The Circle Method on the Binary Goldbach Conjecture

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1 The Goldbach Conjecture

The Goldbach Conjecture, appears to be very simple at first glance. It can be stated as thus: Every even number can be represented by the sum of two prime numbers. Or in mathematical notation:

$$2n = p_1 + p_2 \tag{1.1}$$

$\forall n \in \mathbb{N}$, and where p_1 and p_2 depend on n.

However, it has shown itself to be quite difficult to prove. This conjecture is sometimes called the "binary Goldbach problem because a similar problem, sometimes called the "ternary Goldbach problem"

$$(2n + 1) = p_1 + p_2 + p_3 \tag{1.2}$$

for sufficiently large $n \in \mathbb{N}$
or "every odd number can be represented as the sum of three primes" was proven by I.M. Vinogradov in 1937.

We will approach the binary case by examining the methods employed in the proof of the ternary problem. Let M be the odd number which we are trying to represent as the sum of three primes. Consider the sum

$$F_M(x) = \sum_{p \leq M} (\log p) e(px) \tag{1.3}$$

where $e(\alpha) = exp(2\pi i \alpha)$ and p is prime.
Note: We will be using this notation throughout the remainder of this paper.

Now consider the integral

$$\begin{aligned} R_3(M) &= \int_0^1 F_M^3(x) e(-Mx) dx \\ &= \int_0^1 \left(\sum_{p \leq M} (\log p) e(px) \right)^3 e(-Mx) dx \end{aligned}$$

Observe that since our function $e(\alpha x) = exp(2\pi i \alpha x)$

$$\int_0^1 e(\alpha x) dx = 0$$

for any value of $\alpha \in \mathbb{Z}$ such that $\alpha \neq 0$. Note that the integral is 1 if $\alpha = 0$.

Since $p_i \in \mathbb{N}$ and $M \in \mathbb{N}$, $(p_1 + p_2 + p_3 - M) \in \mathbb{Z}$, so the only terms that contribute to the integral $R_3(M)$ are the terms

$$(\log p_1 \log p_2 \log p_3) e\left((p_1 + p_2 + p_3 - M)x\right)$$

where $p_1 + p_2 + p_3 - M = 0$.
Or

$$p_1 + p_2 + p_3 = M$$

So continuing on, we see that

$$\begin{aligned} R_3(M) &= \int_0^1 \sum_{p_1, p_2, p_3 \leq M} (\log p_1)(\log p_2)(\log p_3) e\left((p_1 + p_2 + p_3 - M)x\right) dx \\ &= \sum_{p_1 + p_2 + p_3 = M} \log p_1 \log p_2 \log p_3 \end{aligned} \tag{1.4}$$

This integral $R_3(M)$ is then a weighted sum of the number of representations of our odd number M as the sum of three primes. Note that

$$C(M) = \sum_{\substack{p_1, p_2, p_3 \leq M \\ p_1 + p_2 + p_3 = M}} 1 \tag{1.5}$$

is simply the count of different ways that each M can be represented as the sum of three primes. It is also evident that when $R_3(M) > 0$, by partial summation we have that $C(M) > 0$ and hence we have a representation of M.

Through similar calculations we can derive $R_2(N)$ where N is an even number, and find it to be

$$\begin{aligned} R_2(N) &= \int_0^1 F_N^2(x) e(-Nx) \\ &= \sum_{p_1 + p_2 = N} \log p_1 \log p_2 \end{aligned} \tag{1.6}$$

where

$$F_N(x) = \sum_{p \leq N} (\log p) e(px) \tag{1.7}$$

In general, $R_s(N)$ determines the weighted count of the number of ways that N is the sum of s number of primes.

2 Decomposition into Major and Minor Arcs

Now let us return to the ternary problem. To evaluate the integral $R_3(N)$, we can employ the circle method by decomposing the unit interval, over which the intgral is evaluated, into major and minor arcs.

The major arcs are defined as the following:
Let $B > 0$ and

$$Q = [(\log N)^B] \tag{2.8}$$

where $[\alpha]$ is the greatest integer less than α.
Consider all pairs (a, q) such that

$$\begin{aligned} 0 \leq a \leq q, \\ 1 \leq q \leq Q, \\ (a, q) = 1 \end{aligned} \tag{2.9}$$

Each major arc $\mathcal{M}_{a,q}$ is defined as the interval of $x \in \mathbb{R}$ such that

$$\left| x - \frac{a}{q} \right| \leq \frac{Q}{N}$$

and the set of major arcs $\mathcal{M}$ is

$$\mathcal{M} = \bigcup_{q=1}^{Q} \bigcup_{(a,q)=1}^{q} \mathcal{M}_{q,a} \tag{2.10}$$

Note that for N sufficiently large, the major arcs become disjoint.

The set of minor arcs is defined to be the complement of the set of all major arcs in the unit interval. More clearly,

$$m = [0, 1] - \mathcal{M} \tag{2.11}$$

From this definition of the major arcs, we can find an elementary upper bound for the measure of the set of major arcs, $\mathcal{M}$.

For N sufficiently large that the major arcs are disjoint, we have

$$\begin{aligned}|\mathcal{M}| &= \left|\bigcup_{q=1}^{Q}\bigcup_{(a,q)=1}^{q}\mathcal{M}_{a,q}\right| = \sum_{q=1}^{Q}\sum_{(a,q)=1}^{q}\mu(\mathcal{M}_{a,q}) \\ &= \sum_{q=1}^{Q}\sum_{(a,q)=1}^{q}\frac{2Q}{N} \leq \sum_{q=1}^{Q}\frac{2Qq}{N}\end{aligned}$$

Since $\frac{2}{N}$ does not depend on q

$$= \frac{2}{N}\sum_{q\leq Q} Qq = \frac{Q^2(Q+1)}{N} \approx \frac{Q^3}{N} \tag{2.12}$$

as N grows large.
Observe also that as $N \to \infty$,

$$\lim_{N\to\infty} |\mathcal{M}| = 0$$

So as N grows large, the mass of the major arcs decreases and we see that the unit interval consists mostly of minor arcs. However, it has been shown that despite this density of minor arcs, the contribution of

$$\int_m F_M^3(x)e(-Mx)dx$$

is negligible and the main term of

$$\int_0^1 F_M^3(x)e(-Mx) = \int_{\mathcal{M}} F_M^3(x)e(-Mx)dx + \int_m F_M^3(x)e(-Mx)dx \tag{2.13}$$

comes from the integral over the major arcs.

This is not apparent for the binary case. It becomes much more difficult to show that

$$\int_m F_N^2(x)e(-Nx)dx$$

is neglible compared to the term

$$\int_{\mathcal{M}} F_N^2(x)e(-Nx)dx$$

The contribution of the integral over the minor arcs is precisely the motivation for this investigation. We hope to see that the the contribution of $F_N^2(x)$ from the major arcs is large, while the contribution from the minor arcs is small.

3 Properties of the Integral over the Major Arcs

The integral over the major arcs for the binary problem

$$R_{\mathcal{M}}(N) = \int_{\mathcal{M}} F_N^2(x)e(-Nx)dx \tag{3.14}$$

is not trivial to evaluate so we will first find a function similar to it which is easy to evaluate and formulate $R_{\mathcal{M}}(N)$ in terms of this function. This section will be devoted to deriving a more accessible form of $R_{\mathcal{M}}(N)$.
We begin by defining this function similar to $F_N(x)$ by

$$u(\beta) = \sum_{m=1}^{N} e(m\beta) \tag{3.15}$$

Now consider the function

$$J(N) = \int_0^1 u^2(\beta)e(-N\beta)d\beta \tag{3.16}$$

Theorem 3.1. *As* $N \to \infty, \qquad J(N) \approx N$

Proof:

Since, as before, when integrated from 0 to 1, our function $e(\alpha) = exp(2\pi i\alpha)$ gives a contribution of 0 for all $\alpha \in \mathbb{Z}$ such that $\alpha \neq 0$, and a contribution of 1 when $\alpha = 0$, we have that

$$\begin{aligned} J(N) &= \int_0^1 u^2(\beta)e(-N\beta)d\beta \\ &= \int_0^1 \sum_{m_1=1}^{N} \sum_{m_2=1}^{N} e\left((m_1 + m_2 - N)\beta\right) d\beta \end{aligned}$$

since $1 \leq m_2 < N$ there are $N - 1$ values of m_2, and for each value of m_2 there is a value of $1 \leq m_1 \leq N$ that satisfies

$$m_1 + m_2 - N = 0$$

except when $m_2 = N$. Therefore there are $N-1$ pairs (m_1, m_2) that give a non-zero contribution to $J(N)$. Since each time $m_1 + m_2 = N$,

$$e((m_1 + m_2 - N)\beta) = 1$$

we conclude that

$$J(N) = N - 1 = N + O(1)$$

and as N grows large

$$J(N) \approx N \tag{3.17}$$

and we are finished.

It is not immediately apparent why this result is significant but we shall use this it later on as a lemma to a more important theorem approximating the function $F_N(x)^2$.

Before continuing, let us first define a few useful functions.
1. Let $\phi(q)$ be the number of integers $b < q$ such that $(b, q) = 1$
2. Let $\mu(q)$ be defined as such:

$$\mu(q) = \begin{cases} 1 & \text{if q=1} \\ 0 & \text{if q is divisible by the square of a prime} \\ (-1)^r & \text{if q is the product of r distinct primes} \end{cases} \tag{3.18}$$

3. Let $c_q(N)$, the Ramanujan Sum, be defined by

$$c_q(N) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(\frac{aN}{q}\right) \tag{3.19}$$

4. Let $\lambda(m)$ be defined by

$$\lambda(m) = \begin{cases} \log p & \text{if } m = p \text{ is prime} \\ 0 & \text{otherwise} \end{cases} \tag{3.20}$$

Theorem 3.2. *(Siegel-Walfisz)*

If $1 \leq q \ll (\log n)^C$, $(a,q) = 1$, *and* $n \geq 2$ *then, for any* $C > B > 0$,

$$\phi(n;a,q) = \sum_{\substack{p \leq n \\ p \equiv a(q)}} \log p = \frac{n}{\phi(q)} + O\left(\frac{n}{(\log n)^C}\right) \tag{3.21}$$

Theorem 3.3.
Let

$$F_n(x) = \sum_{p \leq n} (\log p) e(px) \tag{3.22}$$

Let $B, C > 0$ and $B, C \in \mathbb{R}$.
Then for $1 \leq n \leq N$, and $q \leq Q$,

$$F_n\left(\frac{a}{q}\right) = \frac{\mu(q)}{\phi(q)} n + O\left(\frac{QN}{(\log N)^C}\right) \tag{3.23}$$

Proof:

$$F_n\left(\frac{a}{q}\right) = \sum_{p \leq n} (\log p) e(px) = \sum_{r=1}^{q} \sum_{\substack{p \leq n \\ p \equiv r(q)}} (\log p) e(\frac{pa}{q})$$

Let $p \equiv r(modq)$ Then p divides q if and only if $(r,q) > 1$. To bound the contribution of terms where $p|q$ we have

$$\left| \sum_{\substack{r=1 \\ (r,q)>1}}^{q} \sum_{\substack{p \leq n \\ p \equiv r(q)}} (\log p) e(\frac{pa}{q}) \right| = \sum_{\substack{p \leq n \\ p|q}} (\log p) \left| e(\frac{pa}{q}) \right| \ll \sum_{p|q} \log p \leq \log q \tag{3.24}$$

A weaker, but more intuitive bound can be found by considering that there are less than q integers $r \leq q$ such that $(r,q) > 1$. Each one contributes a term less than or equal to $\log q$ since $|e(\frac{pa}{q})| \leq 1$. So a very crude bound would be

$$\sum_{\substack{r=1 \\ (r,q)>1}}^{q} \sum_{\substack{p \leq n \\ p \equiv r(q)}} (\log p) e(\frac{pa}{q}) \ll q \log q \tag{3.25}$$

We shall use the stronger bound but both are sufficient. Continuing on,

$$\begin{aligned} F_n\left(\frac{a}{q}\right) &= \sum_{\substack{r=1 \\ (r,q)=1}}^{q} \sum_{\substack{p \le n \\ p \equiv r(q)}} (\log p) e\left(\frac{ra}{q}\right) + O(\log q) \\ &= \sum_{\substack{r=1 \\ (r,q)=1}}^{q} e\left(\frac{ra}{q}\right) \sum_{\substack{p \le n \\ p \equiv r(q)}} \log p + O(\log q) \end{aligned}$$

By the Siegel-Walfisz Theorem:

$$\begin{aligned} &= \sum_{\substack{r=1 \\ (r,q)=1}}^{q} e\left(\frac{ra}{q}\right) \theta(n;q,r) + O(\log q) \\ &= \sum_{\substack{r=1 \\ (r,q)=1}}^{q} e\left(\frac{ra}{q}\right) \left(\frac{n}{\phi(q)} + O\left(\frac{n}{(\log n)^C}\right)\right) + O(\log q) \end{aligned}$$

Since $\left|\sum_{r=1}^{q} e\left(\frac{ra}{q}\right)\right| \le q$,

$$\frac{c_q(a)}{\phi(q)} n + O\left(\frac{qn}{(\log n)^C}\right) + O(\log Q)$$

Since $c_q(a) = \mu(a)$ for $(a,q) = 1$, $q \le Q$, and $n \le N$,

$$= \frac{\mu(q)}{\phi(q)} n + O\left(\frac{QN}{(\log N)^C}\right)$$

and we are finished.

Theorem 3.4. *Let $B, C \in \mathbb{R}$ such that $B, C > 0$ and $C > 2B$. If $x \in \mathcal{M}_{a,q}$ and $\beta = x - \frac{a}{q}$, then*

$$F_N(x) = \frac{\mu(q)}{\phi(q)} u(\beta) + O\left(\frac{Q^2 N}{(\log N)^C}\right) \tag{3.26}$$

and

$$F_N^2(x) = \frac{\mu^2(q)}{\phi^2(q)} u^2(\beta) + O\left(\frac{Q^2 N^2}{(\log N)^C}\right) \tag{3.27}$$

Proof:

If $x \in \mathcal{M}_{(a,q)}$ then $x = \frac{a}{q} + \beta$ where $|\beta| \leq \frac{Q}{N}$
For $1 \leq n \leq N$,

$$\begin{aligned} F_N(x) - \frac{\mu(q)}{\phi(q)}u(\beta) &= \sum_{p \leq N} (\log p) e(px) - \frac{\mu(q)}{\phi(q)} \sum_{m=1}^{N} e(m\beta) \\ &= \sum_{m=1}^{N} \lambda(m) e(mx) - \frac{\mu(q)}{\phi(q)} \sum_{m=1}^{N} e(m\beta) \\ &= \sum_{m=1}^{N} \lambda(m) e\left((\frac{ma}{q} + m\beta) \right) - \sum_{m=1}^{N} \frac{\mu(q)}{\phi(q)} e(m\beta) \\ &= \sum_{m=1}^{N} \left(\lambda(m) e(\frac{ma}{q}) - \frac{\mu(q)}{\phi(q)} \right) e(m\beta) \qquad (3.28) \end{aligned}$$

We apply Partial Summation to this last term. Let $A(n)$ be defined by

$$A(n) = \sum_{1 \leq m \leq n} \left(\lambda(m) e(\frac{ma}{q}) - \frac{\mu(q)}{\phi(q)} \right) \qquad (3.29)$$

Since $\frac{\mu(q)}{\phi(q)}$ does not depend on n, and $\mu(q) \leq 1$,

$$A(n) = \sum_{1 \leq m \leq n} \lambda(m) e(\frac{ma}{q}) - \frac{\mu(q)}{\phi(q)} n$$

From the definition of $\lambda(m)$ we can see that the only terms which give a non-zero contribution are when $m = p$ where p is a prime. So we have

$$= \sum_{p \leq n} (\log p) e(\frac{pa}{q}) - \frac{\mu(q)}{\phi(q)} n = F_n \left(\frac{a}{q} \right) - \frac{\mu(q)}{\phi(q)} n$$

By Theorem 3.3,

$$A(n) = O \left(\frac{QN}{(\log p)^C} \right) \qquad (3.30)$$

By using $A(n)$ and Partial Summation again, we can find an upper bound for

$$\begin{aligned}
& F_N(x) - \frac{\mu(q)}{\phi(q)}u(\beta) \\
= \quad & A(N)e(N\beta) - 2\pi i\beta \int_1^N A(n)e(n\beta)dn \\
\ll \quad & |A(N)| + |\beta|\, N max\{A(n) : 1 \leq n \leq N\} \\
\ll \quad & \frac{Q^2 N}{(\log N)^C} \qquad (3.31)
\end{aligned}$$

It follows directly that

$$F_N(x) = \frac{\mu(q)}{\phi(q)}u(\beta) + O\left(\frac{Q^2 N}{(\log N)^C}\right) \qquad (3.32)$$

Let us now derive

$$\begin{aligned}
F_N^2(x) &= \left(\frac{\mu(q)}{\phi(q)}u(\beta) + O\left(\frac{Q^2 N}{(\log N)^C}\right)\right)^2 \\
&= \frac{\mu^2(q)}{\phi^2(q)}u^2(\beta) + 2\left(\frac{\mu(q)}{\phi(q)}u(\beta)\right) O\left(\frac{Q^2 N}{(\log N)^C}\right) + O\left(\frac{Q^4 N^2}{(\log N)^{2C}}\right)
\end{aligned}$$

As $N \to \infty$, we find that

$$\frac{Q^2 N}{(\log N)^C} \ll \frac{Q^4 N^2}{(\log N)^{2C}}$$

so the cross term

$$2\left(\frac{\mu(q)}{\phi(q)}u(\beta)\right) O\left(\frac{Q^2 N}{(\log N)^C}\right)$$

becomes negligible.
Since $Q = (\log N)^B$ and $C > 2B$,

$$\frac{1}{(\log N)^C} \leq \frac{1}{(\log N)^{2B}}$$

so

$$O\left(\frac{Q^4N^2}{(\log N)^{2C}}\right) = O\left(\frac{(\log N)^{2B}Q^2N^2}{(\log N)^{2B}(\log N)^C}\right) = O\left(\frac{Q^2N^2}{(\log N)^C}\right)$$

and we conclude that

$$F_N^2(x) = \frac{\mu^2(q)}{\phi^2(q)}u^2(\beta) + O\left(\frac{Q^2N^2}{(\log N)^C}\right)$$

Note that the error term

$$O\left(\frac{Q^2N^2}{(\log N)^C}\right) = O\left(\frac{N^2}{(\log N)^{C-2B}}\right) \tag{3.33}$$

is much smaller than N^2 for $C > 2B$.
We shall now investigate the contribution from the major arcs.

Theorem 3.5. *Let $B, C \in \mathbb{R}$ such that $B, C > 0$ and $C > 2B$. Let $\epsilon > 0$. Then the weighted sum $R_{\mathcal{M}}(N)$ over the major arcs $\mathcal{M}$ can be represented as*

$$\int_{\mathcal{M}} F_N^2(x)e(-Nx)dx = \mathfrak{S}(N)\frac{N^2}{2} + O\left(\frac{N^2}{(\log N)^{(1-\epsilon)B}}\right) + O\left(\frac{N^2}{(\log N)^{C-5B}}\right) \tag{3.34}$$

where $\mathfrak{S}(N)$ is defined below.

Proof:
First consider this integral:

$$\int_{\mathcal{M}} \left(F_N^2(x) - \frac{\mu^2(q)}{\phi^2(q)}u^2\left(x - \frac{a}{q}\right)\right) e(-Nx)dx$$

$$= \sum_{q \le Q} \sum_{\substack{a=0 \\ (a,q)=1}}^{q} \int_{\mathcal{M}_{(a,q)}} \left(F_N^2(x) - \frac{\mu^2(q)}{\phi^2(q)}u^2\left(x - \frac{a}{q}\right)\right) e(-Nx)dx$$

simply from the definition of the major arcs $\mathcal{M}$.

Then, since the function $e(\alpha x) \leq 1$, and Theorem 3.4,

$$\ll \sum_{q \leq Q} \sum_{\substack{a=0 \\ (a,q)=1}}^{q} \int_{\mathcal{M}_{(a,q)}} \frac{Q^2 N^2}{(\log N)^C} dx$$

Observe that when $q = 1$, $\left|\mathcal{M}_{(a,q)}\right| = \frac{Q}{N}$, and when $q \geq 2$, $\left|\mathcal{M}_{(a,q)}\right| = \frac{2Q}{N}$. Then since $C > 2B$,

$$\begin{aligned} &\ll \sum_{q \leq Q} \sum_{\substack{a=0 \\ (a,q)=1}}^{q} \int_{\mathcal{M}_{(a,q)}} \frac{Q^3 N}{(\log N)^C} dx \\ &\leq \frac{Q^5 N}{(\log N)^C} \\ &\leq \frac{N}{(\log N)^{C-5B}} \end{aligned} \tag{3.35}$$

Since

$$\int_{\mathcal{M}} \left(F_N^2(x) - \frac{\mu^2(q)}{\phi^2(q)} u^2 \left(x - \frac{a}{q} \right) \right) e(-Nx) dx \ll \frac{N}{(\log N^{C-5B})}$$

to determine

$$R_{\mathcal{M}}(N) = \int_{\mathcal{M}} F_N^2(x) e(-Nx) dx$$

it is sufficient to study

$$\begin{aligned} &\int_{\mathcal{M}} \frac{\mu^2(q)}{\phi^2(q)} u^2 \left(x - \frac{a}{q} \right) e(-Nx) dx \\ = &\sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\mathcal{M}_{(a,q)}} \frac{\mu^2(q)}{\phi^2(q)} u^2 \left(x - \frac{a}{q} \right) e(-Nx) dx \\ = &\sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{\mu^2(q)}{\phi^2(q)} \int_{\frac{a}{q} - \frac{Q}{N}}^{\frac{a}{q} + \frac{Q}{N}} u^2 \left(x - \frac{a}{q} \right) e(-Nx) dx \end{aligned}$$

Since $x = \frac{a}{q} + \beta$, and the integrand is periodic,

$$= \sum_{q \leq Q} \frac{\mu^2(q)}{\phi^2(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{-\frac{Q}{N}}^{\frac{Q}{N}} u^2(\beta) e\left(-N(\frac{a}{q} + \beta)\right)$$

$$= \sum_{q \leq Q} \frac{\mu^2(q)}{\phi^2(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-\frac{Na}{q}\right) \int_{-\frac{Q}{N}}^{\frac{Q}{N}} u^2(\beta) e(-N\beta)$$

Recalling the definition of $c_q(N)$, the above is

$$\sum_{q \leq Q} \frac{\mu^2(q) c_q(-N)}{\phi^2(q)} \int_{-\frac{Q}{N}}^{\frac{Q}{N}} u^2(\beta) e(-N\beta) d\beta$$

$$= \mathfrak{S}(N, Q) \int_{-\frac{Q}{N}}^{\frac{Q}{N}} u^2(\beta) e(-N\beta) d\beta \tag{3.36}$$

where

$$\mathfrak{S}(N, Q) = \sum_{q \leq Q} \frac{\mu^2(q) c_q(-N)}{\phi^2(q)} \tag{3.37}$$

Continuing with the proof, observe that if $|\beta| \leq \frac{1}{2}$, then $u(\beta) \ll |\beta|^{-1}$. Since $e(\alpha x) \leq 1$,

$$\int_{\frac{Q}{N}}^{\frac{1}{2}} u^2(\beta) e(-N\beta) d\beta$$

$$\ll \int_{\frac{Q}{N}}^{\frac{1}{2}} u^2(\beta) d\beta$$

$$\ll \int_{\frac{Q}{N}}^{\frac{1}{2}} \beta^{-2} = \frac{N}{Q} - 2 \leq \frac{N}{Q}$$

Implementing the same calculation, we obtain that

$$\int_{-\frac{1}{2}}^{-\frac{Q}{N}} u^2(\beta) e(-N\beta) d\beta \ll \frac{N}{Q}$$

From these approximations, it is easy to see that

$$\int_{-\frac{Q}{N}}^{\frac{Q}{N}} u^2(\beta)e(-N\beta)d\beta = \int_{-\frac{1}{2}}^{\frac{1}{2}} u^2(\beta)e(-N\beta)d\beta$$

up to a small error.
By Theorem 3.1, we have that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} u^2(\beta)e(-N\beta)d\beta = N + O\left(\frac{N}{Q}\right) \tag{3.38}$$

Now consider $\mathfrak{S}(N, Q)$:
Let $\mathfrak{S}(N)$ be defined by

$$\mathfrak{S}(N) = \sum_{q=1}^{\infty} \frac{\mu^2(q)c_q(N)}{\phi^2(q)} \tag{3.39}$$

This is the Singular Series for the Binary Goldbach Problem. We first will show that it converges. Then we will show that for $Q \to \infty$, $\mathfrak{S}(N, Q)$ converges to $\mathfrak{S}(N)$

We know that $c_q(N) \ll \phi(q)$ and that $\phi(q) > q^{1-\epsilon}$ for $\epsilon > 0$ and sufficiently large q. We know that $c_q(N)$ is multiplicative.

Let us, for the moment, use a clearer notation for this discussion. Since N is fixed, we shall denote $c_q(N)$ by $c_N(q)$.
We see that since $c_N(q)$ is multiplicative in q, that if $q = p_1 p_2 \ldots p_r$, that

$$c_N(q) = c_N(p_1 p_2 \ldots p_r) = c_N(p_1)c_N(p_2)\ldots c_N(p_r)$$

Note that each p_i must be distinct since if q is divisible by p_i^k where $k \geq 2$, then $\mu(q) = 0$.
Also note that from the definition of

$$c_q(N) = c_N(q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(\frac{aN}{q}\right)$$

we have

$$c_N(p) = \begin{cases} p-1 & \text{if p divides N} \\ -1 & \text{otherwise} \end{cases} \tag{3.40}$$

and since $p \leq N$, it follows that

$$\frac{\mu^2(q)c_N(q)}{\phi^2(q)} \ll \frac{(N-1)^s}{\phi^2(q)} + \frac{(-1)^{r-s}}{\phi^2(q)} < \frac{(N-1)^s}{q^{2(1-\epsilon)}} + \frac{(-1)^{r-s}}{q^{2(1-\epsilon)}} \tag{3.41}$$

where s is the number of prime factors common to q and N.
Since the numerators do not depend on q, it is obvious that as $q \to \infty$, the Singular Series

$$\mathfrak{S}(N) = \sum_{q=1}^{\infty} \frac{\mu^2(q)c_q(N)}{\phi^2(q)}$$

converges.
Convergence of the Singular Series is also apparent when we consider the bound

$$c_q(N) \leq \phi(N) \leq N \tag{3.42}$$

Then we have

$$\frac{\mu^2(q)c_N(q)}{\phi^2(q)} \leq \frac{N}{\phi^2(q)} < \frac{N}{q^{2(1-\epsilon)}} \tag{3.43}$$

which tends to 0 as $q \to \infty$.
Now, for fixed N, and q square-free, we have

$$\begin{aligned} \mathfrak{S}(N) - \mathfrak{S}(N,Q) &= \sum_{q>Q} \frac{\mu^2(q)c_q(N)}{\phi^2(q)} \\ &\ll \sum_{q>Q} \frac{N}{\phi^2(q)} \ll \sum_{q>Q} \frac{N}{q^{2-\epsilon}} \\ &\ll \frac{N}{Q^{1-\epsilon}} \end{aligned}$$

therefore

$$\mathfrak{S}(N,Q) = \mathfrak{S}(N) + O\left(\frac{N}{Q^{1-\epsilon}}\right) \tag{3.44}$$

but this approximation is of no use since the error term is of the same order as the main term. With theorem 3.5 as motivation, we have now discovered a problem which has not yet been solved. It has not yet been shown that

$$\mathfrak{S}(N, Q)$$

can be approximated by

$$\mathfrak{S}(N)$$

with an error term smaller than the main term. We now digress into a discussion on this problem.

We see that the problem arises in bounding $c_q(N)$ when the common factors between q and N, (q, N) is very large. We will consider the special case where N has few factors relative to its size. In particular, fix an integer $D > 0$. We consider $N \to \infty$ such that the number of factors of N is at most $\log^D(N)$. Let us represent q as the product of two functions, $x(q)y(q)$, where $x(q)$ is the product of factors common to q and N, and $y(q)$ is the product of what remains.

Observe that for each $y(q)$, there are at most $\tau(N) \ll \log^D(N)$ (ref: HW) choices of $x(q)$ that have the same $y(q)$.

Now let's consider the sum

$$S = \sum_{\substack{x(q), y(q) \\ x(q)y(q) > Q}} \frac{C_N(x(q)y(q))}{\phi^2(x(q)y(q))}$$

There are two cases to consider. The first is when $x(q) > Q^{1/2}$ and the second is when $y(q) > Q^{1/2}$. We can split the sum into two sums with these conditions as follows:

$$S = \sum_{\substack{x(q), y(q) \\ x(q) > Q^{1/2}}} \frac{C_N(x(q)y(q))}{\phi^2(x(q)y(q))} + \sum_{\substack{x(q), y(q) \\ y(q) > Q^{1/2}}} \frac{C_N(x(q)y(q))}{\phi^2(x(q)y(q))}$$

Let us first examine the first sum, when $x(q) > Q^{1/2}$. Call it L. Since C_N and ϕ are both multiplicative functions, we can separate the terms. We have,

$$L = \sum_{\substack{x(q),y(q)\\ x(q)>Q^{1/2}}} \frac{C_N(x(q)y(q))}{\phi^2(x(q)y(q))} = \sum_{\substack{x(q),y(q)\\ x(q)>Q^{1/2}}} \frac{C_N(x(q))}{\phi^2(x(q))} * \frac{C_N(y(q))}{\phi^2(y(q))}$$
$$= \sum_{\substack{x(q)\\ x(q)>Q^{1/2}}} \frac{C_N(x(q))}{\phi^2(x(q))} * \sum_{y(q)} \frac{C_N(y(q))}{\phi^2(y(q))}$$

We observe that, by definition, and by the fact that $x(q)|N$,

$$C_N(x(q)) = \sum_{\substack{a=1\\ (a,x(q))=1}}^{x(q)} e\left(\frac{aN}{x(q)}\right) = \sum_{\substack{a=1\\ (a,q)=1}}^{x(q)} 1 = \phi(x(q))$$

And since $y(q) \not| N$,

$$C_N(y(q)) = \sum_{\substack{a=1\\ (a,y(q))=1}}^{y(q)} e\left(\frac{aN}{y(q)}\right) < \sum_{\substack{a=1\\ (a,q)=1}}^{y(q)} 1 = \phi(y(q))$$

We then have that,

$$L < \sum_{\substack{x(q)\\ x(q)>Q^{1/2}}} \frac{1}{\phi(x(q))} * \sum_{y(q)} \frac{1}{\phi(y(q))} < \sum_{\substack{x(q)\\ x(q)>Q^{1/2}}} \frac{1}{x(q)^{1-\epsilon}} * \sum_{y(q)} \frac{1}{\phi(y(q))}$$
$$< \sum_{x(q)} \frac{1}{Q^{(1-\epsilon)/2}} * \sum_{y(q)} \frac{1}{\phi(y(q))} \leq \frac{\tau(N)}{Q^{(1-\epsilon)/2}} * \sum_{y(q)} \frac{1}{\phi(y(q))}$$
$$\ll \frac{\log^D(N)}{Q^{(1-\epsilon)/2}} * \sum_{y(q)} \frac{1}{\phi(y(q))} = O\left(\frac{\log^D(N)}{Q^{(1-\epsilon)/2}}\right)$$

Let us now examine the second sum. We shall call it R.

$$R = \sum_{\substack{x(q),y(q)\\ y(q)>Q^{1/2}}} \frac{C_N(x(q)y(q))}{\phi^2(x(q)y(q))} = \sum_{\substack{x(q),y(q)\\ y(q)>Q^{1/2}}} \frac{C_N(x(q))}{\phi^2(x(q))} * \frac{C_N(y(q))}{\phi^2(y(q))}$$
$$= \sum_{\substack{y(q)\\ (q)>Q^{1/2}}} \frac{C_N(x(q))}{\phi^2(x(q))} * \sum_{x(q)} \frac{C_N(y(q))}{\phi^2(y(q))}$$

By similar reasoning as with L, we find that

$$\begin{aligned} R &< \sum_{\substack{y(q) \\ y(q)>Q^{1/2}}} \frac{1}{\phi(y(q))} * \sum_{x(q)} \frac{1}{\phi(x(q))} \\ &\ll \frac{1}{Q^{(1-\epsilon)/2}} \sum_{x(q)} \frac{1}{\phi(x(q))} \\ &\ll \frac{log^d(N)}{Q^{(1-\epsilon)/2}} \end{aligned}$$

Since D was arbitrary, we see that we can choose D so that for N such that $\tau(N) \ll log^D(N)$, $\mathfrak{S}(N,Q)$ is a good approximation to $\mathfrak{S}$. We end our digression here.
Returning to the proof, we have

$$\begin{aligned} R_{\mathcal{M}}(N) &= \int_{\mathcal{M}} F_N^2(x)e(-Nx)dx \\ &= \mathfrak{S}(N,Q) \int_{-\frac{Q}{N}}^{\frac{Q}{N}} u^2(\beta)e(-N\beta)d\beta + O\left(\frac{N}{(\log N)^{C-5B}}\right) \\ &= \left(\mathfrak{S}(N) + O\left(\frac{1}{Q^{1-\epsilon}}\right)\right)\left(N + O\left(\frac{N}{Q}\right)\right) \\ &+ O\left(\frac{N}{(\log N)^{C-5B}}\right) \\ &= N\mathfrak{S}(N) + O\left(\frac{N}{Q^{1-\epsilon}}\right) + O\left(\frac{N}{(\log N)^{C-5B}}\right) \\ &= N\mathfrak{S}(N) + O\left(\frac{N}{(\log N)^{(1-\epsilon)B}}\right) + O\left(\frac{N}{(\log N)^{C-5B}}\right) \end{aligned}$$

and this is the desired result.

4 The Singular Series

Earlier, we had defined the Singular Series for the Binary Goldbach Problem to be

$$\mathfrak{S}(N) = \sum_{q=1}^{\infty} \frac{\mu^2(q) c_q(N)}{\phi^2(q)}$$

This function has many interesting properties. The first that we will discuss is that it can be represented as an Euler Product.

Theorem 4.1. *The Singular Series has the Euler Product*

$$\mathfrak{S}(N) = \prod_{p \nmid N} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|N} \left(1 + \frac{1}{p-1}\right) \tag{4.45}$$

Proof:
Recall the calculation done in Section 3 that showed that

$$\frac{\mu^2(q) c_q(N)}{\phi^2(q)} \ll \frac{(N-1)^r}{q^{2(1-\epsilon)}}$$

Hence the Singular Series converges. Since the functions $c_q(N)$, $\mu(q)$, and $\phi(q)$ are multiplicative in q, and the product of multiplicative functions is multiplicative, so is the Singular Series. Therefore it has the Euler Product

$$\mathfrak{S}(N) = \prod_{p} \left(1 + \sum_{j=1}^{\infty} \frac{\mu^2(p^j) c_{p^j}(N)}{\phi^3(p^j)}\right)$$

Returning to the definition of the function $\mu(q)$, we see that for $j \geq 2$, $\mu(q) = 0$. So we can write the Singular Series as

$$\mathfrak{S}(N) = \prod_{p} \left(1 + \sum_{j=1}^{\infty} \frac{\mu^2(p) c_p(N)}{\phi^3(p)}\right)$$

Recall that

$$c_p(N) \;=\; \begin{cases} p-1 & \text{if p divides N} \\ -1 & \text{otherwise} \end{cases}$$

It follows that

$$\prod_p \left(1+\sum_{j=1}^{\infty}\frac{\mu^2(p)c_p(N)}{\phi^3(p)}\right) = \prod_p\left(1+\frac{c_p(N)}{\phi^2(p)}\right)$$

$$= \prod_{p\nmid N}\left(1+\frac{c_p(N)}{\phi^2(p)}\right)\prod_{p|N}\left(1+\frac{c_p(N)}{\phi^2(p)}\right)$$

$$= \prod_{p\nmid N}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p|N}\left(1+\frac{1}{p-1}\right)$$

The Singular Series is very significant to the Circle Method. It basically encodes all of the information that restricts us to the problem we are investigating. For example:

If we try to prove that every odd number is the sum of two primes, we observe that in order for this to happen, one of the primes has to be 2. This is because 2 is the only even prime and the sum of two odd numbers is even. Now, looking back at the Euler Product for our Singular Series for two primes, since N is odd, our first prime 2 does not divide N, and so we have

$$\left(1-\frac{1}{(2-1)^2}\right)\prod_{\substack{p\nmid N\\ p>2}}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p|N}\left(1+\frac{1}{p-1}\right)$$

$$= 0*\prod_{\substack{p\nmid N\\ p>2}}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p|N}\left(1+\frac{1}{p-1}\right) = 0$$

and so the weighted sum of the number of ways our odd N can be represented as the sum of two primes, $R(N)$ is 0. This reflects that it is not true that all odd numbers can be represented as the sum of two primes. We can find a simple counterexample by considering the number 27.

For the sake of interest, let us now turn to the Singular Series for the ternary Goldbach Problem. It follows from similar analysis that

$$\begin{aligned}\mathfrak{S}(M) &= \sum_{q=1}^{\infty} \frac{\mu(q)c_q(M)}{\phi^3(q)} \\ &= \prod_{p\nmid M}\left(1+\frac{1}{(p-1)^3}\right)\prod_{p|M}\left(1-\frac{1}{(p-1)^2}\right) \qquad (4.46)\end{aligned}$$

If we violate the condition that M is odd and set M as even, we see immediately that the Singular Series forces the weighted sum $R_3(M)$ to equal zero since 2 divides M

$$\prod_{p\nmid M}\left(1+\frac{1}{(p-1)^3}\right)\prod_{p|M}\left(1-\frac{1}{(2-1)^2}\right) = \prod_{p\nmid M}\left(1+\frac{1}{(p-1)^3}\right) * 0 = 0$$

This is very interesting since if the Singular Series allowed us to show that every even number is the sum of three primes then we would be able to prove the binary Goldbach problem from that result. Let us see why.
Let it be true that

$$2n = p_1 + p_2 + p_3$$

$\forall n \in \mathbb{N}$ where p_1, p_2, p_3 depend on n.
Now, there are three primes (p_1, p_2, p_3) and each can be even or odd. This gives us eight different combinations. The four such that $p_1+p_2+p_3$ is even are
1. if all of them are even
or
2. if one is even while the other two are odd.

If all of them are even, they must all be 2 since 2 is the only even prime. The sum is then 6 and we are limited to this specific case. If we consider one of the primes, say p_1, to be even, and hence 2, and the other two to be odd, then we have

$$2n = 2 + p_2 + p_3$$

Furthermore, we have

$$2(n-1) = p_2 + p_3$$

and for $n \geq 2$ we have proven that every even number greater than 2 can be represented as the sum of two primes.

This demonstrates the power of $\mathfrak{S}(N)$ and why the rigidity of the Singular Series is so important. It is a very precise expression.

5 The Integral Over the Minor Arcs

Now that we have established the value of the integral over the Major Arcs within a certain error term, we must investigate the contribution of the integral over the Minor Arcs. Recall that our weighted sum, $R_2(N)$, of the number of different ways the number N can be the sum of two primes can be represented by

$$\begin{aligned} \mathbb{R}(N) &= \int_0^1 F_N^2(x)e(-Nx)dx \\ &= \int_{\mathcal{M}} F_N^2(x)e(-Nx)dx + \int_m F_N^2(x)e(-Nx)dx \end{aligned}$$

Vinogradov showed that in the case of the ternary Goldbach problem, the integral over the minor arcs is bounded by

$$\int_m F_M^3(x)e(-Nx)dx \ll \frac{M^2}{(\log M)^{(B/2)-5}} \tag{5.47}$$

From this he showed that for $A > 0$

$$R(M) = \int_0^1 F_M^3(x)e(-Mx)dx = \mathfrak{S}(M)\frac{M^2}{2} + O\left(\frac{M^2}{(\log M)^A}\right) \tag{5.48}$$

which is bounded away from 0 since as $M \to \infty$, $\log M \to \infty$ and so

$$\frac{M^2}{(\log M)^A} \ll \mathfrak{S}(M)\frac{M^2}{2}$$

Therefore the function $R(M) > 0$.
Note that the function

$$R(M) = \sum_{\substack{p_1,p_2 \leq M \\ p_1+p_2=M}} \log p e(px)$$

is greater than 0 if and only if the function

$$C(M) = \sum_{\substack{p_1, p_2 \le M \\ p_1 + p_2 = M}} 1$$

is greater than 0. Therefore, since $R(M) > 0$, the number of ways M can be represented as the sum of three primes is greater than 0. Since the function $C(M)$ only takes integer values, $C(M) \geq 1$, and the ternary problem is proven.

Let us investigate some elementary bounds for the integral over the minor arcs. For the ternary case, it was possible to determine bounds for the integral over the minor arcs. If we simply take

$$\begin{aligned} \int_m F_M^3(x) e(-Nx) dx &\ll \int_m |F_M(x)|^3 \, dx \\ &\ll max\{|F_M(x)| : x \in m\} \int_m |F_M(x)|^2 \, (x) dx \end{aligned}$$

then it is possible to find a bound for the maximum of the function F_M on the minor arcs and the integral

$$\int_0^1 |F_M(x)|^2 = \sum_{p \le M} (\log p)^2 \le \log M \sum_{p \le M} \log p$$

and by Chebyshev's Theorem,

$$\ll M \log M$$

And Vinogradov gives us a bound for the function $F_M(x)$.

$$\begin{aligned} F_M(x) &\ll \left(\frac{N}{q^{1/2}} + N^{4/5} + N^{1/2} q^{1/2} \right) (\log N)^4 \\ &\ll \frac{N}{(\log N)^{(B/2)-4}} \end{aligned} \tag{5.49}$$

From above, we have

$$
\begin{aligned}
& max\{|F_M(x)| : x \in m\} \int_m |F_M(x)|^2 (x)dx \\
\ll\ & \frac{N}{(\log N)^{(B/2)-4}} \int_0^1 |F_M(x)|^2 dx \\
\ll\ & \frac{N^2}{(\log N)^{(B/2)-5}}
\end{aligned}
$$

When we try to apply this method to the binary Goldbach problem, we find that we cannot bound $R(N)$ away from 0 since we cannot determine the contribution from the minor arcs to be small enough that were it to be negative, it still would not contribute enough to diminish the value of $R(N)$ to less than 0. We have

$$
\begin{aligned}
R(N) &= N\mathfrak{S}(N) + O\left(\frac{N}{(\log N)^{(1-\epsilon)}B}\right) + O\left(\frac{N}{(\log N)^{C-5B}}\right) \\
&+ \int_m F_N^2(x)e(-Nx)dx \qquad (5.50)
\end{aligned}
$$

and we cannot yet show that

$$
\left|O\left(\frac{N}{(\log N)^{(1-\epsilon)}B}\right)\right| + \left|O\left(\frac{N}{(\log N)^{C-5B}}\right)\right| + \left|\int_m F_N^2(x)e(-Nx)dx\right| < N\mathfrak{S}(N)
$$

If we try to bound the integral over the minor arcs in the same way that we did for the ternary case we find that

$$
\begin{aligned}
\int_m F_N^2(x)e(-Nx)dx &\ll \int_m |F_N(x)|^2 dx \\
&\ll max\{|F_N(x)|\} \int_m |F_N(x)| dx
\end{aligned}
$$

but it is not easy to bound

$$\int_m |F_N(x)|\,dx$$

enough. As we saw earlier,

$$\int_0^1 |F_N(x)|\,dx \ll N\log N$$

but looking back at the contribution from the major arcs we see that the term from the minor arcs is of larger order.

$$N\mathfrak{S}(N) + O\left(\frac{N}{(\log N)^{(1-\epsilon)B}}\right) + O\left(\frac{N}{(\log N)^{C-5B}}\right) + O\left(N\log N\right)$$

is not necessarily larger than 0.

Let us try another method to bound the contribution from the minor arcs. We employ the Cauchy-Schwartz inequality. Again, we have

$$\begin{aligned} & max\{|F_N(x)|\}\int_m |F_N(x)|\,dx = max\{|F_N(x)|\}\int_m |F_N(x)| * 1dx \\ \le \quad & max\{|F_N(x)|\}\left(\int_m |F_N(x)|^2\right)^{1/2}\left(\int_m |1|^2\right)^{1/2} \ll \frac{N}{(\log N)^{(B/2)-4}}(N\log N)^{1/2} \\ = \quad & \frac{N^{3/2}}{(\log N)^{(B/2)-(7/2)}} \qquad (5.51) \end{aligned}$$

which is still larger than the term contributed by the major arcs.

We now turn to computational methods to determine the behavior of the integral over the minor arcs for the binary problem.

6 Minor Arc Behavior

As we saw in the previous sections, it is not a simple task to theoretically bound the contribution from the minor arcs. Consider the weighted function

$$F_N(x) = \sum \log p e(2\pi ipx) = \sum(\cos 2\pi px + i\sin 2\pi px) \qquad (6.52)$$

For the sake of simplicity we only investigate the behavior of the real part

$$\sum(\cos 2\pi px) \tag{6.53}$$

of $F_N(x)$ in this paper. The main questions we have to ask in bounding the contribution from the minor arcs are:

1: What is the maximum value attained by the function?

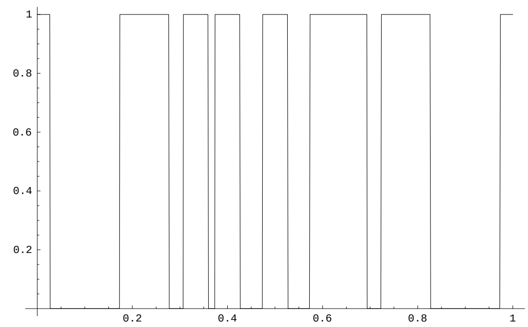
2: How often is this maximum value attained, and what is the measure of the set that it attains this maximum value on?

3: How small is the function away from the peaks?

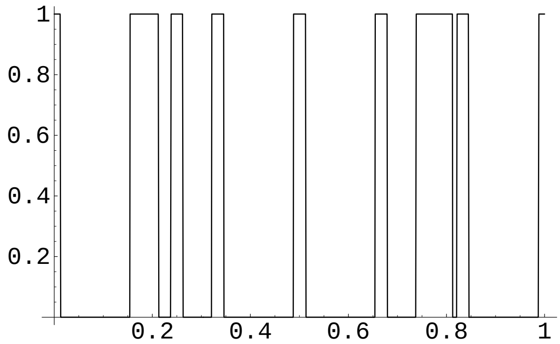
By observing the graphs, we can get a good sense for how the function looks and the liklihood that Goldbach's Conjecture is true.

Before examining the behavior of the function on the minor arcs, let us first take a look at a plot of the major arcs themselves. Note that this is not a plot of the function on the major arcs, but simply an illustration of where the major arcs lie and their spacings.

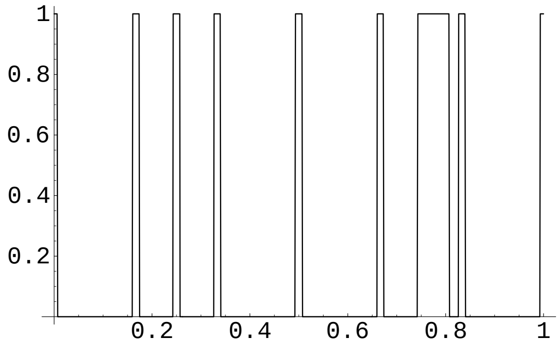
This particular case is the major arc intervals for the first two hundred primes.



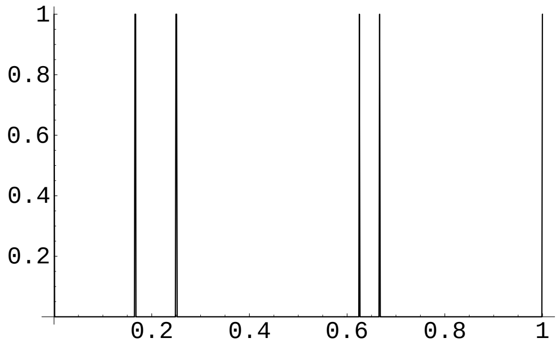
For the first five hundred primes, we have



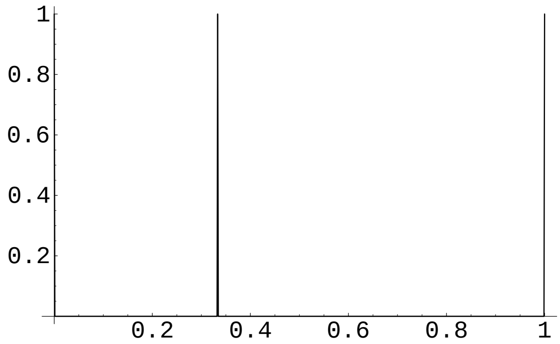
For the first thousand primes, we have



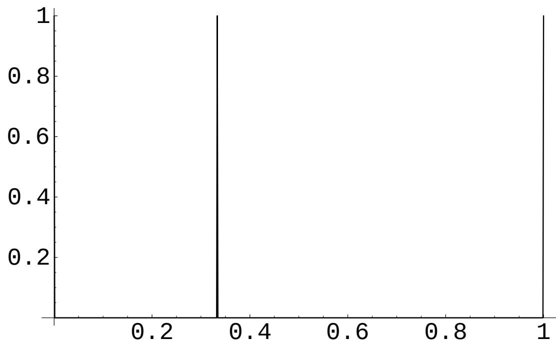
For the first ten thousand primes, we have



For the first one hundred thousand primes, we have

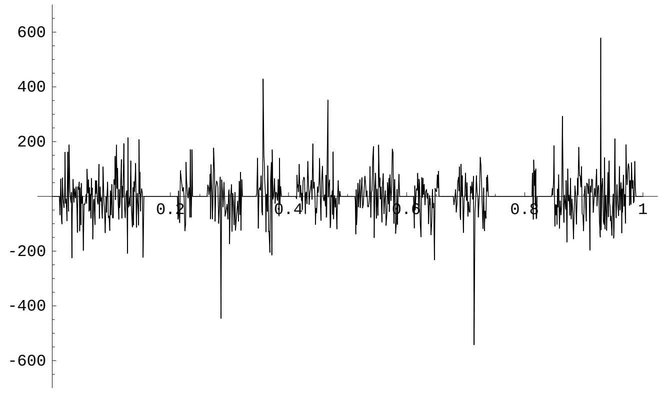


For the first million primes, we have



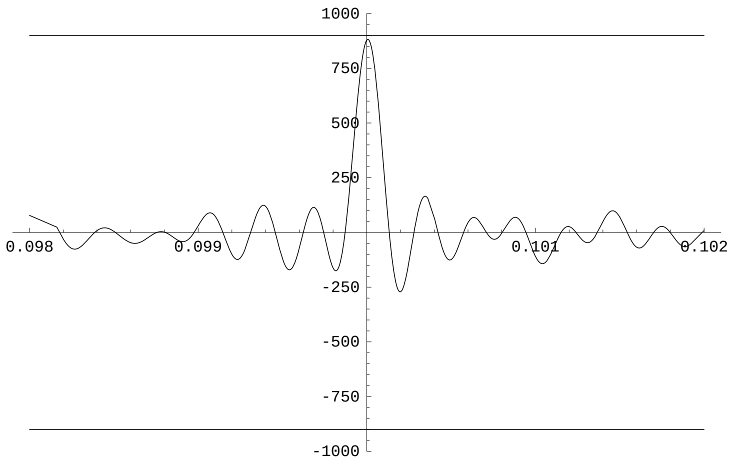
Note how thin and sparse the major arcs become as the number of primes increase. Now let us investigate the behavior of the function on the minor arcs. Note also that we are only investigating the cases where $Q = \log^B(N)$ where $B = 1$. This is because for larger values of B, the major arcs are so thick that they do not become distinct until N is sufficiently high. We have shown behavior up to the first one hundred thousand primes, but for N to take on any higher values would require an extremely long time to compute with the software we have available.

We begin by observing the graph of the real part of the function over the minor arcs in the unit interval for the first five hundred primes.



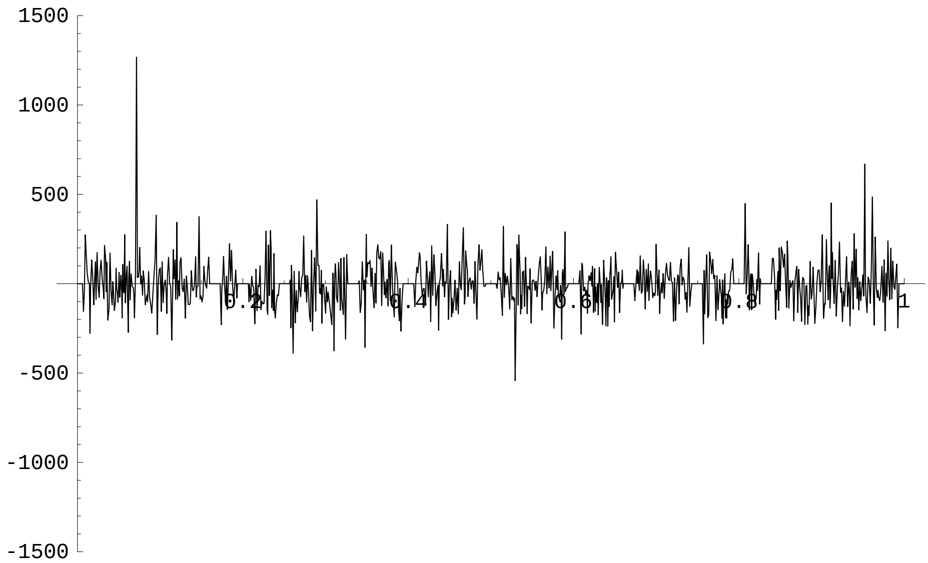
For the first five hundred primes, the major arcs are still apparent and we can see them here as the intervals where the function over the minor arcs vanishes. We now focus in on a smaller interval around one of the peaks to observe its behavior in more detail.

The interval is $(0.098, 0.102)$ where the peak appears to be the highest. The constant plots are at -900 and 900, just as visual aid for determining exactly where the maximums of the peaks lie.

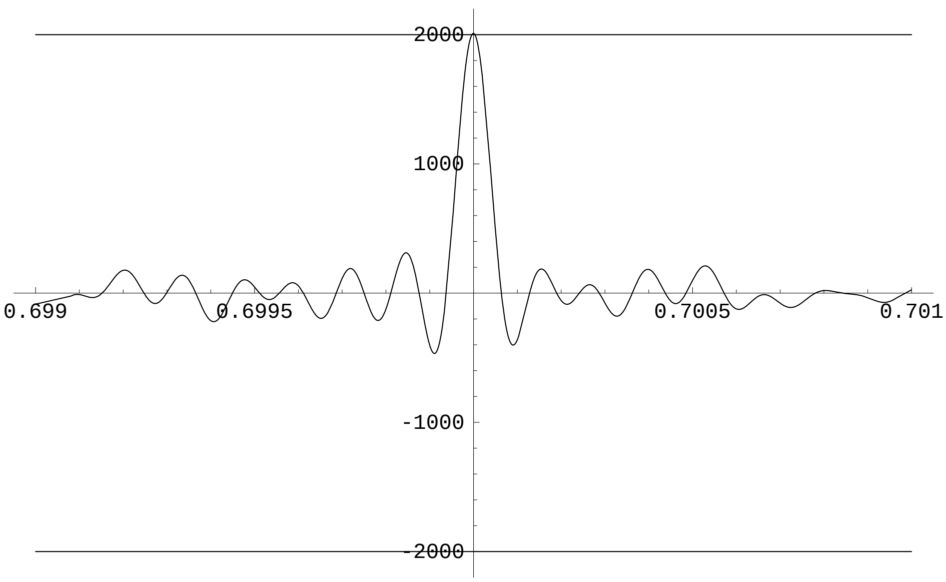


We can see that the maximum of the function over the minor arcs is slightly less than 900.

For the first thousand primes, we have:

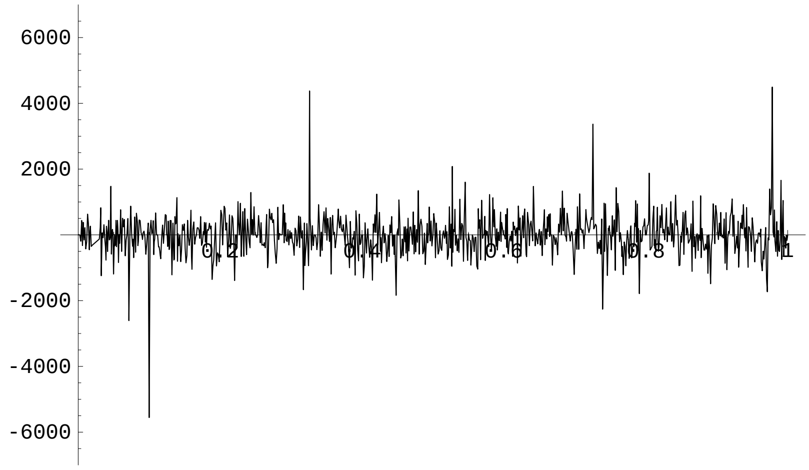


And in the interval (0.689, 0.701), with the constants at magnitude 2000,

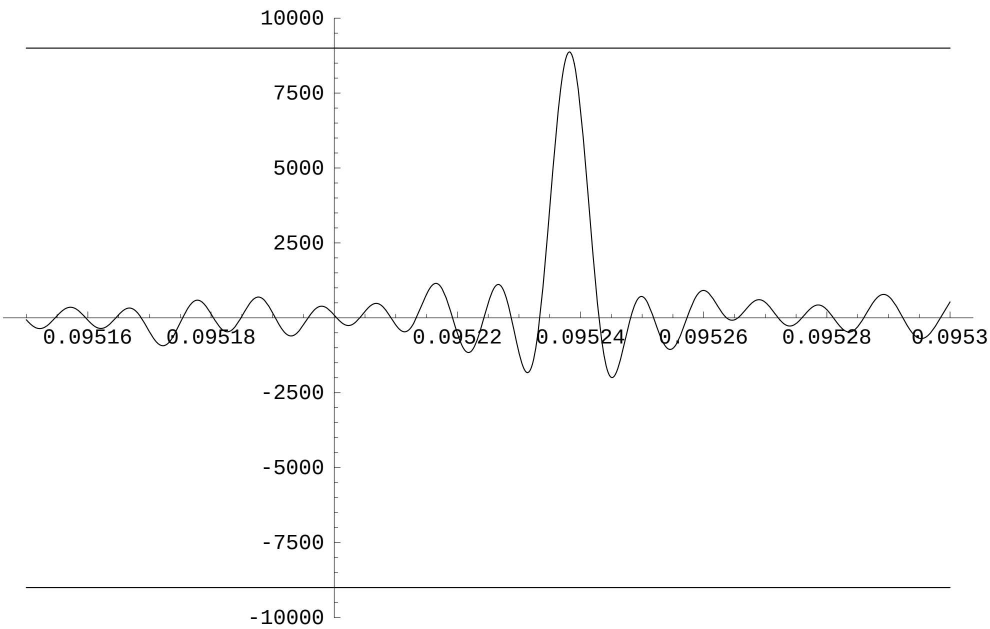


And we see that the maximum is approximately 2000.

For the first ten thousand primes, we have:

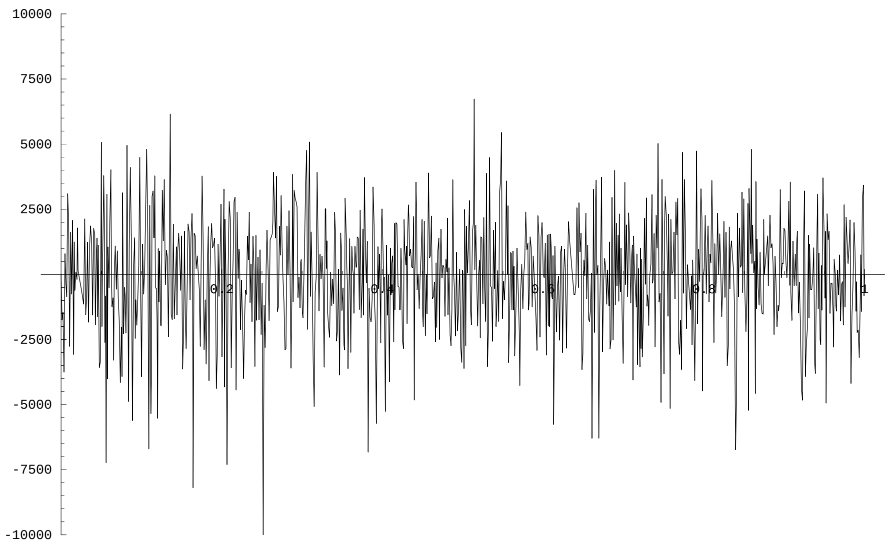


And in the interval (0.09515,0.09530), with the constants at magnitude 9000,

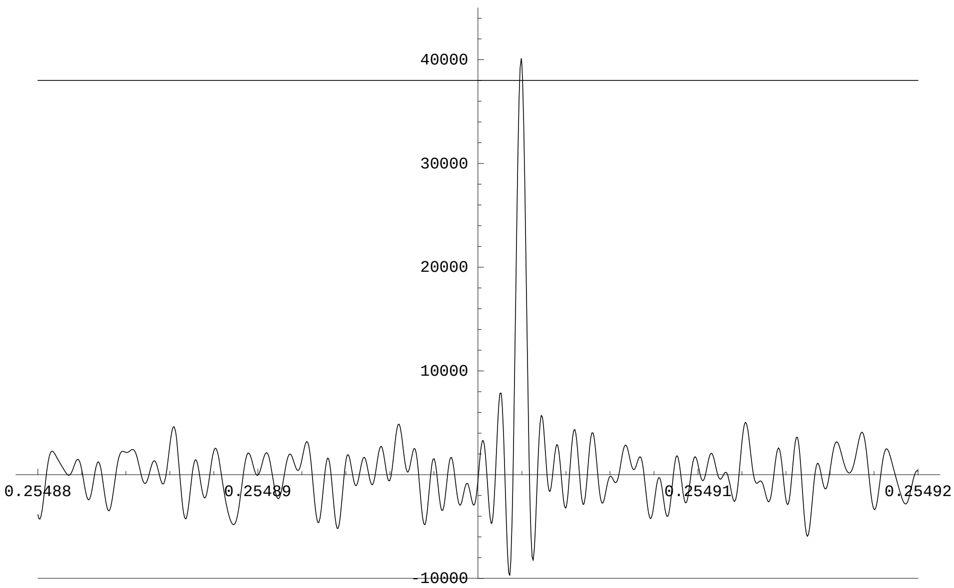


And we see that the maximum is approximately 9000.

For the first one hundred thousand primes, we have:



And in the interval (0.25488, 0.25492), with the constants at −10000 and 38000,



And we see that the maximum is approximately 40000.

It is clear from the graphs that although the peaks are still relatively high, they are extremely thin and rare. It is also promising to observe that aside from the peaks, the function is otherwise extremely small on the minor arcs. Vinogradov only gave a bound for the maximum of the function, which we now see to be significantly larger than the rest of the function. Vinogradov's crude bound using the maximum of the function was not sufficient to prove the binary case, but now that we see that the maximum of the function most likely is far larger than the average of the function over the minor arcs, the binary case could very well be true.

7 Corrections that have yet to be incorporated

7.1 First

remember, y(q) is relatively prime with N – THIS is what allows you to conclude that $|c_N(y(q))| = 1$.

for the y(q) sum, there are two pieces:

Case 1: y(q) ¿ sqrt(Q): then clearly the sum over y(q) will be small

Case 2: x(q) ¿ sqrt(Q): in this case, you are correct and all one can say is the y(q) sum is O(1); however, there will be enormous savings from the x(q) sum.

Note case 1 and 2 are NOT mutually exclusive.

7.2 Second

the problem with your bound for L is that, by using the week bound, you now have a y-sum of 1/phi(y(q)). This sum is NOT O(1). USING THE BOUND THAT y(q) is $\gg y(q)^{1-epsilon}$, you are now summing something like $1/y(q)^{1-epsilon}$, and this DOES NOT CONVERGE!!!

This is why your L bound is now WRONG – you CANNOT use the weak bound,you MUST use $c_N(y(q)) \ll 1$.

Your bound for the x(q)-part of L is okay: x(q) is large so you have some cancellation, and you have at most $\tau(N) \ll log^D N$ terms. But your y(q) sum is no longer O(1) – you've made it infinite!

then see my comments for the R bound.

7.3 Third

page 17: it is S ¡= L + R, not S = L + R. Remember, you are double¿ ¿ counting the terms where both x(q) and y(q) ¿ sqrt(Q). This is minor.

on page 18, your bound for L is wrong. The problem is you are using $C_N(y(q)) < phi(y(q))$ You should instead use $|C_N(y(q))| = 1$, so it is bounded by 1. Remember, since x(q) ¿ sqrt(Q), it is possible for y(q) to be small – y(q) can start at 1. If you use a bound of phi(y(q)),then you do NOT get a summable series for y – you'll have $1/y^{1-epsilon}$ and this blows up. Putting in the correct bound for $C_N(y(q))$, the sum over y is bounded by a constant depending on epsilon (since we have something that is $\ll sum_y 1/y^{2-2epsilon}$ Then, for the x-sum, note that there are at most $\tau(N) \gg > log^D N$ choices for x; as each is at least sqrt(Q), then $1/phi(x(q)) \ll 1/Q^{(1-epsilon)/2}$.

7.4 Fourth

Bottom of 18 / top of 19: Again, you have completely destroyed the savings on $C_N(y(q))$. You were so concerned before about using "=" instead of "¡=" or "¡¡", that I find this surprising. We know $|C_N(y(q)| = 1$ – why aren't you using this bound? To finish the proof, all you need remark is that for each y, there are at most $\tau(N) \gg log^D N$ choices of x(q). Each gives a ¿ contribution of 1/phi(x(q)). Trivially saying this is ¡= 1 suffices for the proof. Thus, in this case, the sum over x(q) gives at most $log^D N$. The sum over y(q) is bounded by

$$\sum_{y \geq \sqrt{Q}} 1/phi(y(q))^2 \ll \sum_{y \geq \sqrt{Q}} 1/y^{2-2epsilon} \gg Q^{(1-2epsilon)/2}$$

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