Application Puzzlers SWIM Princeton 2009

Instructions

Here we present three mathematical puzzles. As part of your application to SWIM Princeton 2009 you are asked to submit your work on **two** of these three problems: Dividing a Pie, Playing 20 Questions, and Arithmetic Means.

These problems are meant to be quite puzzling, and may take you days, possibly spread out over a couple weeks, to solve. When you write up your work for submission, we are interested not just in the final answer, but also in seeing how you think. For example, you could show us how you worked on several special cases, and indicate how this helped you develop the intuition to tackle the more general case as stated in the problem. Then you could show us how you approached the general case, and if you reach the final answer, show us how you did so and how you proved that it was correct. Even if you can't reach the final answer, we are still interested in your submission, and you should write up your work as far as you were able to proceed.

As a guide to show you the sort of submission we are looking for, we also give a fourth sample problem, for which we provide a variety of possible answers. We hope that this will help you in the process of documenting your work.

Good luck on these problems—we hope you enjoy working on them, and we look forward to reading your submission.

Problem 1: Dividing a Pie

A woman is having some guests over, and has baked a pie for the occasion. She knows she will have either p or q guests, where $p, q \ge 1$ are integers. She would like to cut the pie into pieces (not necessarily all of the same size) before the guests arrive. What is the smallest number of pieces she can pre-cut the pie into so that all her guests will get an equal share of the pie, the whole pie gets eaten, and she doesn't have to make any further cuts, regardless of whether p or q guests show up? (Note: the hostess herself prefers ice cream and will not partake of the pie.) Give an answer in each of the following two cases:

Question (a) When p and q are relatively prime.

Question (b) When p and q have greatest common divisor d > 1.

Terminology: Two positive integers p and q are said to be relatively prime if the only positive integer that divides both p and q is 1. More generally, the greatest common divisor (gcd) of p and q is the largest positive integer d such that d divides p and d divides q; thus p and q are relatively prime if their gcd is 1.

Problem 2: Playing 20 Questions with a Liar

Larry the Liar is thinking of an integer between 1 and 2048 (1 and 2048 included). You are allowed to ask Larry yes or no questions to try to determine which integer he is thinking of. Larry may lie to one of your questions, but he is only allowed to lie at most one time. (For example, if Larry is thinking of the number 455, and you ask three times, "Are you thinking of the number 455?" then he has to respond "yes" at least twice.)

Question (a) How many yes or no questions do you have to ask in order to determine the integer between 1 and 2048 which Larry is thinking of? Write down the questions you would ask.

Question (b) Prove that the number of questions you gave in part (a) is minimal. (That is, explain why it is not possible to determine an integer between 1 and 2048 if you could only ask Larry one fewer number of questions than the number you give in part (a)).

Hint: First try to determine the minimal number of questions needed to determine an integer between 1 and 16 (when Larry is still allowed to lie up to one time).

Problem 3: Arithmetic Means

For which positive integers $n \ge 1$ can the *n* numbers 1, 2, 3, ... n be ordered in such a way that for any two numbers in your ordered sequence, their arithmetic mean does not stand between them? Prove your answer.

Hint: As a first step, give such an ordering for n = 8 and n = 16.

Terminology: The arithmetic mean of a and b is (a+b)/2.

Sample Problem

To aid you in preparing your solutions to two of the above problems, we also provide several sample solutions to the following problem:

You are given a sequence of real numbers $\{c_n\}_{n=1}^{\infty}$ satisfying the relation:

$$c_{n+1} = |1 - |1 - 2c_n||$$

for all $n \ge 1$, and with the initial condition $0 \le c_1 \le 1$.

Question (a) Prove that if (and only if) c_1 is rational, the sequence $\{c_n\}_{n=1}^{\infty}$ becomes periodic after some entry c_N , for some N.

Question (b) Find out how many values c_1 can have such that after some term c_N (the threshold N may be different for different c_1), the sequence becomes periodic with period T. Answer this for each value of T = 2, 3, 4, ...

Terminology: A sequence is said to be periodic of period T after c_N , if for all $n \ge N$, $c_n = c_{n+T}$. A sequence is said to be periodic after some c_N if is periodic of period T after c_N for some T.