L_{[P'']_0[P']_0} = [T^R_{P''P'}] \otimes \epsilon : C_{P'} \otimes A \to W''.$$

Since $\epsilon(1_A) = 0$, it follows that $\beta_1(P, S_1, S_2) = \tilde{T}^L_{[P'']_0[P']_0}Q_{[P']_0[P]_0} = 0$. Next consider the form shown in table (40). From diagrams (23) and (19) for type C2 and W3 saddles, we have that

$$\tilde{T}^L_{[P']_1[P'']_1} = [T^R_{P'P''}] \otimes \eta : W'' \to C_{P'} \otimes A, \qquad Q_{[P]_1[P']_1} = Q_{PP'}(S_1) \otimes 1_A : C_{P'} \otimes A \to C_P \otimes A.$$

Since the image of $Q_{[P]_1[P']_1}\tilde{T}^L_{[P']_1[P'']_1}$ lies in $C_P \otimes 1_A$, it follows that $\alpha_2(P, S_1, S_2) = Q_{[P'']_1[P']_1}\tilde{T}^L_{[P']_1[P]_1} = 0$. The arguments for the remaining forms are similar. □

Lemma 10.3. *All the reduction terms obtained from $\partial_{\overline{T}^+}$ occur in ∂_{T^+}.*

Proof. Lemma 6.5 states that terms of $\partial_{\overline{T}^+}$ corresponding to saddles of type (1) and (2) yield a reduction factor $\beta_1(P) = \tilde{\partial}^{c_0}_P$ for each tangle type P of type P1A and a reduction factor $\alpha_2(P) = \tilde{\partial}^{c_1}_P$ for each tangle type P of type P1B.

We now consider the additional reduction factors that are obtained if we include saddles of type (3). By Lemma 10.2, we can ignore terms of $\partial^+_{\overline{T}^+}$ corresponding to pairs of saddles that are both of type (3). Type (3) saddles $[p]_0 \to [p']_0$ and $[p]_1 \to [p']_1$ obtained from the resolution of $\overline{T}$ are induced by a type-changing saddle $S : p \to p'$ obtained from the resolution of T. Using the results of Sections 7 and 8, for each type-changing saddle $S : p \to p'$ obtained from the resolution of T we indicate the corresponding reduction factors:

For a type W2 saddle $S : p_{n+1}(r) \leftrightarrow p'_{n-1}(r)$:

$$\text{P1B} \xrightarrow{\text{W2}} \text{P1A}'.$$
$$\beta_1(P, S) = Q_{P'P}(S) \qquad \alpha_2(P', S) = Q_{P'P}(S)$$

For a type C1 saddle $S : p_n(r) \leftrightarrow p'_n(r+1)$:

$$\begin{array}{ccccc} \text{P1A} & \xrightarrow{\text{C1}} & \text{P1A}', & \qquad \text{P1A} & \xleftarrow{\text{C1}} & \text{P1A}', \\ \beta_1(P,S) = \tilde{T}^L_{P'P}(S) + \tilde{T}^R_{P'P}(S) & & 0 & \qquad 0 & & \beta_1(P',S) = \tilde{T}^L_{PP'}(S) + \tilde{T}^R_{PP'}(S) \end{array}$$

$$\begin{array}{ccccc} \text{P1B} & \xrightarrow{\text{C1}} & \text{P1B}', & \qquad \text{P1B} & \xleftarrow{\text{C1}} & \text{P1B}'. \\ 0 & & \alpha_2(P',S) = \tilde{T}^L_{P'P}(S) + \tilde{T}^R_{P'P}(S) & \qquad \alpha_2(P,S) = \tilde{T}^L_{PP'}(S) + \tilde{T}^R_{PP'}(S) & & 0 \end{array}$$

For a type C2 saddle $S : p_n(r) \leftrightarrow p'_n(r+1)$:

$$\begin{array}{ccccc} \text{P1A} & \xrightarrow{\text{C2}} & \text{P2}', & \qquad \text{P1B} & \xleftarrow{\text{C2}} & \text{P2}'. \\ \beta_1(P,S) = \tilde{T}^L_{P'P}(S) + \tilde{T}^R_{P'P}(S) & & 0 & \qquad \alpha_2(P,S) = \tilde{T}^L_{PP'}(S) + \tilde{T}^R_{PP'}(S) & & 0 \end{array}$$

At the head and tail of each arrow we indicate the type of the corresponding planar tangle. Below the head (tail) of each arrow we indicate the reduction factor α_2 (β_1) obtained via reduction on the corresponding tangle type. Saddles that do not yield reduction factors are not included in the list.

We find that for each type W2 saddle $S : p_{n+1}(r) \to p'_{n-1}(r)$ we obtain the reduction term

$$\tilde{\partial}^{c_0}_{P'} Q_{P'P}(S) + Q_{P'P}(S)\tilde{\partial}^{c_1}_{P} = \tilde{\partial}^{L}_{P'} Q_{P'P}(S) + Q_{P'P}(S)\tilde{\partial}^{L}_{P},$$

where the equality was shown in the proof of Lemma 9.4. We also obtain the following reduction terms for a pair of saddles (S_1, S_2) of type (W2,C1) and (W2,C2):

$$\tilde{T}^L_{P''P'}(S_2)Q_{P'P}(S_1) + \tilde{T}^R_{P''P'}(S_2)Q_{P'P}(S_1), \qquad Q_{P''P'}(S_2)\tilde{T}^L_{P'P}(S_1) + Q_{P''P'}(S_2)\tilde{T}^R_{P'P}(S_1).$$

We can view $S_1 : p \to p'$ and $S_2 : p' \to p''$ as two sides of a commuting square of saddles. Let $\overline{S}_1 : \overline{p} \to p''$ and $\overline{S}_2 : p \to \overline{p}$ denote the sides opposite to S_1 and S_2. If S_1 and S_2 constitute two sides of a commuting square of interleaved saddles, then

$$\tilde{T}^R_{P''P'}(S_2)Q_{P'P}(S_1) = \tilde{T}^L_{P''\overline{P}}(\overline{S}_1)Q_{\overline{P}P}(\overline{S}_2), \qquad Q_{P''P'}(S_2)\tilde{T}^R_{P'P}(S_1) = Q_{P''\overline{P}}(\overline{S}_1)\tilde{T}^L_{\overline{P}P}(\overline{S}_2).$$

If S_1 and S_2 constitute two sides of a commuting square of nested saddles, then

$$\tilde{T}^R_{P''P'}(S_2)Q_{P'P}(S_1) = Q_{P''\overline{P}}(\overline{S}_1)\tilde{T}^L_{\overline{P}P}(\overline{S}_2), \qquad Q_{P''P'}(S_2)\tilde{T}^R_{P'P}(S_1) = \tilde{T}^L_{P''\overline{P}}(\overline{S}_1)Q_{\overline{P}P}(\overline{S}_2).$$

If S_1 and S_2 constitute two sides of a commuting square of disjoint saddles, then the contributions of (S_1, S_2) and $(\overline{S}_1, \overline{S}_2)$ cancel. In each case, the residual terms we obtain occur in ∂_{T^+}. □

Lemmas 10.1 and 10.3 give

Lemma 10.4. *All the residual and reduction terms of ∂_{red} occur in ∂_{T^+}.*

11. Spectral sequence

In this section we use 1-tangle diagrams in the annulus to construct a spectral sequence that converges to reduced Khovanov homology. If we forget the bigrading of the chain complex $(C_{T^\pm}, \partial_{T^\pm})$, we obtain a chain complex $(C_T, \partial^\pm_T)$, where $\partial_{T^\pm} = \partial^0_T + \partial^\pm_T$; note in particular that $C_T = C_{T^+} = C_{T^-}$ as ungraded vector spaces and $\partial^0_T = \partial^0_{T^+} = \partial^0_{T^-}$ as maps of ungraded vector spaces. We define a $\mathbb{Z}$-grading on C_T by

$$C_T = \bigoplus_s C^s_T,$$

where

$$C^s_T = \bigoplus \{A^{c(T_i)} \mid i \in I \text{ such that } r(i) = s\}.$$

We refer to the grading s as the *resolution degree*. We indicate that a homogeneous vector $v \in C_T$ has resolution degree s by using a superscript: $v^{(s)}$. The resolution degrees of the maps ∂^0_T and $\partial^\pm_T$ that appear in the differential $\partial_{T^\pm} = \partial^0_T + \partial^\pm_T$ are $(\partial^0_T)^{(1)}$ and $(\partial^\pm_T)^{(2)}$. The fact that $(\partial_T)^2 = 0$ holds in each resolution degree thus implies:

Lemma 11.1. *We have $(\partial^0_T)^2 = (\partial^0_{T^\pm})^0 = 0$.*

We can define a filtration $\cdots \supset K_T^{-1} \supset K_T^0 \supset K_T^1 \supset \cdots$ of the chain complex C_T by

$$K_T^r = \bigoplus_{s \geq r} C_T^s.$$

Using Theorem 3.1, we thus obtain Theorem 1.2 from the Introduction:

Theorem 11.2. *There is a spectral sequence with E_2 page $(H_{\partial_T^0}(C_T), d_2^{\pm})$, where $d_2^{\pm}([x]) = [\partial_T^{\pm} x]$, that converges to the reduced Khovanov homology of the link $T^{\pm}$.*

12. Example

We now illustrate Theorem 3.1 with an example. Consider the oriented disk 2-tangle diagram T_D shown in Figure 10. Let T denote the corresponding oriented annular 1-tangle diagram, as shown in Figure 1. We resolve the crossings of T to obtain the following planar tangles and saddles:

$$\begin{array}{ccccc} & & p_{-1}(0) & & \\ & \nearrow & & \searrow & \\ p_1(0) & & & & p_{-1}(1). \\ & \searrow & & \nearrow & \\ & & p_{-1}(0) & & \end{array}$$

For T^+ we have $n_+(T^+) = 2$ and $n_-(T^+) = 1$, so the chain complex $(C_{T^+}, \partial_{T^+})$ is

$$\begin{array}{ccccc} & & \mathbb{F}[0,0] & & \\ & \overset{0}{\nearrow} & & \overset{\dot\eta}{\searrow} & \\ \mathbb{F}[0,2] & & \xrightarrow{\quad\eta\quad} & & A[1,1], \\ & \underset{0}{\searrow} & & \underset{\dot\eta}{\nearrow} & \\ & & \mathbb{F}[0,0] & & \end{array}$$

and we obtain the reduced Khovanov homology for the unknot:

$$\mathrm{Khr}(T^+) = \mathbb{F}[0,0].$$

For T^- we have $n_+(T^-) = 3$ and $n_-(T^-) = 0$, so the chain complex $(C_{T^-}, \partial_{T^-})$ is

$$\begin{array}{ccccc} & & \mathbb{F}[2,6] & & \\ & \overset{0}{\nearrow} & & \overset{\dot\eta}{\searrow} & \\ \mathbb{F}[0,2] & & & & A[3,7], \\ & \underset{0}{\searrow} & & \underset{\dot\eta}{\nearrow} & \\ & & \mathbb{F}[2,6] & & \end{array}$$

and we obtain the reduced Khovanov homology for the right trefoil:

$$\mathrm{Khr}(T^-) = \mathbb{F}[0,2] \oplus \mathbb{F}[2,6] \oplus \mathbb{F}[3,8].$$

The E_2 page of the spectral sequence for T is

$$H_{\partial_T^0}(C_T) = \mathbb{F} \cdot \{[1]^{(0)},\ [(1,1)]^{(1)},\ [1_A]^{(2)}\}.$$

For T^+, the differential is d_2^+ is given by

$$d_2^+([1]) = [1_A], \qquad d_2^+([(1,1)]) = 0, \qquad d_2^+([1_A]) = 0.$$

For T^-, the differential is $d_2^- = 0$.

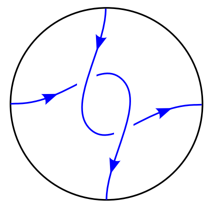

FIGURE 10. Example disk 2-tangle diagram T_D.

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