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Dear Cathleen,

I have been looking at the question of L^∞ norms of eigenfunctions and symmetries along the lines that we discussed. Below I describe what I understand and also update and correct some points in the paper [I-S].

Let (X, g) be a compact Riemannian manifold of dimension n and let $\Delta = \Delta_g$ denote the Laplacian acting on functions on X. The first connection between the L^∞ norms and symmetries comes from multiplicities. For $\lambda \geq 0$ let $\mu(\lambda, X)$ be the multiplicity of the eigenvalue λ, ie

$\mu(\lambda, X) = \dim V_\lambda(X)$ where

$$V_\lambda(X) = \{\phi : \Delta\phi + \lambda\phi = 0\}. \tag{1}$$

Let $m(\lambda, X)$ be the maximum value of an eigenfunction ϕ_λ with $\|\phi_\lambda\|_2 = 1$,

$$m(\lambda, X) := \max_{x \in X} \{|\phi(x)| : \phi \in V_\lambda(X), \|\phi\|_2 = 1\}. \tag{2}$$

Then

$$\mu(\lambda, X) \leq \mathrm{Vol}(X)\, m^2(\lambda, X). \tag{3}$$

PROOF: $\displaystyle\int_X \sum_{j=1}^{\mu} |\phi_j(x)|^2 \, dV(x) = \mu$

where $\phi_1, \phi_2, \ldots, \phi_\mu$ are an orthonormal basis for V_λ. Hence there is an $x_0 \in X$ s.t.

$$\sum_{j=1}^{\mu} |\phi_j(x_0)|^2 \geq \mu / \mathrm{Vol}(X).$$

Let $\psi \in V_\lambda$ be given by

$$\psi(x) = \sum_{j=1}^{\mu} \overline{\phi_j(x_0)}\, \phi_j(x).$$

Then $\displaystyle\|\psi\|_2^2 = \sum_{j=1}^{\mu} |\phi_j(x_0)|^2$ and

$$\psi(x_0) = \sum_{j=1}^{\mu} |\phi_j(x_0)|^2 .$$

Hence

$$\frac{\|\psi\|_\infty}{\|\psi\|_2} \geq \sqrt{\sum_{j=1}^{\mu} |\phi_j(x_0)|^2} \geq \sqrt{\frac{\mu}{Vol(X)}} .$$

Now let A be a commutative ring of operators acting on functions on X which also commutes with Δ. Denote by $m_A(\lambda, X)$ the corresponding L^∞-norm of a joint eigenfunction of A;

$$m_A(\lambda, X) = \max_{x \in X} \Big\{ |\phi(x)| : \Delta\phi + \lambda\phi = 0, \ \phi \text{ an eigenfn of } A, \ \|\phi\|_2 = 1 \Big\}. \qquad (4)$$

Clearly $m_A(\lambda, X) \leq m(\lambda, X)$ and if $\mu(\lambda, X) = 1$ then these are equal while if $\mu(\lambda, X) > 1$ we expect that $m_A(\lambda, X)$

might be smaller. The general upper bound for $m(\lambda, X)$ is (see [Do] for a recent proof)

$$m(\lambda, X) \ll \lambda^{(n-1)/4} \qquad (5)$$

the implied constant depending on X.

(5) and (3) are sharp for $X = S^n$ the round n-sphere. The eigenfunctions on S^n achieving (5) are the zonal harmonics, ie the spherical harmonics which are invariant under rotations of S^n about a given axis. The problem of establishing a 'subconvex' estimate (by which I mean an improvement of (5) in the exponent) if X satisfies some simple geometric condition such as being strictly negatively curved remains (as you well know) elusive.

If $X = X_1 \times X_2 \times \cdots \times X_r$ then $\Delta = \Delta_1 + \Delta_2 + \cdots + \Delta_r$ where Δ_j is the Laplacian on X_j. The Δ_j's commute with each other and with Δ. Let A be the ring of polynomials in $\Delta_1, \ldots, \Delta_r$. An eigenfunction of A is of the form $\phi_1 \cdot \phi_2 \cdots \phi_r$ with $\phi_j(x) = \phi_j(x_j)$ an eigenfunction of Δ_j. Hence for X we have

$$m_A(\lambda, X) \ll \lambda^{\frac{1}{4}\sum_{j=1}^{r}(\dim X_j - 1)} = \lambda^{(\dim X - r)/4}. \qquad (6)$$

Thus in this case each of the differential operators Δ_j gains a 1 in the exponent. From this product example one might expect that if $\Delta = P_1, P_2, \ldots, P_r$ are a commuting family of differential (or pseudo differential) operators satisfying a nondegeneracy condition as in [CdV] and $A = \mathrm{Poly}[P_1, \ldots, P_r]$ that $m_A(\lambda, X)$ would satisfy (6). However the example of $X = S^2$ and P being the differential operator generating rotations about the N-S axis and $A = \mathrm{Poly}[\Delta, P]$ shows that the improved bound (6) need not hold. Even generically the gain (6) is not achieved as is exemplified in the completely integrable case $(r = n)$ studied in [To-Ze]. In this case (6) asserts that the joint eigenfunctions are uniformly bounded. However in [To-Ze] it is shown that unless $X = \mathbb{R}^n/L$ is a flat torus, there is a sequence of A-joint eigenfunctions ϕ_j for which $\|\phi_{\lambda_j}\| \gg \lambda_j^{1/8}$. There should be some conditions under which (6) is true. At least for X a locally symmetric space, it is valid as we now discuss.

Recall that a (globally) symmetric S is a Riemannian space for which inversion in geodesics about any $x \in S$ is a global isometry. A simply connected such S decomposes as $M_0 \times M_+ \times M_-$ where M_0 is Euclidean, M_+ is of compact type (positive curvature) and M_- is of noncompact type (negatively curved) see [He]. A locally symmetric X is of the form $X = \Gamma \backslash S$ with Γ

a discrete subgroup of the isometric motions $G(S)$, of S. We consider the compact and noncompact types separately [the Euclidian case is straight forward; $X = \mathbb{R}^n / L$ with L a lattice in $\mathbb{R}^n$. The eigenfunctions on X are joint eigenfunctions of $A =$ Poly$\left[i\frac{\partial}{\partial x_1}, \ldots, i\frac{\partial}{\partial x_n} \right]$ and hence are of the form $e^{2\pi i \langle x, \xi \rangle}$ where $\xi \in L^*$ the lattice dual to L. Thus they are uniformly bounded, the rank of $\mathbb{R}^n$ is n and so (6) is valid].

The ring $\mathcal{D}(S)$ of $G(S)$ invariant differential operators on functions on S is commutative and finitely generated [Se];

and it contains Δ. Let $D_1, D_2, \ldots, D_r$ be a minimal set of generators of $\mathcal{D}(S)$. Here r is the rank of S which geometrically is the dimension of the largest totally geodesic Euclidian submanifold of S [He]. The action of $\mathcal{D}(S)$ descends to functions on $X = \Gamma \backslash S$. With this notation we have

$$m_{\mathcal{D}(S)}(\lambda, X) \ll \lambda^{(\dim X - \operatorname{rank}(X))/4} \qquad (7)$$

and if S is of the compact type then (7) is sharp.

The following proof of (7) is closely related to your most recent proof of (5) when $X = \Gamma \backslash \mathbb{H}^2$, $\mathbb{H}^2$ being the hyperbolic plane.

If ϕ is an eigenfunction of $\mathcal{D}(S)$, let

$\beta = (\beta_1, \beta_2, \ldots \beta_r)$ where

$$D_j \phi = \beta_j \phi \qquad \text{—————} (8).$$

β is the spectral parameter of ϕ and we let $\lambda = \lambda(\beta)$ be the corresponding Laplace eigenvalue. Denote by $V_\beta(S)$ and $V_\beta(X)$ the spaces

$$\left.\begin{aligned} &V_\beta(S) = \{\phi \text{ on } S : D_j\phi = \beta_j \phi\} \\ \text{and}\quad &V_\beta(X) = \{\phi \text{ on } X : D_j\phi = \beta_j\phi\}. \end{aligned}\right\} (9)$$

For $x_0 \in S$, let $\omega_\beta(x; x_0)$ be the (unique) spherical function with spectral parameter β ([Se]). That is $\omega_\beta(x_0; x_0) = 1$ and $\omega_\beta(kx; x_0) = \omega_\beta(x; x_0)$ for $k \in K_{x_0} = \{g \in G(S) : g x_0 = x_0\}$. For $x \in S$ and $\delta > 0$ we let $B_\delta(x)$ ~~be~~ the geodesic ball about x of radius δ. For any $x_0 \in S$ and $\delta > 0$ (small but fixed) we have the inequality

For any $\phi \in V_\beta(S)$

$$|\phi(x_0)|^2 \leq \nu(\delta, \beta) \int_{B_\delta(x_0)} |\phi(x)|^2 dV(x) \quad \text{——} (9)$$

where
$$\nu(\delta, \beta) = \left(\int_{B_\delta(x_0)} |\omega_\beta(x; x_0)|^2 dV(x) \right)^{-1}.$$

Moreover we have equality in (9) iff $\phi(x) = t\,\omega_\beta(x; x_0)$ with $t \in \mathbb{C}$.

The proof of (9) is straight forward. It uses the uniqueness of β-spherical functions. We have that for $k \in K_{x_0}$

$$\int_{B_\delta(x_0)} \phi(x) \overline{\omega_\beta(x; x_0)} dV(x) = \int_{B_\delta(x_0)} \phi(kx) \overline{\omega_\beta(kx; x_0)} dV(x)$$

$$= \int_{B_\delta(x_0)} \phi(kx) \overline{\omega_\beta(x; x_0)} dV(x)$$

Integrating both sides w.r.t. normalized Haar measure on K_{x_0} gives

$$\int_{B_\delta(x_0)} \phi(x)\,\overline{\omega_\beta(x;x_0)}\,dV(x) = \int_{B_\delta(x_0)} \left(\int_{K_{x_0}} \phi(kx)\,dk \right) \overline{\omega_\beta(x;x_0)}\,dV(x)$$

$$= \int_{B_\delta(x_0)} \phi(x_0)\,|\omega_\beta(x;x_0)|^2\,dV(x)$$

$$= \phi(x_0)\,V(\delta,\beta)$$

Hence by Cauchy Schwartz

$$|\phi(x_0)|^2 \le \frac{1}{V^2(\delta,\beta)} \int_{B_\delta(x_0)} |\phi(x)|^2\,dV(x) \int_{B_\delta(x_0)} |\omega_\beta(x;x_0)|^2\,dV(x)$$

$$= \frac{1}{V(\delta,\beta)} \int_{B_\delta(x_0)} |\phi(x)|^2\,dV(x).$$

The behavior $V(\delta,\beta)$ for δ fixed and $\beta \to \infty$ is a question about asymptotics of spherical functions at a fixed value as the order goes to infinity. It can be studied using stationary phase methods as is done in [D-K-V] when S is of noncompact type (when S is of compact type one can also proceed using an analysis of multiplicities as outlined below). Using their results specifically [Va pp 232] one sees that

$$\nu(\delta, \beta) \gg \lambda(\beta)^{-(\dim S - \operatorname{rank} S)/2} \qquad (10)$$

This combined with (9) yields for $\phi \in V_\beta(S)$

$$|\phi(x_0)|^2 \ll \lambda(\beta)^{(\dim S - \operatorname{rank} S)/2} \int_{B_\delta(x_0)} |\phi(x)|^2 dv(x) \qquad (11)$$

(the implied constant depending only on S and δ).

Turning to $\underline{X} = \Gamma \backslash S$, let $\phi \in V_\beta(\underline{X})$. For $y \in \underline{X}$ let $x_0 \in S$ with $\pi(x_0) = y$ ($\pi : S \to \underline{X}$ the projection). There is $\delta > 0$ (depending only on $\underline{X}$ since the latter is compact) such that $\gamma(B_\delta(x_0)) \cap B_\delta(x_0)$ is empty for $\gamma \in \Gamma$, $\gamma \neq 1$. Hence

$$|\phi(y)|^2 = |\phi(x_0)|^2 \ll \lambda^{(\dim S - \operatorname{rank} S)/2} \int_{B_\delta(x_0)} |\phi(x)|^2 \, dV(x)$$

$$\leq \lambda^{(\dim S - \operatorname{rank} S)/2} \cdot \int_{\underline{X}} |\phi(y)|^2 \, dV(y).$$

This proves (7).

In the case that $\underline{X} = S$ is a globally symmetric space of compact type not only is (7) sharp (that is for suitable eigenfunctions the L^∞ norm is as large up to a fixed constant, as the right hand side) but so are the multiplicity bounds. More precisely, let $\mu(\beta, S) = \dim V_\beta(S)$ and let $m(\beta, S) = \max_{x \in S} \{ |\phi(x)| \,;\, \phi \in V_\beta(S) \,;\, \|\phi\|_2 = 1 \}$. As in (3) we have

$$\mu(\beta, S) \leq \mathrm{Vol}(S)\, m^2(\beta, S). \qquad (12)$$

In this case $G(S)$ acts transitively on S and so if $\{\phi_j\}_{j=1}^{\mu(\beta,S)}$ is an o.n.b. of $V_\beta(S)$ and

$$K_\beta(x, y) = \sum_{j=1}^{\mu(\beta,S)} \phi_j(x)\, \overline{\phi_j(y)} \qquad (13)$$

then $K_\beta(x, x)$ is independent of x. Integrating w.r.t. x shows that

$$K_\beta(x, x) = \frac{\mu(\beta, S)}{\mathrm{Vol}(S)}. \qquad (14)$$

Hence from (13) it follows that

$$m(\beta, S)^2 \leq \frac{\mu(\beta, S)}{\mathrm{Vol}(S)}. \qquad \text{——— (15)}$$

(12) and (15) imply that in fact

$$m(\beta, S)^2 = \frac{\mu(\beta, S)}{\mathrm{Vol}(S)}. \qquad \text{——— (16)}$$

Thus the L^∞ question for $\mathcal{D}(S)$ is the same as the question of multiplicities for symmetric spaces of compact type. These multiplicities are equal to the degrees of the irreducible K-spherical representations of the compact group $G(S)$ and these in turn can be computed using Weyl's character formula [We]. Using this explicit formula one can in this way give another proof of (7) for S of compact type. The story for these is therefore much the same as the special case $S = S^n$.

(7) should be considered as the convexity bound for X. An interesting problem is to establish a subconvex bound when ϕ is an eigenfunction of further operators. If Γ is a 'congruence group' [Sa1], [Sa2] then there is are full commutative Hecke algebras $H(\Gamma)$ which contain $\mathcal{D}(S)$ and which act as normal operators on $L^2(X_\Gamma)$. In many situations the spaces of $H(\Gamma)$ joint eigenfunctions are one dimensional so that multiplicity is not an issue as far as $m_{H(\Gamma)}(\lambda, X_\Gamma)$ goes. A subconvex estimate in this context reads as follows: Fix X_Γ as above, there is $\delta = \delta(X_\Gamma) > 0$ s.t.

$$m_{H(\Gamma)}(\lambda, X_\Gamma) \ll \lambda^{\frac{\dim X - \operatorname{rank} X}{4} - \delta} \qquad \text{—— (17).}$$

The technique in [I-S] which establishes (17) for $S = \mathbb{H}^2$ and Γ an arithmetic lattice coming from a quaternion algebra is quite general and should be investigated in this setting (see [Ko], [Vdk]).

I turn to the question of lower bounds for $m_{H(\Gamma)}(\lambda, X_\Gamma)$. In [I-S] it is shown that for $\Gamma \backslash \mathbb{H}^2$, Γ coming from a quaternion algebra, the eigenfunctions are not uniformly bounded as $\lambda \to \infty$ [precisely that $m_{H(\Gamma)}(\lambda, X_\Gamma) = \Omega(\sqrt{\log\log \lambda})$ and in [By] this was improved in special cases to $\Omega(\sqrt{\log \lambda})$]. The proof exploits special points $z \in X_\Gamma$ at which the Hecke correspondences have a large number of fixed points. One can take better advantage of this feature and give sharper lower bounds by optimizing quadratic forms associated to weights formed ~~out~~ of the Hecke eigenvalues. This was observed recently by Soundararajan (2004 lecture at the Newton Institute) in another context — that of exhibiting large values of L-functions on the critical line.

Let X_Γ be as above and assume that the Hecke algebra is generated by operators T_m, $m \in I$ (typically the m's are integers). The joint eigenfunctions satisfy

$$\left.\begin{aligned} \Delta\phi + \lambda\phi &= 0 \\ T_m \phi &= \lambda_\phi(m)\phi \,. \end{aligned}\right\} \qquad (18)$$

Let $\{\phi_j\}$ be an o.n.b (of joint eigen-functions) of $L^2(X_\Gamma)$.

Using the pretrace formula and trace formula one can establish the following

Weyl type laws:

There is $\delta = \delta(X_\Gamma) > 0$ s.t. if $M \leq \lambda^\delta$ and $a_m \in \mathbb{C}$ with $a_m = O_\varepsilon(|m|^\varepsilon)$ for $|m| \leq M$, then

$$\sum_{\lambda_{\phi_j} \leq \lambda} \Big| \sum_{|m| \leq M} a_m \lambda_{\phi_j}(m) \Big|^2 \sim B(a)\, \lambda^{n/2} \qquad (19)$$

and for $z \in X_\Gamma$ and $a_m \geq 0$

$$\sum_{\lambda_{\phi_j} \leq \lambda} \Big| \sum_{|m| \leq M} a_m \lambda_{\phi_j}(m) \Big|^2 |\phi_j(z)|^2 \geq B_z(a)\, \lambda^{n/2} \qquad (20)$$

where $B(a)$ and $B_z(a)$ are explicit quadratic forms in a_m, $|m| \le M$. $B(a)$ is essentially diagonal while

$$B_z(a) = \sum_{m,n}{}' D_z(m,n)\, a_m \overline{a_n} \qquad \text{——(21)}$$

with $D_z(m,n) \ge 0$ being essentially the number of elements of the mn-th correspondence which fix z. (19) and (20) are weighted sums over the spectrum with weights

$$W_{\phi_j} = \Big| \sum_{|m| \le M}{}' a_m \lambda_{\phi_j}(m) \Big|^2 . \qquad \text{——(22)}$$

It follows that if

$$\gamma_M = \max_{\{a(m)\}_{m \le M}} \frac{B_z(a)}{B(a)} \qquad \text{——(23)}$$

then

$$\max_{\lambda_j \le \lambda} |\phi_j(z)|^2 \ge \gamma_M \, ; \qquad \text{——(24)}$$

and hence that

$$m_{H(\Gamma)}(\lambda, X_\Gamma) = \Omega\left(\gamma_{M(\lambda)}\right). \qquad (25)$$

(23) is an explicit eigenvalue problem for an $M \times M$ matrix. The largest eigenvalue can be estimated and for the purpose of lower bounds a good test vector a_M suffices.

We examine this in two cases, the modular surface $X(1) = SL(2,\mathbb{Z})\backslash\mathbb{H}^2$ (see [Sa1]) and the Picard hyperbolic 3-manifold $X_{PIC} = SL(2, \mathbb{Z}[\sqrt{-1}])\backslash\mathbb{H}^3$. These are noncompact and as far as the behavior of the eigenfunctions in the cusp extra care is needed, as is described in (46) below. If $z = \sqrt{-1} \in X(1)$ (and a similar analysis applies to any point z satisfying $az^2+bz+c=0$, $a,b,c \in \mathbb{Z}$)

then (say the a's are real)

$$B(a) = \sum_{t,u,v \geq 1} \sigma_1(t)\, a_{tu^2}\, a_{tv^2} \qquad (26)$$

where for $s \in \mathbb{C}$

$$\sigma_s(t) = \sum_{d \mid t} d^s \qquad (27).$$

On the other hand

$$B_z(a) = \sum_{d,m_1,m_2 \geq 1} d\, a_{dm_1}\, a_{dm_2}\, R_z(m_1 m_2) \qquad (28)$$

where

$$R_z(n) = \#\left\{ \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{array}{l} \gamma z = z \\ ad-bc = n, \\ a,b,c,d \in \mathbb{Z} \end{array} \right\}. \qquad (29)$$

Hence for $z = \sqrt{-1}$

$$R_{\sqrt{-1}}(n) = \#\{(x,y) : x^2+y^2=n,\ x,y \in \mathbb{Z}\}. \quad (30)$$

and

$$B_{\sqrt{-1}}(a) = \sum_{t \geq 1} c(t) \left| \sum_{n} a_{tn} R_{\sqrt{-1}}(n) \right|^2 \quad \text{——(31)}$$

where

$$c(t) = \sum_{df=t} d\, \chi_4(f) \mu(f). \quad \text{——(32)}$$

$\chi_4(f)$ is 1 if $f \equiv 1(4)$, -1 if $f \equiv -1(4)$ and is 0 if f is even, and μ is the Mobius function.

Changing variable

$$b(m) = \frac{a(m)}{m}, \quad 1 \leq m \leq M$$

yields

$$B(b) = \sum_{t} \frac{\sigma_{-1}(t)}{t} \left| \sum_{n} \frac{b(tn^2)}{n^2} \right|^2 \quad \text{——(33)}$$

and

$$B_{\sqrt{-1}}(b) = \sum_{t} \frac{c_1(t)}{t} \left| \sum_{n} \frac{b(tn) R_{\sqrt{-1}}(n)}{n} \right|^2 \quad \text{——(34)}$$

where

$$c_1(t) = \sum_{f \mid t} \frac{\chi_4(f) \mu(f)}{f} \quad \text{——(35)}$$

(note $c_1(t)$, $\sigma_{-1}(t) \geq 0$).

- Fix $k \geqslant 1$ and choose for example

$$b(n) = \tau_k(n) := \sum_{\substack{x_1 x_2 \cdots x_k = n \\ x_j \geqslant 1}} 1$$

($b(n) = R(n)^k$ also works). ——(36).

This choice is not optimal but a standard calculation shows that for this b, as $M \to \infty$

- $$\frac{B_{\sqrt{-1}}(b)}{B(b)} \gg_k (\log M)^k$$

——(37).

Since $M = S \log \lambda$ it follows that for any fixed k there is a subsequence $\lambda_j \to \infty$ such that

$$|\Phi_{\lambda_j}(\sqrt{-1})| \gg (\log \lambda_j)^{k/2}.$$

——(38).

- The above argument when applied to a compact $X_\Gamma = \Gamma \backslash \mathbb{H}^2$ coming from a quaternion algebra shows that

$$m_{H(\Gamma)}(\lambda, X_\Gamma) = \Omega_k((\log\lambda)^k) \qquad (39)$$

for any k.

The basic conjecture [I-S] for these compact surfaces is that $m_{H(\Gamma)}(\lambda, X_\Gamma) = O_\varepsilon(\lambda^\varepsilon)$ for any $\varepsilon > 0$. Since we also believe that for these X's $\mu(\lambda, X) = O_\varepsilon(\lambda^\varepsilon)$, the stronger conjecture is that $m(\lambda, X_\Gamma) = O_\varepsilon(\lambda^\varepsilon)$. It is implicit in [I-S] that this should also hold for the noncompact quotient $X(1)$. However we had overlooked a feature concerning the transition region for the asymptotics of the K-Bessel function which shows that a cusp form $\phi_{\lambda_j}(z)$ will get quite large for $y = \mathrm{Im}(z)$ near t_j, where $\lambda_j = t_j^2 + \frac{1}{4}$. More precisely such a cusp form has a Fourier expansion

$$\phi_j(z) = \sum_{n \neq 0} \rho_j(n)\, y^{1/2} K_{it_j}(2\pi |n| y)\, e(nx) \qquad (40)$$

where if $\|\phi_j\|_2 = 1$ we have [H-L], [I]

$$t_j^{-\varepsilon} \ll |\rho_j(1)|^2 e^{+\pi t_j} \ll t_j^{\varepsilon} \qquad (41)$$

for any $\varepsilon > 0$.

Also

$$\rho_j(n) = \rho_j(1)\, \lambda_j(n) / \sqrt{n} \qquad (42)$$

The asymptotics of $K_{it}(y)$ in the transition range $|t - y| < t^{1/3}$ gives

$$K_{it}(y) \approx t^{-1/3} e^{-\frac{\pi}{2} t} \sin(t \text{ phase}(y)) \qquad (43)$$

If $2\pi y$ is chosen near t_j so that the sin (t phasey) is not small (which can be done) we see that

$$\max_{0 \le x \le 1} |\phi_j(x+iy)|^2 \geq \int_0^1 |\phi(x+iy)|^2 dx$$

$$\geq |\rho_j(1)|^2 y |K_{it_j}(2\pi y)|^2$$

$$\gg t_j^{1-\frac{2}{3}-\varepsilon} = t_j^{\frac{1}{3}-\varepsilon}. \qquad (44)$$

So before $\phi_j(z)$ dies in the cusp (which it does once $2\pi y \geq t_j + t_j^{1/3}$) it climbs to a value of about $t_j^{1/6}$ for y near t_j. In particular

$$\|\phi_j\|_\infty \gg_\varepsilon \lambda_j^{\frac{1}{12}-\varepsilon} \qquad (45)$$

Thus in this noncompact surface case

the L^∞ $O_\varepsilon(\lambda^\varepsilon)$ conjecture is formally false. We modify the formulation as follows:

X_Γ a noncompact surface as above then for any $\varepsilon > 0$ and $K \subset X_\Gamma$ a fixed compact set with nonempty interior, we have

$$\max_{z \in K} |\phi_\lambda(z)| \ll \lambda^\varepsilon \sqrt{\int_K |\phi_\lambda(z)|^2 \, dA(z)} \,. \tag{46}$$

In this form the conjecture also applies to the continuous spectrum, that is the unitary Eisenstein series and for $X(1)$ the latter implies the Lindeloff Hypothesis for the Riemann Zeta function.

The argument in Lemma A.1 of [I-S] is flawed, in part because it ignores the above transition region for the asymptotics of the K-Bessel function. I have enclosed a corrigendum by Iwaniec and myself concerning this Lemma.

I turn to the analysis of (19) and (20) for the case of $X_{PIC} = \Gamma \backslash \mathbb{H}^3$ where $\Gamma = SL(2, \mathcal{O}) \leq SL(2, \mathbb{C})$ and $\mathcal{O} = \mathbb{Z}[\sqrt{-1}]$ and $\mathbb{H}^3 = SL(2,\mathbb{C})/SU(2)$ is hyperbolic 3-space. The Hecke operators T_m are indexed by integers $m \in \mathcal{O}$.

The quadratic forms corresponding to (19) and (20) are similar to the $X(1)$ case.

$a(m)$ is supported in $N(m) = m\bar{m} \le M$,

$$B(a) = \sum_{t,u,v \in \mathcal{O}} \sigma_1(t)\, a_{tu^2}\, a_{tv^2} \tag{47}$$

where

$$\sigma_s(t) = \sum_{d \mid t} N(d)^s .$$

$$B_z(a) = \sum_{d, m_1, m_2} N(d)\, a_{dm_1}\, a_{dm_2}\, R_z(m_1 m_2) \tag{48}$$

Where for $n \in \mathcal{O}$

$$R_z(n) = \#\left\{ \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix},\ \begin{array}{l} ad - bc = n \\ a, b, c, d \in \mathcal{O} \\ \gamma z = z \end{array} \right\}. \tag{49}$$

Consider the point $z_I = I \cdot SU(2)$ in $\mathbb{H}^3 / SU(2)$. $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $ad - bc = m$

Satisfies $\gamma z_I = z_I$ iff $\gamma \gamma^* = N(m)^{1/2} I$.

In particular if $m \in \mathbb{Z}$, $m \ge 1$ then

$$\gamma = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \quad \text{with } |a|^2 + |b|^2 = m, \ a, b \in \mathcal{O}$$

satisfies $\gamma z_I = z_I$. Hence for $m \in \mathbb{Z}$

$$R_{z_I}(m) = r_4(m) = \#\left\{(x_1, x_2, x_3, x_4) : \begin{array}{l} x_j \in \mathbb{Z} \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 = m \end{array}\right\} \tag{50}$$

For m large $r_4(m)$ is of size m and as a consequence the form B_{z_I} can be made large. For example ~~take~~

$$a(m) = \begin{cases} N(m)^{-1/2} & \text{if } m \in \mathbb{Z}, \ N(m) \leq M \\ 0 & \text{otherwise} \end{cases} \tag{51}$$

One finds that

$$\left.\begin{array}{l} B(a) \ll_\varepsilon M^{1+\varepsilon} \\ \text{while} \quad B_{z_I}(a) \geq M^2 \end{array}\right\} \tag{52}$$

Since $M = \lambda^{\delta}$ for a fixed $\delta > 0$ it follows that

$$|\phi_j(z_I)| = \Omega(\lambda_j^{\delta_1}) \qquad (53)$$

with $\delta_1 > 0$ as $\lambda_j \to \infty$.

So in this case of the Picard manifold the L^{∞}-norms (on a fixed compact set) of cusp forms grow along a subsequence faster than a power of λ! This phenomenon was already noted in [R-S] for hyperbolic manifolds associated to unit groups of integral quadratic forms in dimensions 3 or more. The proof there is based on theta lifts and it also demonstrated that large values (a power of λ) are assumed at special points in X (like the point z_I in X_{pic}). In fact the subsequence of ϕ_j's that are singularly large are identified as ~~theta~~ lifts from hyperbolic surfaces. In the present analysis one can also detect which eigenfunctions ϕ_j are responsible for being large at say z_I. The weights W_{ϕ_j} in (22) and (51) are

$$W_{\phi_j} = \Big| \sum_{\substack{N(m) \leq M \\ m \in \mathbb{Z}}} \frac{\lambda_{\phi_j}(m)}{N(m)^{1/2}} \Big|^2 \qquad \text{— (25)}$$

One can identify the ϕ_j's for which these weights are large. They are precisely the ϕ_j's on X_{pic} which are base change lifts from the modular surface $X(1)$! Indeed for any ϕ as above the L-function

$$L(s,\phi,\text{Asai}) = \sum_{m=1}^{\infty} \frac{\lambda_\phi(m)}{m^{s+\frac{1}{2}}}$$

was studied in [As I] (note the sum is over rational integers). It has a pole at $s=1$ iff $\lambda_\phi(\overline{m}) = \lambda_\phi(m)$ for all $m \in \mathcal{O}$, that is precisely when ϕ is a base-change lift from $\mathbb{Q}$ (in this case it means from $X(1)$). These eigenfunctions on X_{pic} constitute about λ of all the $\lambda^{3/2}$ eigenfunctions with $\lambda_j \leq \lambda$.

The analysis of lower bounds for L^∞ norms on $X(1)$ and X_{pic}

applies in general and it captures quantitatively the role played by the sizes of the sets $K_Z \cap \{$elements of the m-th correspondence$\}$ in the study of these norms. I hope that eventually one will be able to understand precisely for which ϕ's in these higher dimensional and rank cases, one can expect the strong $O_\varepsilon(\lambda^\varepsilon)$ local L^∞-bound. To get to this point one needs to examine many more examples. It is already clear that the situation will be quite a bit more complicated than the analogous questions about the spectra of such

X_p's — i.e. the generalized Ramanujan Conjectures (see [Sa 2]).

With warmest regards
Peter

References:

[We] H. Weyl, Math. Zeit 23 (1925), 24 (1926).
[As] T. Asai Math. Ann 226 (1977) 81-94
[Do] H. Donnelley J.F.A. 187 (201) 247-261
[B] B.A. Bykouski preprint (1995)
[CdV] Y. Colin de Verdière D.M.J. 46 (1979) 169-182
[D-K-V] H. Duistermaat - J. Kolk - V. Varadarajan Compositio Math 49 (1983) 309-398.
[He] S. Helgason "Differential geometry and symmetric spaces" A.P. (1962)
[I] H. Iwaniec Acta Arith 56 (1990), 65-82
[I-S] H. Iwaniec - P. Sarnak Annals of Math 141 (1995) 301-320
[Ko] S. Koyama D.M.J. 77 (1995) 799-817.
[H-L] J. Hoffstein - P. Lockhardt Annals of Math 140 (1994) 161-181
[R-S] Z. Rudnick - P. Sarnak C.M.P. 161 (1994) 195-213
[Sa 1] J. P. Sarnak BAMS 40 2003, (441-478)
[Sa 2] P. Sarnak Fields Inst Lectures 2003 "Generalized Ramanujan Conj"
[Se] A. Selberg J. Indian Math Soc B 20 (1956) 47-87.
[To-Ze] J. Toth - S. Zelditch Contemp. Math 320 (2004) AMS 435-438
[VdK] J. VanderKam IMRN (1997) 329-347.
[Var] V.S. Varadarajan pp. 167-242 in "Algebraic and analytic methods in representation [illegible]" A.P. [illegible] Math

CORRIGENDUM TO "L^∞ NORMS OF EIGENFUNCTIONS OF ARITHMETIC SURFACES"

The bound of Lemma A.1 in Appendix as well as its proof are not correct in the whole range of the upper half plane. They should be replaced by

LEMMA A.1'. Let $\phi(z)$ be a cusp form for the modular group with eigenvalue $\lambda = \frac{1}{4}+r^2$, $r>0$, which is also an eigenfunction of all the Hecke operators. Then for any $z = x+iy$ with $y>0$ we have

$$\text{(A.1')} \qquad \phi(z) \ll r^{\varepsilon}\left(y^{-\frac{1}{2}} r^{\frac{1}{2}} + r^{\frac{1}{6}}\right) \|\phi\|_2$$

for any $\varepsilon > 0$, the implied constant depending only on ε.

Remarks. In the original version the factor r^{ε} and the term $r^{\frac{1}{6}}$ are not present, nevertheless the above alterations have no effect on the use of this lemma in the paper. The additional term $r^{\frac{1}{6}+\varepsilon}$ is in fact necessary. Indeed, if $2\pi y = r + O(1)$ then the first term in the Fourier development (A.1) dominates giving

$$\phi(z) = \rho(1)\, y^{\frac{1}{2}} K_{ir}(2\pi y) \cos(2\pi x) + O(e^{-r})$$

for ϕ even (for ϕ odd this holds with $\sin(2\pi x)$ in place of $\cos(2\pi x)$). Using the asymptotic for K-Bessel function in the transitional

region one can find $2\pi y = r + O(1)$ such that $K_{ir}(2\pi y) \asymp r^{-\frac{1}{3}} e^{-\frac{\pi}{2} r}$.
Hence for $z = \frac{1}{4} + iy$ we have

$$\phi(z) \asymp \rho(1)\, r^{\frac{1}{6}} e^{-\frac{\pi}{2} r}$$

Normalize $\|\phi\|_2 = 1$. Using the estimates

(A.2′) $$r^{-\varepsilon} \ll |\rho(1)| e^{-\frac{\pi}{2} r} \ll r^{\varepsilon}$$

we conclude

(A.3′) $$r^{\frac{1}{6}-\varepsilon} \ll |\phi(z)| \ll r^{\frac{1}{6}+\varepsilon}.$$

These estimates reveal that the bound (08) which is conjectured for co-compact arithmetic groups does not hold for the modular group. Retrospectively this fact is not surprising; the large values of $\phi(z)$ as above appear for a purely analytic reason.

Proof of Lemma A.1′. We use the following bounds (cf. [Er])

$$K_{ir}(y) \ll |y^2 - r^2|^{-\frac{1}{4}} e^{-\frac{\pi}{2} r}$$
$$K_{ir}(y) \ll r^{-\frac{1}{3}} e^{-\frac{\pi}{2} r}$$
$$K_{ir}(y) \ll y^{-\frac{1}{2}} e^{-y}$$

which are uniform in r and y. Hence one derives

$$\sum_{n=1}^{\infty} |K_{ir}(2\pi n y)|^2 \ll (y^{-1} + r^{-\frac{2}{3}}) e^{-\pi r}$$

The Fourier series (A.1) can be reduced to

$$\phi(z) = y^{\frac{1}{2}} \sum_{|n| \leq N} \rho(n) K_{ir}(2\pi |n| y) e(nx) + O(e^{-r})$$

for $N = r y^{-1}$ by estimating the tail trivially. Hence applying Cauchy's inequality

$$|\phi(z)|^2 \ll (1 + y r^{-\frac{2}{3}}) e^{-\pi r} \sum_{|n| \leq N} |\rho(n)|^2$$

The last sum is bounded by $|\rho(1)|^2 N r^{\varepsilon}$ (see (1.43)) and $\rho(1)$ satisfies (A.2'). Combining these estimates we arrive at (A.1').

Nov 2007

purity

Dear Cathleen ,Elon,Erez, Akshay ,Lior and Djorde

Here is a further paragraph to add to my letter to Cathleen . It is based on talking to all of you and thinking a bit further in general terms about this problem

For X a compact n-dimensional Riemannian manifold define the exponents of L-infinity growth of eigenfunctions E(X) to be the set of accumulation points of the numbers

(log ||phi_lambda||) /log lambda

where || || denotes L_infinity norm and phi_lambda is an eigenfunction of L^2 norm one and lambda its eigenvalue.

The convexity bound asserts that E(X) is contained in [0,(n-1)/4]. For example if X is S^n then E(X) is the entire interval.Also E satisfies some obvious relations of containments for products of such manifolds.

Now restrict to X to be locally symmetric ,irreducible and arithmetic and phi_lambda's which are eigenfunctions of the full ring of invariant differential operators as well as of the Hecke ring.Then the convexity bound asserts that E(X) is contained in the interval [0,(n-r)/4] where r is the rank.Subconvexity is statement that E is contained in this interval but open at the right hand side.

The following conjecture is consistent with the few examples that we know coming from theta lifts ,base changes and mollification weights and it is the analogue of what I like to call the purity (which itself is the analogue of the arithmetic geometric purity of Deligne) consequences of Arthur's conjectures concerning the spectra of these manifolds.

Conjecture(purity): For X as above then E(X) is contained in Z/4 (here Z is the integers)

Combined with convexity this would mean that E(X) is contained in Z/4 intersect [0,(n-r)/4] and combined with subconvexity it would imply that E(X) is contained in [0,(n-r)/4) intersect Z/4.

In this language we are a bit like the situation of Automorphic Duals in that the techniques we have may lead to (hopefully in general) nontrivial upper bounds as well as in examples where E(X) is not {0} ,lower bounds on E(X). I dont see right now how to analyze restrictions of eigenfunctions to say closed totally geodesic submanifolds Y of X ,but if one could one might be able to infer useful relations between E(X) and E(Y) as one does in the theory of automorphic duals . The latter is much simpler since it is a purely L^2 theory.

As with the general Ramanujan Conjectures one would like to have an explicit description of the set E(X) for each X .THis will require understanding which distinguished forms (eg lifts) correspond to the various nonzero exponents in E(X).At least in some examples I expect one may be able to make precise conjectures but in general it is ofcourse as least as complicated as what Arthur is trying to describe.

Any proofs ,comments are most welcome. Less welcome but obviously necessary would be a counterexample to the above purity or reasons not to believe it.

best regards
Peter

Dear All,
Erez has now computed the exact form of his and Jacquet's relation between the value at a point of a base change lift of a cusp form f on GL(n)/Q to F on GL(n)/K where K is a quadratic imaginary extension with corresponding character eta.If f is everywhere unramified and if I is a special point in the symmetric space on which F lives (CM like point for which we assume for simplicity the correspoinding orbit consists of this alone -in general there will be a finite number of points but this wont effect things much)ands if F is L^2 normalized ,then |F(I)|^2 is equal to a constant independent of f times

```
    L (1, fxf^x eta)
    --------------------------------
    L(1,eta) Residue at s=1 of L(s,fx f^)
```

It is a lovely and simple formula.

The L functions above are conatain the archimedian factors.
If f has (tempered) parameters (mu_1,mu_2 ,.....,mu_n) (so all are purely imaginary) then as this papremeter gets large the archimedian factors in the above are essentially

the product over i<j of | mu_i +mu_j bar| ^(1/2)

So if mu goes to infinity keeping away from the walls of the Weyl Chambers then this behaves like lambda ^(n(n-1)/8) . Moreover the finite part of the L-functions are essentially bounded from above and below (I will follow this up in another e mail with a complete analysis of what is proven here) so that the behaviour is what we expected -especially in terms of purity.
Care must be taken however if we approach the walls of the chambers. In my letter to Cathleen I was a bit crude here as I was looking for the convexity bound in the generic position and which is simply related to to the Laplace eigenvalue. I hadnt pointed out that near the walls one can give a better convex upper bound -in fact it matches exactly the factor appearing above.So not surprisingly ,near the walls the correct way to maesure the size of the eigenfunction is not the Lpalace eigenvalue but this modified norm. Doing so restores all order and in particular the purity conjecture is on the nose.
I hope that with a a few more iterations we will have a reliable conjecture.
Best
Peter

Now 2004, Peter
Sarnak wrote:

>
>
>
> Dear Cathleen ,Elon,Erez, Akshay ,Lior and Djorde
>