Sample Problem Solution

(a) As long as $0 \le c_n \le 1$, we can write the sequence $\{c_n\}$ as

$$c_{n+1} = \begin{cases} 2c_n & \text{if } c_n \le \frac{1}{2} \\ 2-2c_n & \text{if } c_n > \frac{1}{2} \end{cases} \qquad (*)$$

So if $0 \le c_n \le \frac{1}{2} \Rightarrow c_{n+1} \in [0,1]$

and if $\frac{1}{2} < c_n \le 1 \Rightarrow c_{n+1} \in [0,1)$

$\Rightarrow$ If $0 \le c_n \le 1$, then $0 \le c_{n+1} \le 1$

Since we require that $0 \le c_1 \le 1$, then by induction, all numbers in the sequence satisfy $0 \le c_n \le 1$, and sequence $\{c_n\}$ can be described by relation $(*)$.

We see from $(*)$ that every number in the sequence is related to c_1 by:

$$c_n = 2i_n \pm 2^n c_1 \qquad (**)$$

where i_n is some integer

(Every time we use $(*)$ to get the next number in the sequence, the previous value is multiplied by ± 2, for a total multiplication by $\pm 2^n$ with c_1. The extra integer is introduced when $c_n > \frac{1}{2}$, and remains in the expression from that point onward.)

We will use $(*)$ and $(**)$ to show that

c_1 is rational $\Leftrightarrow$ $\{c_n\}$ becomes periodic.

Proof: If $\{c_n\}$ becomes periodic, c_1 is rational.

Suppose $\{c_n\}$ becomes periodic. This means that there is some integer N, that if we take any $n \geq N$,

$$c_{n+T} = c_n$$

T is the period of the sequence.

From relation (**), we rewrite the above equation as

$$2i_{n+T} \pm 2^{n+T} c_1 = 2i_n \pm 2^n c_1$$

$$2(i_{n+T} - i_n) = \pm c_1 (2^n \pm 2^{n+T})$$

$$c_1 = \pm \frac{2(i_{n+T} - i_n)}{2^n \pm 2^{n+T}}$$

So c_1 is rational! (Both $2(i_{n+T} - i_n)$ and $\pm 2^n \pm 2^{n+T}$ are integers)

Proof: If c_1 is rational, $\{c_n\}$ becomes periodic.

Suppose c_1 is rational — there are integers p_1 and q_1 with $c_1 = \frac{p_1}{q_1}$ (p_1, q_1 relatively prime)

If we use relation (**), then

$$c_n = \frac{p_n}{q_1}$$

where $p_n = 2i_n q_1 \pm 2^n p_1$ and is an integer

Also recall that $0 \leq c_n \leq 1$, so $0 \leq p_n \leq q_1$

So for any number in the sequence, c_n can only take values $0, \frac{1}{q_1}, \frac{2}{q_1}, \ldots, \frac{q_1 - 1}{q_1}, 1$

Then at some point, there is as $c_{n+T} = c_n$

$\Rightarrow \{c_n\}$ is periodic

Sample Problem Solution

(b) We know that c_1 must be rational, so let's consider some examples of sequences:

c_1
$\downarrow$

$0, 0, 0, \ldots$ $T=1$

$1, 0, 0, 0, \ldots$ $T=1$

$\frac{1}{2}, 1, 0, 0, \ldots$ $T=1$

$\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \ldots$ $T=1$

$\frac{1}{4}, \frac{1}{2}, 1, 0, 0, \ldots$ $T=1$

$\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{2}{5}, \ldots$ $T=2$

$\frac{3}{5}, \frac{4}{5}, \frac{2}{5}, \frac{4}{5}, \ldots$ $T=2$

$\frac{1}{6}, \frac{2}{6} = \frac{1}{3} \ldots$ $T=1$

$\frac{5}{6}, \frac{2}{6} = \frac{1}{3}, \ldots$ $T=1$

quickly reduces to $?/3$

$\frac{1}{7}, \frac{2}{7}, \frac{4}{7}, \frac{6}{7}, \frac{2}{7}, \ldots$ $T=3$

$\frac{3}{7}, \frac{6}{7}, \frac{2}{7}, \ldots$ $T=3$

$\frac{5}{7}, \frac{4}{7}, \frac{6}{7}, \ldots$ $T=3$

$\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 0, 0, \ldots$ $T=1$

$\frac{3}{8}, \frac{6}{8} = \frac{3}{4}, \ldots$ $T=1$

$\frac{5}{8}, \frac{3}{4} \ldots$ $T=1$

$\frac{7}{8}, \frac{3}{4}, \ldots$ $T=1$

reduces to $?/4$, then $?/2$

$\frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{8}{9}, \frac{2}{9}, \ldots$ $T=3$

$\frac{5}{9}, \frac{8}{9}, \ldots$ $T=3$

$\frac{7}{9}, \frac{2}{9}, \ldots$ $T=3$

As long as there exists a c_1 that creates a sequence that is periodic with period T, there are infinitely many c_1 that generate a sequence eventually with period T.

For example $c_1 = \frac{1}{5}$ gives a sequence with $T=2$

But also if $c_1 = \frac{1}{10} \Rightarrow \{c_n\} = \frac{1}{10}, \frac{1}{5}, \dots$

and if $c_1 = \frac{1}{20} = \frac{1}{2^2} \cdot \frac{1}{5} \Rightarrow \{c_n\} = \frac{1}{20}, \frac{1}{10}, \frac{1}{5}, \dots$

both sequences have period 2 (after $n=3$ for $c_1 = \frac{1}{10}$ and $n=4$ for $c_1 = \frac{1}{20}$).

We'll denote a c_1 that creates a sequence with period T as c_1^T.

So $(\frac{1}{2})^m \cdot c_1^T$ also creates a sequence with period T, where m is any positive integer.

Now we just need to check that for each value of $T = 2, 3, 4, \dots$ there is some c_1 to create a sequence of that period.

By looking at examples, a pattern emerges:

$c_1 = \frac{1}{3} \Rightarrow T=1$ $\quad \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \dots$

$c_1 = \frac{1}{5} \Rightarrow T=2$ $\quad \frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{2}{5}, \frac{4}{5}, \dots$

$c_1 = \frac{1}{9} \Rightarrow T=3$ $\quad \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{8}{9}, \frac{2}{9}, \frac{4}{9}, \frac{8}{9}, \dots$

$c_1 = \frac{1}{17} \Rightarrow T=4$ $\quad \frac{1}{17}, \frac{2}{17}, \frac{4}{17}, \frac{8}{17}, \frac{16}{17}, \frac{2}{17}, \dots$

We show below that $c_1 = \frac{1}{2^k+1}$ gives a sequence with $T=k$ in general.

In part (a), we showed that the sequence value doubles as long as $c_n \le \frac{1}{2}$

So the general sequence is:

$$\frac{1}{2^k+1}, \frac{2}{2^k+1}, \frac{2^2}{2^k+1}, \dots, \frac{2^k}{2^k+1}, \left|1 - \left|1 - \frac{2 \cdot 2^k}{2^k+1}\right|\right|$$

$$= \left|1 - \left|\frac{2^k+1 - 2 \cdot 2^k}{2^k+1}\right|\right|$$

$$= \left|\frac{2^k+1 + 2^k+1 - 2 \cdot 2^k}{2^k+1}\right| = \left|\frac{2 \cdot 2^k + 2 - 2 \cdot 2}{2^k+1}\right.$$

$$= \frac{2}{2^k+1}$$

k+1 term is the same as 2nd term in sequence

which has period k.

For each value of $T = 2, 3, 4, \ldots$ there is at least one c_1 such that after some term c_N, the sequence becomes periodic. So for each value of $T = 2, 3, 4, \ldots$, there are infinitely many c_1 that give eventually periodic sequences of period T. For each T, some possible c_1^T are

$$c_1^T = \left(\frac{1}{2^m}\right) \cdot \frac{1}{2^T + 1}$$

with m an arbitrary integer greater than 0