The mathematics of voting, power and sharing - Part 2

The power index

In everything discussed so far in this unit, all voters had equal standing. This is not true in all voting situations, as shown by the following examples:

Examples:

- shareholders: their vote is proportional to the number of shares they hold.
- Electoral College: many states require that all their delegates vote for the same presidential candidate; as a result, states function like voters with unequal weights, and thus unequal importance in the end result.
- County boards: some townships have more representatives than others. Assuming that they all vote the same way, this gives the different townships unequal power.

How can one measure this power? It is not simply proportional to the number of votes:

Example: In a shareholders' meeting, there are 3 participants. A has 47% of the shares, B has 48%, and C the remaining 5%. A majority of 51% is needed to pass any measure. Any group of 2 can force the measure to pass over the opposition of the third. So A, B, C have equal power - despite their unequal number of shares.

There exist several schemes to try to measure this "power" of the participants.

One of the most widely accepted (among people who know about this - not as many as there should be!) is the Banzhaf power index.

Motivation: When do you have "power"? When your decision matters. That is, when whether you vote one way or the other makes a difference in the outcome, or, when your vote is a "swing" vote.

So let us define your power index as the fraction

$$\frac{\text{Number of coalitions where you are a swing vote}}{\text{Total number of coalitions}}$$

Example:

- In the case above (A:47%, B:48%, C:5%) the possible coalitions are

1.	ABC		2.	AB	C	3.	AC	B	4.	BC	A
5.	A	BC	6.	B	AC	7.	C	AB	8.		ABC

$$\left.\begin{array}{l}\text{In cases 1,8: nobody is a swing vote}\\ \text{In cases 2,7: } A, B \text{ are both swing votes}\\ \text{In cases 3,6: } A, C \text{ are both swing votes}\\ \text{In cases 4,5: } B, C \text{ are both swing votes}\end{array}\right\} \begin{array}{l}\text{It follows that } A, B \text{ and } C \text{ have the same}\\ \quad\text{power index } \ 4/8 = .5\end{array}$$

- Whether you are a swing vote or not depends not only on your number of shares, but also on what majority is needed to reach a decision.

  If a measure can be passed in the example above only when it has 53% of the votes or more, than the situation changes (fill in the answers):

1 : B swing vote	5 :
2 :	6 :
3 :	7 :
4 :	8 :

$\Rightarrow$ power index of $A =$

power index of $B =$

power index of $C =$

Notation:

$$\underset{\substack{\text{quorum}\\ \text{needed to pass}\\ \text{a measure}}}{\underset{\uparrow}{[\,q:}} \quad \underset{\substack{\text{weights of}\\ \text{the voters}}}{\underset{\nwarrow\uparrow\nearrow}{w_1, w_2, w_3, \ldots]}} \quad = \quad \underset{\substack{\text{power indices}\\ \text{of the voters}}}{\underset{\nwarrow\uparrow\nearrow}{(p_1,\ p_2,\ p_3, \ldots)}}$$

In the two examples above: $[51 : 47, 48, 5] = (.5, .5, .5)$
$[53 : 47, 48, 5] = (\quad,\quad,\quad)$

An example with 4 voters:

Example: $[51 : 40, 30, 20, 10] = (\ ?\ ,\ ?\ ,\ ?\ ,\ ?\ )$

circle votes that are swing votes	A 40	B 30	C 20	D 10	$Votes$	$Pass/Fail$
	$+$	$+$	$+$	$+$	100	P
	$+$	$+$	$+$	$-$	90	P
	$+$	$+$	$-$	$+$	80	P
	$+$	$-$	$+$	$+$		
	$-$	$+$	$+$	$+$		
	$+$	$+$	$-$	$-$		
	$+$	$-$	$+$	$-$		
	$+$	$-$	$-$	$+$		
	$-$	$+$	$+$	$-$		
	$-$	$+$	$-$	$+$		
	$-$	$-$	$+$	$+$		
	$+$	$-$	$-$	$-$		
	$-$	$+$	$-$	$-$		
	$-$	$-$	$+$	$-$		
	$-$	$-$	$-$	$+$		
	$-$	$-$	$-$	$-$		

$\Rightarrow [51 : 40, 30, 20, 10] = (\qquad ,\qquad ,\qquad ,\qquad)$

Steven Bram's book, Rational Politics, contains an interesting discussion of how the power index of the small country of Luxembourg went through some yo-yo motions as the European community expanded:

Country	**1958–1973** Weight	Power	**1973–1981** Weight	Power	**1981–1985** Weight	Power
France	4	0.313	10	0.199	10	0.189
Germany	4	0.313	10	0.199	10	0.189
Italy	4	0.313	10	0.199	10	0.189
Belgium	2	0.187	5	0.109	5	0.100
Netherlands	2	0.187	5	0.109	5	0.100
Luxembourg	1	<u>0.000</u>	2	<u>0.008</u>	2	<u>0.049</u>
England	—	—	10	0.199	10	0.189
Denmark	—	—	3	0.078	3	0.049
Ireland	—	—	3	0.078	3	<u>0.049</u>
Greece	—	—	—	—	5	0.100

Note: Underscored values illustrate the paradox of new members (Luxembourg's voting power increases as its proportion of votes falls on the post-1973 and post-1981 councils) and the paradox that, on the post-1981 council, countries with different voting weights (Luxembourg versus Denmark and Ireland) may have exactly the same voting power. The decision rules (that is, qualified majorities) for the three councils are as follows: 12 out of 17 (1958–1973); 41 out of 58 (1973–1981); and 45 out of 63 (1981–1985).

$$\begin{array}{ll} \text{before } 73: & [12 : 4, 4, 4, 2, 2, 1] \\ 73 - 81: & [41 : 10, 10, 10, 5, 5, 2, 10, 3, 3] \\ 81 - 85: & [45 : 10, 10, 10, 5, 5, 2, 10, 3, 3, 5] \end{array}$$

One can use the Banzhaf power index to analyze other systems as well, where different players have unequal importance, but where this may not be expressed in weights.

Examples:

- A committee of 6 people, where the chair's vote is decisive when there is a tie; otherwise majority rule holds.
- A congress of 5 people and 1 president, where the president can veto, but his veto can be overridden by 2/3 of the congress; normally the majority rule holds.
  Try it also with a congress of 9 people.

Fair Division

Examples:

- splitting a cake
- dividing up an estate among the heirs
- splitting up the assets when a company breaks up

Two Players (division of a "cake" between 2 people)

One cuts, the other chooses.

Implicit assumptions:

- each player is able to divide cake in such a way that either of the two pieces would be OK with that player
- given any division of the cake, each player would find at least one piece acceptable.

Three or more players

Less easy ...

One possibility: last diminisher method. (40's – Banach & Knaster)

- First player (of group of N players) "cuts" a piece that looks fair to that player.
- The piece gets examined by the other players, 2 through N, successively.
  Each of these players can choose to "trim" the piece if they think it is too large for a fair share.
- After everybody has inspected it and possibly trimmed it, the piece goes to the last player who chose to diminish it, or to player 1 if nobody did.
- The procedure can be repeated with the remainder of the cake (+ trimmings) for the remaining N-1 players.

Try it out?

Problem with this (and many other) methods: it is not envy-free.

What is an envy-free solution?

A solution in which, after every player has his/her piece, nobody thinks that someone else is better off.

This is not guaranteed in the above procedure: when the first "piece" is allocated, the player who receives it may be happy with it, but he may change his mind when he sees that later players get much bigger pieces after he has left the division game.

Making fair division envy-free is <u>much</u> harder.

An envy-free division for <u>three</u> players.
(1960; found independently by John Conway and John Selfridge).

- player 1 cuts cake in three pieces that look equal to that player, and hands over to player 2.
- player 2 may, if she wishes, trim the piece that she thinks is largest so that it is equal to the next-largest, in her perception. The trimming T is set aside for the moment.
- player 3 chooses the piece he thinks is largest.
- next player 2 chooses. If 2 did trim in the second step, and if 3 did <u>not</u> take the trimmed piece, then 2 must take the trimmed piece.
- 1 gets the remaining piece.

So far, they are all happy, and there is no envy:

- 3 chose first
- 2 chose and got one of the two pieces she considered to be a tie for largest
- 1 got one of the pieces that he cut, and everybody else got (in their eyes) the same or less

Now what to do with the trimming?

Whatever happens with it, player 1 will never envy the player who received the trimmed piece in the first round, because, for player 1, trimmed piece + trimming only make up as much as he (1) got in the first round anyway.

Let's call the player who got the trimmed piece in the first round *Tr* (*Tr* is either 2 or 3) and the other one (of 2 and 3) *Untr*.

Now *Untr* will cut the trimming into three equal pieces (from his/her point of view). Then the other players choose: first *Tr*, then 1, then *Untr* takes the last piece of the trimming.

Result:

- *Tr* is happy, and envies no one, because *Tr* chose first.
- 1 does not envy *Tr* (see above)
  1 does not envy *Untr* because he chose ahead of *Untr*
- *Untr* does not envy anyone, because *Untr* did the cutting.

The funny thing is that there is no easy way to generalize this to 4 or more players. Until 1992, no envy-free solution was known for 4 players, but then one was found by Brams and Taylor (see the list of final projects).

Remarks:

- you can also use this to divide up a list of chores!
- this can be extended to more complicated problems, such as dividing up an estate.

Dividing up an estate, or property settlement in a divorce

Divorce: usually only two parties.

It is possible to end up with a situation where each party ends up with what they perceive as more than their fair share!

Example: Alice and Bob are divorcing. They have only two major assets, which need to be divided. First, each of them is asked to allocate points to the two assets, out of a total of 100, according to what they value most. Alice is a city person, and places premium value on the small NYC appartment that the couple owns. Bob is retired and likes to spend his time fishing; he values their nice shore house much more than the apartment. This results in the following table:

	Alice	Bob
shore house	30	70
N.Y.C. apt.	70	30

In this case, it makes sense to give Alice the apartment, and Bob the shore house.

In practice, the situation is usually more complicated, with more assets:

Example: 1991 Trump divorce. In this case, the point allocation table is not known, of course. Based on the negotiations, one can make the following guess:

Asset	Donald	Ivana
Connecticut estate	10	38
Palm Beach mansion	40	20
Trump Plaza apartment	10	30
Trump Tower triplex	38	10
Cash & Jewelry	2	2

step 1: Give each party the big items that they like most.

Donald:	Palm Beach mansion	40
	Trump Tower triplex	38
		78 points
Ivana:	Connecticut estate	38
	Trump Plaza apartment	30
		68 points

step 2: Give the remaining "small" things to the party who has the fewest points, to even out the result as much as possible. In this case, Ivana gets the cash and jewelry, and has now 70 points.

step3: The situation is not even. We need to transfer a bit from Donald to Ivana.

Since Ivana values the Palm Beach mansion much more than the triplex, while Donald values these two about equally, it makes sense to transfer part of the Palm Beach mansion. How much?

If we give x % of the Palm Beach mansion to Ivana, this leaves (100-x) % of the Palm Beach mansion for Donald.

How many points does each of the parties have then?

Donald	$40 \times \frac{100-x}{100} + 38 = 78 - .4x$
Ivana	$20 \times \frac{x}{100} + 70 = 70 + .2x$
To make things even, we require	$78 - .4x = 70 + .2x$
Solving this we find	$x = \frac{8}{.6} = 13.3$

In practice:

- Donald did get the Trump Towers triplex
- Ivana did get the Trump Plaza apartment, the Connecticut estate, cash & jewelry
- Donald got the Palm Beach mansion 11 months/year
- Ivana got the Palm Beach mansion 1 month/year, which corresponds to 8.34%.

Dividing an Estate

In other examples, where more than 2 people are involved, the situation is more complicated

The following example illustrates the Knaster procedure, which works if parties have a cash reserve as well.

Example: Alice, Bob, Carol and David have to split up their inheritance, which consists of one big asset: the house that their parents owned.

To start, they each put a subjective value on the house, which takes into account the market value, as well as their own sentimental attachment.

Alice	Bob	Carol	David
120,000	200,000	180,000	140,000

subjective value of house

Since Bob valued the house most highly, he gets it. He then puts 200,000 x 3/4 = 150,000 in a "pot"; this represents, according to his own evaluation, the money to buy out his three siblings for their share of the house.

Now each of the other three gets 1/4 of their own estimate of the total value of the house:

Alice takes 1/4 x 120,000 = 30,000 from the pot.

Carol takes 1/4 x 180,000 = 45,000 from the pot.

David takes 1/4 x 140,000 = 35,000 from the pot.

After all this, the pot is not empty; the remainder is: 40,000

Finally, this gets divided into equal shares, so that there is 10,000 for each

What is the total result?

From Alice's point of view: she got 40,000 even though she thought there was only 120,000 to divide, of which she expected 30,000

From Bob's point of view: he got the house, and spent 140,000; according to his estimate he therefore got 200,000 - 140,000 = 60,000 even though he thought there was only 200,000 to divide, of which he expected 50,000

From Carol's point of view: she got 55,000, although she expected 1/4 x 180,000 = 45,000 only.

From David's point of view: he got 45,000, although he expected 1/4 x 140,000 = 35,000 only.

However, the whole procedure is not envy-free: after all, Carol got more money than David, who got more than Alice!

This procedure is also possible only if Bob has a 140,000 cash reserve!