HOLONOMY PERTURBATIONS OF THE CHERN-SIMONS FUNCTIONAL FOR LENS SPACES

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Abstract. We describe a scheme for constructing generating sets for Kronheimer and Mrowka's singular instanton knot homology for the case of knots in lens spaces. The scheme involves Heegaard-splitting a lens space containing a knot into two solid tori. One solid torus contains a portion of the knot consisting of an unknotted arc, as well as holonomy perturbations of the Chern-Simons functional used to define the homology theory. The other solid torus contains the remainder of the knot. The Heegaard splitting yields a pair of Lagrangians in the traceless $SU(2)$-character variety of the twice-punctured torus, and the intersection points of these Lagrangians comprise the generating set that we seek. We illustrate the scheme by constructing generating sets for several example knots. Our scheme is a direct generalization of a scheme introduced by Hedden, Herald, and Kirk for describing generating sets for knots in S^3 in terms of Lagrangian intersections in the traceless $SU(2)$-character variety of the 2-sphere with four punctures.

Contents

1. Introduction

Singular instanton homology was introduced by Kronheimer and Mrowka to describe knots in 3-manifolds [17, 18, 19]. Singular instanton homology is defined in terms of gauge theory, but has important implications for Khovanov homology, a knot homology theory that categorifies the Jones polynomial and that can be defined in a purely combinatorial fashion. Specifically, given a knot K in S^3, Kronheimer and Mrowka showed that there is a spectral sequence whose E_2 page is the reduced Khovanov homology of the mirror knot $\bar{K}$, and that converges to the singular instanton homology of K. Using this spectral sequence, Kronheimer and Mrowka proved a key property of Khovanov homology: a knot in S^3 is the unknot if and only if its

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reduced Khovanov homology has rank 1. This result is obtained by proving the analogous result for singular instanton homology and then using the rank inequality implied by the spectral sequence.

Calculations of singular instanton homology are generally difficult to carry out, though some results are known. For example, Kronheimer and Mrowka showed that the singular instanton homology of an alternating knot in S^3 is isomorphic to the reduced Khovanov homology of its mirror (modulo grading), since for such knots the spectral sequence collapses at the E_2 page. Hedden, Herald, and Kirk have described a scheme for producing generating sets for singular instanton homology for a variety of knots in S^3, which can sometimes be used to compute the singular instanton homology itself [13]. Their scheme works as follows.

Singular instanton homology is defined in terms of the Morse complex of a perturbed Chern-Simons functional. The unperturbed Chern-Simons functional is typically not Morse, so to obtain a well-defined homology theory it is necessary to include a small perturbation. For the case of knots in S^3, Hedden, Herald, and Kirk show how a suitable perturbation can be constructed. Their scheme involves Heegaard-splitting S^3 into a pair of solid 3-balls B_1 and B_2. The ball B_1 contains a portion of the knot K consisting of two unknotted arcs, together with a specific holonomy perturbation of the Chern-Simons functional. The ball B_2 contains the remainder of the knot. The Heegaard splitting of S^3 yields a "perturbed" Lagrangian L_1^π and an "unperturbed" Lagrangian L_2 in the traceless $SU(2)$-character variety of the 2-sphere with four punctures $R(S^2, 4)$, a symplectic orbifold known as the *pillowcase* that is homeomorphic to the 2-sphere. In many cases, the points of intersection of L_1^π and L_2 constitute a generating set for singular instanton homology. The essential idea of the scheme is to confine the perturbation to a standard ball B_1 corresponding to a perturbed Lagrangian L_1^π that can be described explicitly. The problem of constructing a generating set for a particular knot then reduces to describing the Lagrangian L_2, a task that is facilitated by the fact that the Chern-Simons functional is unperturbed on the ball B_2. In further work, Hedden, Herald, and Kirk define *pillowcase homology* to be the Lagrangian Floer homology of the pair (L_1^π, L_2) [14]. They conjecture that pillowcase homology is isomorphic to singular instanton homology, and compute some examples that support this conjecture.

In the present paper we generalize the scheme of Hedden, Herald, and Kirk to the case of knots in lens spaces. We Heegaard-split a lens space Y containing a knot K into two solid tori U_1 and U_2. The solid torus U_1 contains a portion of the knot consisting of an unknotted arc, together with a specific holonomy perturbation. The solid torus U_2 contains the remainder of the knot. From the Heegaard splitting of Y we obtain a pair of Lagrangians L_1^π and L_2 in the traceless $SU(2)$-character variety of the twice-punctured torus $R(T^2, 2)$, and in many cases the points of intersection of L_1^π and L_2 constitute a generating set for the reduced singular instanton homology $I^\natural(Y, K)$.

To explain the details of our scheme, we must first define several character varieties related to the Chern-Simons functional. Critical points of the unperturbed Chern-Simons functional are flat connections. Gauge-equivalence classes of flat connections correspond to conjugacy classes of homomorphisms $\rho : \pi_1(Y - K \cup H \cup W) \to SU(2)$, where H is a small loop around K and W is an arc connecting K to H, as shown in Figure 1, and the homomorphisms are required to take loops around K and H to traceless matrices and loops around W to -1. The set of such conjugacy classes form a character variety that we will denote by $R^\natural(Y, K)$. We will refer to $H \cup W$ as an *earring* that has been added to the knot K. The conditions on ρ involving the earring are imposed in order to avoid reducible connections; such connections prevent us from obtaining a chain complex for singular instanton homology, with a differential that squares to zero. It is also useful to define a character variety $R(Y, K)$ in which we do not impose these conditions, and which consists of conjugacy classes of homomorphisms $\rho : \pi_1(Y - K) \to SU(2)$ that take loops around K to traceless matrices.

The character variety $R^\natural(Y, K)$ is typically degenerate, in which case the unperturbed Chern-Simons functional is not Morse. We can render the Chern-Simons functional Morse by introducing a suitable holonomy perturbation term that vanishes outside of a small solid torus obtained by thickening a loop $P \subset Y$. The perturbation modifies the corresponding character variety: the critical points of the perturbed Chern-Simons functional correspond to conjugacy classes of homomorphisms $\rho : \pi_1(Y - K \cup H \cup W \cup P) \to SU(2)$, where ρ obeys the same conditions as for $R^\natural(Y, K)$ as well as an additional condition involving the loop P that we will describe in Section 3.3. We will denote the character variety corresponding to the perturbed Chern-Simons functional by $R_\pi^\natural(Y, K)$. The character variety $R_\pi^\natural(Y, K)$ is a finite set of points that under certain conditions generates the reduced singular instanton homology $I^\natural(Y, K)$.

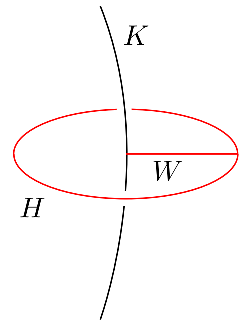


FIGURE 1. The knot K, loop H, and arc W.

Example 1.1. For the trefoil K in S^3, one can show that

$$R(S^3,K) = \{2 \text{ points}\}, \qquad R^\natural(S^3,K) = \{1 \text{ point}\} \amalg S^1, \qquad R^\natural_\pi(S^3,K) = \{3 \text{ points}\},$$

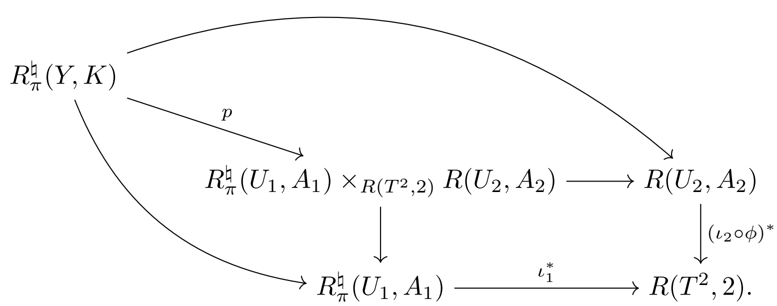
where the perturbation used to define $R^\natural_\pi(S^3,K)$ is as described in Section 6.1.

Our goal, then, is to devise an effective means of calculating $R^\natural_\pi(Y,K)$. We will view (Y,K) as the result of gluing together two solid tori $U_1 = S^1 \times D^2$ and $U_2 = S^1 \times D^2$. The solid torus U_1 contains an unknotted arc A_1, the earring $H \cup W$, and the holonomy perturbation loop P, as shown in Figure 6. The solid torus U_2 contains a (possibly knotted) arc A_2. We glue the two tori together via a homeomorphism $\phi : (\partial U_1, \partial A_1) \to (\partial U_2, \partial A_2)$ to obtain (Y,K).

We define character varieties $R^\natural_\pi(U_1,A_1)$ and $R(U_2,A_2)$ in analogy with $R^\natural_\pi(Y,K)$ and $R(Y,K)$. The character variety $R^\natural_\pi(U_1,A_1)$ consists of conjugacy classes of homomorphisms $\rho : \pi_1(U_1 - A_1 \cup H \cup W \cup P) \to SU(2)$ that take loops around A_1 and H to traceless matrices and loops around W to -1, and satisfy an additional requirement involving P as described in Section 3.3. The character variety $R(U_2,A_2)$ consists of conjugacy classes of homomorphisms $\rho : \pi_1(U_2 - A_2) \to SU(2)$ that take loops around A_2 to traceless matrices. We define a torus $T^2 := \partial U_1$ containing points $\{p_1,p_2\} = \partial A_1$, and we define a corresponding character variety $R(T^2,2)$ that consists of conjugacy classes of homomorphisms $\rho : \pi_1(T^2 - \{p_1,p_2\}) \to SU(2)$ that take loops around p_1 and p_2 to traceless matrices.

We define a map $R^\natural_\pi(U_1,A_1) \to R(T^2,2)$ by pulling back along the inclusion $\iota_1 : (\partial U_1, \partial A_1) \hookrightarrow (U_1,A_1)$. We define a map $R(U_2,A_2) \to R(T^2,2)$ by pulling back along the composition of the inclusion map $\iota_2 : (\partial U_2, \partial A_2) \hookrightarrow (U_2,A_2)$ with the gluing map $\phi : (\partial U_1, \partial A_1) \to (\partial U_2, \partial A_2)$. We similarly define maps $R^\natural_\pi(Y,K) \to R^\natural_\pi(U_1,A_1)$ and $R^\natural_\pi(Y,K) \to R(U_2,A_2)$ by pulling back along inclusions. We have a commutative diagram:

(1)



Here p is an induced map from $R^\natural_\pi(Y,K)$ to the fiber product $R^\natural_\pi(U_1,A_1) \times_{R(T^2,2)} R(U_2,A_2)$. The character variety $R(T^2,2)$ is a symplectic orbifold that generalizes the pillowcase, and the images of the maps $R^\natural_\pi(U_1,A_1) \to R(T^2,2)$ and $R(U_2,A_2) \to R(T^2,2)$ define Lagrangians L_1^π and L_2 in $R(T^2,2)$. We want to use diagram (1) to describe $R^\natural_\pi(Y,K)$ in terms of the intersection points of these Lagrangians. Our first task is to obtain an explicit description of the character variety $R(T^2,2)$. We prove:

Theorem 1.2. *The character variety $R(T^2,2)$ contains a compact subset P_3 with open, dense complement P_4. The piece P_3 deformation retracts onto the pillowcase. The piece P_4 is homeomorphic to $S^2 \times S^2 - \bar{\Delta}$,* *where* $\bar{\Delta} = \{(\hat{r}, -\hat{r})\}$ *is the antidiagonal. (The pieces* P_3 *and* P_4 *are described explicitly in Theorems 3.8 and 3.6.)*

Remark 1.3. The character variety $R(T^2, 2)$ is homeomorphic to the moduli space $M^{ss}(X, 2)$ of semistable parabolic bundles over an elliptic curve X, which is known to have the structure of an algebraic variety isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$ (see [4, 21]). It follows that $R(T^2, 2)$ is homeomorphic to $S^2 \times S^2$, although this does not seem to be easy to show from our description of this space. We use the character variety $R(T^2, 2)$, rather than the moduli space $M^{ss}(X, 2)$, since it is only for $R(T^2, 2)$ that we can explicitly describe the Lagrangians L_1^π and L_2.

Our next task is to explicitly describe the perturbed Lagrangian L_1^π. We prove:

Theorem 1.4. *The character variety* $R_\pi^\natural(U_1, A_1)$ *is homeomorphic to* S^2*. The map* $R_\pi^\natural(U_1, A_1) \to R(T^2, 2)$ *is an injective immersion away from the points of* $R_\pi^\natural(U_1, A_1)$ *corresponding to the north and south pole of* S^2*, which are mapped to the same point. All representations in the image* L_1^π *of the map are nonabelian. (An explicit parameterization of* L_1^π *is given in Theorem 3.17.)*

Theorem 1.4 is closely analogous to a corresponding result in the scheme of Hedden, Herald, and Kirk. In their scheme, the perturbed Lagrangian analogous to our Lagrangian L_1^π is the image of S^1 under an injective immersion. Their perturbed Lagrangian has a single transverse double-point, and thus forms a figure-8 in the pillowcase $R(S^2, 4)$. It is unclear whether it is possible, either in their scheme or ours, to construct perturbed Lagrangians without double-points by choosing different holonomy perturbations. Theorem 1.4 also yields:

Corollary 1.5. *The map* $p : R_\pi^\natural(Y, K) \to R_\pi^\natural(U_1, A_1) \times_{R(T^2,2)} R(U_2, A_2)$ *in diagram (1) is injective.*

Proof. Consider a point $([\rho_1], [\rho_2])$ in $R_\pi^\natural(U_1, A_1) \times_{R(T^2,2)} R(U_2, A_2)$, so ρ_1 and ρ_2 pull back to the same homomorphism $\rho_{12} : \pi_1(T^2 - \{p_1, p_2\}) \to SU(2)$. One can show (see [13] Lemma 4.2) that the fiber $p^{-1}([\rho_1], [\rho_2])$ is homeomorphic to the double coset space $\mathrm{Stab}(\rho_1)\backslash \mathrm{Stab}(\rho_{12})/\mathrm{Stab}(\rho_2)$, where

$$\mathrm{Stab}(\rho) = \{g \in SU(2) \mid g\rho(x)g^{-1} = \rho(x) \text{ for all } x \text{ in the domain of } \rho\}.$$

The center of $SU(2)$ is $Z(SU(2)) = \{\pm 1\}$. By Theorem 1.4 we have that $\mathrm{Stab}(\rho_{12}) = Z(SU(2))$, and $Z(SU(2)) \subseteq \mathrm{Stab}(\rho_i) \subseteq \mathrm{Stab}(\rho_{12})$, so $\mathrm{Stab}(\rho_i) = \mathrm{Stab}(\rho_{12}) = Z(SU(2))$ and thus the fibers of p are points. □

By introducing a suitable holonomy perturbation, we obtain a finite character variety $R_\pi^\natural(Y, K)$, each point of which corresponds to a gauge-orbit of connections that are critical points of the perturbed Chern-Simons functional. In order for $R_\pi^\natural(Y, K)$ to serve as a generating set for singular instanton homology, each point in $R_\pi^\natural(Y, K)$ must be nondegenerate; that is, at each connection representing a point in $R_\pi^\natural(Y, K)$ we want the Hessian of the perturbed Chern-Simons functional to be nondegenerate when restricted to a complement of the tangent space to the gauge-orbit of that connection. We show that there is a simple criterion for determining when a point in $R_\pi^\natural(Y, K)$ is nondegenerate. Recall that we defined the Lagrangian L_2 to be the image of $R(U_2, A_2) \to R(T^2, 2)$. If $R(U_2, A_2) \to R(T^2, 2)$ is injective and $[\rho] \in L_1^\pi \cap L_2 \subset R(T^2, 2)$ is not the double-point of L_1^π, then by Corollary 1.5 the point $[\rho]$ is the image of a unique point $[\tilde{\rho}] \in R_\pi^\natural(Y, K)$ under the pullback map $R_\pi^\natural(Y, K) \to R(T^2, 2)$. We prove:

Theorem 1.6. *Suppose* $R(U_2, A_2) \to R(T^2, 2)$ *is an injective immersion and* $[\rho] \in L_1^\pi \cap L_2$ *is the image of a regular point of* $R(U_2, A_2)$ *and is not the double-point of* L_1^π*. Then the unique preimage* $[\tilde{\rho}]$ *of* $[\rho]$ *under the pullback map* $R_\pi^\natural(Y, K) \to R(T^2, 2)$ *is nondegenerate if and only if the intersection of* L_1^π *with* L_2 *at* $[\rho]$ *is transverse.*

Collecting these results, we find if the hypotheses of Theorem 1.6 are satisfied for every point in $L_1^\pi \cap L_2$, then every point in $R_\pi^\natural(Y, K)$ is nondegenerate and the pullback map $R_\pi^\natural(Y, K) \to R(T^2, 2)$ is injective with image $L_1^\pi \cap L_2$. Thus we obtain:

Corollary 1.7. *If the hypotheses of Theorem 1.6 are satisfied for every point in* $L_1^\pi \cap L_2$*, then* $R_\pi^\natural(Y, K)$ *is a generating set for* $I^\natural(Y, K)$ *consisting of* $|L_1^\pi \cap L_2|$ *generators.*

Our scheme is particularly well-suited for $(1, 1)$-knots. By definition, a $(1, 1)$-knot is a knot K in a lens space Y that has a Heegaard splitting into a pair of solid tori $U_1, U_2 \subset Y$ such that the components $U_1 \cap K$

and $U_2 \cap K$ of the knot in each solid torus are unknotted arcs. It is known that $(1,1)$-knots include all torus knots and 2-bridge knots.

We can construct $(1,1)$-knots by taking (U_2, A_2) to be a copy of (U_1, A_1) without the earring $H \cup W$ or the perturbation loop P, and we can explicitly describe the corresponding Lagrangian L_2 as follows. We first define a character variety $R(U_1, A_1)$ that consists of conjugacy classes of homomorphisms $\pi_1(U_1 - A_1) \to SU(2)$ that take loops around A_1 to traceless matrices. We define a map $R(U_1, A_1) \to R(T^2, 2)$ by pulling back along the inclusion $(\partial U_1, \partial A_1) \hookrightarrow (U_1, A_1)$. The image of this map defines a Lagrangian L_1 in $R(T^2, 2)$. We can view $R(U_1, A_1)$ and L_1 as "unperturbed" and "unearringed" versions of $R^\natural_\pi(U_1, A_1)$ and L^π_1. Since $(T^2, \{p_1, p_2\}) := (\partial U_1, \partial A_1)$ and there is a natural identification $(\partial U_2, \partial A_2) \xrightarrow{\sim} (T^2, \{p_1, p_2\})$, the gluing map $\phi : (\partial U_1, \partial A_1) \to (\partial U_2, \partial A_2)$ defines an element $[\phi]$ of the mapping class group $\mathrm{MCG}_2(T^2)$ of the twice-punctured torus. The group $\mathrm{MCG}_2(T^2)$ acts on $R(T^2, 2)$ from the right in a way that we explicitly describe in Section 5, and the Lagrangian L_2, which we defined to be the image of the map $R(U_2, A_2) \to R(T^2, 2)$, is given by $L_2 = L_1 \cdot [\phi]$. We prove results that explicitly describe the character variety $R(U_1, A_1)$ and the Lagrangian L_1:

Theorem 1.8. *The character variety $R(U_1, A_1)$ is homeomorphic to the closed disk D^2. The map $R(U_1, A_1) \to R(T^2, 2)$ is injective and is an immersion on the interior of $R(U_1, A_1)$. (An explicit parameterization of the image L_1 of the map is given in Theorem 3.13.)*

Theorem 1.9. *The character variety $R(U_1, A_1)$ is regular on its interior.*

From Theorems 1.8 and 1.9, we obtain a Corollary to Theorem 1.6 for the special case of $(1,1)$-knots. Recall that for a $(1,1)$ knot K we can describe the Lagrangian L_2 as the image $L_2 = L_1 \cdot [\phi]$ of L_1 under the action of the mapping class group element $[\phi] \in \mathrm{MCG}_2(T^2)$, where $\phi : (\partial U_1, \partial A_1) \to (\partial U_2, \partial A_2)$ is the gluing map. We then have:

Corollary 1.10. *For a $(1,1)$-knot K, if L^π_1 intersects $L_2 = L_1 \cdot [\phi]$ transversely away from the double-point of L^π_1, then $R^\natural_\pi(Y, K)$ is a generating set for $I^\natural(Y, K)$ consisting of $|L^\pi_1 \cap L_2|$ generators.*

Since we have explicit descriptions of the character variety $R(T^2, 2)$, the Lagrangians L^π_1 and L_1, and the action of the mapping class group $\mathrm{MCG}_2(T^2)$ on $R(T^2, 2)$, Corollary 1.10 yields a practical scheme for calculating a generating set for $I^\natural(Y, K)$ for any $(1,1)$-knot K in any lens space Y. In Section 6 we illustrate this scheme by calculating generating sets for several example $(1,1)$-knots. We first rederive known results for knots in S^3: we produce generating sets for the unknot (one generator) and trefoil (three generators), which allow us to compute the singular instanton homology for these knots. Next we consider knots in lens spaces $L(p, 1)$. We prove:

Theorem 1.11. *If p is not a multiple of 4, then the rank of $I^\natural(L(p, 1), U)$ for the unknot U in the lens space $L(p, 1)$ is at most p.*

A knot K in a lens space $L(p, q)$ is said to be *simple* if the lens space can be Heegaard-split into solid tori U_1 and U_2 with meridian disks D_1 and D_2 such that D_1 intersects D_2 in p points and $K \cap U_i$ is an unknotted arc in disk D_i for $i = 1, 2$ (see [12]). Up to isotopy, there is exactly one simple knot in each nonzero homology class of $H_1(L(p, q); \mathbb{Z}) = \mathbb{Z}_p$. We prove:

Theorem 1.12. *If K is the unique simple knot representing the homology class $1 \in \mathbb{Z}_p = H_1(L(p, 1); \mathbb{Z})$ of the lens space $L(p, 1)$, then the rank of $I^\natural(L(p, 1), K)$ is at most p.*

To our knowledge, Theorems 1.11 and 1.12 give the first rank bounds on instanton homology for knots in 3-manifolds other than S^3. For a simple knot K in the lens space $Y = L(p, q)$, the knot Floer homology $\widehat{HFK}(Y, K)$ has rank p (see [12]). Thus, Theorem 1.12 is consistent with Kronheimer and Mrowka's conjecture that for a knot K in a 3-manifold Y, the ranks of $I^\natural(Y, K)$ and $\widehat{HFK}(Y, K)$ are the same (see [16] Section 7.9). There is a combinatorial method for computing the Floer homology for arbitrary $(1,1)$-knots in S^3 (see [10]), and it would be interesting to test Kronheimer and Mrowka's conjecture by using our results to derive rank bounds for the singular instanton homology of such knots.

As a more ambitious application of our results, we hope to generalize Khovanov homology to links in lens spaces. In recent work [15], Hedden, Herald, Hogancamp, and Kirk show that Bar Natan's functor from the tangle cobordism category for 2-tangles in the 3-ball to chain complexes can be factored through the twisted Fukaya category for the pillowcase $R(S^2, 4)$. We conjecture that an analogous factorization result holds for the twisted Fukaya category of $R(T^2, 2)$; if so, the corresponding functor would provide a natural way of generalizing Khovanov homology to links in lens spaces. Using the results presented here, we can already describe some of the relevant structure of the Fukaya category of $R(T^2, 2)$. The information we have obtained regarding this Fukaya category suggests that the factorization result of [15] does indeed generalize, but we have not proven this. Based on clues provided by this information, we have shown that the reduced Khovanov homology of a link in S^3 can be recovered from a chain complex constructed from a description of the link in terms of a 1-tangle diagram in the annulus [5]. We hope that further investigation of the Fukaya category of $R(T^2, 2)$ will enable us to construct a homology theory for links in lens spaces that generalizes Khovanov homology.

2. The group $SU(2)$

Here we briefly review some basic facts about the group $SU(2)$. We define $SU(2)$-matrices $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ by

$$\mathbf{i} = -i\sigma_x, \qquad \mathbf{j} = -i\sigma_y, \qquad \mathbf{k} = -i\sigma_z,$$

where σ_x, σ_y, and σ_z are the Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The matrices $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ satisfy the quaternion multiplication laws $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. Any $SU(2)$-matrix A can be uniquely expressed as

$$A = t + x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k},$$

where $(t, x, y, z) \in S^3 = \{(t, x, y, z) \in \mathbb{R}^4 \mid t^2 + x^2 + y^2 + z^2 = 1\}$, and thus we may identify $SU(2)$ with the space of unit quaternions. We will refer to t and $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ as the *scalar* and *vector* parts of the matrix A, respectively. Note that $\mathrm{tr}(A) = 2t$, so traceless $SU(2)$-matrices are precisely those for which the scalar part is zero. It follows that traceless $SU(2)$-matrices are parameterized by unit vectors in $\mathbb{R}^3$, and we will frequently pass back and forth between traceless matrices $a = a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k} \in SU(2)$ and their corresponding unit vectors $\hat{a} = (a_x, a_y, a_z) \in S^2$.

We can define a surjective group homomorphism $SU(2) \to SO(3)$ by $g \mapsto (\hat{v} \mapsto \hat{v}')$, where the unit vectors $\hat{v} = (v_x, v_y, v_z)$ and $\hat{v}' = (v'_x, v'_y, v'_z)$ are related by

$$g(v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k})g^{-1} = v'_x\,\mathbf{i} + v'_y\,\mathbf{j} + v'_z\,\mathbf{k}.$$

In general, conjugating an arbitrary $SU(2)$-matrix preserves the scalar part of the matrix and rotates the vector part of the matrix:

$$g(t + r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k})g^{-1} = t + r'_x\,\mathbf{i} + r'_y\,\mathbf{j} + r'_z\,\mathbf{k},$$

where (r'_x, r'_y, r'_z) is given by multiplying (r_x, r_y, r_z) by the $SO(3)$-matrix corresponding to $g \in SU(2)$. We will thus sometimes describe conjugation in terms of the corresponding rotation performed on the vector part of an $SU(2)$-matrix. We use the conjugation action to prove the following result:

Lemma 2.1. *Any $SU(2)$-matrix can be expressed as a product of two traceless $SU(2)$-matrices.*

Proof. Define a product map $SU(2) \times SU(2) \to SU(2)$, $(A, B) \mapsto AB$. If we restrict the product map to traceless $SU(2)$-matrices, represent each traceless $SU(2)$-matrix as a point in S^2, and represent their product as a point in S^3, we obtain a map $S^2 \times S^2 \to S^3$ given by

$$(\hat{a},\, \hat{b}) \mapsto (-\hat{a} \cdot \hat{b},\, \hat{a} \times \hat{b}).$$

This map is surjective, since

$$(\hat{x},\, \hat{x}\cos\theta + \hat{y}\sin\theta) \mapsto (-\cos\theta,\, \hat{z}\sin\theta)$$

and the product map is equivariant under conjugation. □

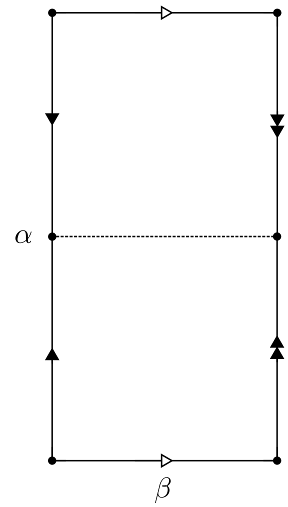


FIGURE 2. The pillowcase $R(T^2)$. The black dots indicate the four pillowcase points.

3. Character varieties

3.1. **The character variety $R(T^2, 2)$.** Our first task is to understand the structure of $R(T^2, 2)$, the traceless $SU(2)$-character variety of the twice-punctured torus. In general, we make the following definition:

Definition 3.1. Given a surface S with n distinct marked points $p_1, \cdots, p_n \in S$, we define the character variety $R(S, n)$ to be the space of conjugacy classes of homomorphisms $\rho : \pi_1(S - \{p_1, \cdots, p_n\}) \to SU(2)$ that take loops around the marked points to traceless $SU(2)$-matrices.

Before examining the space $R(T^2, 2)$, we first consider the simpler space $R(T^2) := R(T^2, 0)$, which is known as the *pillowcase*. We have the following well-known result:

Theorem 3.2. *The pillowcase $R(T^2)$ is homeomorphic to S^2.*

Proof. The fundamental group of T^2 is $\pi_1(T^2) = \langle A, B \mid ABA^{-1}B^{-1} = 1\rangle$, where A and B are represented by the two fundamental cycles. A homomorphism $\rho : \pi_1(T^2) \to SU(2)$ is uniquely determined by the pair of matrices $(\rho(A), \rho(B))$, which for simplicity we will also denote by (A, B). Since A and B commute, any conjugacy class $[\rho] \in R(T^2)$ has a representative (A, B) of the form

$$A = \cos\alpha + \sin\alpha\,\mathbf{k}, \qquad B = \cos\beta + \sin\beta\,\mathbf{k}$$

for $(\alpha, \beta) \in \mathbb{R}^2$. These equations are invariant under the replacements $\alpha \to \alpha + 2\pi$ and $\beta \to \beta + 2\pi$, and we can simultaneously flip the signs of α and β by conjugating by $\mathbf{i}$, so we obtain the following identifications:

$$(\alpha, \beta) \sim (\alpha + 2\pi, \beta), \qquad (\alpha, \beta) \sim (\alpha, \beta + 2\pi), \qquad (\alpha, \beta) \sim (-\alpha, -\beta).$$

We can thus restrict to a domain in which $(\alpha, \beta) \in [0, 2\pi] \times [0, \pi]$, with edges identified as shown in Figure 2. From Figure 2 it is clear that this space is homeomorphic to S^2. □

We will refer to the four points $[A, B] = [\pm 1, \pm 1] \in R(T^2)$ as *pillowcase points*. One can show that the character variety $R(S^2, 4)$ that is used in the work of Hedden, Herald, and Kirk is also described by a rectangle with edges identified as shown in Figure 2 (see, for example, [13] Section 3.1), so both $R(T^2)$ and $R(S^2, 4)$ are called the *pillowcase*.

We now consider the space $R(T^2, 2)$. The fundamental group of the twice-punctured torus is

$$\pi_1(T^2 - \{p_1, p_2\}) = \langle A, B, a, b \mid ABA^{-1}B^{-1}ab = 1\rangle,$$

where p_1 and p_2 denote the puncture points, A and B denote the fundamental cycles of the torus, and a and b denote loops around the punctures p_1 and p_2, as shown in Figure 3. As above, we will use the same notation for generators of the fundamental group and their images under ρ; for example, we denote $\rho(A)$ by A. A homomorphism $\rho : \pi_1(T^2 - \{p_1, p_2\}) \to SU(2)$ is thus specified by $SU(2)$-matrices (A, B, a, b) such

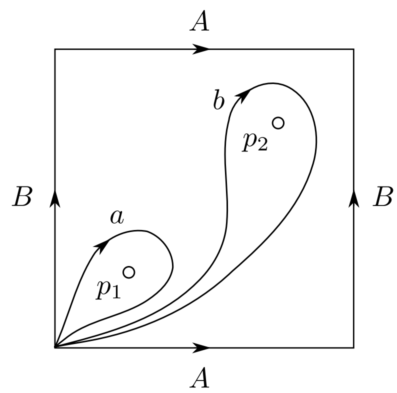


FIGURE 3. Cycles corresponding to the generators A, B, a, b of the fundamental group $\pi_1(T^2 - \{p_1, p_2\})$.

that a and b are traceless and $ABA^{-1}B^{-1}ab = 1$, and we will sometimes denote a homomorphism ρ by the corresponding list of matrices (A, B, a, b).

The structure of $R(T^2, 2)$ can be understood by considering the fibers of the map $\mu : R(T^2, 2) \to [-1, 1]$ defined by

$$\mu([A, B, a, b]) = (1/2)\,\mathrm{tr}(ABA^{-1}B^{-1}) = (1/2)\,\mathrm{tr}((ab)^{-1}).$$

In particular, it is convenient to decompose $R(T^2, 2)$ into the disjoint union of an open piece $P_4 = \mu^{-1}([-1, 1))$ and a closed piece $P_3 = \mu^{-1}(1)$. The notation for these pieces is motivated by the fact that, as we will see, the piece P_4 is four-dimensional and the piece P_3 is three-dimensional. We will describe the topology of the pieces P_3 and P_4 and define coordinate systems on each piece that are useful for performing calculations.

3.1.1. *The piece $P_4 \subset R(T^2, 2)$.* We define the piece $P_4 \subset R(T^2, 2)$ to be the set of conjugacy classes $[\rho] \in R(T^2, 2)$ such that $\mu([\rho]) \in [-1, 1)$. For any representative (A, B, a, b) of a given conjugacy class $[\rho] \in P_4$, the matrices A and B do not commute. This fact can be used to choose a canonical representative of each conjugacy class in P_4:

Lemma 3.3. *Any conjugacy class $[\rho] \in P_4$ has a unique representative (A, B, a, b) for which*

$$A = r\cos\alpha + \sqrt{1 - r^2}\,\mathbf{i} + r\sin\alpha\,\mathbf{k}, \qquad B = \cos\beta + \sin\beta\,\mathbf{k}, \tag{2}$$

where $\alpha \in [0, 2\pi)$, $\beta \in (0, \pi)$, and $r \in [0, 1)$.

Proof. Since $[\rho] \in P_4$, for any representative of $[\rho]$ the matrices A and B do not commute. Given an arbitrary representative, first conjugate so that the coefficients of $\mathbf{i}$ and $\mathbf{j}$ in B are zero and the coefficient of $\mathbf{k}$ is positive, then rotate about the z-axis so the coefficient of $\mathbf{j}$ in A is zero and the coefficient of $\mathbf{i}$ in A is positive. The restrictions on the ranges of β and r follow from the fact that the matrices A and B do not commute. The uniqueness of the resulting representative follows from the fact that the coefficients of $\mathbf{i}$ in A and $\mathbf{k}$ in B are nonzero. □

We can use the canonical representatives of conjugacy classes in P_4 to define maps $q_1 : P_4 \to SU(2) \times SU(2)$ and $q_2 : P_4 \to S^2 \times S^2$:

$$q_1([\rho]) = (A, B), \qquad q_2([\rho]) = (\hat{a}, \hat{b}),$$

where (A, B, a, b) is the canonical representative of $[\rho]$, and $\hat{a} = (a_x, a_y, a_z)$ and $\hat{b} = (b_x, b_y, b_z)$ are the unit vectors corresponding to the traceless matrices a and b:

$$a = a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k}, \qquad b = b_x\,\mathbf{i} + b_y\,\mathbf{j} + b_z\,\mathbf{k}.$$

Note that we cannot extend the maps q_1 and q_2 to all of $R(T^2, 2)$, since our choice of canonical representative relies on the fact that the matrices A and B do not commute.

To describe the piece P_4, we will show that the map $q_2 : P_4 \to S^2 \times S^2$ is injective and identify its image. This requires two Lemmas that describe the image of $q_1 : P_4 \to SU(2) \times SU(2)$ on the fibers of $\mu : P_4 \to [-1, 1)$:

Lemma 3.4. *The space* $q_1(\mu^{-1}(-1))$ *consists of the single point* $(\mathbf{i}, \mathbf{k})$*.*

Proof. Consider a point $[\rho] \in \mu^{-1}(-1)$. From equation (2) for the canonical representative (A, B, a, b) of $[\rho]$, we find that

$$\mu([\rho]) = -1 = (1/2)\,\mathrm{tr}(ABA^{-1}B^{-1}) = \cos 2\beta + r^2(1 - \cos 2\beta).$$

Thus $r = 0$ and $\beta = \pi/2$. Substituting these values into equation (2), we obtain the desired result. □

Lemma 3.5. *For* $t \in (-1, 1)$ *we can define a map* $q_1(\mu^{-1}(t)) \to S^2$*,* $(A, B) \mapsto \hat{v}$*, where the unit vector* $\hat{v} = (v_x, v_y, v_z)$ *is the direction of the vector part of* $ABA^{-1}B^{-1}$*:*

$$ABA^{-1}B^{-1} = t + \sqrt{1 - t^2}\,(v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k}).$$

This map is a homeomorphism.

Proof. Consider a point $[\rho] \in \mu^{-1}(t)$ for $t \in (-1, 1)$. From equation (2) for the canonical representative (A, B, a, b) of $[\rho]$, we find that

$$\mu([\rho]) = t = (1/2)\,\mathrm{tr}(ABA^{-1}B^{-1}) = \cos 2\beta + r^2(1 - \cos 2\beta). \tag{3}$$

We solve equation (3) for r to obtain

$$r = \left(\frac{t - \cos 2\beta}{1 - \cos 2\beta}\right)^{1/2}. \tag{4}$$

From equation (4), we see that for a fixed value of $t \in (-1, 1)$ the parameter β must lie in the range $[\beta_0, \pi - \beta_0]$, where we have defined

$$\beta_0 := (1/2)\cos^{-1} t \in (0, \pi/2). \tag{5}$$

Using equations (2), (4), and (5), we find that the matrices A and B can be expressed as

$$A = \left(\frac{\cos 2\beta_0 - \cos 2\beta}{1 - \cos 2\beta}\right)^{1/2}(\cos\alpha + \sin\alpha\,\mathbf{k}) + \left(\frac{1 - \cos 2\beta_0}{1 - \cos 2\beta}\right)^{1/2}\mathbf{i}, \qquad B = \cos\beta + \sin\beta\,\mathbf{k}, \tag{6}$$

where $(\alpha, \beta) \in [0, 2\pi) \times [\beta_0, \pi - \beta_0]$. Define a space

$$X = \{(\alpha, \beta) \in [0, 2\pi] \times [\beta_0, \pi - \beta_0]\}/\!\sim,$$

where the equivalence relation $\sim$ is defined such that the bottom edge of the rectangle $[0, 2\pi] \times [\beta_0, \pi - \beta_0]$ is collapsed to a point, the top edge is collapsed to a point, and the left and right edges are identified:

$$(\alpha, \beta_0) \sim (0, \beta_0), \qquad (\alpha, \pi - \beta_0) \sim (0, \pi - \beta_0), \qquad (0, \beta) \sim (2\pi, \beta).$$

We note that $\mu^{-1}(t)$ is nonempty for all $t \in (-1, 1)$, since Lemma 2.1 shows that we can always find traceless $SU(2)$-matrices a and b such that $ABA^{-1}B^{-1}ab = 1$. Define a map $X \to q_1(\mu^{-1}(t))$, $[\alpha, \beta] \mapsto (A, B)$, where A and B are given by equation (6). From equation (6), it is clear that this map is well-defined and is a homeomorphism.

Using equations (2) and (4), a calculation shows that

$$ABA^{-1}B^{-1} = t + \sqrt{1 - t^2}\,(v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k}),$$

where the unit vector $\hat{v} = (v_x, v_y, v_z) \in S^2$ is given by

$$\hat{v} = (\sqrt{1 - z(\beta)^2}\,\sin(\alpha + \beta),\ -\sqrt{1 - z(\beta)^2}\,\cos(\alpha + \beta),\ z(\beta)) \tag{7}$$

and we have defined a diffeomorphism $z : [\beta_0, \pi - \beta_0] \to [-1, 1]$ by

$$z(\beta) = -\sqrt{\frac{1 - t}{1 + t}}\,\cot\beta.$$

Define a map $X \to S^2$, $[\alpha, \beta] \mapsto \hat{v}$, where $\hat{v}$ is given by equation (7). From equation (7), it is clear that this map is well-defined and is a homeomorphism. Composing the inverse of the map $X \to q_1(\mu^{-1}(t))$ with the map $X \to S^2$, we obtain the desired result. □

We can now describe the topology of the piece P_4:

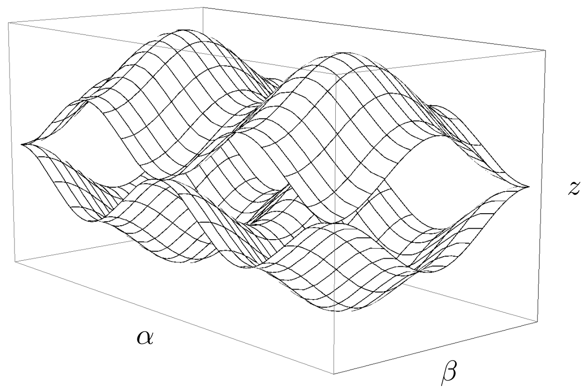


FIGURE 4. The space Y, which is homeomorphic to the piece P_3, is the region between the pair of surfaces. The vertical faces are identified as described in Definition 3.7.

Theorem 3.6. *The space P_4 is homeomorphic to $S^2 \times S^2 - \bar{\Delta}$, where $\bar{\Delta} = \{(\hat{r}, -\hat{r})\}$ is the antidiagonal. All representations in P_4 are nonabelian.*

Proof. Consider the map $q_2 : P_4 \to S^2 \times S^2$. Clearly the image of q_2 lies in $S^2 \times S^2 - \bar{\Delta}$, since if $q_2([\rho]) \in \bar{\Delta}$ then $b = a^{-1}$, which implies that $\mu([\rho]) = (1/2)\,\mathrm{tr}((ab)^{-1}) = 1$ and hence $[\rho] \notin P_4$. We can define an inverse map $S^2 \times S^2 - \bar{\Delta} \to P_4$ as follows. Given a point $(\hat{a}, \hat{b}) \in S^2 \times S^2 - \bar{\Delta}$, define traceless matrices $a = a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k}$ and $b = b_x\,\mathbf{i} + b_y\,\mathbf{j} + b_z\,\mathbf{k}$ corresponding to $\hat{a} = (a_x, a_y, a_z)$ and $\hat{b} = (b_x, b_y, b_z)$. Then

$$ab = t - v_x\,\mathbf{i} - v_y\,\mathbf{j} - v_z\,\mathbf{k},$$

where $t := -\hat{a} \cdot \hat{b}$ and $\vec{v} = (v_x, v_y, v_z) := -\hat{a} \times \hat{b}$. If $t = -1$ then map $(\hat{a}, \hat{b})$ to $[\mathbf{i}, \mathbf{k}, a, b]$, otherwise map $(\hat{a}, \hat{b})$ to $[A, B, a, b]$, where A and B are determined from t and $\hat{v} := \vec{v}/|\vec{v}| \in S^2$ via the homeomorphism $q_1(\mu^{-1}(t)) \to S^2$ defined in Lemma 3.5. By Lemmas 3.4 and 3.5, this inverse map is well-defined. The fact that all representations in P_4 are nonabelian is clear from the definition of the space P_4. □

Our main application of Theorem 3.6 will be to use $(\hat{a}, \hat{b})$ as coordinates on the piece P_4.

3.1.2. *The piece $P_3 \subset R(T^2, 2)$.* We define the piece $P_3 \subset R(T^2, 2)$ to be the set of conjugacy classes $[\rho] \in R(T^2, 2)$ such that $\mu([\rho]) = 1$. For any representative (A, B, a, b) of a given conjugacy class $[\rho] \in P_3$, the matrices A and B commute. We can therefore define a map $q : P_3 \to R(T^2)$ by

$$q([A, B, a, b]) = [A, B].$$

We will describe the topology of the piece P_3 by considering the fibers of the map q. In particular, we will show that P_3 is homeomorphic to the following space:

Definition 3.7. We define a space Y by

$$Y = \{(\alpha, \beta, z) \mid \alpha \in [0, 2\pi],\ \beta \in [0, \pi],\ |z| \leq \sin^2 \alpha + \sin^2 \beta\}/\sim,$$

where the equivalence relation $\sim$ is defined such that

$$(\alpha, 0, z) \sim (2\pi - \alpha, 0, -z), \qquad (\alpha, \pi, z) \sim (2\pi - \alpha, \pi, -z), \qquad (0, \beta, z) \sim (2\pi, \beta, z).$$

The space Y is depicted in Figure 4.

Theorem 3.8. *The space P_3 is homeomorphic to Y. Representations on the boundary of P_3 are abelian, and representations on the interior of P_3 are nonabelian.*

Proof. We first determine the fibers of the map $q : P_3 \to R(T^2)$. Given a conjugacy class $[\rho] \in P_3$, we can always choose a representative of the form

$$A = \cos\alpha + \sin\alpha\,\mathbf{k}, \qquad B = \cos\beta + \sin\beta\,\mathbf{k}, \qquad a = \cos\gamma\,\mathbf{i} + \sin\gamma\,\mathbf{k}, \qquad b = a^{-1}, \tag{8}$$

where $(\alpha, \beta) \in [0, 2\pi] \times [0, \pi]$ and $\gamma \in [-\pi/2, \pi/2]$. For $(A, B) = (\pm 1, \pm 1)$, we can conjugate so as to force $\gamma = 0$. From these considerations it follows that the fibers of q are points ($\gamma = 0$) over the four pillowcase points $[A, B] = [\pm 1, \pm 1]$, and intervals ($\gamma \in [-\pi/2, \pi/2]$) over all other points. We can thus define a homeomorphism $P_3 \to Y$ by

$$(\alpha,\, \beta,\, \gamma) \mapsto (\alpha,\, \beta,\, (2\gamma/\pi)(\sin^2\alpha + \sin^2\beta)),$$

where $(\alpha, \beta) \in [0, 2\pi] \times [0, \pi]$, and $\gamma \in [-\pi/2, \pi/2]$ are chosen such that equations (8) are satisfied. The statement regarding abelian and nonabelian representations is clear from equation (8). ☐

Our main application of Theorem 3.8 will be to use (α, β, γ) as coordinates on P_3, subject to the identifications

$$(\alpha,\beta,\gamma) \sim (\alpha + 2\pi, \beta, \gamma), \qquad (\alpha,\beta,\gamma) \sim (\alpha, \beta + 2\pi, \gamma), \qquad (\alpha,\beta,\gamma) \sim (-\alpha, -\beta, -\gamma),$$

and if $(\alpha, \beta) \in \{(0,0),\, (0,\pi),\, (\pi,0),\, (\pi,\pi)\}$, corresponding to the four pillowcase points of $R(T^2)$, then $(\alpha,\beta,\gamma) \sim (\alpha,\beta,0)$. Note that P_3 deformation retracts onto $P_3 \cap \{\gamma = 0\}$, which may be identified with the pillowcase $R(T^2)$. Theorems 3.6 and 3.8 imply Theorem 1.2 from the Introduction.

Remark 3.9. The character variety $R(T^2, 2)$ is smooth away from the reducible locus ∂P_3. We note that ∂P_3 is homeomorphic to T^2; a specific homeomorphism $T^2 \to \partial P_3$ is given by $(e^{i\alpha}, e^{i\beta}) \mapsto [A, B, a, b]$, where

$$A = \cos\alpha + \sin\alpha\,\mathbf{k}, \qquad B = \cos\beta + \sin\beta\,\mathbf{k}, \qquad a = b^{-1} = \mathbf{k}.$$

3.2. **The unperturbed character variety $R(U_1, A_1)$ and Lagrangian $L_1 \subset R(T^2, 2)$.** Our next task is to determine the Lagrangian L_1 in $R(T^2, 2)$ that corresponds to a solid torus $U_1 = S^1 \times D^2$ containing an unknotted arc A_1 connecting distinct points $p_1, p_2 \in \partial U_1$. We first define and describe a character variety $R(U_1, A_1)$ for (U_1, A_1). We then define the Lagrangian L_1 to be the image of a pullback map $R(U_1, A_1) \to R(T^2, 2)$.

Definition 3.10. We define the *unperturbed* character variety $R(U_1, A_1)$ to be the space of conjugacy classes of homomorphisms $\rho : \pi_1(U_1 - A_1) \to SU(2)$ that map loops around the arc A_1 to traceless matrices.

Theorem 3.11. *The space $R(U_1, A_1)$ is homeomorphic to the closed unit disk D^2. Representations on the boundary of $R(U_1, A_1)$ are abelian, and representations on the interior of $R(U_1, A_1)$ are nonabelian.*

Proof. The fundamental group of $U_1 - A_1$ is given by

$$\pi_1(U_1 - A_1) = \langle A, B, a, b \mid B = 1,\, b = a^{-1} \rangle,$$

where A and B are the longitude and meridian of the boundary of the solid torus and a and b are loops in the boundary encircling the points p_1 and p_2, respectively.

We now consider homomorphisms $\rho : \pi_1(U_1 - A_1) \to SU(2)$ that satisfy the requirements described in Definition 3.10 for $R(U_1, A_1)$. As usual, we use the same notation for generators of the fundamental group and their images under ρ; for example, we denote $\rho(A)$ by A. Given an arbitrary representative of a conjugacy class $[\rho] \in R(U_1, A_1)$, we will argue that we can always conjugate so as to obtain a representative of the form

$$A = \cos\chi + \sin\chi\,\mathbf{k}, \qquad B = 1, \qquad a = b^{-1} = \cos\psi\,\mathbf{i} + \sin\psi\,\mathbf{k}, \tag{9}$$

where $(\chi, \psi) \in [0, \pi] \times [-\pi/2, \pi/2]$. We first conjugate so the coefficients of $\mathbf{i}$ and $\mathbf{j}$ in A are zero and the coefficient of $\mathbf{k}$ is nonnegative, and then rotate about the z-axis so the coefficient of $\mathbf{j}$ in a is zero and the coefficient of $\mathbf{i}$ is nonnegative. We have thus obtained a representative of the form given in equation (9). If $\chi \in (0, \pi)$, then it is clear from these equations that the representative is unique. If $\chi \in \{0, \pi\}$ then $A = \pm 1$ and we can conjugate so that $a = b^{-1} = \mathbf{i}$, so we obtain the identifications $(0, \psi) \sim (0, 0)$ and $(\pi, \psi) \sim (\pi, 0)$. It follows that $R(U_1, A_1)$ is homeomorphic to the square $[0, \pi] \times [-\pi/2, \pi/2]$ with the left and right edges each collapsed to a point, and this space is homeomorphic to the closed disk D^2. The statement regarding abelian and nonabelian representations is clear from equation (9). ☐

Given a representation of $\pi_1(U_1 - A_1)$, we can pull back along the inclusion $T^2 - \{p_1, p_2\} \hookrightarrow U_1 - A_1$ to obtain a representation of $\pi_1(T^2 - \{p_1, p_2\})$. This induces a map $R(U_1, A_1) \to R(T^2, 2)$.

Definition 3.12. We define the *unperturbed* Lagrangian L_1 to be the image of $R(U_1, A_1) \to R(T^2, 2)$, and we denote the image in $R(T^2, 2)$ of the point in $R(U_1, A_1)$ with coordinates (χ, ψ) by $L_1(\chi, \psi)$.

The following Theorem gives an explicit description of the Lagrangian L_1 in terms of the coordinates $(\chi, \psi) \in [0, \pi] \times [-\pi/2, \pi/2]$:

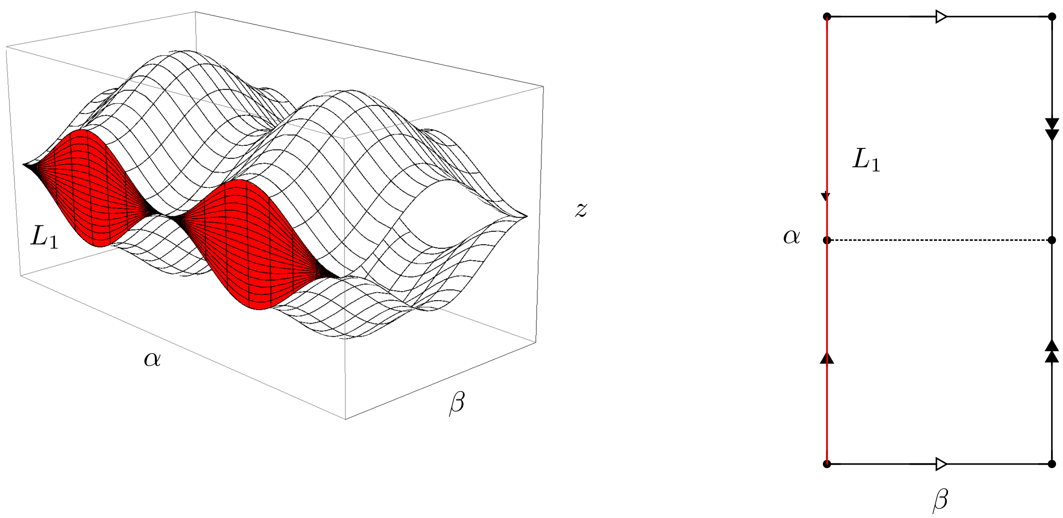


FIGURE 5. (Left) The unperturbed Lagrangian L_1 in the piece P_3. (Right) The intersection of the unperturbed Lagrangian L_1 with the pillowcase $P_3 \cap \{\gamma = 0\}$.

Theorem 3.13. *The map $R(U_1, A_1) \to R(T^2, 2)$ is injective and is an immersion on the interior of $R(U_1, A_1)$. The image $L_1(\chi, \psi) = [A, B, a, b] \in R(T^2, 2)$ of the point in $R(U_1, A_1)$ with coordinates (χ, ψ) is given by*

$$A = \cos\chi + \sin\chi\,\mathbf{k}, \qquad B = 1, \qquad a = b^{-1} = \cos\psi\,\mathbf{i} + \sin\psi\,\mathbf{k}.$$

The image L_1 of the map lies entirely in the piece P_3, and the (α, β, γ) coordinates of $L_1(\chi, \psi)$ are

$$\alpha(L_1(\chi, \psi)) = \chi, \qquad \beta(L_1(\chi, \psi)) = 0, \qquad \gamma(L_1(\chi, \psi)) = \psi.$$

Representations on the boundary of L_1 are abelian, and representations on the interior of L_1 are nonabelian.

Proof. The representative (A, B, a, b) of $L_1(\chi, \psi)$ follows directly from equation (9), and the statement regarding abelian and nonabelian representations is clear from the form of this representative. The (α, β, γ) coordinates of $L_1(\chi, \psi)$ can be read off from equation (8). It is clear from these expressions that the map $R(U_1, A_1) \to R(T^2, 2)$ is injective and is an immersion on the interior of $R(U_1, A_1)$. □

We plot the Lagrangian L_1 in Figure 5. Theorems 3.11 and 3.13 imply Theorem 1.8 from the Introduction.

3.3. **The perturbed character variety $R^\natural_\pi(U_1, A_1)$ and Lagrangian $L_1^\pi \subset R(T^2, 2)$.** We now want to modify the character variety $R(U_1, A_1)$ in order to address the technical issues described in the Introduction. Specifically, we want to (1) eliminate reducible connections, and (2) introduce a suitable holonomy perturbation so as to render the Chern-Simons functional Morse. These modifications yield a perturbed character variety $R^\natural_\pi(U_1, A_1)$. We define a corresponding perturbed Lagrangian L_1^π that is given by the image of a pullback map $R^\natural_\pi(U_1, A_1) \to R(T^2, 2)$.

We eliminate reducible connections by adding an *earring* consisting of a small loop H around A_1 and an arc W connecting A_1 to H, as shown in Figure 6. We require that representations take loops around A_1 and H to traceless matrices and loops around W to -1. One can show that representations satisfying these requirements must be nonabelian, corresponding to irreducible connections.

We render the Chern-Simons functional Morse by adding a holonomy perturbation term [17, 18]. We choose a perturbation that vanishes outside of a small solid torus obtained by thickening the loop P shown in Figure 6. The net effect of the perturbation is to impose an additional requirement on the representations. Specifically, letting $\lambda_P = h^{-1}A$ and $\mu_P = B$ denote the homotopy classes of the longitude and meridian of the solid torus obtained by thickening P, we require that if $\rho(\lambda_P)$ has the form

$$\rho(\lambda_P) = \cos\phi + \sin\phi\,(r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k}) \tag{10}$$

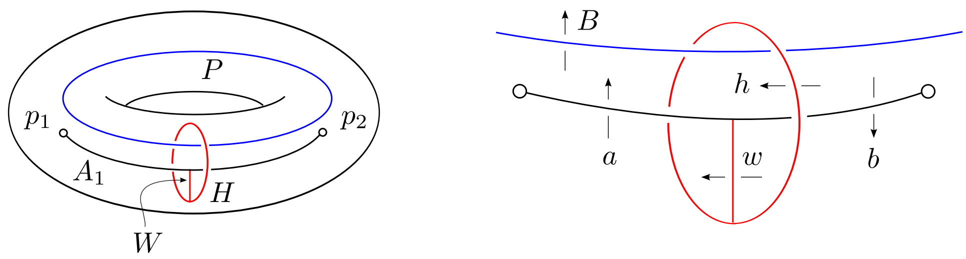


FIGURE 6. (Left) Solid torus U_1 used to define $R^\natural_\pi(S^1 \times D^2, A_1)$. Shown are the arc A_1, the loop H and arc W, and the perturbation loop P. (Right) Loops B, a, b, h, and w.

for some angle ϕ and some unit vector $\hat{r} = (r_x, r_y, r_z) \in S^2$, then $\rho(\mu_P)$ must have the form

$$\rho(\mu_P) = \cos\nu + \sin\nu\,(r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k}), \tag{11}$$

where $\nu = \epsilon f(\phi)$. Here $\epsilon > 0$ is a small parameter that controls the magnitude of the perturbation and $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f(-x) = -f(x)$, f is 2π-periodic, and $f(x)$ is zero if and only if x is a multiple of π. We will usually take $f(\phi) = \sin\phi$.

We define a character variety $R^\natural_\pi(U_1, A_1)$ that includes both of these modifications to $R(U_1, A_1)$:

Definition 3.14. We define the *perturbed* character variety $R^\natural_\pi(U_1, A_1)$ to be the space of conjugacy classes of homomorphisms $\rho : \pi_1(U_1 - A_1 \cup H \cup W \cup P) \to SU(2)$ that take loops around A_1 and H to traceless matrices and loops around W to -1, and are such that if $\rho(\lambda_P)$ has the form given in equation (10) then $\rho(\lambda_P)$ must have the form given in equation (11).

Theorem 3.15. *For $\epsilon > 0$ sufficiently small, the space $R^\natural_\pi(U_1, A_1)$ is homeomorphic to S^2. All representations in $R^\natural_\pi(U_1, A_1)$ are nonabelian.*

Proof. We define homotopy classes of loops A, B, a, b, and h as shown in Figure 6, and read off relations from Figure 6 to obtain a presentation of $\pi_1(U_1 - A_1 \cup H \cup W \cup P)$:

$$\pi_1(U_1 - A_1 \cup H \cup W \cup P) = \langle A, B, a, b, h, w \mid hwaB = aBh,\ b = ha^{-1}w^{-1}h^{-1} \rangle$$

We now consider homomorphisms $\rho : \pi_1(U_1 - A_1 \cup H \cup W \cup P) \to SU(2)$ that satisfy the requirements described in Definition 3.14 for $R^\natural_\pi(U_1, A_1)$. As usual, we use the same notation for generators of the fundamental group and their images under ρ; for example, we denote $\rho(A)$ by A. Given an arbitrary representative of a conjugacy class $[\rho] \in R^\natural_\pi(U_1, A_1)$, we will argue that we can always conjugate so as to obtain a unique representative of the form given by

$$\begin{aligned}
A &= h(\cos\phi + \sin\phi\,(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})), & B &= \cos\nu + \sin\nu\,(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}),\\
a &= \mathbf{k}, & b &= -ha^{-1}h^{-1},\\
h &= (\cos^2\nu + \sin^2\nu\sin^2\theta)^{-1/2}(\cos\nu\,\mathbf{i} + \sin\nu\sin\theta\,\mathbf{k}), & w &= -1,
\end{aligned}$$

where $\nu = \epsilon\sin\phi$ and $(\phi, \theta) \in [0, \pi] \times [0, 2\pi]$ are spherical-polar coordinates on S^2. We first conjugate so that $a = \mathbf{k}$. Next, we rotate about the z-axis so that the coefficient of $\mathbf{j}$ in h is zero. After these operations have been performed, we can express λ_P as

$$\lambda_P = \cos\phi + \sin\phi\,(r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k})$$

for some angle ϕ and some unit vector $\hat{r} = (r_x, r_y, r_z) \in S^2$. The relationship between λ_P and μ_P described in equations (10) and (11) then implies that

$$B = \mu_P = \cos\nu + \sin\nu\,(r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k}),$$

where $\nu = \epsilon\sin\phi$. We also find that

$$A = h\lambda_P = h(\cos\phi + \sin\phi\,(r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k})).$$

Since $w = -1$, the relation $b = ha^{-1}w^{-1}h^{-1}$ implies that $b = -ha^{-1}h^{-1}$, and the relation $hwaB = aBh$ implies that aB and h anticommute. Since $a = \mathbf{k}$, the fact that aB and h anticommute implies that $r_z = 0$, so $\hat{r} = (\cos\theta, \sin\theta, 0)$ for some angle θ. The fact that aB and h anticommute, in conjunction with the fact that the coefficient of $\mathbf{j}$ in h is zero, further implies that h must have the form

$$h = \pm(\cos^2\nu + \sin^2\nu\sin^2\theta)^{-1/2}(\cos\nu\,\mathbf{i} + \sin\nu\sin\theta\,\mathbf{k}). \tag{12}$$

In fact, we can assume that the plus sign obtains in equation (12), since if not then we can conjugate by $\mathbf{k}$ and redefine $\theta \mapsto \theta + \pi$; this operation flips the signs of h and A and leaves B, a, b, and w invariant. We have thus obtained a representative of the desired form. Since $a = \mathbf{k}$ and the coefficient of $\mathbf{i}$ in h is nonzero for ϵ sufficiently small, this representative is unique and nonabelian.

We note that the unique representative is invariant under the transformations

$$(\phi,\theta) \mapsto (\phi + 2\pi, \theta), \qquad (\phi,\theta) \mapsto (\phi, \theta + 2\pi), \qquad (\phi,\theta) \mapsto (-\phi, \theta + \pi).$$

By invariance under the first transformation we can assume that $\phi \in [-\pi,\pi]$, by invariance under the third transformation we can further assume that $\phi \in [0,\pi]$, and by invariance under the second transformation we can assume that $\theta \in [0, 2\pi]$. From the equations defining the unique representative, it is clear that the map $S^2 \to R^\natural_\pi(U_1, A_1)$, $(\phi,\theta) \mapsto [\rho]$ is a homeomorphism, where (ϕ,θ) are spherical-polar coordinates on S^2. ☐

Given a representation of $\pi_1(U_1 - A_1 \cup H \cup W \cup P)$, we can pull back along the inclusion $U_1 - A_1 \cup H \cup W \cup P \hookrightarrow T^2 - \{p_1, p_2\}$ to obtain a representation of $\pi_1(T^2 - \{p_1, p_2\})$. This induces a map $R^\natural_\pi(U_1, A_1) \to R(T^2, 2)$.

Definition 3.16. We define the *perturbed* Lagrangian L_1^π to be the image of $R^\natural_\pi(U_1, A_1) \to R(T^2, 2)$, and we denote the image in $R(T^2, 2)$ of the point in $R^\natural_\pi(U_1, A_1)$ with coordinates (ϕ,θ) by $L_1^\pi(\phi,\theta)$.

We can view the Lagrangian L_1^π as a perturbation of L_1, which we defined to be the image of $R(U_1, A_1) \to R(T^2, 2)$. The following Theorem gives an explicit description of the Lagrangian L_1^π in terms of the spherical-polar coordinates $(\phi,\theta) \in [0,\pi] \times [0, 2\pi]$:

Theorem 3.17. *The map $R^\natural_\pi(U_1, A_1) \to R(T^2, 2)$ is an injective immersion except at the north pole ($\phi = 0$) and south pole ($\phi = \pi$), both of which get mapped to the same point $(\alpha,\beta,\gamma) = (\pi/2, 0, 0)$ in the piece P_3. The image $L_1^\pi(\phi,\theta) = [A, B, a, b] \in R(T^2, 2)$ of the point in $R^\natural_\pi(U_1, A_1)$ with coordinates (ϕ,θ) is given by*

$$\begin{aligned}
A &= (\cos^2\nu + \sin^2\nu\sin^2\theta)^{-1/2}(\cos\nu\,\mathbf{i} + \sin\nu\sin\theta\,\mathbf{k})(\cos\phi + \sin\phi\,(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})),\\
B &= \cos\nu + \sin\nu\,(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}),\\
a &= \mathbf{k},\\
b &= (\cos^2\nu + \sin^2\nu\sin^2\theta)^{-1}(\sin 2\nu\sin\theta\,\mathbf{i} - (\cos^2\nu - \sin^2\nu\sin^2\theta)\mathbf{k}),
\end{aligned}$$

where $\nu = \epsilon\sin\phi$ and $\epsilon > 0$ is a small control parameter that determines the strength of the perturbation. Points $L_1^\pi(\phi,\theta)$ with $\phi \in (0,\pi)$, $\theta \notin \{0,\pi\}$ lie in the piece P_4, and the $(\hat{a}, \hat{b})$ coordinates of such points are

$$\begin{aligned}
\hat{a}(L_1^\pi(\phi,\theta)) &= (\sin(\phi+\nu),\, -\cos(\phi+\nu),\, 0),\\
b_x(L_1^\pi(\phi,\theta)) &= -(\cos^2\nu + \sin^2\nu\sin^2\theta)^{-1}(\cos^2\nu\cos^2\theta\sin(\phi+\nu) + \sin^2\theta\sin(\phi-\nu)),\\
b_y(L_1^\pi(\phi,\theta)) &= (\cos^2\nu + \sin^2\nu\sin^2\theta)^{-1}(\cos^2\nu\cos^2\theta\cos(\phi+\nu) + \sin^2\theta\cos(\phi-\nu)),\\
b_z(L_1^\pi(\phi,\theta)) &= (1/2)(\cos^2\nu + \sin^2\nu\sin^2\theta)^{-1}\sin(2\nu)\sin(2\theta)
\end{aligned}$$

for $\theta \in (0,\pi)$, and

$$\begin{aligned}
\hat{a}(L_1^\pi(\phi,\theta)) &= (-\sin(\phi+\nu),\, \cos(\phi+\nu),\, 0),\\
b_x(L_1^\pi(\phi,\theta)) &= (\cos^2\nu + \sin^2\nu\sin^2\theta)^{-1}(\cos^2\nu\cos^2\theta\sin(\phi+\nu) + \sin^2\theta\sin(\phi-\nu)),\\
b_y(L_1^\pi(\phi,\theta)) &= -(\cos^2\nu + \sin^2\nu\sin^2\theta)^{-1}(\cos\nu^2\cos^2\theta\cos(\phi+\nu) + \sin^2\theta\cos(\phi-\nu)),\\
b_z(L_1^\pi(\phi,\theta)) &= (1/2)(\cos^2\nu + \sin^2\nu\sin^2\theta)^{-1}\sin(2\nu)\sin(2\theta)
\end{aligned}$$

for $\theta \in (\pi, 2\pi)$. Points $L_1^\pi(\phi,\theta)$ with $\theta \in \{0,\pi\}$ lie in the piece P_3, and the (α,β,γ) coordinates of such points are

$$\begin{aligned}
\alpha(L_1^\pi(\phi,0)) &= \phi + \pi/2, & \beta(L_1^\pi(\phi,0)) &= \nu = \epsilon\sin\phi, & \gamma(L_1^\pi(\phi,0)) &= 0,\\
\alpha(L_1^\pi(\phi,\pi)) &= \phi - \pi/2, & \beta(L_1^\pi(\phi,\pi)) &= \nu = \epsilon\sin\phi, & \gamma(L_1^\pi(\phi,\pi)) &= 0.
\end{aligned}$$

All representations in L_1^π are nonabelian.

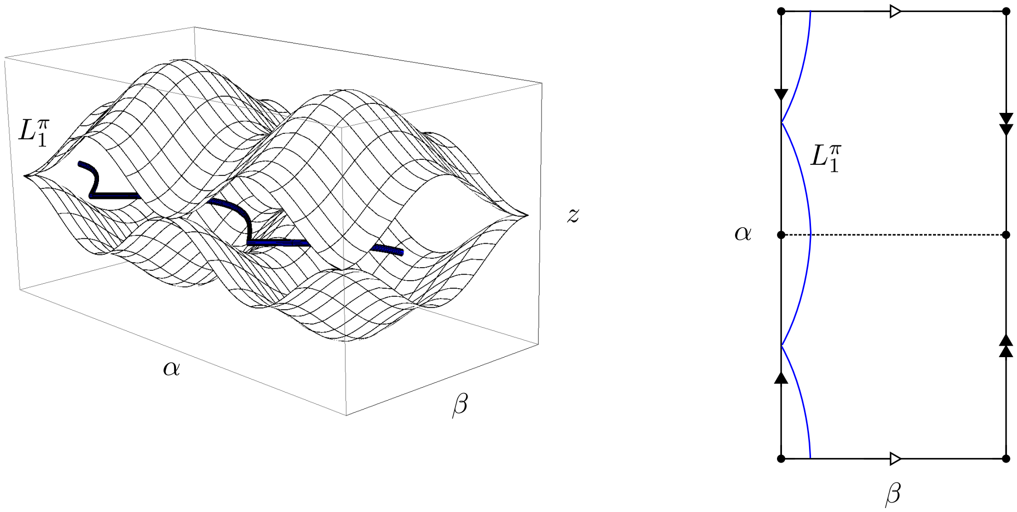


FIGURE 7. (Left) The intersection of the perturbed Lagrangian L_1^π with the piece P_3. (Right) The intersection of the perturbed Lagrangian L_1^π with the pillowcase $P_3 \cap \{\gamma = 0\}$.

Proof. The representative (A, B, a, b) of $L_1^\pi(\phi, \theta)$ follows directly from the proof of Theorem 3.15. The fact that all representations in L_1^π are nonabelian follows from the fact that $a = \mathbf{k}$ and the coefficient of either $\mathbf{i}$ or $\mathbf{j}$ in B is nonzero. We find the $(\hat{a}, \hat{b})$ coordinates for points $L_1^\pi(\phi, \theta) \in P_4$ by conjugating the representative of $L_1^\pi(\phi, \theta)$ so that A and B have the form given in equation (2), then reading off $\hat{a} = (a_x, a_y, a_z)$ and $\hat{b} = (b_x, b_y, b_z)$ from $a = a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k}$ and $b = b_x\,\mathbf{i} + b_y\,\mathbf{j} + b_z\,\mathbf{k}$. We find the (α, β, γ) coordinates for points $L_1^\pi(\phi, \theta) \in P_3$ by substituting $\theta = 0$ and $\theta = \pi$ into the representative of $L_1^\pi(\phi, \theta)$ and then conjugating the resulting equations so they have the form given in equation (8).

We will prove that $R_\pi^\natural(U_1, A_1) \to R(T^2, 2)$ is an injective immersion on $\phi \in (0, \pi)$, $\theta \neq \{0, \pi\}$ by showing that the coordinates (ϕ, θ) can be recovered from certain functions defined on $R(T^2, 2)$. Define functions $h_1 : R(T^2, 2) \to \mathbb{R}$ and $h_2 : R(T^2, 2) \cap \{\operatorname{tr} Aa \neq 0\} \to \mathbb{R}$ by

$$h_1([A, B, a, b]) = -\operatorname{tr} AB - (i/2)(\operatorname{tr} B)(\operatorname{tr} Aa), \qquad h_2([A, B, a, b]) = -\frac{\operatorname{tr} Ab}{\operatorname{tr} Aa}.$$

A calculation shows that

$$h_1(L_1^\pi(\phi, \theta)) = \frac{2\cos\nu \sin(\phi + \nu) e^{i\theta}}{\sqrt{\cos^2\nu + \sin^2\nu \sin^2\theta}}.$$

We note that if $\phi \in (0, \pi)$ then $h_1(L_1^\pi(\phi, \theta)) \neq 0$ and $\operatorname{Arg}(h_1(L_1^\pi(\phi, \theta))) = \theta$. A calculation shows that $(\operatorname{tr} Aa)(L_1^\pi(\phi, \theta)) \neq 0$ for $\phi \in (0, \pi)$, $\theta \neq \{0, \pi\}$, and for such values of (ϕ, θ) we have

$$h_2(L_1^\pi(\phi, \theta)) = \frac{\sin(\phi - \nu)}{\sin(\phi + \nu)} = \frac{\sin(\phi - \epsilon \sin\phi)}{\sin(\phi + \epsilon \sin\phi)}.$$

Define $\tilde{h}_2(\phi)$ to be the right-hand-side of this equation. It is straightforward to show that if ϵ is sufficiently small then $\tilde{h}_2'(\phi) > 0$ for all $\phi \in (0, \pi)$, hence $\tilde{h}_2 : (0, \pi) \to \mathbb{R}$ is a diffeomorphism onto its image. We conclude that $R_\pi^\natural(U_1, A_1) \to R(T^2, 2)$ is an injective immersion on $\phi \in (0, \pi)$, $\theta \neq \{0, \pi\}$.

We similarly prove that $R_\pi^\natural(U_1, A_1) \to R(T^2, 2)$ is an immersion on $\phi \in (0, \pi)$, $\theta \in \{0, \pi\}$ by using the functions $h_1 : R(T^2, 2) \to \mathbb{R}$ and $\alpha : P_3 \to \mathbb{R}$. The statements regarding the injectivity of the map for $\theta \in \{0, \pi\}$ are clear from the expressions for the (α, β, γ) coordinates. □

We plot the intersection of the Lagrangian L_1^π with the piece P_3 in Figure 7. Theorems 3.15 and 3.17 imply Theorem 1.4 from the Introduction.

Remark 3.18. One can also define a character variety $R^\natural(U_1, A_1)$ that includes the earring but not the holonomy perturbation. It is straightforward to show that $R^\natural(U_1, A_1)$ is homeomorphic to S^3 and all representations in $R^\natural(U_1, A_1)$ are nonabelian. We will not use the character variety $R^\natural(U_1, A_1)$ here.

4. Nondegeneracy

In this section, we adapt an argument from [1] to obtain a simple criterion for determining when a point $[\rho] \in R^\natural_\pi(Y,K)$ is nondegenerate; namely, it is nondegenerate if and only if the Lagrangians L_1^π and L_2 in $R(T^2,2)$ corresponding to the Heegaard splitting of (Y,K) intersect transversely at the image of $[\rho]$ under the pullback map $R^\natural_\pi(Y,K) \to R(T^2,2)$. The argument relies on several results involving group cohomology and the regularity of character varieties, which we discuss first.

4.1. **Constrained group cohomology.** Consider a finitely presented group $\Gamma = \langle S \mid R\rangle$ with generators $S = \{s_1, \cdots, s_n\}$ and relations $R = \{r_1, \cdots, r_m\}$. In defining character varieties, we often want to consider a space $X(\Gamma) \subseteq \mathrm{Hom}(\Gamma, SU(2))$ consisting of homomorphisms that satisfy certain constraints; for example, we may require the homomorphisms to map certain generators to traceless matrices. Provided the constraints are algebraic, the space $X(\Gamma)$ has the structure of a real algebraic variety, and we can define a corresponding scheme $\mathcal{X}(\Gamma)$ whose set of closed points is $X(\Gamma)$. The group $SU(2)$ acts on the variety $X(\Gamma)$ by conjugation, and we define the character variety $R(\Gamma)$ and character scheme $\mathcal{R}(\Gamma)$ to be the GIT quotients $X(\Gamma)//SU(2)$ and $\mathcal{X}(\Gamma)//SU(2)$. Generalizing a result due to Weil for the unconstrained case [22], we have that the Zariski tangent space $T_{[\rho]}\mathcal{R}(\Gamma)$ of the character scheme $\mathcal{R}(\Gamma)$ at a closed point $[\rho]$ can be identified with the constrained group cohomology $H^1_c(\Gamma; \mathrm{Ad}\,\rho)$, which we define here.

Roughly speaking, the constrained group cohomology $H^1_c(\Gamma; \mathrm{Ad}\,\rho)$ describes infinitesimal deformations of homomorphisms $\rho : \Gamma \to SU(2)$ that satisfy the relevant constraints, modulo deformations that can be obtained by the conjugation action of $SU(2)$. The precise definition of $H^1_c(\Gamma; \mathrm{Ad}\,\rho)$ that we will use is as follows. Define a function $F_r : \mathrm{Hom}(\langle S\rangle, SU(2)) \to SU(2)^m$, where $\langle S\rangle$ is the free group on S, by

$$F_r(\rho) = (\rho(r_1), \cdots, \rho(r_m)).$$

Thus $F_r(\rho) = (1, \cdots, 1)$ if and only if $\rho : \langle S\rangle \to SU(2)$ preserves all the relations in R and thus descends to a homomorphism $\rho : \Gamma \to SU(2)$. Given a homomorphism $\rho : \Gamma \to SU(2)$ and a function $\eta : S \to \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of $SU(2)$, define a homomorphism $\rho_t : \langle S\rangle \to SU(2)$ such that

$$\rho_t(s_k) = e^{t\eta(s_k)}\rho(s_k).$$

Note that we can view η as a vector in $\mathfrak{g}^{\oplus n}$. We define a linear map $c_r : \mathfrak{g}^{\oplus n} \to \mathfrak{g}^{\oplus m}$ by

$$c_r(\eta) = \frac{d}{dt} F_r(\rho_t)|_{t=0}.$$

Thus $c_r(\eta) = 0$ if and only if η describes an infinitesimal deformation of ρ that is a homomorphism $\Gamma \to SU(2)$.

Homomorphisms $\Gamma \to SU(2)$ that represent points in a character variety may be required to satisfy certain constraints; for example, that they take particular generators to traceless matrices. Define a function $F_c : \mathrm{Hom}(\langle S\rangle, SU(2)) \to \mathbb{R}^q$ such that $F_c(\rho) = 0$ if and only if ρ satisfies these constraints; for example, if we require that ρ take the generator s_1 to a traceless matrix, we would define $F_c : \mathrm{Hom}(\langle S\rangle, SU(2)) \to \mathbb{R}$ by

$$F_c(\rho) = \mathrm{tr}(\rho(s_1)).$$

We define a linear map $c_c : \mathfrak{g}^{\oplus n} \to \mathbb{R}^q$ by

$$c_c(\eta) = \frac{d}{dt} F_c(\rho_t)|_{t=0}.$$

Thus $c_c(\eta) = 0$ if and only if η describes an infinitesimal deformation of ρ that satisfies the constraints.

We now combine the linear maps for the relations and constraints to obtain a linear map $c : \mathfrak{g}^{\oplus n} \to \mathfrak{g}^{\oplus m} \oplus \mathbb{R}^q$, $c(\eta) = (c_r(\eta), c_c(\eta))$. Given a homomorphism $\rho : \Gamma \to SU(2)$ that satisfies the constraints, we define the space of 1-cocycles to be

$$Z^1_c(\Gamma; \mathrm{Ad}\,\rho) = \ker c,$$

so a vector $\eta \in \mathfrak{g}^{\oplus n}$ is a 1-cocycle if and only if it describes an infinitesimal deformation of ρ that is a homomorphism that preserves the constraints. We define the space of 1-coboundaries to be infinitesimal deformations of ρ that are obtained via the conjugation action of $SU(2)$:

$$B^1_c(\Gamma; \mathrm{Ad}\,\rho) = \{\eta : S \to \mathfrak{g} \mid \text{there exists } u \in \mathfrak{g} \text{ such that } \eta(s_k) = u - \mathrm{Ad}_{\rho(s_k)}\, u \text{ for } k = 1, \cdots, n\}.$$

Here $\mathrm{Ad}_g u := gug^{-1}$ for $g \in SU(2)$ and $u \in \mathfrak{g}$. We define the constrained group cohomology $H^1_c(\Gamma; \mathrm{Ad}\,\rho)$ to be

$$H^1_c(\Gamma; \mathrm{Ad}\,\rho) = Z^1_c(\Gamma; \mathrm{Ad}\,\rho)/B^1_c(\Gamma; \mathrm{Ad}\,\rho).$$

4.2. **Regularity.** We define the *local dimension* $\dim_{[\rho]} R(\Gamma)$ of $R(\Gamma)$ at $[\rho] \in R(\Gamma)$ to be the maximal dimension of the irreducible components of $R(\Gamma)$ containing $[\rho]$. We say that a point $[\rho]$ of a character variety $R(\Gamma)$ is *regular* if

$$\dim_{[\rho]} R(\Gamma) = \dim H^1_c(\Gamma; \mathrm{Ad}\,\rho).$$

We define $R'(\Gamma)$ to be the set of regular points of $R(\Gamma)$. The set $R'(\Gamma)$ has the structure of a smooth manifold, and the smooth tangent space at a point $[\rho] \in R'(\Gamma)$ is $T_{[\rho]} R(\Gamma) = H^1_c(\Gamma; \mathrm{Ad}\,\rho)$. We will prove theorems that describe the regular points of the character varieties $R(U_1, A_1)$, $R^\natural_\pi(U_1, A_1)$, and $R(T^2, 2)$:

Theorem 4.1. *The character variety $R(U_1, A_1)$ is regular at all points represented by nonabelian homomorphisms.*

Proof. Using results from the proof of Theorem 3.11, we find that we can take the set of generators of the fundamental group Γ to be $S = \{A, a\}$, with no relations, and we can take the constraint function $F_c : \mathrm{Hom}(\langle S\rangle, SU(2)) \to \mathbb{R}$ to be

$$F_c(\rho) = \mathrm{tr}(\rho(a)).$$

Using the expressions for the homomorphisms $\rho : \Gamma \to SU(2)$ given in the proof of Theorem 3.11, we obtain a linear map $c : \mathbb{R}^6 \to \mathbb{R}$. A straightforward calculation shows that $\dim H^1_c(\Gamma; \mathrm{Ad}\,\rho) = \dim R(U_1, A_1) = 2$ for all $[\rho] \in R(U_1, A_1)$ such that ρ is nonabelian. □

We would next like to determine the regular points of the perturbed character variety $R^\natural_\pi(U_1, A_1)$, but there are two difficulties that must be overcome. The first difficulty involves the function $f(\phi)$ that defines the perturbation. Recall that points $[\rho] \in R^\natural_\pi(U_1, A_1)$ are constrained by the requirement that if $\rho(\lambda_P)$ has the form $\rho(\lambda_P) = \cos\phi + \sin\phi\,(r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k})$, then $\rho(\mu_P)$ must have the form $\rho(\mu_P) = \cos\nu + \sin\nu\,(r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k})$, where $\nu = \epsilon f(\phi)$. In order to give $R^\natural_\pi(U_1, A_1)$ the structure of a real algebraic variety, and to define the corresponding character scheme, this constraint must be algebraic. We will therefore choose $f(\phi)$ to be

$$f(\phi) = \frac{1}{\epsilon}\sin^{-1}(\epsilon \sin\phi). \tag{13}$$

Then the constraint on ρ becomes

$$\epsilon\,\mathrm{tr}(\rho(\lambda_P)\mathbf{i}) = \mathrm{tr}(\rho(\mu_P)\mathbf{i}), \qquad \epsilon\,\mathrm{tr}(\rho(\lambda_P)\mathbf{j}) = \mathrm{tr}(\rho(\mu_P)\mathbf{j}), \qquad \epsilon\,\mathrm{tr}(\rho(\lambda_P)\mathbf{k}) = \mathrm{tr}(\rho(\mu_P)\mathbf{k}). \tag{14}$$

In fact, the constraint given in equation (14) yields a variety with two connected components, one with $\rho(\mu_P)$ near 1 and one with $\rho(\mu_P)$ near -1, and only the first component corresponds to $R^\natural_\pi(U_1, A_1)$. To calculate the constrained group cohomology, however, we consider only infinitesimal deformations of homomorphisms, hence the extraneous second component is irrelevant.

A second difficulty in determining the regular points of $R^\natural_\pi(U_1, A_1)$ is that a direct calculation of the constrained group cohomology for $R^\natural_\pi(U_1, A_1)$ does not appear to be practical, because the perturbed representations, as described in Theorem 3.15, are rather complicated. Instead, we will apply the following theorem, which simplifies the necessary calculations by allowing us to extrapolate from unperturbed representations:

Theorem 4.2. *Consider a character variety $R_\epsilon(\Gamma)$ in which the homomorphisms are required to satisfy an algebraic constraint that depends on a control parameter $\epsilon \in \mathbb{R}$. Given a homomorphism $\rho_\epsilon : \Gamma \to SU(2)$ representing a point $[\rho_\epsilon] \in R_\epsilon(\Gamma)$, let $c_\epsilon : \mathfrak{g}^{\oplus n} \to \mathfrak{g}^{\oplus m} \oplus \mathbb{R}^q$ denote the corresponding linear map used to define the constrained group cohomology. Define $c_0, c_1 : \mathfrak{g}^{\oplus n} \to \mathfrak{g}^{\oplus m} \oplus \mathbb{R}^q$ such that $c_\epsilon = c_0 + \epsilon c_1 + \cdots$. The following string of inequalities holds for $\epsilon > 0$ sufficiently small:*

$$\dim Z^1_c(\Gamma; \mathrm{Ad}\,\rho_\epsilon) \le \dim(\ker c_0 \cap \ker c_1) + \dim(c_1(\ker c_0) \cap \mathrm{im}\, c_0) \le \dim Z^1_c(\Gamma; \mathrm{Ad}\,\rho_0). \tag{15}$$

Proof. Since the dimension of the Zariski tangent space is upper semi-continuous, for $\epsilon > 0$ sufficiently small we have that

$$\dim(\ker c_\epsilon) = \dim Z^1_c(\Gamma; \mathrm{Ad}\,\rho_\epsilon) \leq \dim Z^1_c(\Gamma; \mathrm{Ad}\,\rho_0) = \dim(\ker c_0).$$

Thus any vector $w_\epsilon \in \ker c_\epsilon$ must have the form $w_\epsilon = w_0 + \epsilon w_1 + \cdots$, where

$$c_\epsilon(w_\epsilon) = c_0(w_0) + \epsilon(c_0(w_1) + c_1(w_0)) + \cdots = 0. \tag{16}$$

The space V of vectors $w_0 \in \mathfrak{g}^{\oplus n}$ that satisfy equation (16) up to first order in ϵ is

$$V = \{w_0 \in \ker c_0 \mid c_1(w_0) \in \mathrm{im}\, c_0\}.$$

Since $\ker c_\epsilon = Z^1_c(\Gamma; \mathrm{Ad}\,\rho_\epsilon)$ is the space of vectors that satisfies equation (16) to all orders in ϵ, it follows that $Z^1_c(\Gamma; \mathrm{Ad}\,\rho_\epsilon) \subseteq V \subseteq \ker c_0 = Z^1_c(\Gamma; \mathrm{Ad}\,\rho_0)$, and we have the string of inequalities

$$\dim Z^1_c(\Gamma; \mathrm{Ad}\,\rho_\epsilon) \leq \dim V \leq \dim Z^1_c(\Gamma; \mathrm{Ad}\,\rho_0).$$

Equation (15) now follows from the fact that

$$\dim V = \dim(\ker c_0 \cap \ker c_1) + \dim(c_1(\ker c_0) \cap \mathrm{im}\, c_0).$$

□

Example 4.3. Take $\Gamma = \mathbb{Z}$, and consider the character varieties $R^i_\epsilon(\Gamma)$ for $i = 1, 2, 3$ with constraint functions $F^i_c : \mathrm{Hom}(\Gamma, SU(2)) \to \mathbb{R}$ given by

$$F^1_c(\rho) = \epsilon \,\mathrm{tr}\,\rho(1), \qquad F^2_c(\rho) = \epsilon(\mathrm{tr}\,\rho(1))^2, \qquad F^3_c(\rho) = \epsilon(\mathrm{tr}\,\rho(1))^2 + \epsilon^2 \,\mathrm{tr}\,\rho(1).$$

The character varieties are given by

$$R^1_\epsilon(\Gamma) = R^2_\epsilon(\Gamma) = R^3_\epsilon(\Gamma) = \begin{cases} S^2 & \text{if } \epsilon \neq 0, \\ S^3 & \text{if } \epsilon = 0. \end{cases}$$

Consider the homomorphism $\rho_\epsilon : \mathbb{Z} \to SU(2)$, $\rho_\epsilon(1) = \mathbf{k}$. Then $\dim Z^1_c(\Gamma; \mathrm{Ad}\,\rho_\epsilon)$ and $\dim(\ker c_0 \cap \ker c_1) + \dim(c_1(\ker c_0) \cap \mathrm{im}\, c_0)$ are given by

	F^1_c	F^2_c	F^3_c
$\dim Z^1_c(\Gamma; \mathrm{Ad}\,\rho_\epsilon)$	2	3	2
$\dim(\ker c_0 \cap \ker c_1) + \dim(c_1(\ker c_0) \cap \mathrm{im}\, c_0)$	2	3	3

From the expressions for $\dim Z^1_c(\Gamma; \mathrm{Ad}\,\rho_\epsilon)$, we find that for $\epsilon \neq 0$ the character schemes $\mathcal{R}^1_\epsilon(\Gamma)$ and $\mathcal{R}^3_\epsilon(\Gamma)$ are reduced, and the character scheme $\mathcal{R}^2_\epsilon(\Gamma)$ is not reduced. We can use Theorem 4.2 to show that $\mathcal{R}^1_\epsilon(\Gamma)$ is reduced, but not that $\mathcal{R}^3_\epsilon(\Gamma)$ is reduced.

Theorem 4.4. *The character variety $R^\natural_\pi(U_1, A_1)$ is regular everywhere.*

Proof. Using results from the proof of Theorem 3.15, we find that we can take the set of generators for the fundamental group Γ to be $S = \{a, A, B, h\}$, the relations function $F_r : \mathrm{Hom}(\langle S\rangle, SU(2)) \to SU(2)$ to be

$$F_r(\rho) = -\rho([h, aB]),$$

and the constraint function $F_c : \mathrm{Hom}(\langle S\rangle, SU(2)) \to \mathbb{R}^6$ to be

$$F_c(\rho) = (\mathrm{tr}(\rho(a)),\ \mathrm{tr}(\rho(ha^{-1}h^{-1})),\ \mathrm{tr}(\rho(h)),\ f(\rho, \mathbf{i}),\ f(\rho, \mathbf{j}),\ f(\rho, \mathbf{k})),$$

where

$$f(\rho, q) = \epsilon \,\mathrm{tr}(\rho(h^{-1}A)q) - \mathrm{tr}(\rho(B)q).$$

Using the expressions for the homomorphisms $\rho_\epsilon : \Gamma \to SU(2)$ given in the proof of Theorem 3.15, we obtain a linear map $c_\epsilon : \mathbb{R}^{12} \to \mathbb{R}^9$. We now apply Theorem 4.2. A straightforward, but rather lengthy, calculation shows that $\dim(\ker c_0 \cap \ker c_1) + \dim(c_1(\ker c_0) \cap \mathrm{im}\, c_0) = 5$ for all homomorphisms representing points in $R^\natural_\pi(U_1, A_1)$. Since these homomorphisms are all nonabelian, we conclude that $\dim H^1_c(\Gamma; \mathrm{Ad}\,\rho) = \dim R^\natural_\pi(U_1, A_1) = 2$ for all $[\rho] \in R^\natural_\pi(U_1, A_1)$, and thus $R^\natural_\pi(U_1, A_1)$ is regular everywhere. □

Theorem 4.5. *The image L^π_1 of the immersion $R^\natural_\pi(U_1, A_1) \to R(T^2, 2)$ lies in the regular locus of $R(T^2, 2)$.*

Proof. Using results from Section 3.1, we find that we can take the set of generators for the fundamental group Γ to be $S = \{a, A, B\}$, with no relations, and we can take the constraint function $F_c : \mathrm{Hom}(\langle S\rangle, SU(2)) \to \mathbb{R}^2$ to be

$$F_c(\rho) = (\mathrm{tr}(\rho(a)),\ \mathrm{tr}(\rho(ABA^{-1}B^{-1}a))).$$

Using results from the proof of Theorem 3.15, we obtain a linear map $c_\epsilon : \mathbb{R}^9 \to \mathbb{R}^2$ for homomorphisms representing points in L_1^π. A straightforward, but rather lengthy, calculation shows that $\dim(\ker c_0 \cap \ker c_1) + \dim(c_1(\ker c_0) \cap \mathrm{im}\, c_0) = 7$ for all homomorphisms representing points in L_1^π. Since these homomorphisms are all nonabelian, we conclude that $\dim H_c^1(\Gamma; \mathrm{Ad}\,\rho) = \dim R(T^2, 2) = 4$. □

We conjecture that $R(T^2, 2)$ is in fact regular at all points represented by nonabelian homomorphisms, but Theorem 4.5 will suffice for our purposes.

4.3. **Transversality.** We are now ready to prove our key result that relates nondegeneracy to transversality. Recall that we defined the Lagrangian L_2 to be the image of $R(U_2, A_2) \to R(T^2, 2)$. If $R(U_2, A_2) \to R(T^2, 2)$ is injective, and $[\rho] \in L_1^\pi \cap L_2 \subset R(T^2, 2)$ is not the double-point of L_1^π, then by Corollary 1.5 the point $[\rho]$ is the image of a unique point in $R_\pi^\natural(Y, K)$ under the pullback map $R_\pi^\natural(Y, K) \to R(T^2, 2)$. The following is a restatement of Theorem 1.6 from the Introduction:

Theorem 4.6. *Suppose $R(U_2, A_2) \to R(T^2, 2)$ is an injective immersion and $[\rho] \in L_1^\pi \cap L_2$ is the image of a regular point of $R(U_2, A_2)$ and is not the double-point of L_1^π. Then the unique preimage of $[\rho]$ under the pullback map $R_\pi^\natural(Y, K) \to R(T^2, 2)$ is nondegenerate if and only if the intersection of L_1^π with L_2 at $[\rho] \in L_1^\pi \cap L_2$ is transverse.*

Proof. We introduce the notation $K' = K \cup W \cup H \cup P$, $Y' = Y - K'$, $U_i' = U_i - K'$, and $\Sigma' = T^2 - \{p_1, p_2\}$. We have the following Mayer-Vietoris sequence:

$$\cdots \longrightarrow H_c^0(\Sigma'; \mathrm{Ad}\,\rho) \longrightarrow H_c^1(Y'; \mathrm{Ad}\,\rho) \longrightarrow H_c^1(U_1'; \mathrm{Ad}\,\rho) \oplus H_c^1(U_2'; \mathrm{Ad}\,\rho) \longrightarrow H_c^1(\Sigma'; \mathrm{Ad}\,\rho) \longrightarrow \cdots.$$

Here $H_c^0(\Sigma'; \mathrm{Ad}\,\rho)$ is

$$H_c^0(\Sigma'; \mathrm{Ad}\,\rho) = \{x \in \mathfrak{g} \mid [\rho(\lambda), x] = 0 \text{ for all } \lambda \in \pi_1(\Sigma')\},$$

and $H_c^1(Y'; \mathrm{Ad}\,\rho)$, $H_c^1(U_1'; \mathrm{Ad}\,\rho)$, $H_c^1(U_2'; \mathrm{Ad}\,\rho)$, $H_c^1(\Sigma'; \mathrm{Ad}\,\rho)$ are the constrained group cohomology for the character varieties $R_\pi^\natural(Y, K)$, $R_\pi^\natural(U_1, A_1)$, $R(U_2, A_2)$, and $R(T^2, 2)$, respectively. For notational simplicity, we are using ρ to denote a homomorphism representing a point in $R_\pi^\natural(Y, K)$, as well as its pullbacks to homomorphisms representing points in $R_\pi^\natural(U_1, A_1)$, $R(U_2, A_2)$, and $R(T^2, 2)$. From Theorem 3.17 we have that all points in L_1^π are represented by nonabelian homomorphisms, thus $H_c^0(\Sigma'; \mathrm{Ad}\,\rho) = 0$. From Theorems 4.4 and 4.5, we have the identifications

$$H_c^1(U_1'; \mathrm{Ad}\,\rho) = T_{[\rho]} R_\pi^\natural(U_1, A_1), \qquad H_c^1(\Sigma'; \mathrm{Ad}\,\rho) = T_{[\rho]} R(T^2, 2).$$

Since we have assumed that $[\rho] \in R(U_2, A_2)$ is regular, we have the identification

$$H_c^1(U_2'; \mathrm{Ad}\,\rho) = T_{[\rho]} R(U_2, A_2).$$

By Theorem 3.17, the map $R_\pi^\natural(U_1, A_1) \to R(T^2, 2)$ is an immersion (with image L_1^π), and we have assumed that $R(U_2, A_2) \to R(T^2, 2)$ is an immersion (with image L_2), so we can identify

$$T_{[\rho]} R_\pi^\natural(U_1, A_1) = T_{[\rho]} L_1^\pi, \qquad T_{[\rho]} R(U_2, A_2) = T_{[\rho]} L_2.$$

We conclude that the constrained group cohomology $H_c^1(Y'; \mathrm{Ad}\,\rho)$ is given by

$$H_c^1(Y'; \mathrm{Ad}\,\rho) = T_{[\rho]} \mathcal{R}_\pi^\natural(Y, K) = T_{[\rho]} L_1^\pi \cap T_{[\rho]} L_2.$$

The constrained group cohomology $H_c^1(Y'; \mathrm{Ad}\,\rho)$ is zero if and only if $[\rho]$ is nondegenerate (see [7] Section 2.5.4). Thus $[\rho]$ is nondegenerate if and only if L_1^π intersects L_2 transversely at $[\rho]$. □

Example 4.7. Consider the algebraic functions $f, g : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, $g(x) = x^3$. The schemes corresponding to the critical loci of f and g are $\mathrm{Spec}\, F = \{(0)\}$ and $\mathrm{Spec}\, G = \{(x)\}$, where

$$F = \mathbb{R}[x]/(f'(x)) = \mathbb{R}, \qquad G = \mathbb{R}[x]/(g'(x)) = \mathbb{R}[x]/(x^2).$$

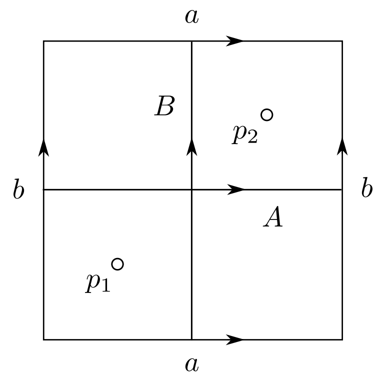


FIGURE 8. Cycles a, A, b, and B corresponding to generators T_a, T_A, T_b, and T_B of $\mathrm{MCG}_2(T^2)$.

The fact that 0 is a nondegenerate critical point of f, but a degenerate critical point of g, is reflected in the fact that F is reduced, but G is nonreduced, which in turn is reflected in the fact that $T_{(0)} \operatorname{Spec} F = 0$, but $T_{(x)} \operatorname{Spec} G = \mathbb{R}$.

Since nondegeneracy is a stable property, for sufficiently small $\epsilon > 0$ we can use the function $f(\phi) = \sin\phi$ to define the perturbation, rather than the function $f(\phi)$ given in equation (13).

5. The group $\mathrm{MCG}_2(T^2)$ and its action on $R(T^2, 2)$

An important property of the character variety $R(T^2, 2)$ is that it admits an action of the mapping class group $\mathrm{MCG}_2(T^2)$. Here we describe the group $\mathrm{MCG}_2(T^2)$ and its action on $R(T^2, 2)$.

5.1. The mapping class group $\mathrm{MCG}_2(T^2)$.

Definition 5.1. Given a surface S and n distinct marked points $p_1, \cdots, p_n \in S$, we define the *mapping class group* $\mathrm{MCG}_n(S)$ to be the group of isotopy classes of orientation-preserving homeomorphisms of S that fix $\{p_1, \cdots, p_n\}$ as a set.

Presentations for mapping class groups are described in [6, 9, 20]. The mapping class group $\mathrm{MCG}_2(T^2)$ for the twice-punctured torus is generated by Dehn twists T_a, T_A, T_b, and T_B around the simple closed curves a, A, b, and B shown in Figure 8, together with a π-rotation ω of the square shown in Figure 8. The mapping class group $\mathrm{MCG}(T^2) := \mathrm{MCG}_0(T^2)$ for the unpunctured torus is generated by the Dehn twists T_a and T_b.

It is useful to relate the mapping class groups $\mathrm{MCG}_2(T^2)$ and $\mathrm{MCG}(T^2)$ to the braid group $B_2(T^2)$, which we define as follows:

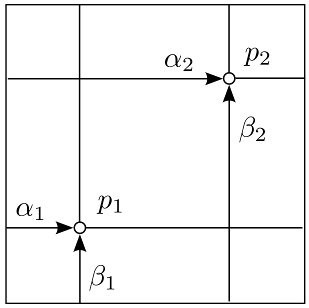
Definition 5.2. Given a surface S, we define the *configuration space for ordered points* $\mathrm{Conf}'_n(S)$ to be the space $\{(p_1, \cdots, p_n) \in S^n \mid p_i \neq p_j \text{ if } i \neq j\}$. We define the *configuration space for unordered points* $\mathrm{Conf}_n(S)$ to be the space $\mathrm{Conf}'_n(S)/\Sigma_n$, where the fundamental group on n letters Σ_n acts on $\mathrm{Conf}'_n(S)$ by permutation.

Definition 5.3. Given a surface S and n distinct marked points $p_1, \cdots, p_n \in S$, we define the *braid group* $B_n(S)$ to be the fundamental group of $\mathrm{Conf}_n(S)$ with base point $[(p_1, \cdots, p_n)]$.

Presentations for braid groups are described in [2]. The braid group $B_2(T^2)$ for the twice-punctured torus is generated by braids α_i and β_i for $i = 1, 2$ that drag marked the point p_i rightward and upward around a cycle, together with a braid σ that interchanges the marked points p_1 and p_2 via a counterclockwise π-rotation. These generators are depicted in Figure 9.

The braid group $B_2(T^2)$ and the mapping class groups $\mathrm{MCG}_2(T^2)$ and $\mathrm{MCG}(T^2)$ are related by the Birman exact sequence [3]:

$$1 \longrightarrow \pi_1(\mathrm{Homeo}_0(T^2)) \longrightarrow B_2(T^2) \xrightarrow{p} \mathrm{MCG}_2(T^2) \xrightarrow{g} \mathrm{MCG}(T^2) \longrightarrow 1. \tag{17}$$



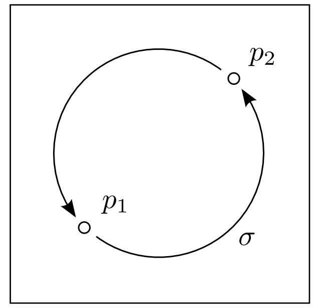


FIGURE 9. (Left) Generators α_1, α_2, β_1, and β_2 of $B_2(T^2)$. (Right) Generator σ of $B_2(T^2)$.

Here $\mathrm{Homeo}_0(T^2)$ is the group of orientation-preserving homeomorphisms of T^2 that are isotopic to the identity. The group $\mathrm{Homeo}_0(T^2)$ deformation retracts onto T^2 [11], so $\pi_1(\mathrm{Homeo}_0(T^2)) = \pi_1(T^2) = \mathbb{Z}^2$. The two free abelian generators of $\pi_1(\mathrm{Homeo}_0(T^2))$ can be identified with the elements $\alpha_1\alpha_2$ and $\beta_1\beta_2$ of $B_2(T^2)$ under the injection $\pi_1(\mathrm{Homeo}_0(T^2)) \to B_2(T^2)$. The push homomorphism $p : B_2(T^2) \to \mathrm{MCG}_2(T^2)$ is given by

$$p(\alpha_1) = p(\alpha_2)^{-1} = T_a T_A^{-1}, \qquad p(\beta_1) = p(\beta_2)^{-1} = T_b T_B^{-1}, \qquad p(\sigma) = (T_a T_b^{-1} T_a)^2 \omega.$$

The forgetful homomorphism $g : \mathrm{MCG}_2(T^2) \to \mathrm{MCG}(T^2)$ is given by

$$g(T_a) = g(T_A) = T_a, \qquad g(T_b) = g(T_B) = T_b, \qquad g(\omega) = (T_a T_b^{-1} T_a)^2.$$

In what follows we will use the generators of $B_2(T^2)$ to also denote their images in $\mathrm{MCG}_2(T^2)$ under $p : B_2(T^2) \to \mathrm{MCG}_2(T^2)$.

We will use elements of the group $\mathrm{MCG}_2(T^2)$ to describe gluing data for constructing $(1,1)$-knots. By definition, a $(1,1)$-knot K in a lens space Y can be obtained by gluing together two copies of a solid torus containing an unknotted arc via a homeomorphism that represents an element $f \in \mathrm{MCG}_2(T^2)$. The Birman sequence is useful for understanding the relationship between elements $f \in \mathrm{MCG}_2(T^2)$ and the corresponding pairs (Y, K). The lens space Y can be recovered from the image of f under $g : \mathrm{MCG}_2(T^2) \to \mathrm{MCG}(T^2)$, so this map can be viewed as forgetting the part of the gluing data used to construct the knot and preserving the part of the data used to construct the lens space. If we multiply f by an element in the image of the map $p : B_2(T^2) \to \mathrm{MCG}_2(T^2)$, the resulting element $f' \in \mathrm{MCG}_2(T^2)$ yields a pair (Y, K') consisting of a potentially different knot K' in the same lens space Y. The braid group $B_2(T^2)$ is thus useful for constructing different knots in a fixed lens space.

5.2. **The action of** $\mathrm{MCG}_2(T^2)$ **on** $R(T^2, 2)$**.** We will define an action of the group $\mathrm{MCG}_2(T^2)$ on the character variety $R(T^2, 2)$ via a homomorphism from $\mathrm{MCG}_2(T^2)$ to $\mathrm{Out}(\pi_1(T^2 - \{p_1, p_2\}))$, the group of outer automorphisms of $\pi_1(T^2 - \{p_1, p_2\})$. In general, we define a group homomorphism from $\mathrm{MCG}_n(T^2)$ to $\mathrm{Out}(\pi_1(T^2 - \{p_1, \cdots, p_n\}))$, the group of outer automorphisms of $\pi_1(T^2 - \{p_1, \cdots, p_n\})$, as follows. Define $X = T^2 - \{p_1, \cdots, p_n\}$. Choose a base point $x_0 \in X$ and consider the fundamental group $\pi_1(X, x_0)$. Given an element $[\phi] \in \mathrm{MCG}_n(X)$ represented by a homeomorphism $\phi : X \to X$, there is an induced isomorphism $\phi_* : \pi_1(X, x_0) \to \pi_1(X, \phi(x_0))$, $[\alpha] \mapsto [\phi \circ \alpha]$. Choose a path $\gamma : I \to X$ from x_0 to $\phi(x_0)$; this induces an isomorphism $\gamma_* : \pi_1(X, \phi(x_0)) \to \pi_1(X, x_0)$, $[\alpha] \mapsto [\gamma\alpha\bar{\gamma}]$. We now define a map $\mathrm{MCG}_n(T^2) \to \mathrm{Out}(\pi_1(X, x_0))$ by $[\phi] \mapsto [\gamma_*\phi_*]$. One can show that this map is well-defined and is a homomorphism (see [8] Chapter 8.1). In particular, the map is independent of the choice of the path γ, since if we choose a different path γ' then the automorphisms $\gamma_*\phi_*$ and $\gamma'_*\phi_*$ of $\pi_1(X, x_0)$ differ by the inner automorphism corresponding to conjugation by $[\gamma'\bar{\gamma}] \in \pi_1(X, x_0)$.

Remark 5.4. A version of the Dehn-Nielsen-Baer theorem states that the homomorphism $\mathrm{MCG}_n(T^2) \to \mathrm{Out}(\pi_1(T^2 - \{p_1, \cdots, p_n\}))$ is injective (see [8] Theorem 8.8), and one can use this result to obtain the expressions for the homomorphisms p and g in the Birman sequence (17).

We define a right action of $\mathrm{MCG}_2(T^2)$ on the character variety $R(T^2, 2)$ by

$$[\rho] \cdot f = [\rho \circ \tilde{f}],$$

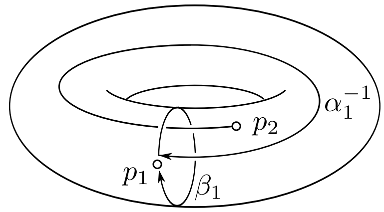


FIGURE 10. The trefoil in S^3 is constructed by gluing together (U_1, A_1) and (U_2, A_2) using the mapping class group element $f = s\beta_1\alpha_1^{-1}$.

where $[\rho] \in R(T^2, 2)$, $f \in \mathrm{MCG}_2(T^2)$, and $\tilde{f} \in \mathrm{Aut}(\pi_1(T^2 - \{p_1, p_2\}))$ is a representative of the image of f under the homomorphism $\mathrm{MCG}_2(T^2) \to \mathrm{Out}(\pi_1(T^2 - \{p_1, p_2\}))$. We find that the action of $\mathrm{MCG}_2(T^2)$ on $R(T^2, 2)$ is given by

$$\begin{aligned}
[A,\, B,\, a,\, b] \cdot T_a &= [A,\, BA,\, a,\, b], \\
[A,\, B,\, a,\, b] \cdot T_b &= [AB,\, B,\, a,\, b], \\
[A,\, B,\, a,\, b] \cdot T_A &= [A,\, aAB,\, a,\, Aaba^{-1}A^{-1}], \\
[A,\, B,\, a,\, b] \cdot T_B &= [a^{-1}BA,\, B,\, a,\, a^{-1}BbB^{-1}a], \\
[A,\, B,\, a,\, b] \cdot \omega &= [A^{-1},\, B^{-1},\, B^{-1}A^{-1}bAB,\, A^{-1}B^{-1}aBA].
\end{aligned}$$

The action of $\mathrm{MCG}_2(T^2)$ on $R(T^2, 2)$ fixes the reducible locus ∂P_3 of $R(T^2, 2)$ as a set. The homomorphism $p : B_2(T^2) \to \mathrm{MCG}_2(T^2)$ in the Birman sequence (17) induces a right action of $B_2(T^2)$ on $R(T^2, 2)$.

6. Examples

We will now consider several examples in which we use our scheme to compute a generating set for the reduced singular instanton homology $I^\natural(Y, K)$ of a $(1,1)$-knot K in a lens space Y. As described in the Introduction, we Heegaard-split (Y, K) into a pair of handlebodies (U_1, A_1) and (U_2, A_2). The handlebodies are glued together via a homeomorphism $\phi : (\partial U_1, \partial A_1) \to (\partial U_2, \partial A_2)$, which defines an element $f = [\phi]$ of the mapping class group $\mathrm{MCG}_2(T^2)$. We define a character variety $R(T^2, 2)$ corresponding to the Heegaard surface $(T^2, \{p_1, p_2\}) := (\partial U_1, \partial A_1)$, and we define Lagrangians L_1^π and $L_2 = L_1 \cdot f$ in $R(T^2, 2)$ corresponding to the handlebodies (U_1, A_1) and (U_2, A_2). To obtain a generating set for $I^\natural(Y, K)$, we count the intersection points $L_1^\pi \cap L_2$ and show that the intersection is transverse at each point. The calculations needed to accomplish this task rely on the parameterizations $L_1^\pi(\phi, \theta)$ and $L_1(\chi, \psi)$ of the Lagrangians L_1^π and L_1 given in Theorems 3.17 and 3.13, together with the description of the action of $\mathrm{MCG}_2(T^2)$ on $R(T^2, 2)$ given in Section 5.2. To describe the intersection, we will use the coordinates $(\hat{a}, \hat{b})$ that we defined on the piece $P_4 \subset R(T^2, 2)$ in Section 3.1.1, and the coordinates (α, β, γ) that we defined on the piece $P_3 \subset R(T^2, 2)$ in Section 3.1.2.

6.1. **Trefoil in S^3.** As shown in Figure 10, we can construct the trefoil in S^3 by gluing the two handlebodies together using the mapping class group element $f = s\beta_1\alpha_1^{-1} \in \mathrm{MCG}_2(T^2)$, where $s := T_a T_b^{-1} T_a$ exchanges the longitude and meridian of T^2. We first prove a Lemma that constrains the possible intersection points of L_1^π and $L_2 = L_1 \cdot f$:

Lemma 6.1. *If $L_1^\pi(\phi, \theta) = L_2(\chi, \psi)$, then $\chi = \pi/2$ and either $\theta \in \{\pi/2, 3\pi/2\}$ or $\phi \in \{0, \pi\}$*

Proof. Define functions $h_1, h_2 : R(T^2, 2) \to \mathbb{R}$ by

$$h_1([A, B, a, b]) = \mathrm{tr}\, A, \qquad h_2([A, B, a, b]) = \mathrm{tr}\, Ba.$$

We evaluate the functions h_1 and h_2 at the points $L_1^\pi(\phi, \theta)$ and $L_2(\chi, \psi)$. If we require that each function give the same value at both points, we obtain the desired result. □

Theorem 6.2. *The rank of $I^\natural(S^3, K)$ for the trefoil K in S^3 is at most 3.*

Proof. From Lemma 6.1, we know that if $L_1^\pi(\phi,\theta) = L_2(\chi,\psi)$ then $\chi = \pi/2$. A calculation shows that $L_2(\pi/2,\psi) = L_1(\pi/2,\psi)\cdot f = [A,B,a,b]$, where

$$A = \mathbf{i}, \qquad B = \sin 3\psi + \cos 3\psi\,\mathbf{k}, \tag{18}$$

$$a = -\cos 2\psi\,\mathbf{i} + \sin 2\psi\,\mathbf{j}, \qquad b = -\cos 4\psi\,\mathbf{i} - \sin 4\psi\,\mathbf{j}. \tag{19}$$

We will first show that the intersection $L_1^\pi \cap L_2$ takes place entirely in the piece P_4. Suppose $L_2(\pi/2,\psi)$ lies in the piece P_3. Then the matrices A and B in equation (18) must commute, so $\cos 3\psi = 0$, corresponding to $\psi \in \{\pm\pi/6, \pm\pi/2\}$. From equations (18) and (19), we find that

$$\gamma(L_2(\pi/2,\pm\pi/6)) = -\pi/6, \qquad \gamma(L_2(\pi/2,\pm\pi/2)) = \pi/2.$$

But Theorem 3.17 states that all of the points in $L_1^\pi \cap P_3$ have $\gamma = 0$. It follows that L_1^π does not intersect L_2 in the piece P_3.

We now consider the intersection $L_1^\pi \cap L_2$ in the piece P_4. Using equations (18) and (19), we find that the $(\hat{a},\hat{b})$ coordinates of $L_2(\pi/2,\psi)$ are

$$\hat{a}(L_2(\pi/2,\psi)) = (-\cos 2\psi,\, \sin 2\psi,\, 0), \qquad \hat{b}(L_2(\pi/2,\psi)) = (-\cos 4\psi,\, -\sin 4\psi,\, 0) \tag{20}$$

for $\psi \in (-\pi/6, \pi/6)$, and

$$\hat{a}(L_2(\pi/2,\psi)) = (-\cos 2\psi,\, -\sin 2\psi,\, 0), \qquad \hat{b}(L_2(\pi/2,\psi)) = (-\cos 4\psi,\, \sin 4\psi,\, 0) \tag{21}$$

for $\psi \in (-\pi/2,-\pi/6) \cup (\pi/6,\pi/2)$. From Lemma 6.1, we know that either $\theta \in \{\pi/2, 3\pi/2\}$ or $\phi \in \{0,\pi\}$. But $\phi = 0$ and $\phi = \pi$ correspond to the double-point of L_2, which lies in P_3, and we have already shown that L_1^π does not intersect L_2 in P_3. Thus $\theta \in \{\pi/2, 3\pi/2\}$. Substituting $\theta = \pi/2$ and $\theta = 3\pi/2$ into the expressions for the $(\hat{a},\hat{b})$ coordinates of $L_1^\pi(\phi,\theta)$ given in Theorem 3.17, we find that

$$\hat{a}(L_1^\pi(\phi,\pi/2)) = (-\sin(\phi+\nu),\, -\cos(\phi+\nu),\, 0), \quad \hat{b}(L_1^\pi(\phi,\pi/2)) = (\sin(\phi-\nu),\, \cos(\phi-\nu),\, 0), \tag{22}$$

$$\hat{a}(L_1^\pi(\phi,3\pi/2)) = (\sin(\phi+\nu),\, \cos(\phi+\nu),\, 0), \quad \hat{b}(L_1^\pi(\phi,3\pi/2)) = (-\sin(\phi-\nu),\, -\cos(\phi-\nu),\, 0). \tag{23}$$

From equations (20)–(23), it follows that the intersection $L_1^\pi \cap L_2$ fact takes place in a torus $T^2 - \bar{\Delta} \subset S^2 \times S^2 - \bar{\Delta}$, where $\bar{\Delta} \subset T^2$ is the antidiagonal. In Figure 11 we use equations (20)–(23) to plot the intersection of L_1^π and L_2 in $T^2 - \bar{\Delta}$. We see that L_1^π and L_2 intersect in three points.

We will now show that the intersection is transverse at each of these three points. A calculation shows that at each point we have

$$\partial_\phi h_1(L_1^\pi(\phi,\theta)) = 0, \quad \partial_\theta h_1(L_1^\pi(\phi,\theta)) \neq 0, \quad \partial_\chi h_1(L_2(\chi,\psi)) = 0, \quad \partial_\psi h_1(L_2(\chi,\psi)) = 0,$$
$$\partial_\phi h_2(L_1^\pi(\phi,\theta)) = 0, \quad \partial_\theta h_2(L_1^\pi(\phi,\theta)) = 0, \quad \partial_\chi h_2(L_2(\chi,\psi)) \neq 0, \quad \partial_\psi h_2(L_2(\chi,\psi)) = 0.$$

These equations, together with Figure 11, show that the intersection is transverse at each intersection point. □

For a knot K in S^3, one can show (see [13], Section 12.1) that

$$\operatorname{rank} I^\natural(S^3,K) \geq \sum_i |a_i|,$$

where a_i is the coefficient of t^i in the Alexander polynomial $\Delta_K(t)$ of K:

$$\Delta_K(t) = \sum_i a_i t^i.$$

This inequality, together with Theorem 6.2, gives the singular instanton homology for the trefoil. This result was already known, since, as shown by Kronheimer and Mrowka, the singular instanton homology of an alternating knot in S^3 is isomorphic to the reduced Khovanov homology of its mirror [19].

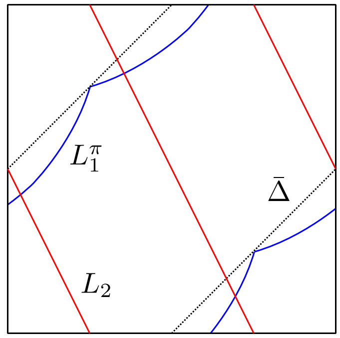


FIGURE 11. The trefoil in S^3. The space depicted is $T^2 - \bar{\Delta} \subset S^2 \times S^2 - \bar{\Delta}$. Shown are the Lagrangian L_1^π, the Lagrangian L_2, and the antidiagonal $\bar{\Delta}$.

6.2. Unknot in $L(p,1)$ for p not a multiple of 4.

We can construct the unknot U in the lens space $L(p,1)$ by gluing the two handlebodies together using the mapping class group element $f = (T_a)^p \in \mathrm{MCG}_2(T^2)$. The following is a restatement of Theorem 1.11 from the Introduction:

Theorem 6.3. *If p is not a multiple of 4, then the rank of $I^\natural(L(p,1), U)$ for the unknot U in the lens space $L(p,1)$ is at most p.*

Proof. A calculation shows that $L_2(\chi,\psi) = L_1(\chi,\psi) \cdot f = [A, B, a, b]$, where

$$A = \cos\chi + \sin\chi\,\mathbf{k}, \qquad B = \cos p\chi + \sin p\chi\,\mathbf{k}, \qquad a = b^{-1} = \cos\psi\,\mathbf{i} + \sin\psi\,\mathbf{k}. \tag{24}$$

Since A and B commute, the Lagrangian L_2 lies in the piece P_3. From equation (24), it follows that the (α,β,γ) coordinates of the point $L_2(\chi,\psi)$ are

$$\alpha(L_2(\chi,\psi)) = \chi, \qquad \beta(L_2(\chi,\psi)) = p\chi, \qquad \gamma(L_2(\chi,\psi)) = \psi.$$

Comparing with the parameterization of L_1^π in P_3 given in Theorem 3.17, we find that the intersection $L_1^\pi \cap L_2$ in fact takes place in the pillowcase $P_3 \cap \{\gamma = 0\}$. In Figure 12 we plot the intersection of L_1^π with L_2 in the pillowcase $P_3 \cap \{\gamma = 0\}$ for $p = 1, 2, 3$. We find that if p is not a multiple of 4 then we obtain a generating set with p generators. If p is a multiple of 4 then $L_1^\pi \cap L_2$ contains the double-point $(\alpha,\beta,\gamma) = (\pi/2, 0, 0)$ of L_1^π, and thus our scheme for counting generators fails.

We will now show that the intersection is transverse at each intersection point. Define functions

$$h_1([A,B,a,b]) = \operatorname{tr} Aa, \qquad h_2([A,B,a,b]) = \operatorname{tr} Ba.$$

A straightforward calculation shows that at each point of $L_1^\pi \cap L_2$ we have that

$$\begin{aligned}
&\partial_\phi h_1(L_1^\pi(\phi,\theta)) = 0, && \partial_\theta h_1(L_1^\pi(\phi,\theta)) \neq 0, && \partial_\chi h_1(L_2(\chi,\psi)) = 0, && \partial_\psi h_1(L_2(\chi,\psi)) \neq 0,\\
&\partial_\phi h_2(L_1^\pi(\phi,\theta)) = 0, && \partial_\theta h_2(L_1^\pi(\phi,\theta)) = 0, && \partial_\chi h_2(L_2(\chi,\psi)) = 0, && \partial_\psi h_2(L_2(\chi,\psi)) \neq 0.
\end{aligned}$$

These equations, together with Figure 12, show that the intersection is transverse at each point of $L_1^\pi \cap L_2$. $\square$

For the case $p = 1$ we have that $L(p,1) = S^3$, and our results imply that the unknot in S^3 has a generating set with a single generator. Since there is a single generator, there are no differentials, and this result amounts to a calculation of the singular instanton homology.

Remark 6.4. It is interesting to note that for the unknot U in the lens space $Y = L(p,q)$, the knot Floer homology $\widehat{HFK}(Y,U)$ has rank p (see [12]).

6.3. Simple knot in $L(p,1)$ in homology class $1 \in \mathbb{Z}_p = H_1(L(p,1);\mathbb{Z})$.

Definition 6.5. A knot K in a lens space $L(p,q)$ is said to be *simple* if the lens space has a Heegaard splitting into solid tori U_1 and U_2 with meridian disks D_1 and D_2 such that D_1 intersects D_2 in p points and $K \cap U_i$ is an unknotted arc in disk D_i for $i = 1, 2$ (see [12]).

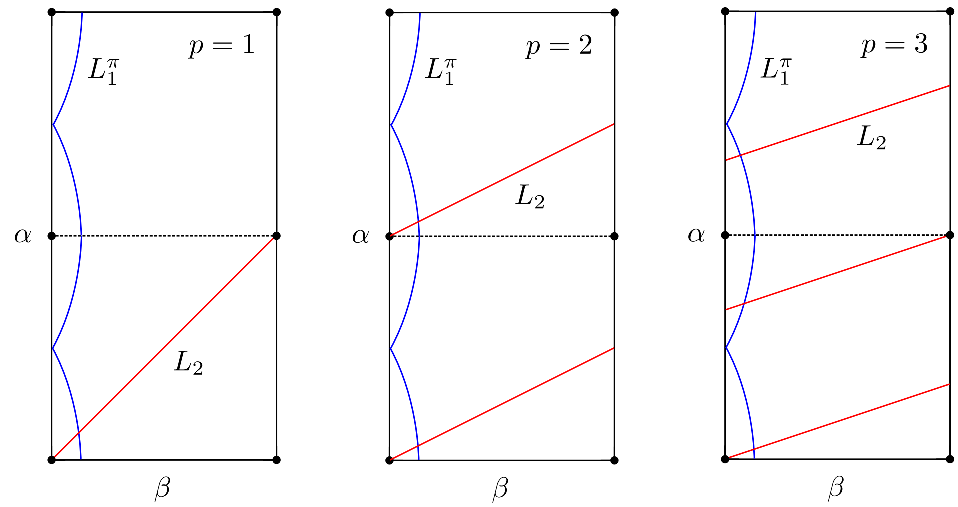


FIGURE 12. The unknot in $L(p,1)$ for $p = 1, 2, 3$. The space depicted is the pillowcase $P_3 \cap \{\gamma = 0\}$. Shown are the Lagrangians L_1^π and L_2.

One can show that there is exactly one simple knot in each nonzero homology class of $H_1(L(p,q);\mathbb{Z}) = \mathbb{Z}_p$ [12]. For the case $q = 1$, we can view the lens space $L(p,1)$ is a circle bundle over S^2, and a loop that winds n times around a circle fiber is a simple knot in homology class $n \in \mathbb{Z}_p = H_1(L(p,1);\mathbb{Z})$. For $p \geq 2$, we can construct the simple knot K in the lens space $L(p,1)$ corresponding to the homology class $1 \in \mathbb{Z}_p = H_1(L(p,1);\mathbb{Z})$ by gluing the two handlebodies together using the mapping class group element $f = \alpha_1^{-1}(T_a)^p \in \mathrm{MCG}_2(T^2)$. We first prove a result that constrains the possible intersection points of L_1^π and $L_2 = L_1 \cdot f$:

Lemma 6.6. *If $L_1^\pi(\phi,\theta) = L_2(\chi,\psi)$ then $\phi = \pi/2$ and $(\chi,\psi) \in \{(\chi_0,\psi_0), \cdots, (\chi_{p-1},\psi_{p-1})\}$, where $\chi_n := (n+1/2)(\pi/p)$ and $\psi_n := (-1)^{n+1}(\pi/2 - \epsilon)$.*

Proof. Define a function $h_1 : R(T^2,2) \cap \{\mathrm{tr}\, Ab \neq 0\} \to \mathbb{R}$ and functions $h_2, h_3 : R(T^2,2) \to \mathbb{R}$ by

$$h_1([A,B,a,b]) = -\frac{\mathrm{tr}\, Aa}{\mathrm{tr}\, Ab}, \qquad h_2([A,B,a,b]) = \mathrm{tr}\, Ba, \qquad h_3([A,B,a,b]) = \mathrm{tr}\, B.$$

Using straightforward calculations, one can show that if $h_3(L_1^\pi(\phi,\theta)) = h_3(L_2(\chi,\psi))$ then $(\mathrm{tr}\, Ab)(L_2(\chi,\psi)) \neq 0$, and thus the function h_1 is defined everywhere on $L_1^\pi \cap L_2$. We evaluate the functions h_1, h_2, and h_3 at the points $L_1^\pi(\phi,\theta)$ and $L_2(\chi,\phi)$. If we require that each function give the same value at both points, we obtain the desired result. ☐

The following is a restatement of Theorem 1.12 from the Introduction:

Theorem 6.7. *If K is the unique simple knot in the lens space $L(p,1)$ representing the homology class $1 \in \mathbb{Z}_p = H_1(L(p,1);\mathbb{Z})$, then the rank of $I^\natural(L(p,1),K)$ is at most p.*

Proof. We will argue that each of the p potential intersection points described by Lemma 6.6 is an actual intersection point. A calculation shows that $L_2(\chi_n,\psi_n) = L_1(\chi_n,\psi_n) \cdot f = [A,B,a,b]$, where

$$A = \cos\chi_n + \sin\chi_n\,\mathbf{i}, \qquad B = \cos\epsilon + \sin\epsilon\,\mathbf{k}, \tag{25}$$

$$a = (-1)^{n+1}(\cos\epsilon\,\mathbf{i} + \sin\epsilon\,\mathbf{j}), \qquad b = (-1)^n\cos\epsilon\,\mathbf{i} + \sin\epsilon\cos\eta_n\,\mathbf{j} + \sin\epsilon\sin\eta_n\,\mathbf{k}, \tag{26}$$

and $\eta_n := (1 + n(p+2))(\pi/p)$. We note that A and B do not commute, since the coefficient of $\mathbf{i}$ in A and the coefficient of $\mathbf{k}$ in B are both nonzero, so the intersection $L_1^\pi \cap L_2$ takes place entirely in the piece P_4. From equations (25) and (26), we find that the $(\hat{a},\hat{b})$ coordinates of $L_2(\chi_n,\psi_n)$ are given by

$$\hat{a}(L_2(\chi_n,\psi_n)) = (-1)^{n+1}(\cos\epsilon,\, \sin\epsilon,\, 0), \qquad \hat{b}(L_2(\chi_n,\psi_n)) = ((-1)^n\cos\epsilon,\, \sin\epsilon\cos\eta_n,\, \sin\epsilon\sin\eta_n). \tag{27}$$

From Lemma 6.6, we know that if $L_1^\pi(\phi,\theta) = L_2(\chi,\psi)$ then $\phi = \pi/2$. Substituting $\phi = \pi/2$ into the expressions for the $(\hat{a},\hat{b})$ coordinates of $L_1^\pi(\phi,\theta)$ given in Theorem 3.17, we find that

$$\hat{a}(L_1^\pi(\pi/2,\theta)) = (\cos\epsilon,\, \sin\epsilon,\, 0), \qquad \hat{b}(L_1^\pi(\pi/2,\theta)) = (-\cos\epsilon,\, -\sin\epsilon\cos\bar{\theta},\, \sin\epsilon\sin\bar{\theta}) \tag{28}$$

for $\theta \in (0,\pi)$, and

$$\hat{a}(L_1^\pi(\pi/2,\theta)) = (-\cos\epsilon,\, -\sin\epsilon,\, 0), \qquad \hat{b}(L_1^\pi(\pi/2,\theta)) = (\cos\epsilon,\, \sin\epsilon\cos\bar{\theta},\, \sin\epsilon\sin\bar{\theta}) \tag{29}$$

for $\theta \in (\pi,2\pi)$, where $\bar{\theta}$ is defined such that

$$\cos\bar{\theta} = \frac{\cos^2\epsilon\cos^2\theta - \sin^2\theta}{\cos^2\epsilon\cos^2\theta + \sin^2\theta}, \qquad \sin\bar{\theta} = \frac{\cos\epsilon\sin 2\theta}{\cos^2\epsilon\cos^2\theta + \sin^2\theta}.$$

It is straightforward to verify that for small enough values of ϵ, the maps $(0,\pi) \to (0,2\pi)$, $\theta \mapsto \bar{\theta}$ and $(\pi,2\pi) \to (0,2\pi)$, $\theta \mapsto \bar{\theta}$ are diffeomorphisms. Thus we can always solve equations (27)–(29) to obtain a unique value of θ such that $L_1^\pi(\pi/2,\theta) = L_2(\chi_n,\psi_n)$. Specifically, if n is even, then $\theta \in (0,\pi)$ is given by $\bar{\theta}(\theta) = \eta_n$, and if n is odd then $\theta \in (\pi,2\pi)$ is given by $\bar{\theta}(\theta) = \pi - \eta_n$. We conclude that L_1^π and L_2 intersect in p points.

We will now show that L_1^π intersects L_2 transversely at each of these p points. A straightforward calculation shows that at each point of $L_1^\pi \cap L_2$ we have

$$\begin{aligned}
&\partial_\phi h_1(L_1^\pi(\phi,\theta)) \neq 0, && \partial_\theta h_1(L_1^\pi(\phi,\theta)) = 0, && \partial_\chi h_1(L_2(\chi,\psi)) = 0, && \partial_\psi h_1(L_2(\chi,\psi)) = 0,\\
&\partial_\phi h_2(L_1^\pi(\phi,\theta)) = 0, && \partial_\theta h_2(L_1^\pi(\phi,\theta)) = 0, && \partial_\chi h_2(L_2(\chi,\psi)) \neq 0, && \partial_\psi h_2(L_2(\chi,\psi)) = 0,\\
&\partial_\phi h_3(L_1^\pi(\phi,\theta)) = 0, && \partial_\theta h_3(L_1^\pi(\phi,\theta)) = 0, && \partial_\chi h_3(L_2(\chi,\psi)) = 0, && \partial_\psi h_3(L_2(\chi,\psi)) \neq 0.
\end{aligned}$$

These equations, together with Theorem 3.17, show that the intersection is transverse at each point. □

For the case $p = 0$, the knot we have constructed is $K = S^1 \times \{pt\}$ in $S^1 \times S^2$, and our above result implies that this knot has a generating set with zero generators. This result holds even in the absence of the perturbation, since there are no homomorphisms $\rho : \pi_1(S^1 \times S^2 - K) \to SU(2)$ that take loops around K to traceless matrices.

For the case $p = 1$, the knot we have constructed is the unknot in S^3, and we have have reproduced the result of Section 6.2 for this knot.

Remark 6.8. It is interesting to note that for a simple knot K in the lens space $Y = L(p,q)$, the knot Floer homology $\widehat{HFK}(Y,K)$ has rank p (see [12]).

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