

Various Tendencies of Non-Poissonian
Distributions Along Subsequences of Certain
Transcendental Numbers

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Abstract

In this paper we examine the tendencies of non-Poissonian distributions that arise from the nearest neighbor spacings of $\{n^2\alpha\}$ for certain transcendental numbers. Tendencies of non-Poissonian distribution means the question of how the *Disc* functions used depend on the starting n and the number of ns used to calculate the distribution. For certain numbers, if we can find infinitely many $\frac{p_n}{q_n}$ s such that $|\alpha - \frac{p_n}{q_n}| < \frac{1}{q_n^{3+\epsilon}}$, i.e. α can be approximated to order $3 + \epsilon$ or greater, then we can show that the nearest neighbor spacings of $\{n^2\alpha\}$ from $n = 1, \dots, N$ for certain values of N give non-Poissonian distributions. From Roth's Theorem we know that these numbers must be transcendental. If we look at $\{n^2\alpha\}$ for n from 1 to q_n and take these in increasing order, calling the j -th of them β_j , we can show that the distribution of $\beta_{j+1} - \beta_j$ is supported on the integers and therefore is non-Poissonian.

The main part of this paper concerns itself with the problem of determining how much we can increase or decrease N , starting from $N = q_n$. A second question looked at in this paper is determining whether there will be a Poissonian distribution or not when n goes from K to $K + N - 1$, where K is a variable number and N is constant throughout the tests.

0.1 Introduction

This paper presents evidence concerning the questions of the tendencies of non-Poissonian distributions. Tendencies of non-Poissonian distributions means the question of how the *Disc* functions used depend on how many consecutive n s used to calculate the distribution and what the first n is. Consider $\{n^2\alpha\}$ for $n = 1 \rightarrow N$. Write these numbers in increasing order, label the j^{th} of them ζ_j . This paper is concerned with the distribution of the spacings between the ζ_j . This is similar to the paper by [Li], but it goes in another direction. In [Li], he is just concerned with showing that as we increase N , the distribution tends to the Poissonian distribution. In this paper we look at certain irrational numbers which have subsequences, N_j , such that along this sequence the distribution never tends to the Poissonian distribution. These certain irrational numbers must be transcendental and be approximated to at least order $3 + \epsilon$.

0.2 Theory

This section concerns itself with some background information about this problem. We need to define the Poissonian distribution and what we mean when we say that neighbor spacings of $\{n^2\alpha\}_{n=1}^N$, where the curly brackets indicate the fractional part of the number enclosed, converge to the Poissonian distribution. We also need to show that certain subsequences, N_j , exist such that the neighbor spacings will not converge to the Poissonian distribution for certain numbers. These subsequences are the main concern of this paper. The question of how much we can increase N , starting from $N = N_j$, until we return to the Poissonian distribution is experimentally analyzed. A second question of looking at the nearest neighbor spacings of $\{n^2\alpha\}_{n=K}^{K+N-1}$ and determining how we can increase K , while keeping N constant, and get a Poissonian distribution is also analyzed. We will follow the route taken by [TMHS], essentially all of Chapter 13 (Poissonian Behavior and $\{n^k\alpha\}$), to derive the Poissonian distribution and prove that there exists subsequences for k^{th} neighbor spacings such that we get non-Poissonian distributions for certain numbers, when the sequences go from $n = 1$ to N . Most of the times the theorems are taken verbatim from [TMHS].

We first need to start off by defining what equidistribution is.

Definition 0.2.1. We say that a sequence $\{\alpha_k\}_{k=1}^{\infty}$ in $[0,1]$ is *equidistributed* if for any interval $[a,b] \subset [0,1]$,

$$\lim_{N \rightarrow \infty} \frac{(\#\{\alpha_k \in [a, b], 1 \leq k \leq N\})}{N} = b - a \quad (1)$$

Both [Ca] and [Li], who says he follows the treatment of [Ca], show that for any integer k , $\{n^k \alpha\}$ is equidistributed.

The Poissonian distribution is defined as a convergence of certain induced probability measures. Induced probability measures make use of the *Dirac-Delta Function*, which is an approximation to the identity at the origin.

Definition 0.2.2 (Approximation to the Identity). *A sequence of functions $g_n(x)$ is an approximation to the identity (at the origin) if*

1. $g_n(x) \geq 0$
2. $\int g_n(x) dx = 1$
3. Given $\epsilon, \delta > 0$ there exists $N > 0$ such that for all $n > N$, $\int_{|x| > \delta} g_n(x) dx < \epsilon$

We can call $\delta(x)$, the *Dirac-Delta function*, the limit of any such family of $g_n(x)$

If $f(x)$ is a nice function then

$$\int f(x) \delta(x) dx = \lim_{n \rightarrow \infty} \int f(x) g_n(x) dx = f(0) \quad (2)$$

Given N points x_1, x_2, \dots, x_N we can form a probability measure

$$\mu_N(x) dx = \frac{1}{N} \sum_{n=1}^N \delta(x - x_n) dx \quad (3)$$

For the case of the random distribution (i.e. each x_n is chosen independently and uniformly in $[0,1)$), μ_N converges to the Poissonian distribution.

Since $\int \mu_N(x) dx = 1$, and if $f(x)$ is a nice function

$$\begin{aligned} \int f(x) \mu_N(x) dx &= \frac{1}{N} \int \sum_{n=1}^N f(x) \delta(x - x_n) dx \\ &= \frac{1}{N} \sum_{n=1}^N \int f(x) \delta(x - x_n) dx \\ &= \frac{1}{N} \sum_{n=1}^N f(x_n) \end{aligned} \quad (4)$$

The final result in Equation 4 looks like a Riemann sum if the x_n s are equidistributed. In general, these x_n s will not be equidistributed, but assume that for any interval $[a, b]$, as $N \rightarrow \infty$, the fraction of x_n s ($1 \leq n \leq N$) in $[a, b]$ tends to $\int_a^b p(x)dx$ for some nice function $p(x)$:

$$\lim_{N \rightarrow \infty} \frac{\#\{n : 1 \leq n \leq N, x_n \in [a, b]\}}{N} \rightarrow \int_a^b p(x)dx \quad (5)$$

In this case, if $f(x)$ is nice, then

$$\begin{aligned} \int f(x)\mu_N(x)dx &= \frac{1}{N} \sum_{n=1}^N f(x_n) \\ &\approx \sum_{k=-\infty}^{\infty} f\left(\frac{k}{N}\right) \frac{\#\{n : 1 \leq n \leq N, x_n \in [\frac{k}{N}, \frac{k+1}{N}]\}}{N} \\ &\rightarrow \int f(x)p(x)dx \end{aligned} \quad (6)$$

Definition 0.2.3 (Convergence to $p(x)$). *If the sequence of points x_n satisfies Equation 5 for some nice function $p(x)$, we say the probability measure $\mu_N(x)dx$ converges to $p(x)dx$*

We can now consider the questions of the distribution of neighbor spacings. Let α_n be a collection of points in $[0, 1)$. We order them by size and relabel them $\beta_1, \beta_2, \dots, \beta_N$ where $\beta_k \geq \beta_l$ if and only if $k \geq l$

We will now look at how the differences $\beta_{j+1} - \beta_j$ are distributed. We will use distance on the wrapped unit interval. The distance between 0.999 and 0.001 on the wrapped unit interval and in *Mod 1* distance is 0.002 instead of 0.998.

Definition 0.2.4 (mod 1 distance). *Let $x, y \in [0, 1)$. We define the mod 1 distance from x to y , $\|x - y\|$, by*

$$\|x - y\| = \min\{|x - y|, 1 - |x - y|\} \quad (7)$$

Any arithmetic operation done in this paper will always be done **mod 1**, and therefore it lives in $[0, 1)$

The maximum value of $\|x - y\|$ is $\frac{1}{2}$. To show this we look at when $|x - y| = 1 - |x - y|$. This gives $|x - y| = \frac{1}{2}$ and therefore $\|x - y\| = \frac{1}{2}$. For $|x - y| < \frac{1}{2} \rightarrow \|x - y\| = |x - y| < \frac{1}{2}$. For $\frac{1}{2} < |x - y| < 1 \rightarrow \|x - y\| = 1 - |x - y| < \frac{1}{2}$. Since $|x - y| < 1$ the max value of $\|x - y\|$ must be $\frac{1}{2}$.

Definition 0.2.5 (Neighbor Spacings). Given a sequence of numbers α_n in $[0, 1)$, fix an N and arrange the numbers α_n ($n \leq N$) in increasing order. Label the new sequence β_j ; note that the ordering will depend on N . Let $\beta_{-j} = \beta_{N-j}$ and $\beta_{N+j} = \beta_j$.

1. The nearest neighbor spacings are the numbers $\beta_{j+1} - \beta_j, j = 1, \dots, N$
2. The k^{th} -neighbor spacings are the numbers $\beta_{j+k} - \beta_j, j = 1, \dots, N$

This paper concerns itself only with non-Poissonian distributions of nearest neighbor spacings, though it is best to derive everything for k^{th} neighbor spacings.

Let us consider N independent random variables x_n . Each variable is chosen in $[0, 1)$; thus the probability that $x_n \in [a, b)$ is $b - a$. Let the y_n s be the x_n s arranged in increasing order. Because we have N objects on the wrapped unit interval we have N nearest neighbor spacings. Thus we expect the average spacing to be $\frac{1}{N}$.

Definition 0.2.6 (Unfolding). Let $z_n = Ny_n$. The numbers $z_n = Ny_n$ have unit mean spacing. Thus, while we expect the average spacing between the adjacent y_n s to be $\frac{1}{N}$ units, we expect the average spacing between adjacent z_n s to be 1 unit.

On the wrapped unit interval the expected nearest neighbor spacing to be independent of j . In other words, we expect $\beta_{j+1} - \beta_j$ to have the same distribution as $\beta_{i+1} - \beta_i$.

When ordered in increasing order, consider $x_1 = y_l$. What is the probability that y_{l+k} is located between $\frac{t}{N}$ and $\frac{t+\Delta t}{N}$ units to the right of y_l ? For this to occur we need exactly $k - 1$ of the x_n s to lie between 0 and $\frac{t}{N}$ units to the right of $x_1 = y_l$, exactly one x_n, y_{l+k} , to lie between $\frac{t}{N}$ and $\frac{t+\Delta t}{N}$ units to the right of y_l , and the remaining x_n s to lie at least $\frac{t+\Delta t}{N}$ units to the right of y_l .

As the x_n s are independently determined, there are $\binom{N-1}{k-1}$ choices of x_n that are at most $\frac{t}{N}$ units to the right of $x_1 = y_l$. There are then $\binom{(N-1)-(k-1)}{1}$ ways to choose the x_n located between $\frac{t}{N}$ and $\frac{t+\Delta t}{N}$ units to the right of x_1 .

Therefore we have,

$$\begin{aligned} & \text{Prob} \left(\|y_{l+k} - y_l\| \in \left[\frac{t}{N}, \frac{t + \Delta t}{N} \right] \right) = \\ & = \binom{N-1}{k-1} \left(\frac{t}{N} \right)^{k-1} \binom{(N-1)-(k-1)}{1} \frac{\Delta t}{N} \left(1 - \frac{t + \Delta t}{N} \right)^{N-(k+1)} \end{aligned}$$

$$\begin{aligned}
&= \frac{(N-1) \cdots (N-1-(k-2)) (N-1)-(k-1)}{N^{k-1}} \frac{t^{k-1}}{(k-1)!} \left(1 - \frac{t+\Delta t}{N}\right)^{N-(k+1)} \Delta t \\
&\quad \rightarrow \frac{t^{k-1}}{(k-1)!} e^{-t} \Delta t \tag{8}
\end{aligned}$$

We have just proven the following theorem:

Theorem 0.2.7. *Consider N independent random variables x_n chosen from the uniform distribution on the wrapped interval $[0, 1)$. For fixed N , arrange the x_n s in increasing order, labeled $y_1 \leq y_2 \leq \cdots \leq y_N$.*

From the induced probability measures from the k^{th} neighbor spacings, as $N \rightarrow \infty$ we have

$$\mu_{N,k}(t)dt = \frac{1}{N} \sum_{n=1}^N \delta(t - N(y_n - y_{n-k})) dt \rightarrow \frac{t^{k-1}}{(k-1)!} e^{-t} dt \tag{9}$$

Using the unfolded z_n with unit spacing, we have

$$\mu_{N,k}(t)dt = \frac{1}{N} \sum_{n=1}^N \delta(t - (z_n - z_{n-k})) dt \rightarrow \frac{t^{k-1}}{(k-1)!} e^{-t} dt \tag{10}$$

For the nearest neighbor spacings, which are the only ones considered in this paper, we have

$$\mu_{N,1}(t)dt = \frac{1}{N} \sum_{n=1}^N \delta(t - (z_n - z_{n-1})) dt \rightarrow e^{-t} dt \tag{11}$$

Definition 0.2.8 (Poissonian Behavior). *A sequence of points exhibits Poissonian behavior if, in the limit as $N \rightarrow \infty$, the induced probability measures $\mu_{N,k}(t)dt$ converge to $\frac{t^{k-1}}{(k-1)!} e^{-t} dt$.*

If α is irrational, the sequence of points $\{n^2\alpha\}_{n=1}^N$ is equidistributed in $[0, 1)$ and the neighbor spacings have been observed to be essentially Poissonian. It is possible to show that, if α can be approximated to order $3 + \epsilon$, there is a certain subsequence of N that gives rise to non-Poissonian behavior.

If α can be approximated to order $3 + \epsilon$ then there we can find infinitely many $\frac{p_n}{q_n}$ such that $|\alpha - \frac{p_n}{q_n}| < \frac{a_n}{q_n^{3+\epsilon}}$ where $a_n \rightarrow 0$ as $n \rightarrow \infty$

If $\beta_l = \{k^2\alpha\}$ ordered in increasing order, we can show that $\gamma_l = \{k^2\frac{p_n}{q_n}\}$, i.e. the ordering of $\{k^2\frac{p_n}{q_n}\}$ is the same as the ordering of $\{k^2\alpha\}$.

$$\begin{aligned} k^2\left|\alpha - \frac{p_n}{q_n}\right| &= \left|k^2\alpha - k^2\frac{p_n}{q_n}\right| \\ &= \left|\{k^2\alpha\} - \left\{k^2\frac{p_n}{q_n}\right\}\right| \\ &< k^2\frac{a_n}{q_n^3} \end{aligned} \tag{12}$$

For $k^2 \leq q_n^2$ we get

$$\left|\{k^2\alpha\} - \left\{k^2\frac{p_n}{q_n}\right\}\right| < k^2\frac{p_n}{q_n^3} \leq \frac{a_n}{q_n} < \frac{1}{2q_n} \tag{13}$$

for n big enough that $a_n < \frac{1}{2}$

From this result we can see that $\{k^2\frac{p_n}{q_n}\}$ is the closest to $\{k^2\alpha\}$. This implies that the orderings are the same.

Theorem 0.2.9. *If α is irrational such that $|\alpha - \frac{p_n}{q_n}| < \frac{a_n}{q_n^3}$ holds infinitely often, where $a_n \rightarrow 0$ as $n \rightarrow \infty$, there exist integers $N_j \rightarrow \infty$ such that $\mu_{N_j,1}(t)$ does not converge to $e^{-t}dt$*

Proof: We just showed that

$$\left|\{k^2\alpha\} - \left\{k^2\frac{p_n}{q_n}\right\}\right| = \|\beta_l - \gamma_l\| < \frac{a_n}{q_n} \tag{14}$$

If we let $N_n = q_n$ and multiply through by $N_n = q_n$ we get

$$\|N_n(\beta_l - \gamma_l)\| < a_n \tag{15}$$

We can now see that

$$\begin{aligned} \|N_n(\beta_l - \gamma_l) - N_n(\beta_{l-1} - \gamma_{l-1})\| &< \|N_n(\beta_l - \gamma_l)\| + \|N_n(\beta_{l-1} - \gamma_{l-1})\| \\ &< a_n + a_n = 2a_n \end{aligned} \tag{16}$$

We can also see that

$$\begin{aligned} \|N_n(\beta_l - \gamma_l) - N_n(\beta_{l-1} - \gamma_{l-1})\| &< \|N_n(\beta_l - \beta_{l-1}) - N_n(\gamma_l - \gamma_{l-1})\| \\ &< 2a_n \end{aligned} \tag{17}$$

Since we know that $a_n \rightarrow 0$ this result must mean the difference between $\|N_n(\beta_l - \beta_{l-1})\|$ and $\|N_n(\gamma_l - \gamma_{l-1})\|$ goes to zero also.

Thus, $\mu_{N_n,1}^{\frac{p_n}{q_n}}(t)dt - \mu_{N_n,1}^{\alpha}(t)dt \rightarrow 0$. If we look at $\mu_{N_n,1}^{\frac{p_n}{q_n}}(t)dt$ we can show that it is supported on the integers and therefore cannot be Poissonian.

$$\mu_{N_n,1}^{\frac{p_n}{q_n}}(t)dt = \frac{1}{N_n} \sum_{l=1}^{N_n} \delta(t - N_n(\gamma_l - \gamma_{l-1})) dt \quad (18)$$

We can see that

$$N_n(\gamma_l - \gamma_{l-1}) = N_n\gamma_l - N_n\gamma_{l-1} \quad (19)$$

We know that $\gamma_l = \{k^2 \frac{p_n}{q_n}\}$,

$$k^2 \frac{p_n}{q_n} = \frac{k^2 p_n}{q_n} = \frac{Tq_n + R}{q_n} = T + \frac{R}{q_n}, \quad (20)$$

where T is an integer, $R < q_n$, and R is an integer. This implies that

$$\{k^2 \frac{p_n}{q_n}\} = \{T + \frac{R}{q_n}\} = \frac{R}{q_n} \quad (21)$$

Therefore,

$$N_n\gamma_l = q_n \frac{R}{q_n} = R, \quad (22)$$

which is an integer, so $N_n(\gamma_l - \gamma_{l-1})$ is an integer and the induced probability measures $\mu_{N_n,1}^{\frac{p_n}{q_n}}(t)dt$ and $\mu_{N_n,1}^{\alpha}(t)dt$ are supported on the integers. Consequently, they do not converge to $e^{-t}dt$ and are not Poissonian.

The main experimental part of this paper is concerned with examining two different families of transcendental numbers and another specific number, all of which can be approximated to at least order $3 + \epsilon$ and therefore exhibit this non-Poissonian behavior for certain N_n . The two families that are being considered are $\sum_{i=1}^{\infty} \frac{1}{k^{i!}}$ and $[k^{1!}, k^{2!}, \dots]$, where the brackets denote the continued fraction of the numbers enclosed. The specific number being examined is the number labeled η in [Li] and constructed in [RSZ].

0.3 Experimental Numbers

In this section we prove that the two families used in the experiments can be approximated arbitrarily well. The number η will also be constructed and shown to be of order $3 + \epsilon$.

We will first show that $\sum_{i=1}^{\infty} \frac{1}{k^{i!}}$ can be approximated arbitrarily well. This construction is also essentially done in [TMHS], in Chapter 12 Section 3 (Constructing Transcendental Numbers), though it is a common construction and it is modified from [TMHS] to show that k can be any integer, not just $k = 10$. If we let $\frac{p_n}{q_n} = \sum_{i=1}^n \frac{1}{k^{i!}}$, then $q_n = \frac{1}{k^{n!}}$ and

$$\begin{aligned} \left| \alpha - \frac{p_n}{q_n} \right| &= \left| \sum_{i=1}^{\infty} \frac{1}{k^{i!}} - \sum_{i=1}^n \frac{1}{k^{i!}} \right| = \sum_{i=n+1}^{\infty} \frac{1}{k^{i!}} \\ &= \frac{1}{k^{(n+1)!}} \left(1 + \frac{1}{k^{n+2}} + \frac{1}{k^{(n+2)(n+3)}} + \dots \right) \\ &< \frac{2}{k^{(n+1)!}} = \frac{2}{(k^{n!})^{n+1}} < \frac{2}{q_n^{n+1}} \leq \frac{2}{q_n^N} \end{aligned} \quad (23)$$

Since we can choose N arbitrarily, we can approximate it arbitrarily well. We will call this number β_k from now on, where the k refers to the k in the infinite sum.

To show that $\alpha = [k^{1!}, k^{2!}, \dots] = [a_0, a_1, \dots]$ can be approximated arbitrarily well, we will again follow a construction in [TMHS], also in Chapter 12 Section 3 (Constructing Transcendental Numbers), though it is again a common construction and again it is modified from [TMHS] to show it is true for all integers, k , not just $k = 10$. Let $\frac{p_n}{q_n} = [k^{1!}, k^{2!}, \dots, k^{n!}]$. Then $a_n = k^{(n+1)!}$ and

$$\left| \alpha - \frac{p_n}{q_n} \right| = \frac{1}{q_n q'_{n+1}} = \frac{1}{q_n (a'_{n+1} q_n + q_{n-1})} < \frac{1}{a_{n+1}} = \frac{1}{k^{(n+1)!}} \quad (24)$$

Since $q_k = a_k q_{k-1} + q_{k-2}$ we see that

$$\frac{q_{k+1}}{q_k} = a_{k+1} + \frac{q_{k-1}}{q_k} < a_{k+1} + 1 \quad (25)$$

Using this inequality for q_n we get

$$\begin{aligned}
q_n = q_0 \frac{q_1}{q_0} \frac{q_2}{q_1} \cdots \frac{q_n}{q_{n-1}} &< (a_0 + 1)(a_1 + 1) \cdots (a_n + 1) \\
&= \left(1 + \frac{1}{a_0}\right) \left(1 + \frac{1}{a_1}\right) \cdots \left(1 + \frac{1}{a_n}\right) a_1 a_2 \cdots a_n \\
&< 2^n a_0 a_1 \cdots a_n = 2^n k^{(1! + \cdots + (n+1)!)} \\
&< k^{2(n+1)!} = a_n^2
\end{aligned} \tag{26}$$

Combining Equations 24 and 26 we get

$$\begin{aligned}
\left| \alpha - \frac{p_n}{q_n} \right| &< \frac{1}{a_{n+1}} = \frac{1}{a_n^{n+1}} \\
&< \left(\frac{1}{a_n^2} \right)^{\frac{n}{2}} < \left(\frac{1}{q_n} \right)^{\frac{n}{2}} = \frac{1}{q_n^{\frac{n}{2}}}
\end{aligned} \tag{27}$$

Therefore this α can be approximated arbitrarily well because all $\frac{p_n}{q_n}$ with $n > 2 * t$ will be approximated to at least order t . We will call this well-approximable number γ_k from now on, where the k refers to the k in the continued fraction representation of the number.

The construction of η used here is the same as in [RSZ] and used in [Li]. To start off we will define a pair of integers (r_m, v_m) by $r_{-1} = v_{-1} = 0, r_0 = v_0 = 1, r_1 = v_1 = 1$ and for $m \geq 1$

$$\begin{aligned}
v_{m+1} &= r_m v_m^2 + v_{m-1} \\
r_{m+1} &= [\log_2(v_{m+1})]
\end{aligned} \tag{28}$$

where the brackets mean the greatest integer less than or equal to the value enclosed. Now we set $a_0 = 1$, and for $m \geq 0$

$$a_{m+1} = r_m^2 v_m^2 + 2r_m v_{m-1} \tag{29}$$

We let η be $[a_0, a_1, \dots] = [1, 1, 3, 6, \dots]$. To show that this number is of type $3 + \epsilon$ we need to show that $q_m = v_m^2$. To do this we look at the recursion relation for $q_{m+1} = a_{m+1} q_m + q_{m-1}$ and use induction

$$\begin{aligned}
q_{m+1} &= a_{m+1}v_m^2 + v_{m-1}^2 = (r_m^2v_m^2 + 2r_mv_{m-1})v_m^2 + v_{m-1}^2 \\
&= (r_mv_m^2 + v_{m-1})^2 = v_{m+1}^2
\end{aligned} \tag{30}$$

and $q_{m+1} = v_{m+1}^2$ as long as $q_m = v_m^2$. Since $q_0 = v_0^2 = 1$ and we can define $q_{-1} = v_{-1}^2 = 0$ it is proved. From the previous equation we can see

$$q_{m+1} \sim r_m^2 q_m^2 \sim q_m^2 (\log(q_m))^2 \tag{31}$$

From basic continued fractions we have

$$\left| \eta - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} \sim \frac{1}{q_n^3 (\log(q_n))^2} \tag{32}$$

And thus η is of order $3 + \epsilon$

0.4 Experimental Data

In this section we will discuss the experimental data and what can be determined from this data about the tendencies of non-Poissonian distributions for nearest neighbor spacings. It is best first to describe the experiments that were done and how they were carried out. As was shown in Theorem 0.2.9, if the number of points $N = q_N$, a non-Poissonian distribution supported on the integers arises. The question of how many additional points does it take, increasing N , to return to the Poissonian distribution, and how many points can we take away before we return again to the Poissonian distribution, is investigated. Throughout our discussion of the first question, the sequence $\{n^{2\alpha}\}$ starts at $n = 1$. It is a separate question whether the distribution of nearest neighbor spacings is different when a sequence of a defined length starts at some different n such as 10^8 or maybe a q_n .

All these experiments were done using *Mathematica*. The experiments were made up of the following steps:

1. Calculate the numbers $\{n^{2\alpha}\}$ for $n = n_{min}, \dots, n_{max}$
2. Sort all the numbers
3. Calculate all the nearest neighbor spacings and make them unit mean spacing by multiplying by $n_{max} - n_{min} + 1$.

4. Sort the nearest neighbor spacings
5. Calculate *Disc* and *Disc2*, the two discrepancy functions used, defined below

Disc is the same discrepancy function used in [Li] to calculate the largest difference between the experimental cumulative probability distribution (the probability that the difference is less than a certain value, t) and the Poissonian cumulative probability distribution, $\int_0^t e^{-x} dx = 1 - e^{-t}$. *Disc2* calculates the average difference between the distributions.

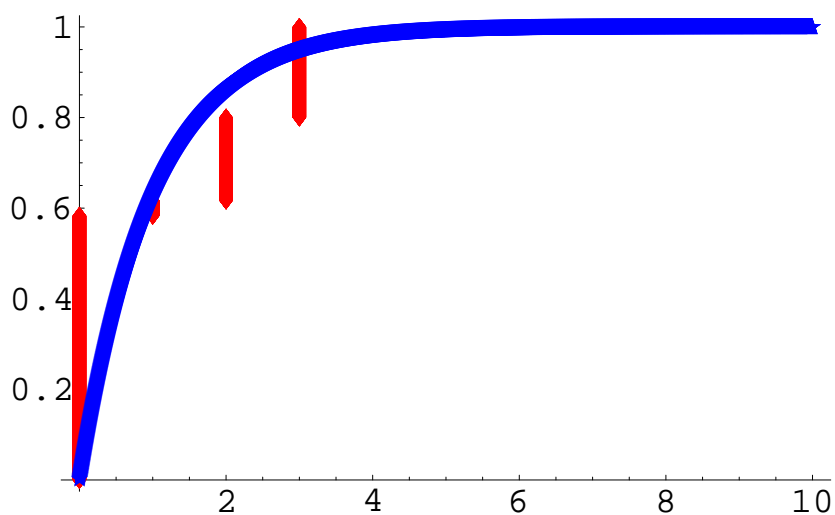
To answer the first question about how far away from $N = q_n$ do we need to go to return to the Poissonian distribution it is best first to see exactly how the numbers behave at $N = q_n$

β_k	q_n	Disc	Disc2
k=2	64	0.8016	0.3594
k=3	729	0.6228	0.2338
k=4	4096	0.8328	0.3598
k=5	15625	0.5833	0.2124
k=6	46656	0.9295	0.4343
k=7	117649	0.5625	0.1829
k=8	262144	0.8333	0.3598
γ_k	q_n	Disc	Disc2
k=2	257	0.4981	0.1525
k=3	6562	0.7339	0.2893
k=4	65537	0.5000	0.1513
k=5	390626	0.7353	0.2900
k=6	1679617	0.7353	0.2899
η	2704	0.8614	0.3940
e	8544	0.0123	0.0041
e	208524	0.0029	0.0011

Looking at this table, we cannot make any specific guesses as to what the *Disc* and *Disc2* would be for a random transcendental number that satisfied the conditions of Theorem 0.2.9 at a random $N = q_n$. For β_k it appeared that, as k is increased, the *Disc* alternates back and forth between values around 0.8 and around 0.6. However, for γ_k it looks like it alternates between 0.73 and 0.5 until

we go from $k = 5$ to $k = 6$, where the the $Disc$ and $Disc2$ stay essentially the same. η seems to have a somewhat high $Disc$ and $Disc2$, only topped by β_6 , but that means little since there is no strong correlation between $N = q_n$ and the discrepancy. The base of the natural logarithm, e , is in this table to show that the big discrepancy observed for the other numbers is not normal for all numbers. More information about the discrepancies of trascendental numbers not of order $3 + \epsilon$ and of algebraic numbers can be found in [Li].

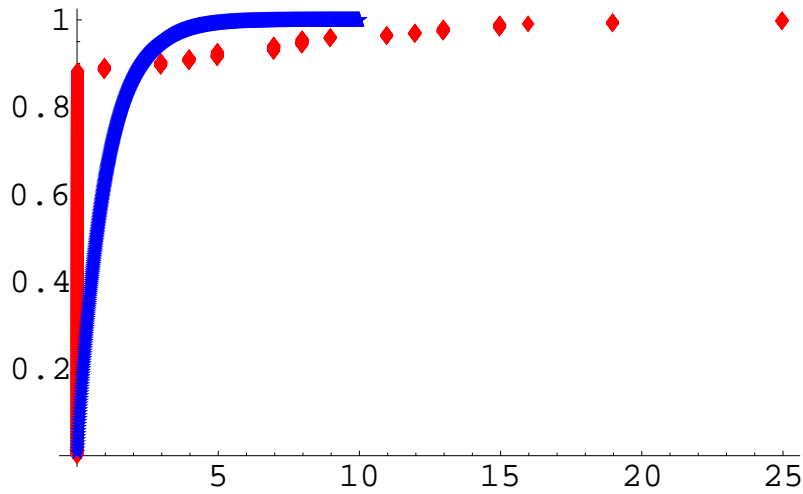
The following graph is a plot of the cumulative probability distributions for β_5 where $N = q_n = 15625$. This plot has a $Disc = 0.5833$ and a $Disc2 = 0.2124$



Plot 1

In this graph, the experimental probability is the red line and the Poissonian theoretical probability is the blue line. The experimental probability is supported on the integers, although the height at each integer roughly follows the theoretical probability.

In contrast to a $Disc$ of 0.5833, the next plot is a plot of $N = q_n = 2704$ for the number η .



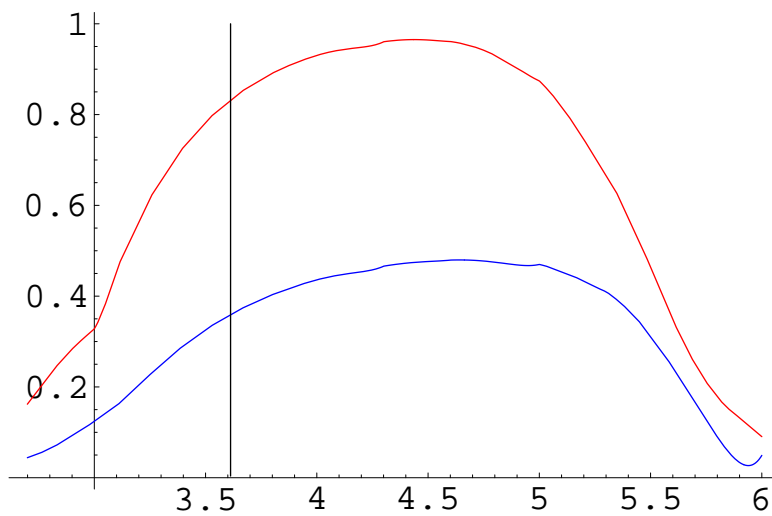
Plot 2

This plot has a *Disc* of 0.8614 and a *Disc2* of 0.3940. η is a different kind of number than β_5 , and it is obvious from this plot. This plot has more different integers on which the experimental curve is supported; it also follows the theoretical plot less closely.

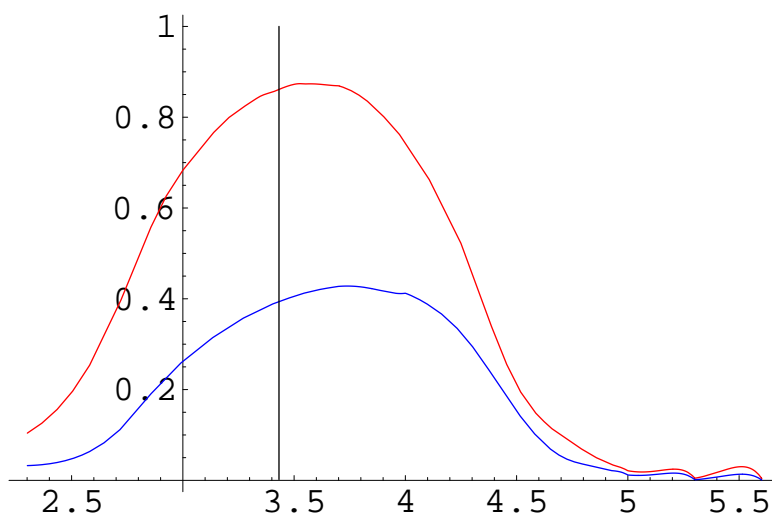
One thing that should be taken from these plots is the origin of the value of *Disc*. For all the plots with $Disc \geq 0.4$ or so the value comes from the points supported on the integer 0. *Disc* is the probability that the we have a difference < 0.5 . Since the differences are supported on the integers it is also the probability we have a difference < 0.4 , < 0.2 , or < 0.05 . The plots given above are at the extremes when it comes to *Disc* and *Disc2* when evaluated at $N = q_n$. If one looks at the plots for all the numbers given in the table above, he will find that the curves do roughly follow the Poissonian distribution, but they are supported on the integers and have a high value supported at 0.

After looking at the how the distributions behave at $N = q_n$ it is now time to look at how the distributions behave away from $N = q_n$. The following two graphs are plots of the values of *Disc* and *Disc2* at different values of N . The x-axis is $\text{Log}_{10}(x)$ and the y-axis is the value of the *Discs*. The vertical line is where $x = q_n$; the red line is the value of *Disc*, and the blue line is the value of *Disc2*. Each plot was created using an interpolation of order 3 to connect between 25 and 30 points run over the range of the plot. Plot 3 is a plot of β_4 and Plot 4 is a plot of η . They were chosen because it was possible in these cases to get points

sufficiently above and below $N = q_n$ such that their distributions were essentially Poissonian. For almost every other number it was only possible to get a good plot either above or below $N = q_n$, but not both.



Plot 3



Plot 4

The first thing to mention is a comment relating to [Li]. In his paper he mentions that it took him up to $N = 50000$ until η started behaving as he felt it should

be behaving, Poissonian-like. If he had looked at $N = 200 \rightarrow 300$ he would have noticed that it was Poissonian for these small N s. It took that long for Lipschitz to notice Poissonian behavior, because η satisfies the requirements of Theorem 0.2.9 and he was in the neighborhood of $N = q_n$. Consequently his graphs felt the effects of the non-Poissonian, integer-supported distribution, and only at $N = 50000$ for a $q_n = 2704$, did the graph return to Poissonian behavior. With a computer with sufficient memory and able to do $N = q_5 = 182,925,625$, he would have noticed that the graph again became non-Poissonian and integer-supported.

To get on with looking at the graphs above, the graph of β_4 has a $q_3 = 4096$ and the graph of η has a $q_4 = 2704$. The next q_n for β_4 , q_4 , is 281474976710656 and the next q_n for η , q_5 , is as previously mentioned 182925625. The first q_n s listed have similar values, both on the order of 10^3 ; however the next q_n s are very different. The q_n for β_4 is on the order of 10^{14} , whereas that for η is on the order of 10^8 . This is what causes the differences in the two graphs. The values for which the distributions seem to return to Poissonian behavior, above or below $N = q_n$, depend on the magnitude of q_n and the magnitude of either q_{n-1} or q_{n+1} depending on whether we are looking at how much N must be increased or decreased to return to Poissonian behavior. The following is a table of discrepancies for different values of N for different numbers above the q_n for each number. Each N is either the value with which $Disc \approx 0.1$ and $Disc2 \approx 0.05$ or the value at 1, 000, 000. 1, 000, 000 is essentially the largest N possible for the machine to run the program in a reasonable amount of time and not run out of memory. The program halted because of lack of memory when it was run with $N = 10,000,000$.

α	q_n	$\approx q_{n+1}$	N	$Disc$	$Disc2$	N/q_n
β_2	64	$1.7 * 10^7$	1400	0.0884	0.0480	21.88
β_3	729	$2.8 * 10^{11}$	45000	0.0931	0.0503	61.73
β_4	4096	$2.8 * 10^{14}$	1000000	0.0905	0.0492	244.14
β_5	15625	$5.9 * 10^{16}$	1000000	0.6503	0.3962	N\A
γ_2	257	$4.3 * 10^9$	105000	0.0935	0.0614	408.56
γ_3	6562	$1.9 * 10^{15}$	1000000	0.9972	0.4981	N\A
η	2704	$1.8 * 10^8$	50000	0.0932	0.0533	18.49

One thing to notice from this table concerns the entries for β_3 and γ_2 . β_3 has a higher q_n and a higher q_{n+1} than γ_2 , yet the value of N that gives a $Disc \approx 0.1$ and a $Disc2 \approx 0.05$ is smaller for β_3 than γ_2 , henceforth the value of N/q_n for β_3 is much smaller than for γ_2 . Also, η has a higher q_n by a factor of ≈ 5 and a lower

q_{n+1} by a factor of $\approx 10^3$ than does β_3 , yet it requires approximately the same value of N to give the *Discs* listed. There is clearly something more going on than is implied by a relation between the q_n s that determines how far away from $N = q_n$ we must go to return to Poissonian behavior. There is not common value of N/q_n from which we can predict a value of N where we will get a Poissonian distribution. The tendency to return to a Poissonian distribution probably also depends on what the value of t in the following equation:

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n^t} \quad (33)$$

β_k, γ_k , and η all have different values of t , which might explain the differences in the N s in the table above. However, exploring this point any further is beyond the scope of this present work..

The previous tables and graphs show only how the *Discs* behave around one q_n which is on the order of 10^3 or 10^4 . Only β_2 gives a small indication at what happens at a second q_n . β_2 has $q_3 = 64$ and $q_4 = 16777216$. The following is a table of discrepancies for a couple values of N for β_2

N	<i>Disc</i>	<i>Disc2</i>
64	0.8016	0.3594
50000	0.0102	0.0027
1000000	0.0845	0.0243
5000000	0.4408	0.1535

One can see from the discrepancies that the distribution is non-Poissonian at $N = q_3 = 64$, then is essentially Poissonian for N between 10^4 and 10^6 . Finally as it approaches $N = q_4 = 16777216$, the distribution is non-Poissonian again. This is the only experimental proof that Theorem 0.2.9 is true for $q_n \geq 1000000$. All the experimental proof obtained in this paper is for single q_n s for a number with $q_n \approx 10^3 \rightarrow 10^4$. In the future, with a more powerful computer, it might be instructive to look at how the distributions behave for several q_n s for a single number.

The next experimental results concern different starting values for n in $\{n^2\alpha\}$. The discrepancies for the same size blocks (i.e. number of numbers in the sequence), but with different starting positions, were compared. In the previous experiments, the starting positions were always $n = 1$. Now we are looking at

starting positions of $n = 10^8$, $n = q_n$, or something similar.

The following is a table of values of the discrepancies for a block of size 2704, with η as our number. A block size of 2704 was chosen because it is a q_n of η . The input value that changes is the minimum n

Min n	<i>Disc</i>	<i>Disc2</i>
1	0.8614	0.3940
2704	0.8332	0.3860
182925625	0.8611	0.3937
q_6	0.8611	0.3937
$2704 + 10^4$	0.7403	0.3604
$182925625 + 10^4$	0.7632	0.3673
$q_6 + 10^4$	0.7632	0.3673
$2704 + 10^5$	0.3312	0.1833
$182925625 + 10^5$	0.3397	0.1871
$q_6 + 10^5$	0.3397	0.1871
$2704 + 10^6$	0.0173	0.0049
$182925625 + 10^6$	0.0149	0.0044
$q_6 + 10^6$	0.0149	0.0044
$182925625 - 10^5$	0.3473	0.1909
$q_6 - 10^5$	0.3473	0.1909

$2704 = q_4$, $182925625 = q_5$, and $q_6 = 5655041790912713329$. The values of the discrepancies are essentially the same at values of Min n that are the same distance from a q_n . For q_5 and q_6 they are the same to 4 significant figures. The reason q_4 is different is probably because it is closer to q_3 , q_2 , and q_1 than the increase required of the Min n to return to the Poissonian distribution and there is probably some influence on the distribution from these other q_n s. Their return to a Poissonian distribution acts differently from looking at different size blocks with the same Min n . In this case there is a certain distance from q_n at which the distribution returns to Poissonian. In the first case we looked at, variable block size and constant Min n , it look longer to return to Poissonian distribution when q_n was larger. To show that the results from this case, constant block size and variable Min n , are not a fluke the following is a table for $\alpha = \beta_5$, block size of $q_3 = 15625$.

Min n	$Disc$	$Disc2$
q_3	0.5831	0.2123
q_4	0.5832	0.2124
$q_3 + 10^6$	0.5749	0.2104
$q_4 + 10^6$	0.5751	0.2104
$q_3 + 5 * 10^7$	0.2471	0.1290
$q_4 + 5 * 10^7$	0.2472	0.1290
$q_3 + 3 * 10^8$	0.1006	0.0462
$q_4 + 3 * 10^8$	0.1006	0.0462

In this table $q_3 = 15625$ as was mentioned, and $q_4 = 59604644775390625$. The distance required to return to the Poissonian distribution is different than that for η , but as was also the case for η , the same distance from a q_n gives the same discrepancies. If we had chosen any other α , we would have the $Discs$ act the same. This is essentially all that we can deduce about the tendencies of non-Poissonian distributions for certain transcendental numbers.

0.5 Conclusion and What Comes Next

Nothing concrete can be deduced from the data taken. What could be seen has already been mentioned in the previous section. It was quite interesting to see that for the sequences of $n = 1 \rightarrow N$, if $N = q_n + R$, that for a larger q_n , a larger R was needed to have a distribution that looked like the Poissonian distribution. This was quite different that if we looked at $n = q_n + R \rightarrow q_n + R + N - 1$. In this case for all different q_n s of the same α , the discrepancies were the same if R is held constant.

As was metioned at the beginning of the Experimental Data section, all the experiments were done using *Mathematica*. The prograns, whose outline is also given in that section, were originally just the same as the *Maple* programs used to get the data in [Li], except changed to run in *Mathematica*, but the programs were eventually modified when needed to change the range of n . The only further experiments for the nearest neighbor distributions that could be done would be to look at larger q_n s than the ones done here. This is impossible until there are much faster computers, with more RAM. It could take anywhere from 3 to 30 minutes to calculate the distribution for $n = 1 \rightarrow 1,000,000$. It would take even longer when looking at $n = q_n + R \rightarrow q_n + R + N - 1$ because more digits of α was

needed. When starting at $n = 1$, 30 digits of α was used. When starting from $\alpha = q_n + R$, the numbers of digits used was based on the highest value of n used such that there would be ≈ 10 digits to the right of the decimal place when looking at $n^2\alpha$.

Because right now it is almost impossible to get any data for higher q_n s for nearest neighbor distributions, the only reasonable things to do next would be to look other k^{th} neighbor distributions and see if the results are similar to the results for nearest neighbor distributions. Only nearest neighbor distributions were considered in this paper and it would be interesting to see if in the higher neighbor distributions it took longer to return to a Poissonian distribution or it took around the same amount of increase on N when starting at $n = 1$ and also when going from $n = q_n + K$ to $n = q_n + K + N - 1$ (i.e. the second question that was begin answered in the paper), do we get the same *Discs* for the same K s as the nearest neighbor distribution. It would not be that hard to modify the programs I used to make them run for k^{th} neighbor distributions. The parts that would need to be changed would be the actual calculation of the neighbor spacings, changed from 1^{st} neighbor spacings to general k^{th} neighbor spacings and the theoretical Poissonian distribution from $\int_0^x e^{-t} dt$ to $\int_0^x \frac{t^{k-1}}{(k-1)!} e^{-t} dt$.

Bibliography

- [Ca] J. W. S. Cassels, *An Introduction to Diophantine Approximations*, Vol II. Second Edition. John Wiley & Sons, Inc., New York-London-Sydney 1971
- [Li] R. Lipshitz, *Numerical Results Concerning the Distribution of $n^2\alpha$* , Junior Research Project 2000-2001
- [RSZ] Zeév Rudnick, P. Sarnak, and A. Zaharescu, *The Distribution of Spacings Between the Fractional Parts of $n^2\alpha$* , *Invent. Math.* 145 (2001), no. 1, 37–57.
- [TMHS] R. Takloo-Bighash, S. J. Miller, H. Helfgott, F. Spinu, *Junior Research Seminar: Diophantine Analysis and Approximations* Lecture Notes, 2002