Distribution of Digits in the Continued Fraction Representations of Seventh Degree Algebraic Irrationals

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Abstract

In this paper, we compare the statistical distribution of the digits that arise in the continued fraction representation of algebraic irrational numbers of degree 7. Representatives are chosen from each of the degree 7 Galois groups, and compared to the "expected" distribution predicted by Kuz'min's Theorem.

Contents

Chapter 1

Theory and Background

1.1 Kuz'min's Theorem

In this section we will discuss the theorem of Kuz'min. As a procursor to a discussion of the continued fraction representations of roots of polynomials which are irreducible over the field of rational numbers a few words should be said about the theorem which motivates this paper.

Following convention set

$$a = [0; a_1, a_2, \ldots, a_n, \ldots]$$

$$r_n = r_n(a) = [a_n; a_{n+1}, a_{n+2}, \ldots]$$

Denote by $m_n(x)$ the measure of the set of numbers $a \in (0, 1)$ for which

$$z_n(a) < x$$

Though his paper was never published Gauss ,claimed in his correspondance with Laplace, that he had shown

$$\lim_{x \to 0} m_n(x) = \frac{\ln(1 + x)}{\ln 2}$$

and that he wanted to have an approximation for

$$m_n(x) - \frac{\ln(1 + x)}{\ln 2}.$$

Kuz'min proved a theorem from which one can find the estimate that Gauss desired. The details of the proof are not needed to motivate our results, as statement and implications of the proof will suffice.

Theorem *(Kuz'min's Theorem)*

Suppose that $f_1(x), f_2(x), \ldots, f_n(x), \ldots$ is a sequence of real functions defined on the interval $(0,1)$ that, on that interval, satisfy the relationship

$$f_{n+1}(x) = \sum_{k=1}^{\infty} \frac{1}{(k+x)^2} f_n\left(\frac{1}{k+x}\right)(n \geq 0)$$

$$If, for 0 \leq x \leq 1, 0 < f_0(x) < M, and \ |f_0'(x)| < \mu,$$

$$then \ f_n(x) = \frac{a}{1+x} + \theta A e^{-\lambda\sqrt{n}} \quad (0 \leq x \leq 1),$$

$$with \ a = \frac{1}{\ln 2}\int_0^{\infty} f_0(z)dz, \quad |\theta| < 1,$$

λ is an absolute positive constand while A is positive but depends on M and μ. The key implication is that

$$\left|\mathcal{ME}\left(\begin{smallmatrix} n \\ k \end{smallmatrix}\right) - \frac{\ln 1 + \frac{1}{k(k+2)}}{ln2}\right| < \frac{A}{k(k+1)} e^{-\lambda\sqrt{n-1}}$$

Where

$$\mathcal{ME}\left(\begin{smallmatrix} n \\ k \end{smallmatrix}\right) = \int_{\frac{1}{k+1}}^{\frac{1}{k}} m_{n-1}'(x)dx$$

This estimate fulfills Gauss' original desire. However, the question is not completed with Kuz'min's proof. His theorem only holds for all but a set of measure 0. One can easily give examples of continued fractions which deviate in very pronounced ways from what the theory would predict. For this reason it would be interesting to choose a set and examine if whether or not Kuz'min's theorem held. Such analysis coupled with an examination of the underlying algebraic structure generating the continued fraction could yield greater insight into the sets for which Kuz'min's theorem does or does not hold.

It would be nice to have a detailed description of this set of measure 0. One could ask what the fundamental structural property of the set of numbers actually is. Are they connected to the roots of a certain set of polynomials. Experimentally one can try and test this hypothesis in order to justify later analytical work.

It is known that degree 1 and degree 2 algebraic numbers do not satisfy Kuz'min's theorem. However the question of how higher degree algrebraic numbers behave still remains. For this reason we have decided to examine degree 7 algebraic nubmers to see if the digits occur in the frequencies predicted by Kuz'min's theorem. To make the search through that set of numbers more systematic we chose to compare the distributions of the digits of the various solutions to polynomials with different galois groups.

1.2 Galois Theory of 7th Degree Polynomials

Every irreducible polynomial of degree 7 has as its Galois group a transitive subgroup of the symmetric group S_7. There are seven such subgroups, listed below according to the naming scheme of Conway, Hulpke, and McKay[**?**]:

Group	Name	Order	Generators
T_1	$C(7)$	7	(0123456)
T_2	$D(7)$	14	$(0123456), (16)(25)(34)$
T_3	$F_{21}(7)$	21	$(0123456), (124)(365)$
T_4	$F_{42}(7)$	42	$(0123456), (132645)$
T_5	$L(7)$	168	$(0123456), (12)(36)$
T_6	A_7	2520	$(012), (023), (034), (045), (056)$
T_7	S_7	5040	$(01), (02), (03), (04), (05), (06)$

The question of whether any transitive permutation group can be realized as the Galois group of some polynomial remains an open one, although much work has been done for polynomials of low degree. Malle and Matzat in their book[**?**] offer sample polynomials in $\mathbb{Q}$ for each of the above groups. In this paper we seek to experiment on a large number of polynomials with each Galois group. Smith [**?**] found polynomials in $\mathbb{Q}(t)$ for each of the above groups. Since Gal $(\mathbb{Q}(\alpha)/\mathbb{Q}) =$ Gal $(\mathbb{Q}(t,\alpha)/\mathbb{Q}(t))$, each of his polynomials, listed below, defines a class of polynomials in $\mathbb{Q}$ with the desired Galois group by selecting a value for the parameter t.

Group	Name	Polynomial
T_1	$C(7)$	$x^7 + (t^3 + 2t^2 - t + 13)x^6 + 3(t^2 - t + 2)(t^3 + t + 9)x^5$ $+ (3t^7 - 9t^6 + 27t^5 - 22t^4 + 6t^3 + 84t^2 - 121t + 75)x^4$ $+ (t^2 - t + 2)(t^7 - 5t^6 + 15t^5 - 32t^4 + 20t^3 + 14t^2 - 113t - 1)x^3$ $- (t^{10} - 5t^9 + 25t^8 - 61t^7 + 126t^6 - 117t^5$ $+ 58t^4 + 155t^3 - 168t^2 + 80t + 44)x^2$ $- (t^{10} - 8t^9 + 30t^8 - 75t^7 + 102t^6 - 89t^5$ $- 34t^4 + 56t^3 - 113t^2 - 42t + 17)x$ $+ t^9 - 7t^8 + 23t^7 - 42t^6 + 28t^5 + 19t^4 - 60t^3 - 2t^2 + 16t - 1$
T_2	$D(7) = 7\colon 2$	$x^7 - (7t^3 + 35t^2 + 21t + 1)(21x^5 + (98t + 70)x^4$ $- (1029t^3 + 4557t^2 + 343t - 105)x^3$ $- 28(7t + 1)(49t^3 + 147t^2 + 63t - 3)x^2$ $+ 7(7t^2 + 42t - 1)(7t^2 + 14t - 5)(7t + 1)^2 x$ $+ 235298t^7 + 1236858t^6 + 1138074t^5$ $+ 562226t^4 + 11270t^3 - 4914t^2 - 322t + 6)$
T_3	$F_{21}(7) = 7\colon 3$	$x^7 - (t^6 + 56t^4 - 7t^3 + 980t^2 - 189t + 5103)$ $(21x^5 + (21t^3 - 49t^2 + 539t - 1323)x^4$ $- (91t^6 + 98t^5 + 4753t^4 + 3822t^3 + 72471t^2 + 31752t + 250047)x^3$ $- (112t^9 - 49t^8 + 8379t^7 - 7154t^6 + 221186t^5 - 293265t^4$ $+ 2343033t^3 - 4167450t^2 + 7501410t - 13502538)x^2$ $+ 7(12t^{11} + 14t^{10} + 1456t^9 + 1687t^8 + 67816t^7 + 79786t^6 + 1492540t^5$ $+ 1847888t^4 + 15067647t^3 + 20994687t^2 + 51902613t + 94517766)tx$ $+ 97t^{15} + 14t^{14} + 14371t^{13} + 798t^{12} + 865494t^{11}$ $- 24087t^{10} + 26944169t^9 - 2680790t^8 + 451657745t^7$ $- 66683316t^6 + 3747134223t^5 - 557965989t^4 + 10380951252t^3$ $- 573857865t^2 - 14765025303t - 4921675101)$
T_4	$F_{42}(7) = 7\colon 6$	$x(x^2 + 7x + 28)(x^2 + 7)^2 + 28(x - 7)(x^2 + 7)t$ $- 7(x^4 + 16x^2 - x + 84)t^2 - 7(5x^2 + 37)t^3 - 30t^4 - t^5$
T_5	$L(7) = L(3,2)$	$x(x^2 - 49)^3 - t(x - 1)^3(x - 9)$
T_6	A_7	$x(x^2 - 49)^3 - t(x^2 - 25)^3$
T_7	S_7	$x^7 - t(x - 6)$

Chapter 2

Experiment

2.1 Calculation

Due to its probabilistic nature, the distribution of digits predicted by Kuz'min's Theorem applies only to "almost all" real numbers; it is possible that there are a large number of exceptions so long as the set of exceptions has measure zero. It is believed by the authors that algebraic numbers of degree > 2 do not constitute such an exception. Results by other members of this laboratory point to this conclusion for degrees smaller than 7. The data we have obtained here suggest the same thing for 7^{th} degree irrationals. Our goal in this paper is to offer evidence in support of this conjecture, and to show that the Galois group is irrelevant to the distribution of digits in the continued fraction.

The classes of polynomials calculated by G. W. Smith, listed above, were taken as representative of their Galois groups. It is possible that peculiarities in his algorithm could obscure facts about the broader set of degree 7 algebraic numbers, but as we were interested in discovering properties of the Galois groups as a whole, restricting attention to these polynomials is sufficient. For each class, we apply the homomorphism $\varphi : t \to k$ for each value of k between 1 and 1000. This gives us a sample of 1000 polynomials with each Galois group. For each polynomial, we computed the first 10000 digits of the continued fraction and built a table telling us the number of occurances of each number in those 10000 digits. All calculations were done in Mathematica using its built-in routined for computing continued fractions. The results for each polynomial corresponded closely, as expected, with Kuz'min's Theorem.

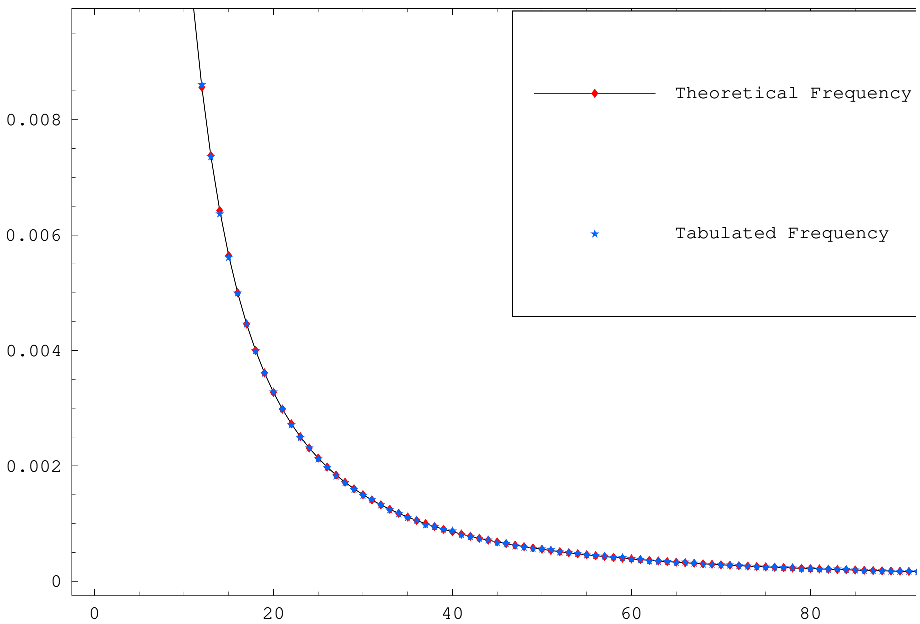
2.2 Data

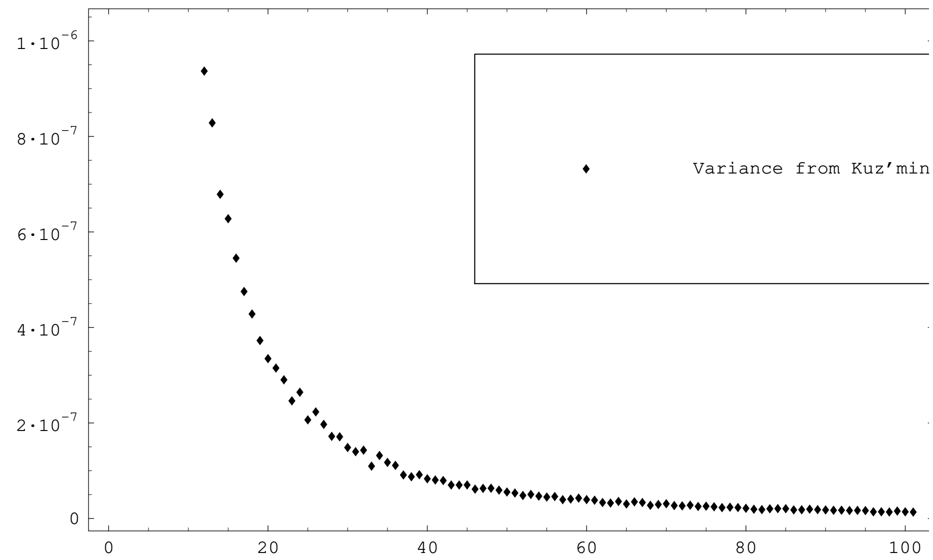
What follows are data tables for each of the Galois groups under investigation. Three graphs are shown for each Galois group.

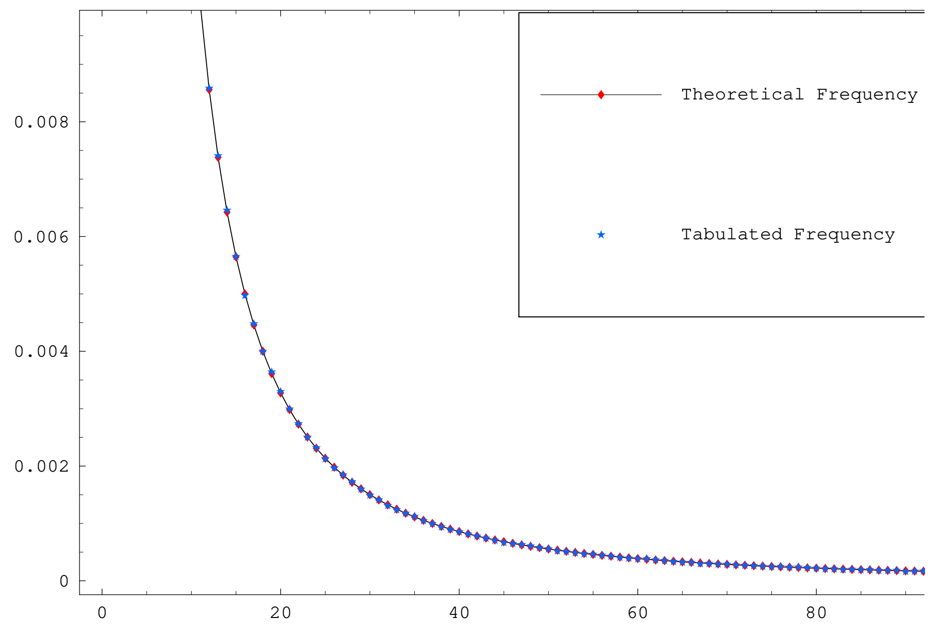
In the first graph, the mean frequency of occurance of each digit is plotted against the value of the digit. Kuz'min's Theorem is that this graph should look like:

$$\log_2\left(1+\frac{1}{k\,(k+2)}\right) \tag{2.1}$$

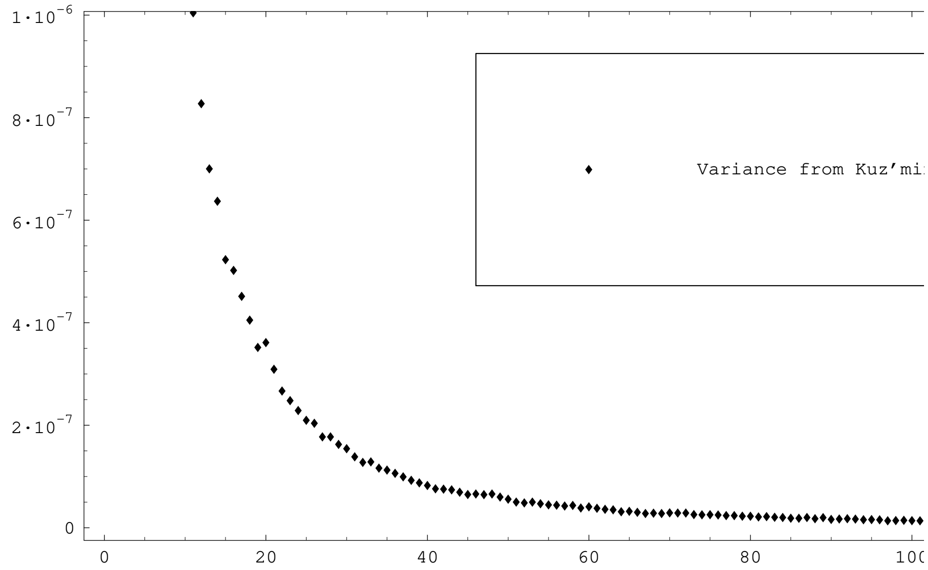
This expected function is also drawn on the graphs, and we can see a clear correlation. This indicates that, on the average, continued fractions from each Galois group obey Kuz'min's Theorem. The second graph is the variance of the frequency of occurance of each digit, also plotted against the value of the digit. As the value of the digit increases, the variance decreases, probably due mostly to the smaller sample size. The most important thing to notice on this graph is the magnitude of the variance; the data are packed extremely closely around their mean. This shows that it is really every number that obeys these predictions, not just the averages. Again, as the next few pages show, this investigation suggests that there is really no distinction between Galois groups that is reflected in the distribution of digits in their continued fraction.

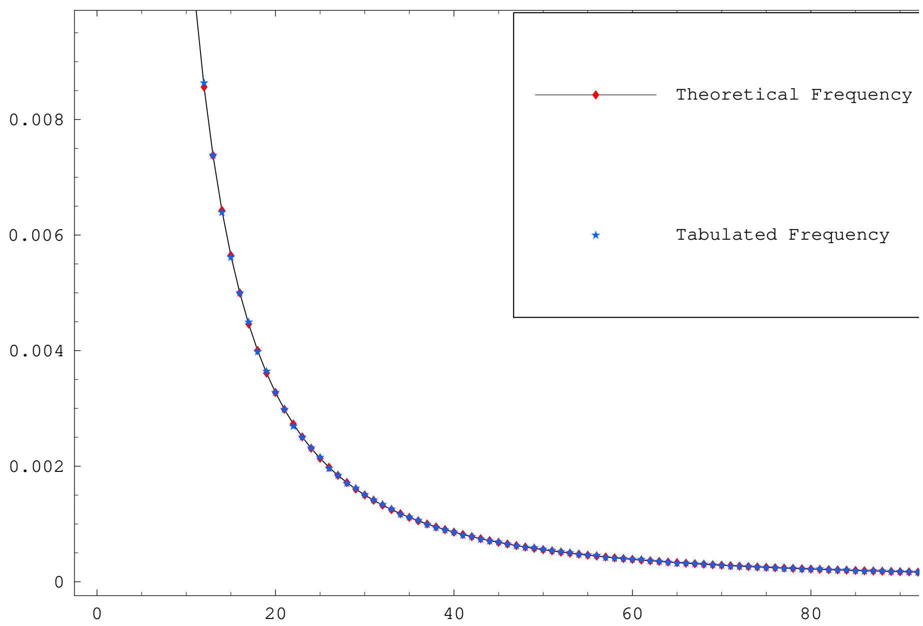

Theoretical Frequency
Tabulated Frequency
0.008
0.006
0.004
0.002
0
0
20
40
60
80

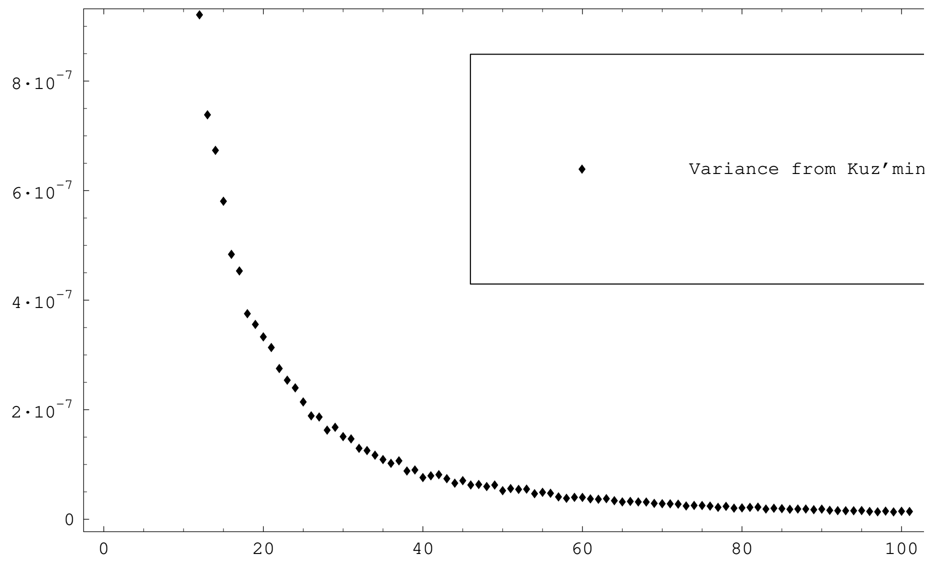

1·10^-6
8·10^-7
6·10^-7
4·10^-7
2·10^-7
0
0
20
40
60
80
100
Variance from Kuz'min

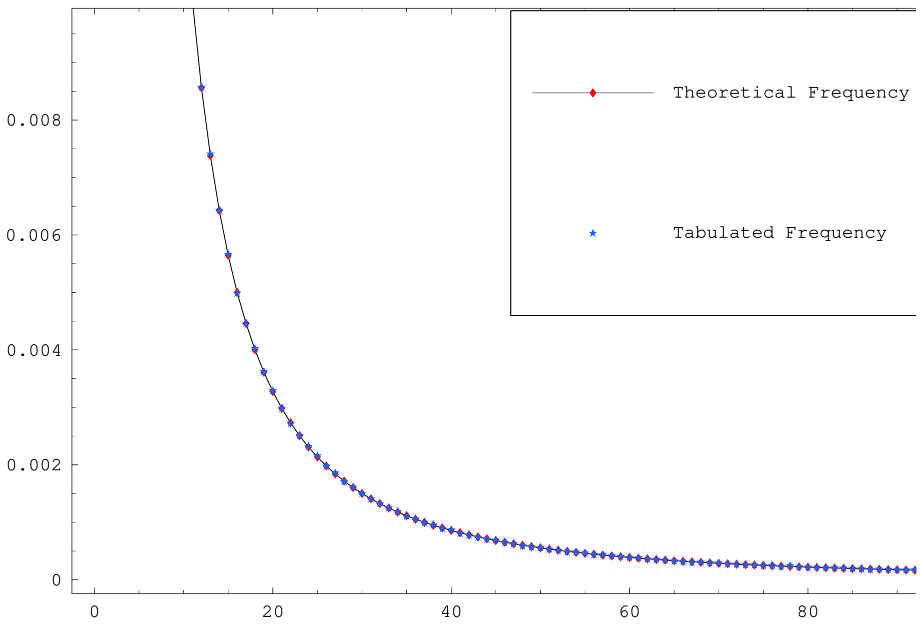


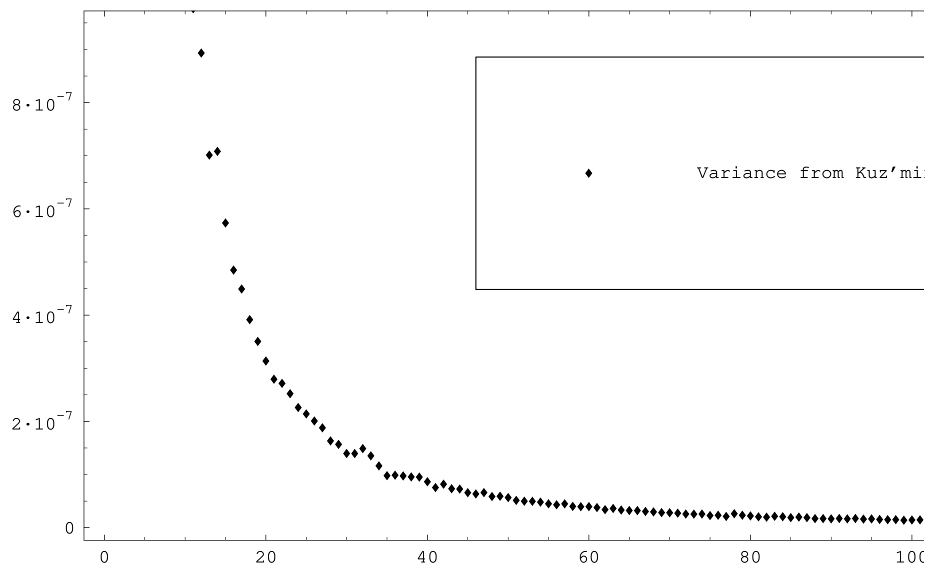
0.008
0.006
0.004
0.002
0
0
20
40
60
80
Theoretical Frequency
Tabulated Frequency

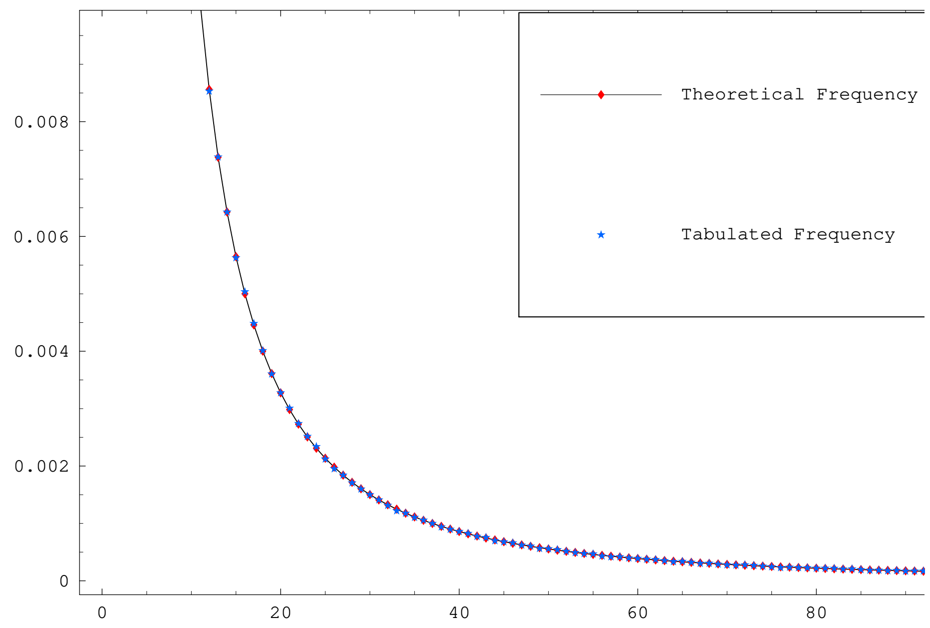

1·10^-6
8·10^-7
6·10^-7
4·10^-7
2·10^-7
0
0
20
40
60
80
Variance from Kuz'mi

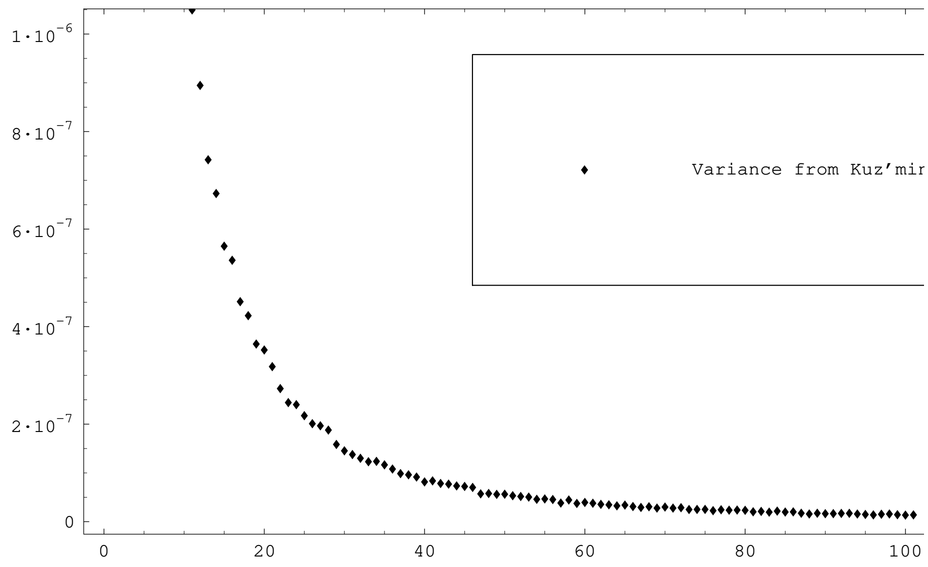

Theoretical Frequency
Tabulated Frequency
0.008
0.006
0.004
0.002
0
0
20
40
60
80


8·10^-7
6·10^-7
4·10^-7
2·10^-7
0
0
20
40
60
80
100
Variance from Kuz'min

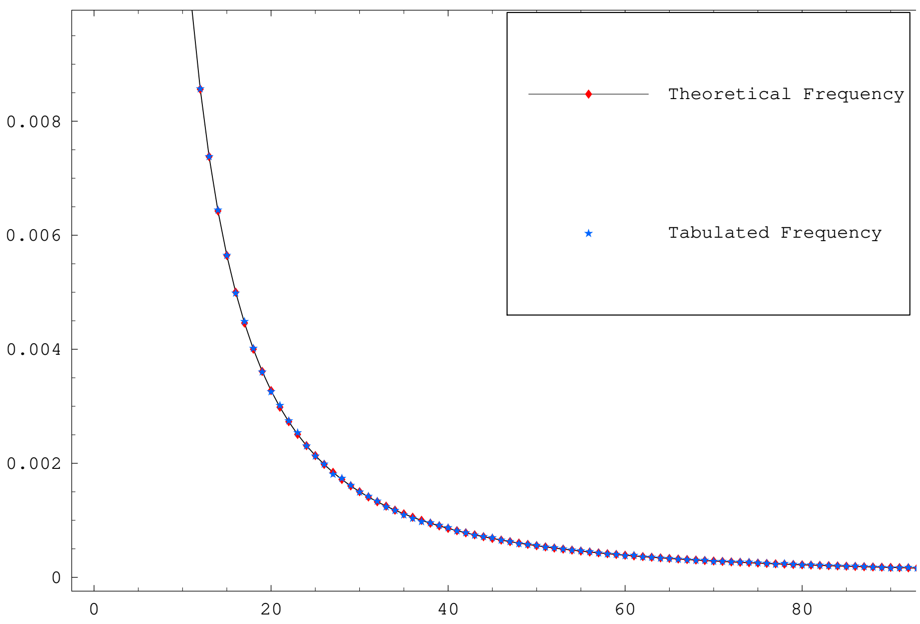

0.008
0.006
0.004
0.002
0
0
20
40
60
80
Theoretical Frequency
Tabulated Frequency


8·10^-7
6·10^-7
4·10^-7
2·10^-7
0
0
20
40
60
80
Variance from Kuz'mi

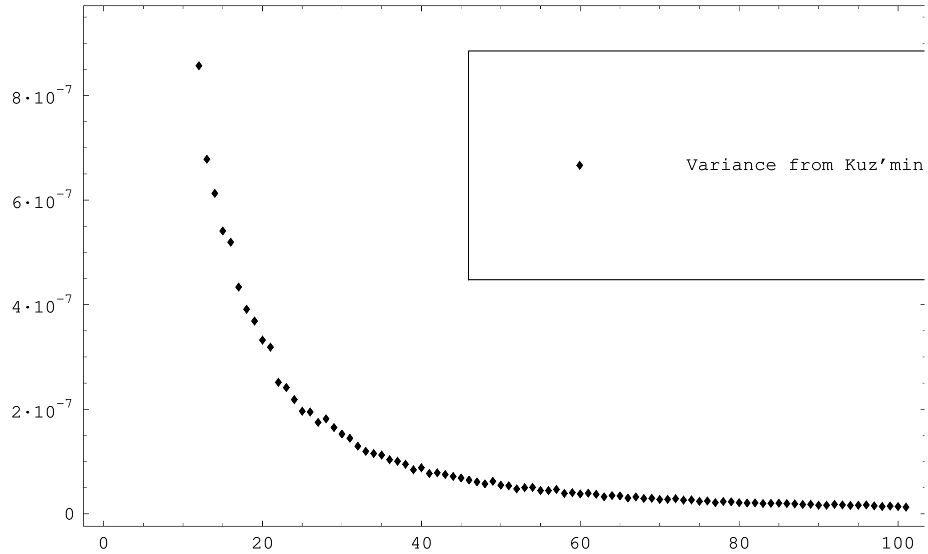

Theoretical Frequency
Tabulated Frequency
0.008
0.006
0.004
0.002
0
0
20
40
60
80



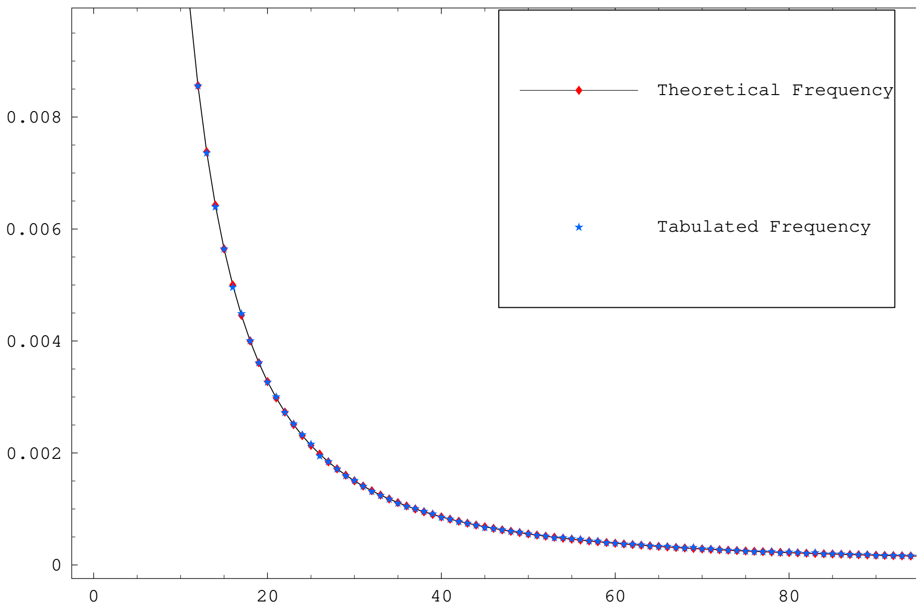
Variance from Kuz'min
$1\cdot10^{-6}$
$8\cdot10^{-7}$
$6\cdot10^{-7}$
$4\cdot10^{-7}$
$2\cdot10^{-7}$
0
0
20
40
60
80
100



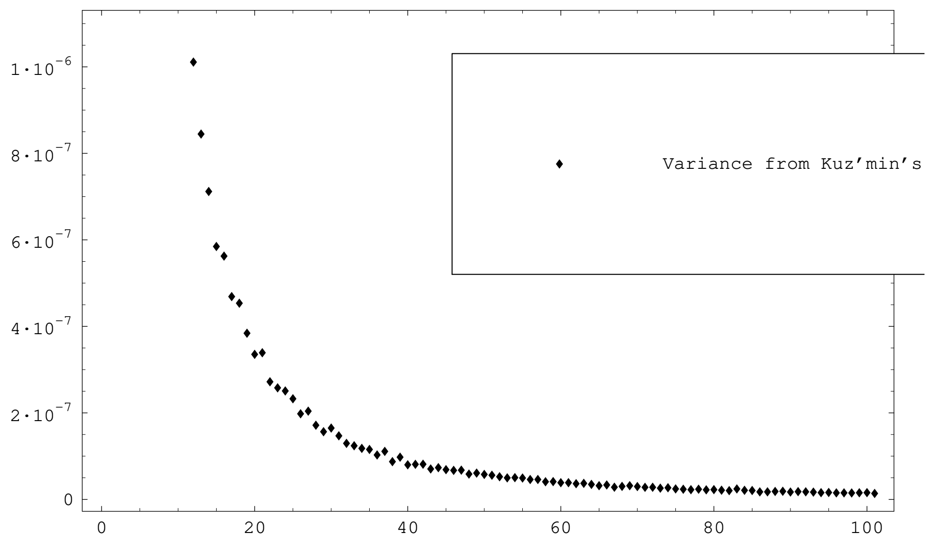
0.008
0.006
0.004
0.002
0
0
20
40
60
80
Theoretical Frequency
Tabulated Frequency



8·10^-7
6·10^-7
4·10^-7
2·10^-7
0
0
20
40
60
80
100
Variance from Kuz'min



Theoretical Frequency
Tabulated Frequency
0.008
0.006
0.004
0.002
0
0
20
40
60
80



1·10^-6
8·10^-7
6·10^-7
4·10^-7
2·10^-7
0
0
20
40
60
80
100
Variance from Kuz'min's

Here are some sections of the tables from which the graphs were created. The rows correspond to a given continued fraction representation while the columns correspond to the number of times that a given digit has occured. The behavior is clearly very close to the prediction of Kuz'min's theorem.

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4136 & 1681 & 956 & 562 & 396 & 304 & 226 & 186 & 152 & 146 \\
4149 & 1682 & 930 & 604 & 378 & 326 & 216 & 189 & 159 & 126 \\
4104 & 1655 & 962 & 611 & 442 & 274 & 228 & 180 & 139 & 113 \\
4174 & 1730 & 870 & 566 & 411 & 325 & 233 & 152 & 153 & 109 \\
4184 & 1696 & 910 & 547 & 400 & 311 & 249 & 184 & 156 & 106 \\
4109 & 1696 & 980 & 523 & 434 & 295 & 262 & 153 & 132 & 126 \\
4106 & 1698 & 987 & 582 & 385 & 307 & 232 & 165 & 156 & 129 \\
4098 & 1728 & 980 & 593 & 384 & 304 & 221 & 193 & 134 & 120
\end{pmatrix}$$

$$\begin{pmatrix} 4149 & 1711 & 925 & 580 & 433 & 296 & 235 & 160 & 137 & 127 \\ 4118 & 1701 & 969 & 550 & 381 & 322 & 239 & 187 & 152 & 127 \\ 4184 & 1697 & 887 & 575 & 415 & 302 & 232 & 186 & 135 & 107 \\ 4129 & 1718 & 911 & 631 & 428 & 292 & 223 & 175 & 145 & 118 \\ 4139 & 1700 & 922 & 587 & 416 & 296 & 271 & 185 & 148 & 102 \\ 4163 & 1691 & 950 & 579 & 395 & 305 & 231 & 178 & 155 & 126 \\ 4152 & 1700 & 911 & 554 & 429 & 327 & 236 & 180 & 140 & 116 \\ 4125 & 1695 & 930 & 586 & 414 & 307 & 232 & 183 & 147 & 113 \\ 4130 & 1684 & 965 & 625 & 405 & 312 & 216 & 165 & 155 & 115 \\ 4219 & 1705 & 899 & 573 & 420 & 302 & 244 & 182 & 133 & 118 \end{pmatrix}$$

$$\begin{pmatrix} 4133 & 1684 & 880 & 625 & 423 & 299 & 227 & 195 & 139 & 134 \\ 4080 & 1683 & 901 & 569 & 440 & 279 & 228 & 195 & 142 & 116 \\ 4107 & 1727 & 945 & 609 & 427 & 265 & 237 & 167 & 144 & 115 \\ 4067 & 1705 & 955 & 594 & 411 & 322 & 218 & 187 & 155 & 134 \\ 4192 & 1645 & 949 & 584 & 433 & 293 & 241 & 145 & 141 & 142 \\ 4151 & 1774 & 896 & 576 & 409 & 293 & 220 & 174 & 165 & 108 \\ 4086 & 1716 & 951 & 562 & 431 & 289 & 256 & 161 & 124 & 124 \\ 4151 & 1728 & 931 & 569 & 377 & 316 & 234 & 160 & 150 & 107 \\ 4215 & 1648 & 943 & 590 & 380 & 290 & 237 & 171 & 145 & 111 \\ 4158 & 1648 & 935 & 578 & 434 & 277 & 231 & 168 & 142 & 120 \end{pmatrix}$$

$$\begin{pmatrix} 4124 & 1690 & 951 & 596 & 407 & 340 & 235 & 180 & 138 & 122 \\ 4177 & 1714 & 897 & 548 & 431 & 289 & 235 & 185 & 163 & 109 \\ 4139 & 1658 & 932 & 577 & 441 & 309 & 236 & 189 & 140 & 100 \\ 4167 & 1681 & 903 & 588 & 424 & 288 & 234 & 169 & 159 & 125 \\ 4105 & 1701 & 898 & 601 & 430 & 325 & 251 & 179 & 139 & 110 \\ 4135 & 1714 & 915 & 598 & 408 & 318 & 241 & 178 & 159 & 120 \\ 4115 & 1759 & 942 & 544 & 418 & 307 & 234 & 178 & 145 & 119 \\ 4258 & 1712 & 901 & 562 & 411 & 295 & 228 & 170 & 135 & 136 \\ 4153 & 1771 & 922 & 585 & 404 & 280 & 227 & 165 & 143 & 97 \\ 4112 & 1627 & 943 & 584 & 449 & 326 & 238 & 197 & 135 & 125 \end{pmatrix}$$

Out[87]//MatrixForm=

$$\begin{pmatrix} 4199 & 1715 & 944 & 559 & 405 & 294 & 214 & 168 & 149 & 121 \\ 4139 & 1681 & 916 & 567 & 448 & 327 & 221 & 158 & 141 & 100 \\ 4136 & 1702 & 931 & 568 & 446 & 308 & 223 & 169 & 130 & 99 \\ 4193 & 1666 & 940 & 588 & 428 & 309 & 226 & 148 & 146 & 124 \\ 4092 & 1717 & 938 & 593 & 431 & 297 & 237 & 182 & 165 & 108 \\ 4147 & 1722 & 888 & 580 & 431 & 269 & 230 & 200 & 161 & 122 \\ 4178 & 1693 & 939 & 580 & 397 & 288 & 244 & 177 & 143 & 120 \\ 4215 & 1668 & 925 & 565 & 390 & 306 & 231 & 156 & 130 & 120 \\ 4294 & 1629 & 930 & 574 & 400 & 300 & 192 & 168 & 154 & 125 \\ 4218 & 1633 & 871 & 617 & 414 & 275 & 219 & 203 & 147 & 119 \end{pmatrix}$$

$$\begin{pmatrix} 4187 & 1716 & 933 & 565 & 392 & 284 & 240 & 180 & 144 & 125 \\ 4051 & 1720 & 990 & 622 & 386 & 305 & 230 & 171 & 148 & 109 \\ 4150 & 1692 & 913 & 612 & 422 & 309 & 228 & 154 & 153 & 131 \\ 4155 & 1641 & 949 & 615 & 416 & 291 & 240 & 151 & 130 & 129 \\ 4111 & 1676 & 949 & 567 & 411 & 314 & 218 & 187 & 152 & 129 \\ 4161 & 1673 & 932 & 608 & 404 & 267 & 234 & 174 & 178 & 103 \\ 4195 & 1687 & 860 & 591 & 383 & 318 & 216 & 182 & 153 & 129 \\ 4217 & 1709 & 937 & 571 & 407 & 300 & 223 & 166 & 147 & 104 \\ 4101 & 1723 & 939 & 586 & 429 & 300 & 213 & 173 & 151 & 123 \\ 4145 & 1646 & 1028 & 614 & 394 & 296 & 208 & 182 & 147 & 122 \end{pmatrix}$$

Out[71]//MatrixForm=

$$\begin{pmatrix} 4098 & 1647 & 932 & 609 & 409 & 309 & 230 & 202 & 143 & 133 \\ 4199 & 1697 & 965 & 582 & 440 & 285 & 211 & 163 & 150 & 106 \\ 4196 & 1690 & 889 & 605 & 389 & 304 & 237 & 178 & 153 & 145 \\ 4081 & 1756 & 955 & 569 & 409 & 282 & 222 & 176 & 146 & 108 \\ 4192 & 1669 & 953 & 582 & 397 & 273 & 204 & 161 & 157 & 123 \\ 4151 & 1691 & 1010 & 535 & 445 & 286 & 213 & 162 & 148 & 116 \\ 4120 & 1716 & 930 & 594 & 401 & 277 & 226 & 188 & 136 & 109 \\ 4149 & 1704 & 944 & 549 & 422 & 276 & 229 & 164 & 150 & 125 \\ 4216 & 1666 & 903 & 577 & 410 & 295 & 231 & 184 & 158 & 113 \\ 4141 & 1706 & 911 & 648 & 358 & 308 & 223 & 159 & 144 & 122 \end{pmatrix}$$

The data above demonstrate that the irrationals that we have selected behave according to Kuz'min's Theorem, and leads us to conjecture that all 7^{th} degree algebraic irrational numbers have a Kuz'minian distribution of digits in their continued fraction. The Galois group that the number belongs to does not correlate with the distrubution of digits, at least not for the representative examples chosen. It has been observed that for lower degree algebraic irrationals one typically does not see deviations from Kuz'min's theorem. We are inclined to believe that algebraic irrationals generally follow the theoretical prediction.

2.3 For Those who come after

One can perform similar computational experiments but the utility of this approach must be considered. As the degree of the algebraic irrational number increases one runs into various difficulties. First the number of galois groups and hence the number irreducible polynimials increase leading to more calculations and also more involved algebra. However the more fundamental question is whether or not there is a better way to guide one's intuition about the measure 0 set for which Kuz'min's theorem fails. Given that a number having the property of being irrational does not seem to cause it to deviate from the theory what is the next class or type of number that should be considered? The fundamental limitations of computers are prevent one from performing the same type of calculation on a transcendental number. For this reason it appears that the next step in this line of research may need to be done analytically. Dispite such computational difficulties it may be the case that as more experiments like this one are performed very subtle deviations from Kuz'min's theorem may be tied to some higher structural similarity. If this is truly the case then not only will more computational experiments be beneficial but but also more detailed experiments will be necessary. Specifically 10000 elements of a continued fraction representation may not be enough to descern the devations from the "expected" distribution. However 1000000 elements may be enough to start to see divergence which is not the result of the inherent finite quality of computations but rather a fundamental difference between theory and experiment.

Bibliography

[KM] J. Klüners and G. Malle, (2000): Explicit Galois realization of transitive groups of degree up to 15, *J. Symbolic Computation* 30, $675 - 716$.

[MM] G. Malle and B. H. Matzat, (1999): *Inverse Galois Theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin.

[BM] G. Butler and J. McKay, (1983): The transitive groups of degree up to eleven, *Comm. in Algebra* 11, $863 - 911$.

[CHM] J. H. Conway, A. Hulpke, and J. McKay, (1998): On transitive permutation groups. *J. Comp. Math.* 1, $1 - 8$.

[Sm2] G. W. Smith, (2000): Some polynomials over $\mathbb{Q}(t)$ and their Galois groups, *Math. Comp.* **69**, $775 - 796$.

[Sm1] G. W. Smith, (1991): Generic cyclic polynomials of odd degree, *Comm. Algebra* **19**, $3367 - 3391$

[ClassNotes] R. Takloo-Bighash, S. J. Miller, H. Helfgott, F. Spinu, 2002: Notes for Princeton Junior Seminar, Fall 2002: Diophantine Analysis and Roth's Theorem, Princeton University.