An Expansion of Keane's Treatment of Interval Exchange Transformations and Haller's Treatment of Rectangle Exchange Transformations

George Perez[1]
Mathematics Department
Princeton University
Princeton, NJ 08544

May 16, 2003

[1]E-mail: gperez@math.princeton.edu

Abstract

Interval exchange transformations, as discussed by Michael Keane(Math. Z. 141, 25-31 (1975)), who is indebted to work done by V. I. Oseledec (Dokl. Akad. Nauk SSSR, 168, 1966, 1009-1011.), will be closely examined. Both details and examples will be added. Induced transformations, especially Rauzy Induction, will also be included. There will be work done with the rectangle exchange transformations studied by Hans Haller (Monatsh. Math. 91, 215-232 (1981)). Proof of his Theorem 1 will be supplied.

Contents

0.1 Reader's Guide

Section 1.1: Here is where the background for the paper is established. In order to follow the rest of the paper it is important to read this section and familiarize oneself with the terminology and notation that will be used in the future, especially if one is not familiar at all with interval exchange transformations.

Section 1.2: Here is where the most essential properties of interval exchange transformations are discussed. Proofs which Keane left out have been added, as well as the proof that powers of interval exchange transformations are interval exchange transfomations. An example of a two-interval exchange is also treated.

Section 1.3: This is the theorem of Keane and Oseledets on minimality is introduced. The proof of the major theorem is broken into two theorems which are proven piecewise and slowly so as to add detail to Keane's treatment and hopefully limit confusion by the reader.

Section 1.4: Here there is more discussion of what can make an interval exchange transformation minimal. Two stronger conditions are introduced, including one of irrationality. There is also another treatment of the two-interval exchange.

Section 1.5: Here the topic of induced transformations is introduced. It flows naturally from the proof of Theorem 1.3.4. Rauzy Induction in particular is treated, and full induction details of one four-interval exchange are provided. Induction can be thought of as a means to map interval exchange transformations to interval exchange transformations.

Section 2.1: Here is where the discussion of rectangle exchanges begins and the notation and terminology is introduced. This is directly from Haller.

Section 2.2: Here the world of intervals is brought into that of recatngles. It is shown that the cross product of two interval exchanges is in fact a rectangle exchange and the simplest partition method is used. The reader may enjoy taking to specific interval exchanges and producing a rectangle exchange in the manner introduced. It would be a good excercise in working with both interval exchanges and rectangle exchanges.

Section 2.3: Here the minimality conditions proposed by Haller for rectangle exchanges are introduced. A proof of his Theorem 1 is provided as Theorem 2.3.1.

Chapter 1

Interval Exchange Transformations

1.1 Notation and Definitions

Begin with $\mathcal{X} = [0, 1)$, that is the half-open interval from 0 to 1.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a probability vector with $m > 1$, $\alpha_i > 0$ and $\sum_{i=1}^{m} \alpha_i = 1$. Define the following points belonging to the set $[0, 1]$ which we refer to as β-points:

$\beta_0 = 0$
$\beta_i = \sum_{j=1}^{i} \alpha_j$

Set $X_i = [\beta_{i-1}, \beta_i)$ that is, it is the half open interval from β_{i-1} to β_i, and let τ be a permutation on the elements $\{1, 2, \ldots, m\}$. Then we define the following:

$$\alpha^\tau = (\alpha_{\tau^{-1}(1)}, \alpha_{\tau^{-1}(2)}, \ldots, \alpha_{\tau^{-1}(m)}).$$

This means that $\alpha_i^\tau = \alpha_{\tau^{-1}(i)}$ and we can define the following points:

$\beta_0^\tau = 0$
$\beta_i^\tau = \sum_{j=1}^{i} \alpha_j^\tau = \sum_{j=1}^{i} \alpha_{\tau^{-1}(j)}$

and the following intervals:

$X_i^\tau = [\beta_{i-1}^\tau, \beta_i^\tau)$.

We may now define the interval exchange transformation T.

Given α and τ we define $T : \mathcal{X} \longrightarrow \mathcal{X}$ by

$$Tx = x - \sum_{j=1}^{i-1} \alpha_j + \sum_{j=1}^{\tau(i)-1} \alpha_j^\tau$$

which is

$$Tx = x - \beta_{i-1} + \beta^\tau_{\tau(i)-1}$$

when $x \in X_i$.

T is entirely determined by α and τ and is called the (α, τ)-interval exchange transformation.

Example 1.1.1. *Let* $\alpha = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ *and* τ *be the permutation* $\binom{123}{312}$ *and note the following:*
$\beta_0 = 0$
$\beta_1 = \frac{1}{4}$
$\beta_2 = \frac{1}{2}$
$\beta_3 = 1$
$X_1 = [0, \frac{1}{4})$
$X_2 = [\frac{1}{4}, \frac{1}{2})$
$X_3 = [\frac{1}{2}, 1)$
$\alpha^\tau = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$
$\beta^\tau_0 = 0$
$\beta^\tau_1 = \frac{1}{4}$
$\beta^\tau_2 = \frac{3}{4}$
$\beta^\tau_3 = 1$
Choose $x = \frac{3}{8} \in X_2$ *then*
$Tx = x - \beta_{2-1} + \beta^\tau_{\tau(2)-1} T(\frac{3}{8}) = \frac{3}{8} - \frac{1}{4} + 0 = \frac{1}{8}$

1.2 Properties of Interval Exchange Transformations

Let T be the (α, τ)-interval exchange transformation, then we observe the following:

Lemma 1.2.1. T *is an order preserving isometry from* X_i *to* $X^\tau_{\tau(i)}$.

Proof: First we must show $T : X_i \longrightarrow X^\tau_{\tau(i)}$ Let $x \in X_i$ then we note the following

$$Tx = x - \beta_{i-1} + \beta^{\tau}_{\tau(i)-1}$$

and so

$$x - \beta_{i-1} = Tx - \beta^{\tau}_{\tau(i)-1}$$

which means that $Tx \in [\beta^{\tau}_{\tau(i)-1}, 1)$. If in fact, $Tx \in X^{\tau}_{\tau(i)} = [\beta^{\tau}_{\tau(i)-1}, \beta^{\tau}_{\tau(i)})$ then we will see that

$$x - \beta_{i-1} = Tx - \beta^{\tau}_{\tau(i)-1} < \beta^{\tau}_{\tau(i)} - \beta^{\tau}_{\tau(i)-1}.$$

Now

$$Tx - \beta^{\tau}_{\tau(i)-1} = x - \beta_{i-1} < \beta_i - \beta_{i-1} = \sum_{j=1}^{i} \alpha_j - \sum_{j=1}^{i-1} \alpha_j = \alpha_i$$

but

$$\beta^{\tau}_{\tau(i)} - \beta^{\tau}_{\tau(i)-1} = \sum_{j=1}^{\tau(i)} \alpha_{\tau^{-1}(j)} - \sum_{j=1}^{\tau(i)-1} \alpha_{\tau^{-1}(j)} = \alpha_{\tau^{-1}(\tau(i))} = \alpha_i.$$

Therefore we have

$$0 \leq x - \beta_{i-1} = Tx - \beta^{\tau}_{\tau(i)-1} < \beta^{\tau}_{\tau(i)} - \beta^{\tau}_{\tau(i)-1}$$

which means that $\beta^{\tau}_{\tau(i)-1} \leq Tx < \beta^{\tau}_{\tau(i)}$ and thus $Tx \in X^{\tau}_{\tau(i)}$.

Second, we will need that T is one-to-one from X_i to $X^{\tau}_{\tau(i)}$. So let $x_1, x_2 \in X_i$. Then when $Tx_1 = Tx_2$ we have

$$x_1 - \beta_{i-1} + \beta^{\tau}_{\tau(i)-1} = x_2 - \beta_{i-1} + \beta^{\tau}_{\tau(i)-1}$$

which implies that $x_1 = x_2$ and shows that T is one-to-one from X_i to $X^{\tau}_{\tau(i)}$.

Finally we need to show that $T : X_i \longrightarrow X^{\tau}_{\tau(i)}$ is onto. In order to help I shall state the following, which shows that T is order preserving from X_i to $X^{\tau}_{\tau(i)}$.

Remark 1.2.2. *If x and $x + \lambda \in X_i$ then $T(x + \lambda) = x + \lambda - \beta_{i-1} + \beta^{\tau}_{\tau(i)-1} = Tx + \lambda$.*□

Assume T is not onto, that is $T : X_i \longrightarrow X' \subset X^\tau_{\tau(i)}$.
Then $\exists y \in X^\tau_{\tau(i)}$ s.t. $\forall x \in X_i \; Tx \neq y$.
Choose $x_h \in X_i$ s.t. $x_h - \beta_{i-1} = \beta_i - x_h = \frac{\alpha_i}{2}$
then $Tx_h - \beta^\tau_{\tau(i)-1} = \frac{\alpha_i}{2}$.
So $\frac{-\alpha_i}{2} \leq y - Tx_h < \frac{\alpha_i}{2}$.
If we let $y - Tx_h = \delta$ then $\beta_{i-1} \leq x_h + \delta < \beta_i$ which means that $x_h + \delta \in X_i$ so we can say $y = Tx_h + \delta = T(x_h + \delta)$.
Therefore, by contradiction, we have that $T : X_i \longrightarrow X^\tau_{\tau(i)}$ is onto. □

Let T be the (α, τ)-interval exchange transformation, then

Lemma 1.2.3. *T is invertible with inverse being the (α^τ, τ^{-1})-interval exchange transformation U.*

Proof: $Tx = x - \beta_{i-1} + \beta^\tau_{\tau(i)-1}$ for $x \in X_i$ and
$Ux = x - \sum_{k=1}^{j-1} \alpha^\tau_k + \sum_{k=1}^{\tau^{-1}(j)-1} \alpha^{\tau^{\tau^{-1}}}_j$ but $\alpha^{\tau^{\tau^{-1}}}_k = \alpha_{\tau\tau^{-1}(k)} = \alpha_k$
and we have $Ux = x - \beta^\tau_{j-1} + \beta_{\tau^{-1}(j)-1}$ when $x \in X^\tau_j$.

Now W.L.O.G. we can rename the parameter j as $\tau(i)$ for i such that $\tau(i) = j$ and it yields
$Ux = x - \beta^\tau_{\tau(i)-1} + \beta_{i-1}$ when $x \in X^\tau_{\tau(i)}$ and then clearly $Ux \in X_i$.
Let $x \in X_i$ so that $Tx \in X^\tau_{\tau(i)}$ then
$UTx = Tx - \beta^\tau_{\tau(i)-1} + \beta_{i-1} = x$.
Next let $x \in X^\tau_{\tau(i)}$ so that $Ux \in X_i$ then
$TUx = Ux - \beta_{i-1} + \beta^\tau_{\tau(i)-1} = x$. □

Let T be the (α, τ) interval exchange transformation, and let $D = \{\beta_1, \beta_2, \ldots, \beta_{m-1}\}$.

Lemma 1.2.4. *T is continuous on $\mathcal{X}$ except at the points in D, but it is continuous from the right at these points.*

Proof: Choose $x \in \mathcal{X}$ but $x \notin D$. So $x \in X_i$ for some i, and we know that $\forall \varepsilon > 0$, $\exists y \in X_i$ s.t. $|x - y| < \varepsilon$. Therefore we choose $x, y \in X_i$ with $|x - y| < \varepsilon$ and it follows that

$$|Tx - Ty| = \left|x - \beta_{i-1} + \beta^\tau_{\tau(i)-1} - y + \beta_{i-1} - \beta^\tau_{\tau(i)-1}\right| = |x - y| < \varepsilon$$

So T is continuous at all points $x \in \mathcal{X} \backslash D$.

Now let $x = \beta_i \in D$ so that $x \in X_{i+1}$. Choose $y \in \mathcal{X}$ s.t. $|y - x| < \varepsilon$ but add the restriction $y - x > 0$. This ensures us that $y, x \in X_{i+1}$ and it follows that

$$Ty - Tx = y - \beta_i + \beta^{\tau}_{\tau(i+1)-1} - x + \beta_i - \beta^{\tau}_{\tau(i+1)-1} = y - x < \varepsilon$$

So T is continuous from the right at points in D.□

Remark 1.2.5. $\lim\limits_{x \uparrow \beta_i} Tx = \beta^{\tau}_{\tau(i)}$ *for* $1 \leq i \leq m$

Let's verify this by looking at Tx for $x = \beta_i - \varepsilon$ when $\varepsilon > 0$ and sufficiently small. So we have $x \in X_i$ and thus

$$Tx = \beta_i - \varepsilon - \beta_{i-1} + \beta^{\tau}_{\tau(i)-1} = \alpha_i - \varepsilon + \beta^{\tau}_{\tau(i)-1} = \beta^{\tau}_{\tau(i)} - \varepsilon.\square$$

Remark 1.2.6. $T\beta_i = \beta^{\tau}_{\tau(i+1)-1}$ *for* $0 \leq i \leq m - 1$

Note that $\beta_i \in X_{i+1}$ and so we have

$$T\beta_i = \beta_i - \beta_{i+1-1} + \beta^{\tau}_{\tau(i+1)-1} = \beta^{\tau}_{\tau(i+1)-1}.\square$$

Lemma 1.2.7. *If T is an interval exchange transformation, T^k is again an interval exchange transformation $\forall k \in \mathbb{Z}$.*

Proof: The starting point is to say that given the (α, τ)-interval exchange transformation T, T^2 is an interval exchange transformation, and to do this we must define a probability vector and a permutation. We know that T determined by (α, τ) will map X_i onto $X^{\tau}_{\tau(i)}$ one-to-one, but we do not know how it affects $X^{\tau}_{\tau(i)}$. We ask the following question, namely for $x \in X_j$ when is $Tx \in X_i$? The answer is when

$$\beta_{i-1} \leq Tx = x - \beta_{j-1} + \beta^{\tau}_{\tau(j)-1} < \beta_i$$

which yields

$$x \in [\beta_{i-1} + \beta_{j-1} - \beta^{\tau}_{\tau(j)-1}, \beta_i + \beta_{j-1} - \beta^{\tau}_{\tau(j)-1}) = J_i.$$

If we fix j and let i vary note that $\bigcup\limits_{i=1}^{n} J_i = X_j$ and $J_i \cap J_m \neq \emptyset \Longrightarrow i = m$.

So the half-open intervals J_i partition X_j and so if we let j vary, we get a new partition of $\mathcal{X}$ and it is finite with at most $2m$ members. In essence we are working backwards, because we have created the β-points first, these are the startpoints of the J_i as well as the point 1. In fact these start points of J_i are merely $D \cup T^{-1}D \cup \{0\}$. From these β-points we can now get our probability vector κ by making $\kappa_i = i^{th}\beta$-point - $(i-1)^{th}\beta$-point.

Our next step is to find the permutation, and we start by noting the following:

$T : J_i \longrightarrow X_i^j \subseteq X_i$ therefore $T^2 : J_i \longrightarrow X_{\tau(i)}^{j\tau} \subseteq X_{\tau(i)}^{\tau}$ and it will be an order preserving isometry, and thus we have created a new set of permuted β-points.

Again we can work backwards. $J_i = [\beta_{q-1}, \beta_q)$ for some m. It will get mapped isomorphically onto $X_{\tau(i)}^{j\tau} = [\beta_{p-1}^{\sigma}, \beta_p^{\sigma})$ for some p, and we define σ to be the permutation such that

$\sum_{l=1}^{p-1} \kappa_{\sigma^{-1}(l)} = \sum_{l=1}^{\sigma(q)-1} \kappa_{\sigma^{-1}(l)}$, namely $\sigma(q) = p$, and q and p are fixed by the previous relation.

Now we have probability vector κ and permutation σ and we can define T^2 by

$T^2x = x - \beta_{q-1} + \beta_{\sigma(q)-1}^{\sigma}$ whenever $x \in [\beta_{q-1}, \beta_q)$.

From this point we will proceed by induction on k for T^k.

Assume T^k is the (κ, σ)-interval exchange transformation and T is the (α, τ)-interval exchange transformation. Then since T^{-1} is an interval exchange transformation T^{-k} is as well. We will have $Y_i = [\gamma_{i-1}, \gamma_i)$ where these γ_i are the β-points determined by κ and so

$T^kx = x - \gamma_{i-1} + \gamma_{\sigma(i)-1}^{\sigma}$ when $x \in Y_i$.

We ask the question: when is $T^kx \in X_i$? We answer

$\beta_{i-1} \leq T^kx < \beta_i$ or equivalently

$$\beta_{i-1} + \gamma_{j-1} - \gamma_{\sigma(j)-1}^{\sigma} \leq x < \beta_i + \gamma_{j-1} - \gamma_{\sigma(j)-1}^{\sigma}.$$

Just like before, we use this equation to determine a new set of β-points and work backwards to get a probability vector ν. Take note that the β-points will belong to $\{0\} \cup D \cup T^{-1}D \cup \cdots \cup T^{-k}D \cup \{0,1\}$. Again we have a new partition where each subinterval is mapped isometrically onto a subinterval of $X_{\tau(i)}^{\tau}$ and use this to work backwards, as before, to get the appropriate permutation π so that we have defined T^{k+1} as the (ν, π)-interval exchange transformation.

This shows that if T is an interval exchange transformation, T^k is an interval exchange transformation$\forall k \in \mathbb{Z}^+$. Recall T^{-1} is an interval exchange transformation, and $(T^{-1})^k = T^{-k}$ to conclude that when T is an interval exchange transformationT^k is an interval exchange transformation $\forall k \in \mathbb{Z}$.□

1.2.1 The Exchange of Two Intervals

Let the probability vector $\alpha = (\lambda, 1-\lambda)$ and the permutation $\tau = \binom{12}{21}$ determine an interval exchange transformation T.

Proposition 1.2.8. *T corresponds to a circle rotation on the circle of circumference 1 that represents $\mathbb{R}\backslash\mathbb{Z}$.*

Proof: $Tx = x + 1 - \lambda$ for $x \in X_1$ or $Tx = x - \lambda$ for $x \in X_2$ On the circle that is $\mathbb{R}\backslash\mathbb{Z}$, we have

$$x + 1 - \lambda \equiv x - \lambda$$

so each x is shifted clockwise along the circumference of the circle by a distance λ. This is a clockwise rotation by an angle $\theta = 2\pi\lambda$.□

Proposition 1.2.9. *T,the irrational interval exchange transformation that is the exchange of two intervals is minimal.*

Proof: The orbit of any point x is infinite, because if not then

$$0 \equiv T^k x - T^n x \mod 1 \equiv x - k\lambda - x + n\lambda \mod 1.$$

This implies that $(n-k)\lambda \in \mathbb{Z}$ which contradicts irrationality of λ. The orbit is also dense in $\mathcal{X}$ by the following observation: cut the circle into $\frac{1}{\varepsilon}$ intervals of length ε. Since these are finitely many intervals, and the orbit of x is infinite, we know by the pidgeon hole principle that we will get $\left|T^n x - T^k x\right| \leq \varepsilon$ for some n and k. Therefore:

$$|x - n\lambda - x + k\lambda| = |(k-n)\lambda| \leq \varepsilon.$$

So if we have x, y where $|T^p x - y| = \delta > \varepsilon$, we continue to apply T^{k-n} or T^{n-k} in order to locate x inside the ε-neighborhood of y. We have that the orbit is dense and so T is minimal.□

1.3 Minimality

A map $f : S \longrightarrow S$ is called minimal if the orbit, under f, of every point in S is dense in S.

Let T be the (α, τ)-interval exchange transformation and define the following for $x \in \mathcal{X}$:

$O^+(x) = \{T^n x : n > 0\}$

$O^-(x) = \{T^n x : n \leq 0\}$

$O(x) = O^+(x) \cup O^-(x)$

These are the orbits of the point x under $T^{n>0}$, $T^{n\leq 0}$, and T^n respectively.

We propose for T the following minimality conditions:

M1) T is aperiodic. So $\forall n \in \mathbb{Z}$ and $\forall x \in \mathcal{X}$ $T^n x \neq x$ or equivalentlty, $\forall n, p \in \mathbb{Z}$ and $\forall x \in \mathcal{X}$ $T^n x \neq T^p x$ which implies $O(x)$ is infinite $\forall x \in \mathcal{X}$.

M2) If F is a finite union of half-open intervals whose endpoints belong to the set

$$D^\infty = \bigcup_{i=0}^{n-1} O(\beta_i) \cup \{1\}$$

then $TF = F \Longrightarrow F = \mathcal{X}$ or $F = \emptyset$.

Proposition 1.3.1. *The set D^∞ is countable.*

Proof: Consider the set $O(\beta_i)$. Define a function $f : O(\beta_i) \longrightarrow \mathbb{Z}$ by $f(T^n \beta_i) = n$. By the definition of $O(x)$, it follows that f is one-to-one and onto. Therefore $O(\beta_i)$ is countable and so D^∞ is countable because it is the finite union of countable sets. □

Theorem 1.3.2. *If $O(x)$ is dense in $\mathcal{X}$ for each $x \in \mathcal{X}$, i.e. T is minimal, then T satisfies the minimality conditions.*

Proof: Assume T fails M1. Then $\exists x \in \mathcal{X}$ such that $T^t x = x$. If $T^t x = x$ then $T^{-t} x = x$ and $T^{s-t} x = T^s x$ so we have $O^+(x) = O^-(x) \cup \{x\}$ and $|O(x)| = t$. But if $O(x)$ is dense in $\mathcal{X}$ then given any $y \in \mathcal{X}$ and $R_\varepsilon = (y - \varepsilon, y - \varepsilon)$ it must be that $|O(x) \cap R_\varepsilon| \geq |\mathbb{Z}|$. Since $O(x)$ is finite, it cannot be dense, so by contradiction we see T satisfies M1.

Suppose T fails M2. Then we know $\exists F \neq \emptyset$ such that $F \subset \mathcal{X}$ and $TF = F$. Therefore $\exists x \in F$ so that $O(x) \subseteq F$ and we need the following remark:

Remark 1.3.3. *Since $F \subset \mathcal{X}$ is a finite union of half-open intervals, F is not dense in $\mathcal{X}$.*

The following must be true: $(x-\varepsilon, x+\varepsilon) \cap F = \emptyset$ for some $x \in \mathcal{X} \backslash F$ If not, then $F = X$ or F is not a finite union. □

It now follows easily that $O(x) \subseteq F$ cannot be dense in $\mathcal{X}$, and by contradiction T must satisfy M2. □

Theorem 1.3.4. *If T satisfies the minimality conditions then $O(x)$ is dense in $\mathcal{X}$ for each $x \in \mathcal{X}$, i.e. T is minimal.*

Proof: If $O(x)$ is not dense for some $x \in \mathcal{X}$ then there is a non-empty open interval $I \subset \mathcal{X}$ with $O(x) \cap I = \emptyset$. W.L.O.G. we can make a small adjustment to I such that it will be half-open.

$O(x) \cap I = \emptyset$ so, $\forall t \in \mathbb{Z}$, $T^t x \notin I$ or equivalently, $\forall t \in \mathbb{Z}$, $x \notin T^{-t} I$. Therefore, if we can show, for T satisfying the minimality conditions and $I \subset \mathcal{X}$ an arbitrary half-open interval of positive length, $\exists k \in \mathbb{Z}$ so that $\bigcup_{t=0}^{k} T^t I = \mathcal{X}$ then we will know that $O(x)$ must be dense because $\forall I \; O(x) \cap I \neq \emptyset$.

Let $I = [a, b)$. D^∞ is countable so we can adjust I just enough so that $a, b \notin D^\infty$.

For each $y \in D \cup \{a, b\}$ we say that $k(y) = \inf \{t \geq 0 : T^{-t} y \in (a, b)\}$.

Use the set $\mathcal{P} = \left\{T^{-k(y)} y : y \in D \cup \{a, b\}\right\}$ to partition I into half-open intervals $I_1, \ldots, I_l$ with $l \leq n+2$ since $|\mathcal{P}| \leq |D \cup \{a, b\}| = n+1$.

Find the minimal $t \geq 1$ such that $T^t \cap I_j \neq \emptyset$ when I_j is fixed and arbitrary. Such a finite t exists by Poincaré Recurrence Theorem. $I_j, TI_j, \ldots, T^t I_j$ are all single half-open intervals as we shall show by the following remark and lemma.

Remark 1.3.5. *If H is a half-open interval such that $H^\circ \cap D = \emptyset$ then TH is a single half-open interval.*

If H is a half-open interval such that $H^\circ \cap D = \emptyset$ then $H \subseteq X_i$ and $TH \subseteq X^\tau_{\tau(i)}$ must be a single half-open interval by continuity of T. □

Lemma 1.3.6. *For n such that $0 \leq n \leq t-1$, $T^n I_j \cap D = \emptyset$.*

Proof: Suppose $\beta_j \in T^n {I_j}^\circ$. Then $T^{-n}\beta_j \in I_j$. Because of the way in which I was partitioned, this implies that $T^{-p}\beta_j$ is an endpoint of I_k with $p < n$. This implies that $T^{n-p} I_j \cap I \neq \emptyset$, so $t = n-p$ and $\beta_j \in T^n I_j$ means $t = n-p \leq n < t-1$

which is a contradiction. Therefore, $T^n I_j{}^\circ \cap D = \emptyset$ for $0 \leq n \leq t-1$. $\square$

By the lemma, we see that $T^n I_j{}^\circ \cap D = \emptyset$, and so by the remark we have that $I_j, TI_j, \ldots, T^t I_j$ are all single half-open intervals.

Lemma 1.3.7. *If $t \geq 1$ is minimal such that $T^t I_j \cap I \neq \emptyset$ then $T^t I_j \subseteq I$.*

Proof: Let's suppose that $T^t I_j^\circ \ni a$ or b. Then $I_j{}^\circ \ni T^{-t}a$ or $T^{-t}b$ which means, because of how I was partitioned, that $T^{-p}a$ or $T^{-p}b$ is an endpoint for some I_k with $p < t$. However, this implies that $T^{t-p}I_j \cap I \neq \emptyset$ and this is an obvious contradiction unless $p = 0$. But $p = 0$ means a or $b \in I_k$ which contradicts the way in which I was partitioned. Thus we are ensured that
$T^t I_j \subset I$. $\square$

Lemma 1.3.8. *For $t \geq 1$ minimal such that $T^t I_j \cap I \neq \emptyset$ the intervals $I_j, TI_j, \ldots, T^{t-1}I_j$ are disjoint.*

Proof: Suppose $m > n$ and $T^n I_j \cap T^m I_j \neq \emptyset$. Then $x \in T^n I_j$ and $x \in T^m I_j$ so $I \supset I_j \ni T^{-n}x \in T^{m-n}I_j$ and so $T^{m-n}I_j \subset I$, so $m - n \geq t$. Therefore if we take $I_j, TI_j, \ldots, T^{t-1}I_j$ we have disjoint intervals. $\square$

So take $F = \bigcup_{j=1}^{l} \bigcup_{i=0}^{t_j-1} T^i I_j$, where t_j is the power of T when I_j first returns to I, a finite union of half-open intervals.

Remark 1.3.9. $F \subseteq \bigcup_{n=0}^{k} T^n I$ *for some $k < \infty$.*

If $x \in F$ then $x \in T^n I_j$ for some n and j. But $I_j \subseteq I$, so $T^n I_j \subseteq T^n I$ and there you have it. $\square$

Remark 1.3.10. $TF = F$.

Let $x \in F$ so $x \in T^n I_j$. If $n < t_j - 1$ then $Tx \in T^{n+1}I_j \subseteq F$. If $n = t_j - 1$ then $Tx \in I \subseteq F$. So, $x \in F$ implies $Tx \in F$ and therefore $TF = F$. $\square$

Remark 1.3.11. *The boundary points of F belong to the set D^∞.*

If $x \notin D^\infty$ is a boundary point, then $T^n x \notin D$ for any $n \in \mathbb{Z}$ and so by continuity of T, x will always be a boundary point. Since there are finite boundary points, x has finite orbit, but this contradicts M1, so it must have been that

$x \in D^{\infty}$. □

So by M2, it must be that $F = \mathcal{X}$ and since $F \subseteq \bigcup_{n=0}^{k} T^n I$, $\bigcup_{n=0}^{k} T^n I = \mathcal{X}$. Therefore we have finally shown that, when T satisfies the minimality conditions, $O(x)$ must be dense in $\mathcal{X}$ for each $x \in \mathcal{X}$. □

1.4 Irrationality

Let us first introduce the notion of irreducibility. We call a permutation τ irreducible if for each $1 \leq j \leq n-1$,

$$\tau(\{1, 2, \ldots, j\}) \neq \{1, 2, \ldots, j\}.$$

Remark 1.4.1. *If τ is not irreducible then T can be decomposed into an interval exchange transformationon $[0, \beta_j)$ and another on $[\beta_j, 1)$.*

First note that the first j coordinates of α^{τ} are a permutaion of $\{\alpha_1, \ldots, \alpha_j\}$. Consider the X_i such that $i \leq j$, they belong to the interval $[0, \beta_j)$. Because τ is not ireducible, we know one of them will go one-to-one onto X_1^{τ}, another will go one-to-one onto X_j^{τ}, and all the others will go one-to-one onto intervals in between and moreover, $X_1^{\tau}, \ldots, X_j^{\tau}$ lie within the interval $[0, \beta_j^{\tau}) = [0, \beta_j)$. Therefore the remark is verified because one piece of T is from $[0, \beta_j)$ onto $[0, \beta_j)$ so there must be another piece from $[\beta_j, 1)$ onto $[\beta_j, 1)$. □

Theorem 1.4.2. *(Oseledets-Keane) If the orbits of the points $D = \{\beta_1, \ldots, \beta_{m-1}\}$ are infinite and distinct, then the minimality conditions are satisfied.*

Proof: Suppose T fails M1, i.e. there is $x \in \mathcal{X}$ such that $T^n x = x$. Then we set

$$\beta = \max\{T^i \beta_j : 0 \leq i \leq n-1, 0 \leq j \leq m-1, T^i \beta_j \leq x\}.$$

If $\beta = x$ then $T^i \beta_j = x = T^n x$ so $\beta_j = T^{n-i} x$. Thus

$$T^n \beta_j = T^n (T^{n-i} x) = T^{n-i} x = \beta_j$$

and the orbit of β_j is not infinite, which is a contradiction, unless $x = T^i 0$. But $T\beta_j = \beta^{\tau}_{\tau(j+1)-1}$, so if $\tau(1) \neq 1$ then $0 = T\beta_j$ and $x = T^{i+1}\beta_j$ for a $\beta_j \in D$. This β_j will have a finite orbit. If $\tau(1) = 1$, then

$$\alpha_1^{\tau} = \alpha_1 = \beta_1 = \beta_1^{\tau} = T\beta_{\tau^{-1}(2)-1}.$$

So $\beta_1 \in TD$ which is a contradiction to distinct orbits.

Clearly the case must have been $\beta < x$, but then consider the interval $[\beta, x]$. Because of how β is defined, $T^p[\beta, x]$ contains no points in D when $1-n \leq p \leq 0$, so $[\beta, x]$ remains intact and order preserved under $T^{-1}, \ldots, T^{-n}$. But $T^{-n}x = x$, so it must be $T^{-n}\beta = \beta$. Since $\beta = T^i\beta_j$, β_j must have a finite orbit. Fnally we have showm that T cannot fail M1.

Let F be a set such that it causes T to fail M2.

Proposition 1.4.3. *If $x \neq 1$ is a boundary point of F, then Tx is a boundary point of F or $x \in D \cup \{0\}$.*

Proof: Let $x \notin D \cup \{0\}$ so that x is a point of continuity. x is a boundary point, so we have $(x - \delta, x) \cap F = \emptyset$ and $(x, x + \delta) \cap F \neq \emptyset$ for $|\delta| > 0$ and arbitrarily small. If $Tx \in F$ is not a boundary point, then $T(x - \delta, x + \delta) \subset F$, but since $TF = F$ this is a contradiction.□

So now we see that if $x \neq 1$ is a boundary point, $x \in D \cup \{0\}$ or Tx is a boundary point. If $T^n x \notin D \cup \{0\}$ for some $n \geq 0$, then $T^n x$ is always a boundary point. Since there are a finite number of boundary points, and $x = T^i\beta_j$ for some $beta_j \in D$, that β_j is periodic which is a contradiction. So $T^n x \in D \cup \{0\}$ for some $n \geq 0$.

Corollary 1.4.4. *If $x \neq 1$ is a boundary point of F, then $T^{-1}x$ is a boundary point, or $T^{-1}x \in D$.*

Proof: This follows from the above proposition because, if $x \notin TD$ then x came from a boundary point, $T^{-1}x$. But if $x \in TD$ then $T^{-1}x \in D$.□

So again because the orbit of β_j is infinite and F has finite boundary points, $T^{-s}x$ must live in D for some s. Let $y = T^{-s}x \in D$, then $T^{n+s}y = T^n x \in D \cup \{0\}$. So if the orbits of the β_j are to remain distinct, then $Ty = 0$, so $n = 0, s = 1, x = 0$. If 0 is the only boundary point of F that is not 1, then $F = \mathcal{X}$. Therefore T does not fail M2.□

Before proceeding to the next lemma, we must make the following definition: the (α, τ)-interval exchange transformation T is called irrational if τ is irreducible and the only rational relations between the numbers $\alpha_1, \ldots, \alpha_m$ are multiples of $\alpha_1 + \cdots + \alpha_m = 1$.

Theorem 1.4.5. *(Oseledets-Keane) If T is irrational, then T satisfies the minimal conditions.*

Proof: Suppose the conditions of the previous lemma are not satisfied. Then we must have for some r, s between 1 and $m-1$ and some $n \geq 0$ that $T^{n+1}\beta_r = \beta_s$.

We let u be such that $0 \leq u \leq n$ and then $T^u\beta_r$ will lie in some interval X_{j_u+1} with $0 \leq j_u \leq m-1$. We define another notation: $k_u = \tau(j_u+1)-1$ for $0 \leq u \leq n$. Thus we have the following: $j_0 = r$ and

$$T\beta_r = \beta_r - \beta_r + \beta^\tau_{k_0} = \beta^\tau_{k_0}$$

and

$$T^{u+1}\beta_r = T(T^u\beta_r) = T^u\beta_r - \beta_{j_u} + \beta^\tau_{k_u}$$

We can now come up with an expression for $T^{n+1}\beta_r$ by expanding that last equation. We obtain the following:

$$T^{n+1}\beta_r = T^n\beta_r - \beta_{j_n} + \beta^\tau_{k_n} = T^{n-1}\beta_r - \beta_{j_n} - \beta_{j_{n-1}} + \beta^\tau_{k_n} + \beta^\tau_{k_{n-1}} = \cdots = \sum_{u=0}^{n} \beta^\tau_{k_u} - \sum_{u=1}^{n} \beta_{j_u}.$$

Now we reindex by letting $l_u = j_u$ if $u \neq 0$ and $l_0 = s$, and we have that:

$$0 = T^{n+1}\beta_r - \beta_s = \sum_{u=0}^{n} \beta^\tau_{k_u} - \sum_{u=1}^{n} \beta_{l_u} - \beta_{l_0} = \sum_{u=0}^{n} (\beta^\tau_{k_u} - \beta_{l_u}).$$

Now we will substitute in the definitions of β_i and β^τ_i to obtain:

$$0 = \sum_{u=0}^{n} \left(\sum_{j=1}^{k_u} \alpha_{\tau^{-1}(j)} - \sum_{j=1}^{l_u} \alpha_j \right) = \sum_{j=1}^{m-1} (c_j \alpha_{\tau^{-1}(j)} - d_j \alpha_j)$$

where we define $c_j = |\{u : 0 \leq u \leq n, k_u \geq j\}|$ and $d_j = |\{u : 0 \leq u \leq n, l_u \geq j\}|$, so that c_j is the number of times that $\alpha_{\tau^{-1}(j)}$ appears and d_j is the number of times that α_j appears. It then follows from the definitions of β_i and β^τ_i that $c_m = d_m = 0$ and: $0 \leq c_{m-1} \leq \cdots \leq c_2 \leq c_1$ and $0 \leq d_{m-1} \leq \cdots \leq d_2 \leq d_1$. When we slightly re-organize our notation, without changing any definition, we happen upon the equation $0 = \sum_{j=1}^{m} (c_{\tau(j)} - d_j)\alpha_i$. The irrationality of T now tells us that $c_{\tau(j)} = d_j$ and we already know that $c_m = d_m = 0$. So, choose the maximal j such that $c_j, d_j \neq 0$ and $c_{j+1}, \ldots, c_m = d_{j+1}, \ldots, d_m = 0$. Because τ is irreducible there must exist $b \geq j+1$ and $a \leq m$ with $\tau(b) \leq j$ and $\tau^{-1}(a) \leq j$. Because if not, then $\tau\{1, \ldots, j\} = \{1, \ldots, j\}$ or $\tau\{j+1, \ldots, m\} = \{j+1, \ldots, m\}$, which would mean τ was reducible. So we have that $c_{\tau(b)} = d_b = 0$ and $d_{\tau^{-1}(a)} = c_a = 0$ which in turn implies that $c_j = d_j = 0$ for each j. Thus, going back to the definitions of c_j and d_j, we have that $k_u = l_u = 0$, which contradicts $l_0 = s \geq 1$. Therefore T must obey the conditions of our previous lemma, and so it satisfies the minimality conditions.$\square$

1.5 Induced Transformations

Let $I \subseteq \mathcal{X}$ be a half-open interval. Partition it into sub-intervals $I_1, \ldots, I_l$ using the method from Theorem 1.3.4. The induced map L is then defined by
$L : I_j \longrightarrow T^{t_j} I_j$ where t_j is the first return of I_j and T is a minimal interval exchange transformation.

Corollary 1.5.1. *The induced transformation L is an interval exchange transformation.*

Proof: The corollary follows immediately from the results obtained in the proof of Theorem 1.3.4 and the following remark:

Remark 1.5.2. $T^{t_j} I_j \cap T^{t_k} I_k = \emptyset$ *when* $j \neq k$.

Proof: Suppose not. If $t_j = t_k$ then $I_j = I_k$ which is a contradiction. If $t_j < t_k$ then $T^{t_k - t_j} I_k \cap I \neq \emptyset$ which is a contradiction. If $t_k < t_j$ then $T^{t_j - t_k} I_j \cap I \neq \emptyset$ which is a contradiction.$\square$

Now we introduce the method of Rauzy Induction, which produces an induced transformation when given an irrational interval exchange transformation T.
So let T be the (α, τ)-interval exchange transformation where α partitions $\mathcal{X}$ into m intervals and $\tau = \binom{1 \cdots m}{p \cdots q}$. There are two possible cases.

Case 1: $\beta^{\tau}_{m-1} < \beta_{m-1}$

We consider the induced transformation on the interval $[0, \beta_{m-1})$. It will be partitioned exactly as before, except that for $\tau(j) = m$ the interval X_j is in two parts: one that goes into $[0, \beta_{m-1})$ and one that does not. The results are a transformation defined for these new intervals with a permutation that may not be the same and a matrix expressing lengths, λ, of old intervals in terms of lengths, λ' of the new intervals.

Case 2: $\beta_{m-1} < \beta^{\tau}_{m-1}$

We consider the induced transformation on the interval $[0, \beta^{\tau}_{m-1})$. This means that our first $m-1$ intervals will be exactly as before, but the new m^{th} interval will be only a fraction of the old one. Again we produce a permutation and a matrix.

Example 1.5.3. *Let T be the minimal (α, τ)-interval exchange transformation, $\tau = \binom{1234}{4321}$ and we will do case 1 of Rauzy Induction. Here is the initial picture*

$$| \quad X_1 \quad | \ X_2 \ | \ X_3 \ | \ X_4 \ |$$

$$| \ X_1^{\tau} \ | \ X_2^{\tau} \ | \ X_3^{\tau} \ | \quad X_4^{\tau} \quad |$$

We then make the inductive cut and produce a new picture under the induced transformation

$$| \quad X_1' \quad | \ X_2' \ | \ X_3' \ | \ X_4' \ |$$

$$| \ X_1^{\sigma} \ | \ X_2^{\sigma} \ | \ X_3^{\sigma} \ | \quad X_4^{\sigma} \quad |$$

So we have the following results: $\sigma = \binom{1234}{4132}$ and

$$\begin{pmatrix} \lambda_1' \\ \lambda_2' \\ \lambda_3' \\ \lambda_4' \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix}$$

Next we proceed to case 2 of Rauzy Induction. Here are the initial and inductive pictures

$$| \ X_1 \ | \ X_2 \ | \ X_3 \ | \quad X_4 \quad |$$

$$| \quad X_1^{\tau} \quad | \ X_2^{\tau} \ | \ X_3^{\tau} \ | \ X_4^{\tau} \ |$$

$$| \quad X_1' \quad | \quad X_2' \quad | \quad X_3' \quad | \qquad X_4' \qquad |$$

$$| \qquad X_1^\sigma \qquad | \quad X_2^\sigma \quad | \quad X_3^\sigma \quad | \quad X_4^\sigma \quad |$$

Thus the induced transformation has given us $\sigma = \binom{1234}{2431}$ *and*

$$\begin{pmatrix} \lambda_1' \\ \lambda_2' \\ \lambda_3' \\ \lambda_4' \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix}$$

For the sake of completeness, I will include for the reader the rest of the Rauzy Induction for $\tau = \binom{1234}{4321}$ *complete with matrices and Rauzy graph.*

$\binom{1234}{4132}$ *by case one goes to* $\binom{1234}{4213}$ *and by case two goes to* $\binom{1234}{3142}$. *It produces the following matrices:*

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$\binom{1234}{2431}$ *by case one goes to* $\binom{1234}{2413}$ *and by case two goes to* $\binom{1234}{3214}$. *It produces the following matrices:*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$\binom{1234}{4213}$ *by case one goes to* $\binom{1234}{4213}$ *and by case two it is not changed. It produces the following matrices:*

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$\binom{1234}{3142}$ *by case one is unchanged and by case two goes to* $\binom{1234}{4132}$*. It produces the following matrices:*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$\binom{1234}{2413}$ *by case one goes to* $\binom{1234}{2431}$ *and by case two it is not changed. It produces the following matrices:*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$\binom{1234}{3241}$ *by case one is not changed and by case two goes to* $\binom{1234}{4321}$*. It produces the following matrices:*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

I will now produce the afore mentioned Rauzy graph.

$$\begin{array}{ccccccc}
 & & \binom{1234}{4321} & \overset{1}{\rightarrow} & \binom{1234}{4132} & \overset{2}{\leftrightarrow} & \binom{1234}{3142} \\
 & \overset{2}{\nearrow} & \overset{2}{\downarrow} & \overset{1}{\nwarrow} & \overset{1}{\downarrow} & & \\
\binom{1234}{3241} & \overset{2}{\leftarrow} & \binom{1234}{2431} & & \binom{1234}{4213} & & \\
 & & \overset{1}{\updownarrow} & & & & \\
 & & \binom{1234}{2413} & & & &
\end{array}$$

Chapter 2

Rectangle Exchange Transformations

2.1 Notations and Definitions

Recall that $\mathcal{X} = [0, 1)$. We now define $\Omega = \mathcal{X} \times \mathcal{X}$, the half-open square in $\mathbb{R}^2$. $B \subset \Omega$ is a (half-open) rectangle if there exist $x_1, x_2, y_1, y_2 \in [0, 1)$ such that $x_1 < x_2, y_1 < y_2$ and $B = [x_1, x_2) \times [y_1, y_2)$. A finite measurable partion $\mathcal{P} = (P_1, \dots, P_m)$ of Ω is called a rectangle partition of length m if it consists of m rectangles $P_1, \dots, P_m$. A bijection that preserves Lebesgue measure in 2 dimensions (λ^2) $S : \Omega \to \Omega$ is called a rectangle exchange transformation, if there are $m \in \mathbb{Z}^+$, a rectangle partition $(P_1, \dots, P_m)$ of length m, and m translations $S_1, \dots, S_m$ of $\mathbb{R}^2$ such that for $i = 1, \dots, m$

$$S|_{P_i} = S_i|_{P_i}.$$

In such a case we call the rectangle partition $\mathcal{P} = (P_1, \dots, P_m)$ admissible for S. A rectangle partition $\mathcal{P}$ of length m is irreducible for some rectangle exchange transformation S if $\mathcal{P}$ is admissible for S and there is no rectangle partition $\mathcal{R}$ with length less than m that is also admissible for S.

2.2 Interval Exchange Transformations in Two Dimensions

It is natural to consider creating a rectangle exchange transformation from two interval exchange transformations. This is in fact valid, as will be shown in the following proposition.

Proposition 2.2.1. *If T_1 and T_2 are interval exchange transformations, then $T_1 \times T_2$ is a rectangle exchange transformation.*

Proof: Let T_1 be the (α, τ)-interval exchange transformation and let T_2 be the (α', σ)-interval exchange transformation. We will now partition Ω with vertical lines $x = \beta_1, \ldots, x = \beta_{m-1}$ and horizontal lines $y = \beta'_1, \ldots, y = \beta'_{m'-1}$. This is a rectangle partition $\mathcal{P}$ of length $(m-1)(m'-1)$, and is nothing more than m columns each themselves divided the same way into m' rectangles. When we let $T_1 \times T_2$ act on this it will permute the columns with T_1 and then within each column T_2 will permute the rectangles. Bijectivity and λ^2 preservation are trivial, and so we are done.$\square$

2.3 A Minimality Condition for Rectangle Exchange Transformations

Let S be a rectangle exchange transformation and let $(P_1, \ldots, P_m)$ be a rectangle partition admissible for S. Suppose that $P_i = [x_1^i, x_2^i) \times [y_1^i, y_2^i)$ for $i = 1, \ldots, m$. Set

$$\delta_i = \{(x, y) \in P_i : x = x_1^i \vee \ y = y_1^i\}$$

so that δ_i forms the left and lower boundary of P_i. We put $\delta = \bigcup_{i=1}^{m} \delta_i$. For $\omega \in \Omega$ we set

$$O(\omega) = \{S^n(\omega) : n \in \mathbb{Z}\}, O_-(\omega) = \{S^n(\omega) : n \leq 0\}$$

$$D^\infty = \bigcup_{\omega \in \delta} O(\omega) \cup \{(x, y) \in [0, 1]^2 : x = 1 \vee \ y = 1\}.$$

We now introduce the following set of minimality conditions:

C1) Each point $\omega \in \Omega$ has infinite orbit projections. This means that if we say $S^n(\omega) = (x_n, y_n)$ for $n \in \mathbb{Z}$ then the sets $\{x_n : n \in \mathbb{Z}\}$ and $\{y_n : n \in \mathbb{Z}\}$ are infinite.

C2) If $F = \bigcup_{j=1}^{r} Q_j$ is the union of a finite number of rectangles $Q_1, \ldots, Q_r$ such that $SF = F$ and all boundary points of F in $[0, 1]^2$ belong to D^∞, then

$F = \Omega$.

C3) If s is a horizontal or vertical line in Ω and Q is a rectangle then there exists $k = k(s, Q)$ such that for each $\omega \in s$

$$O_-(\omega) \cap Q^\circ \neq \emptyset \Rightarrow \{\omega, S^{-1}\omega, \dots, S^{-k}\omega\} \cap Q^\circ \neq \emptyset.$$

Theorem 2.3.1. *If a rectangle exchange S satisfies the minimality conditions, then S is minimal, i.e. for each $\omega \in \Omega$ we have $O(\omega)$ dense in Ω.*

Proof: First I will comment on the necessity of C1. If S fails C1 then we know that for some $\omega \in \Omega$ either $O(\omega)$ does not intersect the area in Ω between the lines $x = a$ and $x = b$ with $|a - b| > 0$, or it does not intersect the area in Ω between the lines $y = c$ and $y = d$ with $|c - d| > 0$. This imlies that we can find a ball $B \subset \Omega$ with radius $\rho = \varepsilon > 0$ and $O(\omega) \cap B = \emptyset$. Thus any minimal S must obey C1.

If S is not minimal, we know that there is some rectangle R with positive measure, such that $O(\omega) \cap R = \emptyset$ for some $\omega \in \Omega$. We will show this cannot happen by showing that for an arbitrary R there is a finite n such that $\bigcup_{n=0}^{k} S^n R \supseteq \Omega$.

W.L.O.G. let $R = [a, b) \times [a, c)$. Now we partition R in the following manner: we consider all of the points in R coresponding to the first pre-image of $\omega \in \delta$,$[a, b]$,, $[b, d]$, $[a, c]$ or $[c, d]$ (where d is the point $|a - c|$ above b) that lies in R°. For all $\omega \in \delta \cup [a, b] \cup [c, d]$ let $k(n) := \ inf\{n \in \mathbb{Z}^+ : S^{-n}\omega \in R^\circ\}$ We concern ourselves only with $k(n) < \infty$.

First consider the points $\omega^{k(n)}$ for $\omega \in [a, c], [b, d]$ or $\omega \in$ vertical δ_i. These form segments (or points, i.e. segement of 0 measure) in R and will form the vertical walls of our partition. By C3 we know that they are finite and a definite bound is m^K where m is the length of the partition on Ω and K is the max of all k that are the finite numbers from C3 that vary depending on which δ_i ω is in, or if $\omega \in [a, c], [b, d,]$.

Now consider $\omega^{k(n)}$ for $\omega \in [a, b], [c, d]$ or $\omega \in$ horizontal δ_i. These form segments (possibly measure 0) in R that will serve as the horizontal walls of our partition. Again C3 guarantees finitude and the bound of m^K where m is the length of the partition of Ω and K is the max of all k that are the finite numbers from C3 that vary depending on which δ_i ω is in, or if $\omega \in [a, b], [c, d]$.

This will produce various line segments within R that are at worst the skeletal walls of our partition. Extend any partition walls so that there are no gaps in the partition, and R is now divided into half-open sub-rectangles $R_1, \ldots, R_l$. This is all perfectly valid because of C3 and the λ^2-preserving bijectivity of S.

So now for each R_j we want to consider the minimal $t \geq 1$ such that $S^t R_j \cap R \neq \emptyset$. This finite t exists by the Poincaré Recurrence Theorem. Our goal now is to show that $R_j, SR_j, \ldots, S^t R_j$ are single half-open rectangles, $R_j, SR_j, \ldots, S^{t-1} R_j$ are disjoint, and $S^t R_j \subseteq R$.

Remark 2.3.2. $R_j, SR_j, \ldots, S^t R_j$ *are single half-open rectangles.*

Because of how R was partitioned, we know that $S^n R_j \cap \delta = \emptyset$ when $n < t$. For if $\omega \in \delta$ and $\omega \in S^n {R_j}^\circ$ (if no $\omega \in S^n {R_j}^\circ$ then the portion removed from R_j has λ^2 measure of 0) then $S^{-p}\omega \in R_i$ for some i and $p < n$. But then $S^{n-p} R_j \cap R \neq \emptyset$ and this is a contradiction. This means that each of $R_j, \ldots, S^{t-1} R_j$ lie entirely within some P_i (not necessarily the same one) and thus it follows that we have single rectangles for each of $R_j, \ldots, S^t R_j$ by the way in which S is defined.□

Remark 2.3.3. $R_j, SR_j, \ldots, S^{t-1} R_j$ *are all disjoint.*

Suppose $m, n < t$ and $m > n$. Then if $S^m R_j \cap S^n R_j$, we know $S^{m-n} R_j \cap R$, therefore $m - n \geq t$ which is a contradiction.□

Remark 2.3.4. $S^t R_j \subseteq R$.

Suppose not. Then $S^t R_j^\circ \ni \omega$ for some $\omega \in [a, b], [a, c], [b, d]$ or $[c, d]$. So $0 < p < t$ such that $S^{-p}\omega$ is on the boundary of some R_k. This implies that $S^{t-p} R_j \cap R \neq \emptyset$ which is a contradiction because $t - p < t$ and t was said to be the first return time.□

Let $F = \bigcup_{j=1}^{l} \bigcup_{i=0}^{t_j - 1} S^i R_j$ where t_j is the power of S when R_j first returns to R. Clearly $F \subseteq \bigcup_{n=0}^{k} S^n R$ with $k < \infty$ and F is a finite union.

Remark 2.3.5. $SF = F$.

If $\omega \in F$ then $S\omega \in S^n R_j$ for some j and $1 \leq n < t_j$ or $S\omega \in R$. Either way, $S\omega \in F$.□

Remark 2.3.6. *The boundary points of F belong to the set D^∞.*

Let $\omega \notin D^\infty$ be a boundary point of F. Since $\omega \notin D^\infty$ we can assure that there is an open ball B with radius ρ such that $S^n\omega \ni B \subset P_i$ for some i, and B, ρ, i depend on ω and n. We know that within this ball is the sequence of points outside F converging to ω as well as the sequence of points inside F converging to ω, so it follows that ω must remain a boundary under any power of S. Since F has finitely many x-coordinates in its boundaries, and finitely many y-coordinates in its boundaries ω would fail C1.□

Now we can apply C2 and get that $F = \Omega$ which in turn yields
$\bigcup_{n=0}^{k} S^n R \supseteq \Omega$.□

Corollary 2.3.7. *There is an induced map on any rectangle $R = [a, b) \times [c, d) \subseteq \Omega$ and it is again a rectangle exchange transformation.*

Proof: Let S be a rectangle exchange and $\mathcal{P}$ an admissible partition for S. The map on R induced by S is the first return map on the partition of R that is exactly like the partition in Theorem 2.3.1. As seen in 2.3.1, all of the rectangles in the partition remain intact and the return in whole. Since S is already measure preserving and bijective, the only thing left to check is that if $t_i \geq 1$ is the first return of R_i then $S^n R_i{}^\circ \cap S^p R_j{}^{circ} = \emptyset$ when $n = t_i$ and $p = t_j$.

Suppose the intersection is non-empty, then if $t_i = t_j$ we have $R_i \cap R_j$ which is a contradiction. If $t_i > t_j$ then $S^{t_i - t_j} R_i \cap R^\circ \neq \emptyset$ which contradicts t_i being the first return. In the final scenario $S^{t_j - t_i} R_j \cap R^\circ \neq \emptyset$ which contradicts t_j being the first return. □

2.4 References

Haller, Hans. *Rectangle Exchange Transformations*. Monatsh. Math. 91, 215-232 (1981).

Keane, Michael. *Interval Exchange Transformations*. Math. Z. 141, 25-31 (1975).

Oseledec, V. I. *The spectrum of ergodic automorphisms*. Dokl. Akad. Nauk SSSR, 168, 1009-10011 (1966).

Rauzy, Gerard. *Échange d'intervalles et transformations induite*. Acta. Arith. 34, no. 4, 315-328 (1979).