THE COMBINATORIAL AND GAUGE-THEORETIC FOAM EVALUATION FUNCTORS ARE NOT THE SAME

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Abstract. Kronheimer and Mrowka used gauge theory to define a functor $J^\sharp$ from a category of webs in $\mathbb{R}^3$ to the category of finite-dimensional vector spaces over the field of two elements. They also suggested a possible combinatorial replacement $J^\flat$ for $J^\sharp$, which Khovanov and Robert proved is well-defined on a subcategory of planar webs. We exhibit a counterexample that shows the restriction of the functor $J^\sharp$ to the subcategory of planar webs is not the same as $J^\flat$.

1. Introduction

Kronheimer and Mrowka have outlined a new approach that could potentially lead to the first non-computer based proof of the four-color theorem [6]. Their approach relies on a functor $J^\sharp$, which they define using gauge theory, from a category of webs in $\mathbb{R}^3$ and foams in $\mathbb{R}^4$ to the category of finite-dimensional vector spaces over the field of two elements $\mathbb{F}$. A *web* is an unoriented trivalent graph and a *foam* is a kind of singular cobordism between webs whose precise form is described in [6].

The four-color theorem can be reformulated as a statement about webs. An edge e of a web is said to be a *bridge* if removing e increases the number of connected components of the web. A *Tait coloring* of a web is a 3-coloring of the edges of the web such that no two edges incident on any given vertex share the same coloring. Given a web K, the *Tait number* $\mathrm{Tait}(K)$ is the number of Tait colorings of K. The four-color theorem is equivalent to the statement that every bridgeless planar web admits at least one Tait coloring.

The functor $J^\sharp$ associates a vector space $J^\sharp(K)$ to a web K in $\mathbb{R}^3$. An edge e of a web K in $\mathbb{R}^3$ is said to be an *embedded bridge* if there is a 2-sphere smoothly embedded in $\mathbb{R}^3$ that transversely intersects K in a single point that lies on e. Kronheimer and Mrowka prove the following nonvanishing theorem:

Theorem 1.1. *(Kronheimer–Mrowka* [6, Theorem 1.1]*) For a web K in $\mathbb{R}^3$, the vector space $J^\sharp(K)$ is zero if and only if K has an embedded bridge.*

Based on some simple examples and general properties of $J^\sharp$, they make the following conjecture, which by Theorem 1.1 implies the four-color theorem:

Conjecture 1.2. *(Kronheimer–Mrowka* [6, Conjecture 1.2]*) For a web K that lies in the plane $\mathbb{R}^2 \subset \mathbb{R}^3$, we have $\dim J^\sharp(K) = \mathrm{Tait}(K)$.*

Kronheimer and Mrowka also suggested a possible combinatorial replacement $J^\flat$ for $J^\sharp$, which they defined via a set of rules that they conjectured would yield a well-defined functor [6, Section 8.3]. Khovanov and Robert later showed that $J^\flat$ is not well-defined for arbitrary webs in $\mathbb{R}^3$ and foams in $\mathbb{R}^4$, but is well defined provided we restrict to the subcategory of webs in $\mathbb{R}^2$ and foams in $\mathbb{R}^3$ [3]. We will call this subcategory the category of *planar* webs. We note that planar webs are precisely those relevant to Conjecture 1.2. Based on results due to Khovanov and Robert [3], and Kronheimer and Mrowka [5], for any planar web K we have

$$\dim J^\flat(K) \leq \mathrm{Tait}(K) \leq \dim J^\sharp(K), \tag{1}$$

and for a special class of *reducible* planar webs (also called *simple* webs in [6]), these three integers coincide:

$$\dim J^\flat(K) = \mathrm{Tait}(K) = \dim J^\sharp(K).$$

A proof that the restriction of $J^\sharp$ to the subcategory of planar webs is indeed the same functor as $J^\flat$ would therefore prove Conjecture 1.2 and hence the four-color theorem.

It is thus of interest to understand the relationship between the functors $J^\flat$ and $J^\sharp$. Some insight into these functors can be gained by considering related functors with different target categories. In [5], Kronheimer

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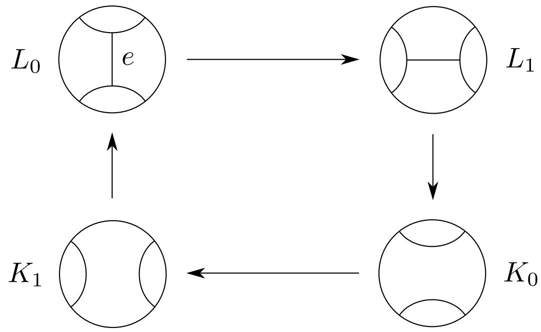

FIGURE 1. Given a planar web L_0 and choice of edge e, we can construct a square of planar webs L_0, L_1, K_0, and K_1 related by standard foam cobordisms in $\mathbb{R}^3$. The webs are identical outside of the indicated 2-balls.

and Mrowka establish the second inequality in equation (1) by introducing a system of local coefficients and defining a functor from the category of webs in $\mathbb{R}^3$ to the category of modules over the ring $\mathbb{F}[T_1^{\pm 1}, T_2^{\pm 1}, T_3^{\pm 1}]$. In [3], Khovanov and Robert extend the ground field $\mathbb{F}$ to the graded ring $R = \mathbb{F}[E_1, E_2, E_3]$, where

$$\deg(E_1) = 2, \qquad \deg(E_2) = 4, \qquad \deg(E_3) = 6,$$

and define a functor from the category of planar webs to the category of modules over R. One can obtain additional functors by base-changing to a ring S via a ring homomorphism $R \to S$. As noted in [1] Corollary 3.1, the first inequality in equation (1) directly follows from [3] Proposition 4.18 by considering such base-changes.

We consider here a base-change from R to the graded ring $\mathbb{F}[E]$, where $\deg(E) = 6$, via the ring homomorphism $R \to \mathbb{F}[E]$ given by

$$E_1, E_2 \mapsto 0, \qquad E_3 \mapsto E.$$

We thereby obtain a functor from the category of planar webs to the category of modules over $\mathbb{F}[E]$. We denote this functor by $\langle - \rangle$. Given a web K, we say that the corresponding $\mathbb{F}[E]$-module $\langle K \rangle$ is the *state space* of K. By [3] Proposition 4.18, the state space $\langle K \rangle$ is a free graded module of rank $\mathrm{Tait}(K)$.

As described in [3] Section 4.3, given a planar web L_0 and a choice of edge e we can construct a square of planar webs L_0, L_1, K_0, and K_1 that are related by standard foam cobordisms in $\mathbb{R}^3$ (see Figure 1). By [3] Lemma 4.11, the image of this square under the functor $\langle - \rangle$ is a 4-periodic complex:

$$\begin{array}{ccc} \langle L_0 \rangle & \xrightarrow{1} & \langle L_1 \rangle \\ {\scriptstyle 1}\uparrow & & \downarrow{\scriptstyle 1} \\ \langle K_1 \rangle & \xleftarrow{2} & \langle K_0 \rangle. \end{array} \tag{2}$$

The integers indicate the degrees of the linear maps. If we further base-change to the ground field $\mathbb{F}$ via the ring homomorphism $\mathbb{F}[E] \to \mathbb{F}$, $E \mapsto 0$, we obtain a 4-periodic complex for the combinatorial functor $J^\flat$:

$$\begin{array}{ccc} J^\flat(L_0) & \xrightarrow{1} & J^\flat(L_1) \\ {\scriptstyle 1}\uparrow & & \downarrow{\scriptstyle 1} \\ J^\flat(K_1) & \xleftarrow{2} & J^\flat(K_0). \end{array} \tag{3}$$

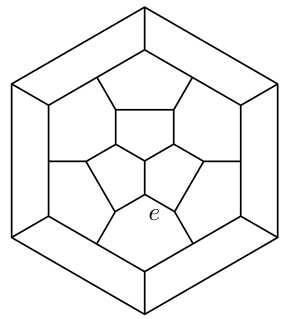

FIGURE 2. Irreducible web W_4. We consider the 4-periodic complex corresponding to the indicated edge e.

As shown in [4] Lemma 10.2, we obtain an analogous 4-periodic complex by applying the gauge-theoretic functor $J^\sharp$ to the square shown in Figure 1:

$$\begin{array}{ccc} J^\sharp(L_0) & \longrightarrow & J^\sharp(L_1) \\ \uparrow & & \downarrow \\ J^\sharp(K_1) & \longleftarrow & J^\sharp(K_0). \end{array} \tag{4}$$

We note that the vector spaces for $J^\flat$ are graded, but the vector spaces for $J^\sharp$ are ungraded. Kronheimer and Mrowka prove the following result:

Theorem 1.3. *(Kronheimer–Mrowka [4, Lemma 10.3]) In the 4-periodic complex (4) for $J^\sharp$, the homology groups at diametrically opposite corners are equal.*

The proof of Theorem 1.3 relies on the fact that the 4-periodic complex (4) for $J^\sharp$ can be extended to an octahedral diagram involving two additional webs that are nonplanar. Since $J^\flat$ is not defined for nonplanar webs, it is natural to ask whether the analog to Theorem 1.3 holds for $J^\flat$. We answer this question in the negative by exhibiting a specific counterexample:

Theorem 1.4. *For the (irreducible) web $L_0 = W_4$ shown in Figure 2 with the indicated choice of edge e, the homology of the complex (3) for $J^\flat$ is zero at K_0 but nonzero at L_0.*

In particular, Theorems 1.3 and 1.4 show that the restriction of the functor $J^\sharp$ to the subcategory of planar webs is not the same as the functor $J^\flat$. We emphasize that this result does not refute Conjecture 1.2, and thus does not invalidate Kronheimer and Mrowka's strategy for proving the four-color theorem.

2. Theoretical results

Consider the situation in which L_0 is irreducible but L_1, K_0, and K_1 are all reducible. Given bases for the state spaces in the complex (2) for $\langle - \rangle$, we can express the linear maps in the complex as $\mathbb{F}[E]$-valued matrices relative to these bases. By performing Smith decompositions of these matrices, we can decompose each state space in the complex into a kernel, denoted by subscript k, and a complement, denoted by subscript c:

$$\begin{array}{ccc} \langle L_0\rangle_k \oplus \langle L_0\rangle_c & \xrightarrow{\ 1\ } & \langle L_1\rangle_k \oplus \langle L_1\rangle_c \\ {\scriptstyle 1}\uparrow & & \downarrow{\scriptstyle 1} \\ \langle K_1\rangle_k \oplus \langle K_1\rangle_c & \xleftarrow{\ 2\ } & \langle K_0\rangle_k \oplus \langle K_0\rangle_c. \end{array} \tag{5}$$

We have:

Theorem 2.1. *In the 4-periodic complex (2), the homology groups are E-torsion.*

Proof. This follows directly from [3] Proposition 4.12 and comments after the statement of Proposition 4.17, which show that the 4-periodic complex (2) becomes exact after localizing $\mathbb{F}[E] \to \mathbb{F}[E, E^{-1}]$. ☐

By Theorem 2.1, the complement for each state space is mapped into a submodule of full rank of the kernel for the subsequent state space.

As described in [1], we have an algorithm for constructing bases of the state spaces for reducible webs, and we can use a computer to construct a submodule M of $\langle L_0 \rangle$ of full rank. The only possible difference between the submodule M and the actual state space $\langle L_0 \rangle$ is that homogeneous generators of M may be shifted upwards in degree by multiples of $\deg(E) = 6$ relative to corresponding homogeneous generators of $\langle L_0 \rangle$. We thus obtain a periodic complex analogous to (5), but with $\langle L_0 \rangle$ replaced by M:

$$\begin{array}{ccc} M_k \oplus M_c & \xrightarrow{1} & \langle L_1 \rangle_k \oplus \langle L_1 \rangle_c \\ {\scriptstyle 1}\uparrow & & \downarrow{\scriptstyle 1} \\ \langle K_1 \rangle_k \oplus \langle K_1 \rangle_c & \xleftarrow{2} & \langle K_0 \rangle_k \oplus \langle K_0 \rangle_c. \end{array} \tag{6}$$

In principle, it need not be the case that the image of $\langle K_1 \rangle \to \langle L_0 \rangle$ is contained in M, but if not we can simply replace M with the module spanned by M and this image. By Theorem 2.1, the complex (6) determines the ranks of the modules in the complex (5). In particular,

$$\operatorname{rank}(\langle L_0 \rangle_k) = \operatorname{rank}(M_k), \qquad \operatorname{rank}(\langle L_0 \rangle_c) = \operatorname{rank}(M_c).$$

By computing the quantum ranks of the modules in the complex (6), we can strongly constrain the possibilities for the complex (5) for $\langle - \rangle$, which in turn strongly constrains the possibilities for the homology of the complex (3) for $J^\flat$.

The relationship between the functors $\langle - \rangle$ and $J^\flat$ is discussed in [1, Section 3]. We briefly summarize the results we will need. Given a planar web K, we define a *half-foam H with boundary K* to be a foam cobordism in $\mathbb{R}^3$ from the empty web to K. A half-foam H with boundary K determines elements $\langle H \rangle \in \langle K \rangle$ and $J^\flat(H) \in J^\flat(K)$. If $\langle H \rangle$ is zero then $J^\flat(H)$ must be zero as well, but in principle it may happen that $J^\flat(H)$ is zero and $\langle H \rangle$ is nonzero, in which case we say that H is a *vanishing* half-foam. The state space $\langle K \rangle$ is freely generated by $\operatorname{Tait}(K)$ half-foams with boundary K. Since some of the generating half-foams may be vanishing, we have

$$\dim J^\flat(K) \le \operatorname{rank}(\langle K \rangle) = \operatorname{Tait}(K), \qquad \operatorname{qdim} J^\flat(K) \le \operatorname{qrank}(\langle K \rangle),$$

with equality holding in the case that there are no vanishing generators. For K reducible, there are no vanishing generators of $\langle K \rangle$. It is an open question as to whether there is a nonreducible web K for which $\langle K \rangle$ has vanishing generators. For every vanishing generator $\langle H_1 \rangle$ of degree d_1, there must be a corresponding vanishing generator $\langle H_2 \rangle$ of degree d_2 such that $d_1 + d_2$ is a positive integer multiple of $\deg(E) = 6$. If there are no vanishing generators of $\langle K \rangle$, then for every generator of $\langle K \rangle$ in degree d there is a corresponding generator in degree $-d$, so $\operatorname{qrank}(\langle K \rangle)$ is symmetric under $q \to q^{-1}$.

Similar considerations apply to the linear maps corresponding to foam cobordisms. Consider a foam cobordism between planar webs K and L and the corresponding linear maps $\langle K \rangle \to \langle L \rangle$ and $J^\flat(K) \to J^\flat(L)$. Suppose H is a half-foam with boundary K such that the image of $\langle H \rangle$ under $\langle K \rangle \to \langle L \rangle$ can be expressed as Ex for $x \in \langle L \rangle$. Then $J^\flat(H)$ maps to zero under $J^\flat(K) \to J^\flat(L)$.

3. Computer results

We take L_0 to be the (irreducible) web W_4 shown in Figure 2 with the indicated choice of edge e. The resulting webs L_1, K_0, and K_1 are all reducible. We use the computer program described in [1] and available on the web [2] to calculate the ranks and quantum ranks of the modules in the complex (6), and we display the results in Table 1. The expressions in parentheses indicate the degrees of vanishing generators that map to zero when we base-change from $\mathbb{F}[E]$ to $\mathbb{F}$ by setting $E = 0$. In particular, there are two vanishing generators for M, one of degree 1 and one of degree 5. Since $\operatorname{Tait}(L_0) = \operatorname{rank}(M) = 180$, the fact that there are two vanishing generators gives us a lower bound of 178 for $\dim J^\flat(L_0)$. There are three possible cases for the state space $\langle L_0 \rangle$:

module	rank	qrank
M	180	$q^{-6}+11q^{-4}+10q^{-3}+29q^{-2}+19q^{-1}+38+19q+29q^2+10q^3+11q^4+q^6+(q+q^5)$
M_k	72	$2q^{-4}+q^{-3}+11q^{-2}+3q^{-1}+19+3q+18q^2+3q^3+9q^4+q^6+(q+q^5)$
M_c	108	$q^{-6}+9q^{-4}+9q^{-3}+18q^{-2}+16q^{-1}+19+16q+11q^2+7q^3+2q^4$
$\langle L_1\rangle$	168	$2q^{-5}+8q^{-4}+12q^{-3}+24q^{-2}+22q^{-1}+32+22q+24q^2+12q^3+8q^4+2q^5$
$\langle L_1\rangle_k$	108	$q^{-5}+q^{-4}+9q^{-3}+9q^{-2}+18q^{-1}+16+19q+15q^2+11q^3+7q^4+2q^5$
$\langle L_1\rangle_c$	60	$q^{-5}+7q^{-4}+3q^{-3}+15q^{-2}+4q^{-1}+16+3q+9q^2+q^3+q^4$
$\langle K_0\rangle$	72	$q^{-5}+q^{-4}+10q^{-3}+3q^{-2}+19q^{-1}+4+19q+3q^2+10q^3+q^4+q^5$
$\langle K_0\rangle_k$	60	$q^{-4}+7q^{-3}+3q^{-2}+15q^{-1}+4+16q+3q^2+9q^3+q^4+q^5$
$\langle K_0\rangle_c$	12	$q^{-5}+3q^{-3}+4q^{-1}+3q+q^3$
$\langle K_1\rangle$	84	$2q^{-5}+q^{-4}+12q^{-3}+3q^{-2}+22q^{-1}+4+22q+3q^2+12q^3+q^4+2q^5$
$\langle K_1\rangle_k$	12	$q^{-3}+3q^{-1}+4q+3q^3+q^5$
$\langle K_1\rangle_c$	72	$2q^{-5}+q^{-4}+11q^{-3}+3q^{-2}+19q^{-1}+4+18q+3q^2+9q^3+q^4+q^5$

TABLE 1. Ranks and quantum ranks of the modules in the complex (6). The expressions in parentheses indicate the degrees of vanishing generators that map to zero when we base-change from $\mathbb{F}[E]$ to $\mathbb{F}$ by setting $E=0$.

(1) No generators are missing. In this case $\langle L_0\rangle = M$ and $\dim J^\flat(L_0)=178$. Since $\dim J^\sharp(L_0)\geq \mathrm{Tait}(L_0)=180$, we already know that this case is not consistent with $J^\flat = J^\sharp$.
(2) A generator of degree -1 is missing. In this case M is a proper submodule of $\langle L_0\rangle$, with the vanishing generator of degree 5 in M shifted up relative to the missing generator of degree -1 in $\langle L_0\rangle$, and $\dim J^\flat(L_0)=180$.
(3) A generator of degree -5 is missing. In this case M is a proper submodule of $\langle L_0\rangle$, with the vanishing generator of degree 1 in M shifted up relative to the missing generator of degree -5 in $\langle L_0\rangle$, and $\dim J^\flat(L_0)=180$.

For each case, we show that the homology of the complex (3) for $J^\flat$ is zero at K_0 but nonzero at L_0, thus proving Theorem 1.4 from the introduction.

3.1. **Case 1: no generators are missing.** From Table 1, it follows that the quantum ranks of the modules for L_0 are

$$\begin{aligned}\mathrm{qrank}(\langle L_0\rangle) &= q^{-6}+11q^{-4}+10q^{-3}+29q^{-2}+19q^{-1}+38+19q+29q^2+10q^3+11q^4+q^6+(q+q^5),\\ \mathrm{qrank}(\langle L_0\rangle_k) &= 2q^{-4}+q^{-3}+11q^{-2}+3q^{-1}+19+3q+18q^2+3q^3+9q^4+q^6+(q+q^5),\\ \mathrm{qrank}(\langle L_0\rangle_c) &= q^{-6}+9q^{-4}+9q^{-3}+18q^{-2}+16q^{-1}+19+16q+11q^2+7q^3+2q^4.\end{aligned}$$

Thus

$$\begin{aligned}&\mathrm{qrank}(\langle L_0\rangle_k)-q\cdot\mathrm{qrank}(\langle K_1\rangle_c)=0, && \mathrm{qrank}(\langle L_1\rangle_k)-q\cdot\mathrm{qrank}(\langle L_0\rangle_c)=q^{-4}-q^2,\\ &\mathrm{qrank}(\langle K_1\rangle_k)-q^2\cdot\mathrm{qrank}(\langle K_0\rangle_c)=0, && \mathrm{qrank}(\langle K_0\rangle_k)-q\cdot\mathrm{qrank}(\langle L_1\rangle_c)=0.\end{aligned}$$

It follows that the quantum dimensions of the homology groups for the complex (3) for $J^\flat$ are

$$\begin{aligned}&\mathrm{qdim}(H(L_0))=q, && \mathrm{qdim}(H(L_1))=q^{-4},\\ &\mathrm{qdim}(H(K_1))=1+q^4, && \mathrm{qdim}(H(K_0))=0,\end{aligned}$$

since we have generators of degrees 0 and 4 in $\langle K_1\rangle_c$ that map to the vanishing generators of $\langle L_0\rangle_k$, and we have a generator of degree 1 in $\langle L_0\rangle_c$ that maps to E times a generator of degree -4 in $\langle L_1\rangle_k$. In particular, the homology is zero at K_0 and nonzero at L_0.

3.2. **Case 2: a generator of degree -1 is missing.** From Table 1, it follows that the ranks and quantum ranks of the modules for L_0 are

$$\begin{aligned}
\mathrm{qrank}(\langle L_0\rangle) &= q^{-6}+11q^{-4}+10q^{-3}+29q^{-2}+20q^{-1}+38+20q+29q^2+10q^3+11q^4+q^6,\\
\mathrm{qrank}(\langle L_0\rangle_k) &= 2q^{-4}+q^{-3}+11q^{-2}+4q^{-1}+19+4q+18q^2+3q^3+9q^4+q^6,\\
\mathrm{qrank}(\langle L_0\rangle_c) &= q^{-6}+9q^{-4}+9q^{-3}+18q^{-2}+16q^{-1}+19+16q+11q^2+7q^3+2q^4.
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathrm{qrank}(\langle L_0\rangle_k)-q\cdot\mathrm{qrank}(\langle K_1\rangle_c)=q^{-1}-q^5, &&\mathrm{qrank}(\langle L_1\rangle_k)-q\cdot\mathrm{qrank}(\langle L_0\rangle_c)=q^{-4}-q^2,\\
&\mathrm{qrank}(\langle K_1\rangle_k)-q^2\cdot\mathrm{qrank}(\langle K_0\rangle_c)=0, &&\mathrm{qrank}(\langle K_0\rangle_k)-q\cdot\mathrm{qrank}(\langle L_1\rangle_c)=0.
\end{aligned}$$

It follows that the quantum dimensions of the homology groups for the complex (3) for $J^\flat$ are

$$\begin{aligned}
&\mathrm{qdim}(H(L_0))=q^{-1}+q, &&\mathrm{qdim}(H(L_1))=q^{-4},\\
&\mathrm{qdim}(H(K_1))=q^4, &&\mathrm{qdim}(H(K_0))=0,
\end{aligned}$$

since we have a generator of degree 4 in $\langle K_1\rangle_c$ that maps to E times a generator of degree -1 in $\langle L_0\rangle_k$ and we have a generator of degree 1 in $\langle L_0\rangle_c$ that maps E times a generator of degree -4 in $\langle L_1\rangle_k$. In particular, the homology is zero at K_0 and nonzero at L_0.

3.3. **Case 3: a generator of degree -5 is missing.** From Table 1, it follows that the ranks and quantum ranks of the modules for L_0 are

$$\begin{aligned}
\mathrm{qrank}(\langle L_0\rangle) &= q^{-6}+q^{-5}+11q^{-4}+10q^{-3}+29q^{-2}+19q^{-1}+38+19q+29q^2+10q^3+11q^4+q^5+q^6,\\
\mathrm{qrank}(\langle L_0\rangle_k) &= q^{-5}+2q^{-4}+q^{-3}+11q^{-2}+3q^{-1}+19+3q+18q^2+3q^3+9q^4+q^5+q^6,\\
\mathrm{qrank}(\langle L_0\rangle_c) &= q^{-6}+9q^{-4}+9q^{-3}+18q^{-2}+16q^{-1}+19+16q+11q^2+7q^3+2q^4.
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathrm{qrank}(\langle L_0\rangle_k)-q\cdot\mathrm{qrank}(\langle K_1\rangle_c)=q^{-5}-q, &&\mathrm{qrank}(\langle L_1\rangle_k)-q\cdot\mathrm{qrank}(\langle L_0\rangle_c)=q^{-4}-q^2,\\
&\mathrm{qrank}(\langle K_1\rangle_k)-q^2\cdot\mathrm{qrank}(\langle K_0\rangle_c)=0, &&\mathrm{qrank}(\langle K_0\rangle_k)-q\cdot\mathrm{qrank}(\langle L_1\rangle_c)=0.
\end{aligned}$$

It follows that the quantum dimensions of the homology groups for the complex (3) for $J^\flat$ are

$$\begin{aligned}
&\mathrm{qdim}(H(L_0))=q^{-5}+q, &&\mathrm{qdim}(H(L_1))=q^{-4},\\
&\mathrm{qdim}(H(K_1))=1, &&\mathrm{qdim}(H(K_0))=0,
\end{aligned}$$

since we have a generator of degree 0 in $\langle K_1\rangle_c$ that maps to E times a generator of degree -5 in $\langle L_0\rangle_k$ and we have a generator of degree 1 in $\langle L_0\rangle_c$ that maps E times a generator of degree -4 in $\langle L_1\rangle_k$. In particular, the homology is zero at K_0 and nonzero at L_0.

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Conflict of interest statement

The author states there is no conflict of interest.

References

[1] D. Boozer. Computer bounds for Kronheimer-Mrowka foam evaluation. *Exp. Math.*, 2021. DOI: 10.1080/10586458.2021.1982078.
[2] D. Boozer. Foam evaluation computer program. `http://www.math.princeton.edu/~adboozer` and `https://www.dropbox.com/s/k8m7dnqd2c2iq96/foam-counterexample.zip?dl=0`, 2023.
[3] M. Khovanov and L. H. Robert. Foam evaluation and Kronheimer–Mrowka theories. *Adv. Math.*, 376:107433, 2021.
[4] P. Kronheimer and T. Mrowka. Exact triangles for $SO(3)$ instanton homology of webs. *J. Topol.*, 9(3):774–796, 2016.
[5] P. Kronheimer and T. Mrowka. A deformation of instanton homology for webs. *Geom. Topol.*, 23(3):1491–1547, 2019.
[6] P. Kronheimer and T. Mrowka. Tait colorings, and an instanton homology for webs and foams. *J. Eur. Math. Soc.*, 21:55–119, 2019.