Algebra Questions

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1 Groups

Question 1.1. *What is a normal subgroup? Can you get some natural map from a normal subgroup? What topological objects can the original group, normal subgroup, and quotient group relate to?*

Question 1.2. *Prove that a subgroup of index two is normal.*

Question 1.3. *Find all normal subgroups of* A_4.

Question 1.4. *Give an interesting example of a non-normal subgroup. Is* $\mathrm{SO}(2)$ *normal inside* $\mathrm{SL}_2(\mathbb{R})$*?*

Question 1.5. *Is normality transitive? That is, is a normal subgroup of a normal subgroup normal in the biggest group?*

Question 1.6. *Define solvable group. Give an example of a solvable nonabelian group. Show A_4 is solvable. Do the Sylow theorems tell you anything about whether this index 3 subgroup of A_4 is normal?*

Question 1.7. *Define lower central series, upper central series, nilpotent and solvable groups.*

Question 1.8. *Define the derived series. Define the commutator. State and prove two nontrivial theorems about derived series.*

Question 1.9. *Prove that $\mathrm{SL}_2(\mathbb{Z})$ is not solvable.*

Question 1.10. *What are all possible orders of elements of $\mathrm{SL}_2(\mathbb{Z})$?*

Question 1.11. *Can you show that all groups of order p^n for p prime are solvable? Do you know how to do this for groups of order $p^r q^s$?*

Question 1.12. *Suppose a p-group acts on a set whose cardinality is not divisible by p (p prime). Prove that there is a fixed point for the action.*

Question 1.13. *Prove that the centre of a group of order p^r (p prime) is not trivial.*

Question 1.14. *Give examples of simple groups. Are there infinitely many?*

Question 1.15. *State and prove the Jordan–Hölder theorem for finite groups.*

Question 1.16. *What's Cayley's theorem? Give an example of a group of order n that embeds in S_m for some m smaller than n. Give an example of a group where you have to use S_n.*

Question 1.17. *Is A_4 a simple group? What are the conjugacy classes in S_4? What about in A_4?*

Question 1.18. *Talk about conjugacy classes in the symmetric group S_n.*

Question 1.19. *When do conjugacy classes in S_n split in A_n?*

Question 1.20. *What is the centre of S_n? Prove it.*

Question 1.21. *Prove that the alternating group A_n is simple for $n \geq 5$.*

Question 1.22. *Prove the alternating group on n letters is generated by the 3-cycles for $n \geq 3$.*

Question 1.23. *Prove that for p prime, S_p is generated by a p-cycle and a transposition.*

Question 1.24. *What is the symmetry group of a tetrahedron? Cube? Icosahedron?*

Question 1.25. *How many ways can you color the tetrahedron with C colors if we identify symmetric colorings?*

Question 1.26. *What is the symmetry group of an icosahedron? What's the stabiliser of an edge? How many edges are there? How do you know the symmetry group of the icosahedron is the same as the symmetry group of the dodecahedron? Do you know the classification of higher-dimensional polyhedra?*

Question 1.27. *Do you know what the quaternion group is? How many elements are there of each order?*

Question 1.28. *What is the group of unit quaternions topologically? What does it have to do with* $\mathrm{SO}(3)$*?*

Question 1.29. *What's the stabiliser of a point in the unit disk under the group of conformal automorphisms?*

Question 1.30. *What group-theoretic construct relates the stabiliser of two points?*

Question 1.31. *Consider* $\mathrm{SL}_2(\mathbb{R})$ *acting on* $\mathbb{R}^2$ *by matrix multiplication. What is the stabiliser of a point? Does it depend which point? Do you know what sort of subgroup this is? What if* $\mathrm{SL}_2(\mathbb{R})$ *acts by Möbius transformations instead?*

Question 1.32. *What are the polynomials in two real variables that are invariant under the action of* D_4*, the symmetry group of a square, by rotations and reflections on the plane that the two variables form?*

Question 1.33. *Give an interesting example of a subgroup of the additive group of the rationals.*

Question 1.34. *Talk about the isomorphism classes of subgroups of* $\mathbb{Q}$*. How many are there? Are the ones you've given involving denominators divisible only by certain primes distinct? So that gives you the cardinality. Are these all of them?*

Question 1.35. *Is the additive group of the reals isomorphic to the multiplicative group of the positive reals? Is the same result true with reals replaced by rationals?*

Question 1.36. *What groups have nontrivial automorphisms?*

Question 1.37. *A subgroup* H *of a finite group* G *that meets every conjugacy class is in fact* G*. Why is that true?*

Question 1.38. *Let* G *be the group of invertible* 3×3 *matrices over* $\mathbb{F}_p$*, for* p *prime. What does basic group theory tell us about* G*? How many conjugates does a Sylow* p*-subgroup have? Give a matrix form for the elements in this subgroup. Explain the conjugacy in terms of eigenvalues and eigenvectors. Give a matrix form for the normaliser of the Sylow* p*-subgroup.*

Question 1.39. *Let's look at* $\mathrm{SL}_2(\mathbb{F}_3)$*. How many elements are in that group? What is its centre? Identify* $\mathrm{PSL}_2(\mathbb{F}_3)$ *as a permutation group.*

Question 1.40. *How many elements does* $\mathrm{GL}_2(\mathbb{F}_q)$ *have? How would you construct representations? What can you say about the 1-dimensional representations? What can you say about simplicity of some related groups?*

Question 1.41. *A subgroup of a finitely-generated free abelian group is? A subgroup of a finitely-generated free group is? Prove your answers.*

Question 1.42. *What are the subgroups of* $\mathbb{Z}^2$*?*

Question 1.43. *What are the subgroups of the free group* F_2*? How many generators can you have? Can you find one with* 3 *generators?* 4 *generators? Countably many generators? Is the subgroup with* 4 *generators you found normal? Why? Can you find a normal one?*

Question 1.44. *Talk about the possible subgroups of* $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$*. Now suppose that you have a subgroup of* $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$*. What theorem tells you something about the structure of the quotient group?*

2 Classification of Finite Groups

Question 2.1. *Given a finite abelian group with at most* n *elements of order divisible by* n*, prove it's cyclic.*

Question 2.2. *Suppose I asked you to classify groups of order* 4*. Why isn't there anything else? Which of those could be realised as a Galois group over* $\mathbb{Q}$*?*

Question 2.3. *State/prove the Sylow theorems.*

Question 2.4. *Classify groups of order* 35*.*

Question 2.5. *Classify groups of order* 21*.*

Question 2.6. *Discuss groups of order* 55*.*

Question 2.7. *Classify groups of order* 14*. Why is there a group of order* 7*? Are all index-*2 *subgroups normal?*

Question 2.8. *How many groups are there of order* 15*? Prove it.*

Question 2.9. *Classify all groups of order* 8*.*

Question 2.10. *Classify all groups of order* p^3 *for* p *prime.*

Question 2.11. *What are the groups of order* p^2*? What about* pq*? What if* q *is congruent to* $1 \pmod{p}$*?*

Question 2.12. *What are the groups of order* 12*? Can there be a group of order* 12 *with* 2 *nonisomorphic subgroups of the same order?*

Question 2.13. *How would you start finding the groups of order* 56*? Is there in fact a way for* $\mathbb{Z}/7\mathbb{Z}$ *to act on a group of order* 8 *nontrivially?*

Question 2.14. *How many abelian groups are there of order* 36*?*

Question 2.15. *What are the abelian groups of order* 16*?*

Question 2.16. *What are the abelian groups of order* 9*? Prove that they are not isomorphic. Groups of order* 27*?*

Question 2.17. *How many abelian groups of order* 200 *are there?*

Question 2.18. *Prove there is no simple group of order* 132*.*

Question 2.19. *Prove that there is no simple group of order* 160*. What can you say about the structure of groups of that order?*

Question 2.20. *Prove that there is no simple group of order* 40*.*

3 Fields and Galois Theory

Question 3.1. *What is the Galois group of a finite field? What is a generator? How many elements does a finite field have? What can you say about the multiplicative group? Prove it.*

Question 3.2. *Classify finite fields, their subfields, and their field extensions. What are the automorphisms of a finite field?*

Question 3.3. *Take a finite field extension* $\mathbb{F}_{p^n}$ *over* $\mathbb{F}_p$*. What is Frobenius? What is its characteristic polynomial?*

Question 3.4. *What are the characteristic and minimal polynomial of the Frobenius automorphism?*

Question 3.5. *What's the field with* 25 *elements?*

Question 3.6. *What is the multiplicative group of* $\mathbb{F}_9$*?*

Question 3.7. *What is a separable extension? Can* $\mathbb{Q}$ *have a non-separable extension? How about* $\mathbb{Z}/p\mathbb{Z}$*? Why not? Are all extensions of characteristic 0 fields separable? Of finite fields? Prove it. Give an example of a field extension that's not separable.*

Question 3.8. *Are there separable polynomials of any degree over any field?*

Question 3.9. *What is a perfect field and why is this important? Give an example of a non-perfect field.*

Question 3.10. *What is Galois theory? State the main theorem. What is the splitting field of* $x^5 - 2$ *over* $\mathbb{Q}$*? What are the intermediate extensions? Which extensions are normal, which are not, and why? What are the Galois groups (over* $\mathbb{Q}$*) of all intermediate extensions?*

Question 3.11. *What is a Galois extension?*

Question 3.12. *Take a quadratic extension of a field of characteristic* 0*. Is it Galois? Take a degree* 2 *extension on top of that. Does it have to be Galois over the base field? What statement in group theory can you think of that reflects this?*

Question 3.13. *Is abelian Galois extension transitive? That is, if K has abelian Galois group over E, E has abelian Galois group over F, and K is a Galois extension of F, is it necessarily true that $\mathrm{Gal}(K/F)$ is also abelian? Give a counterexample involving number fields as well as one involving function fields.*

Question 3.14. *What is a Kummer extension?*

Question 3.15. *Say you have a field extension with only finitely many intermediate fields. Show that it is a simple extension.*

Question 3.16. *Tell me a condition on the Galois group which is implied by irreducibility of the polynomial. What happens when the polynomial has a root in the base field?*

Question 3.17. *What is the discriminant of a polynomial?*

Question 3.18. *If we think of the Galois group of a polynomial as contained in S_n, when is it contained in A_n?*

Question 3.19. *Is $\mathbb{Q}(2^{1/3})$ normal? What is its splitting field? What is its Galois group? Draw the lattice of subfields.*

Question 3.20. *What's the Galois group of x^2+1 over $\mathbb{Q}$? What's the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(i)$?*

Question 3.21. *What's the Galois group of x^2+9?*

Question 3.22. *What is the Galois group of x^2-2? Why is x^2-2 irreducible?*

Question 3.23. *What is the Galois group of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}$?*

Question 3.24. *What is the Galois group of $\mathbb{Q}(\sqrt{n_1}, \sqrt{n_2}, \ldots, \sqrt{n_m})$ over $\mathbb{Q}(\sqrt{n_1} + \ldots + \sqrt{n_m})$?*

Question 3.25. *What are the Galois groups of irreducible cubics?*

Question 3.26. *If an irreducible cubic polynomial has Galois group NOT contained in A_3, does it necessarily have to be all of S_3?*

Question 3.27. *Compute the Galois group of x^3-2 over the rationals.*

Question 3.28. *How would you find the Galois group of x^3+2x+1? Adjoin a root to $\mathbb{Q}$. Can you say something about the roots of x^3+3x+1 in this extension?*

Question 3.29. *Compute the Galois group of x^3+6x+3.*

Question 3.30. *Find the Galois group of x^4-2 over $\mathbb{Q}$.*

Question 3.31. *What's the Galois group of x^4-3?*

Question 3.32. *What is the Galois group of x^4-2x^2+9?*

Question 3.33. *Calculate the Galois group of x^5-2.*

Question 3.34. *Discuss sufficient conditions on a polynomial of degree* 5 *to have Galois group* S_5 *over* $\mathbb{Q}$ *and prove your statements.*

Question 3.35. *Show that if* f *is an irreducible quintic with precisely two nonreal roots, then its Galois group is* S_5*.*

Question 3.36. *Suppose you have a degree* 5 *polynomial over a field. What are necessary and sufficient conditions for its Galois group to be of order divisible by* 3*? Can you give an example of an irreducible polynomial in which this is not the case?*

Question 3.37. *What is the Galois group of* $x^7 - 1$ *over the rationals?*

Question 3.38. *What is the Galois group of the polynomial* $x^n - 1$ *over* $\mathbb{Q}$*?*

Question 3.39. *Describe the Galois theory of cyclotomic extensions.*

Question 3.40. *What is the maximal real field in a cyclotomic extension* $\mathbb{Q}(\zeta_n)/\mathbb{Q}$*?*

Question 3.41. *Compute the Galois group of* $p(x) = x^7 - 3$*.*

Question 3.42. *What Galois stuff can you say about* $x^{2^n} - 2$*?*

Question 3.43. *What are the cyclic extensions of (prime) order* p*?*

Question 3.44. *Can you give me a polynomial whose Galois group is* $\mathbb{Z}/3\mathbb{Z}$*?*

Question 3.45. *Which groups of order* 4 *can be realised as a Galois group over* $\mathbb{Q}$*?*

Question 3.46. *Give a polynomial with* S_3 *as its Galois group.*

Question 3.47. *Give an example of a cubic with Galois group* S_3*.*

Question 3.48. *How do you construct a polynomial over* $\mathbb{Q}$ *whose Galois group is* S_n*? Do it for* $n = 7$ *in particular.*

Question 3.49. *What's a Galois group that's not* S_n *or* A_n*?*

Question 3.50. *Which finite groups are Galois groups for some field extension?*

Question 3.51. *What Galois group would you expect a cubic to have?*

Question 3.52. *Draw the subgroup lattice for* S_3*.*

Question 3.53. *Do you know what the quaternion group is? How many elements are there of each order? Suppose I have a field extension of the rationals with Galois group the quaternion group. How many quadratic extensions does it contain? Can any of them be imaginary?*

Question 3.54. *Suppose you are given a finite Galois extension* $K/\mathbb{Q}$ *by* $f(x) \in \mathbb{Z}[x]$ *such that* $\deg(f) = n$ *and* $\mathrm{Gal}(K/\mathbb{Q}) = S_n$*. What can you say about the roots?*

Question 3.55. *How many automorphisms does the complex field have? How can you extend a simple automorphism (e.g.* $\sqrt{2} \mapsto -\sqrt{2}$*) of an algebraic field into* $\mathbb{C}$*? How can you extend a general subfield isomorphism? What feature of* $\mathbb{C}$ *allows you to?*

Question 3.56. *Can it happen that a proper subfield of $\mathbb{C}$ is isomorphic to $\mathbb{C}$? How?*

Question 3.57. *Consider the minimal polynomial $f(x)$ for a primitive m^{th} root of unity. Prove that if p divides $f(a)$ for some integer a and $(p, m) = 1$ then m divides $p-1$. Use this fact to show that there are infinitely many primes congruent to $1 \pmod{m}$.*

Question 3.58. *What is Dirichlet's theorem about primes in arithmetic progression? What can you say about the density of such primes?*

Question 3.59. *How many irreducible polynomials of degree six are there over $\mathbb{F}_2$?*

Question 3.60. *Can you have a degree 7 irreducible polynomial over $\mathbb{F}_p$? How about a degree 14 irreducible polynomial?*

Question 3.61. *How many irreducible polynomials are there of degree 4 over $\mathbb{F}_2$?*

Question 3.62. *For each prime p, give a polynomial of degree p that is irreducible over $\mathbb{F}_p$. You can do it in a "uniform" way.*

Question 3.63. *Can we solve general quadratic equations by radicals? And what about cubic and so on? Why can't you solve 5^{th} degree equations by radicals?*

Question 3.64. *Talk about solvability by radicals. Why is S_5 not solvable? Why is A_5 simple?*

Question 3.65. *For which n can a regular n-gon be constructed by ruler and compass?*

Question 3.66. *How do you use Galois theory (or just field theory) to prove the impossibility of trisecting an angle? Doubling a cube? Squaring a circle?*

Question 3.67. *Which numbers are constructible? Give an example of a non-constructible number whose degree is nevertheless a power of 2.*

Question 3.68. *State and prove Eisenstein's Criterion.*

Question 3.69. *Why is $(x^p - 1)/(x - 1)$ irreducible over $\mathbb{Q}$?*

Question 3.70. *Can you prove the fundamental theorem of algebra using Galois theory? What do you need from analysis to do so?*

Question 3.71. *What are the symmetric polynomials?*

Question 3.72. *State the fundamental theorem of symmetric polynomials.*

Question 3.73. *Is the discriminant of a polynomial always a polynomial in the coefficients? What does this have to do with symmetric polynomials?*

Question 3.74. *Find a non-symmetric polynomial whose square is symmetric.*

Question 3.75. *Let f be a degree 4 polynomial with integer coefficients. What's the smallest finite field in which f necessarily has four roots?*

Question 3.76. *Define p-adic numbers. What is a valuation?*

Question 3.77. *What's Hilbert's theorem 90?*

Question 3.78. *Consider a nonconstant function between two compact Riemann Surfaces. How is it related to Galois theory?*

4 Normal Forms

Question 4.1. *What is the connection between the structure theorem for modules over a PID and conjugacy classes in the general linear group over a field?*

Question 4.2. *Explain how the structure theorem for finitely-generated modules over a PID applies to a linear operator on a finite dimensional vector space.*

Question 4.3. *I give you two matrices over a field. How would you tell if they are conjugate or not? What theorem are you using? State it. How does it apply to this situation? Why is $k[T]$ a PID? If two matrices are conjugate over the algebraic closure of a field, does that mean that they are conjugate over the base field too?*

Question 4.4. *If two real matrices are conjugate in $\mathrm{M}_n(\mathbb{C})$, are they necessarily conjugate in $\mathrm{M}_n(\mathbb{R})$ as well?*

Question 4.5. *Give the 4×4 Jordan forms with minimal polynomial $(x-1)(x-2)^2$.*

Question 4.6. *Talk about Jordan canonical form. What happens when the field is not algebraically closed?*

Question 4.7. *What are all the matrices that commute with a given Jordan block?*

Question 4.8. *How do you determine the number and sizes of the blocks for Jordan canonical form?*

Question 4.9. *For any matrix A over the complex numbers, can you solve $B^2 = A$?*

Question 4.10. *What is rational canonical form?*

Question 4.11. *Describe all the conjugacy classes of 3×3 matrices with rational entries which satisfy the equation $A^4 - A^3 - A + 1 = 0$. Give a representative in each class.*

Question 4.12. *What 3×3 matrices over the rationals (up to similarity) satisfy $f(A) = 0$, where $f(x) = (x^2+2)(x-1)^3$? List all possible rational forms.*

Question 4.13. *What can you say about matrices that satisfy a given polynomial (over an algebraically closed field)? How many of them are there? What about over a finite field? How many such matrices are there then?*

Question 4.14. *What is a nilpotent matrix?*

Question 4.15. *When do the powers of a matrix tend to zero?*

Question 4.16. *If the traces of all powers of a matrix A are 0, what can you say about A?*

Question 4.17. *When and how can we solve the matrix equation $\exp(A) = B$? Do it over the complex numbers and over the real numbers. Give a counterexample with real entries.*

Question 4.18. *Say we can find a matrix A such that $\exp(A) = B$ for B in $\mathrm{SL}_n(\mathbb{R})$. Does A also have to be in $\mathrm{SL}_n(\mathbb{R})$? Can you take A to be in $\mathrm{SL}_n(\mathbb{R})$?*

Question 4.19. *Is a square matrix always similar to its transpose?*

Question 4.20. *What are the conjugacy classes of $\mathrm{SL}_2(\mathbb{R})$?*

Question 4.21. *What are the conjugacy classes in $\mathrm{GL}_2(\mathbb{C})$?*

5 Matrices and Linear Algebra

Question 5.1. *What is a bilinear form on a vector space? When are two forms equivalent? What is an orthogonal matrix? What's special about them?*

Question 5.2. *What are the possible images of the unit circle under a linear transformation of $\mathbb{R}^2$?*

Question 5.3. *Explain geometrically how you diagonalise a quadratic form.*

Question 5.4. *Do you know Witt's theorem on real quadratic forms?*

Question 5.5. *Classify real division algebras.*

Question 5.6. *Consider the simple operator on $\mathbb{C}$ given by multiplication by a complex number. It decomposes into a stretch and a rotation. What is the generalisation of this to operators on a Hilbert space?*

Question 5.7. *Do you know about singular value decomposition?*

Question 5.8. *What are the eigenvalues of a symmetric matrix?*

Question 5.9. *What can you say about the eigenvalues of a skew-symmetric matrix?*

Question 5.10. *Prove that the eigenvalues of a Hermitian matrix are real and those of a unitary matrix are unitary.*

Question 5.11. *Prove that symmetric matrices have real eigenvalues and can be diagonalised by orthogonal matrices.*

Question 5.12. *To which operators does the spectral theorem for symmetric matrices generalise?*

Question 5.13. *Given a skew-symmetric/skew-Hermitian matrix S, show that $U = (S + I)(S - I)^{-1}$ is orthogonal/unitary. Then find an expression for S in terms of U.*

Question 5.14. *If a linear transformation preserves a nondegenerate alternating form and has k as an eigenvalue, prove that $1/k$ is also an eigenvalue.*

Question 5.15. *State/prove the Cayley–Hamilton theorem.*

Question 5.16. *Are diagonalisable $N \times N$ matrices over the complex numbers dense in the space of all $N \times N$ matrices over the complex numbers? How about over another algebraically closed field if we use the Zariski topology?*

Question 5.17. *For a linear ODE with constant coefficients, how would you solve it using linear algebra?*

Question 5.18. *What can you say about the eigenspaces of two matrices that commute with each other?*

Question 5.19. *What is a Toeplitz operator?*

Question 5.20. *What is the number of invertible matrices over $\mathbb{Z}/p\mathbb{Z}$?*

6 Rings

Question 6.1. *State the Chinese remainder theorem in any form you like. Prove it.*

Question 6.2. *What is a PID? What's an example of a UFD that is not a PID? Why? Is* $k[x]$ *a PID? Why?*

Question 6.3. *Is* $\mathbb{C}[x, y]$ *a PID? Is* $\langle x, y\rangle$ *a prime ideal in it?*

Question 6.4. *Do polynomials in several variables form a PID?*

Question 6.5. *Prove that the integers form a PID.*

Question 6.6. *Give an example of a PID with a unique prime ideal.*

Question 6.7. *What is the relation between Euclidean domains and PIDs?*

Question 6.8. *Do you know a PID that's not Euclidean?*

Question 6.9. *Give an example of a UFD which is not a Euclidean domain.*

Question 6.10. *Is a ring of formal power series a UFD?*

Question 6.11. *Is a polynomial ring over a UFD again a UFD?*

Question 6.12. *What does factorisation over* $\mathbb{Q}[x]$ *say about factorisation over* $\mathbb{Z}[x]$*?*

Question 6.13. *Give an example of a ring where unique factorisation fails.*

Question 6.14. *Factor* 6 *in two different ways in* $\mathbb{Z}[\sqrt{-5}]$*. Is there any way to explain the two factorisations? Factor the ideal generated by* 6 *into prime ideals.*

Question 6.15. *What's the integral closure of* $\mathbb{Z}$ *in* $\mathbb{Q}(i)$*?*

Question 6.16. *Find all primes in the ring of Gaussian integers.*

Question 6.17. *What is a ring of integers? What does "integral over* $\mathbb{Z}$*" mean?*

Question 6.18. *Let* $\mathcal{O}$ *be the ring of integers of* $\mathbb{Q}(\sqrt{d})$*, where* $d > 0$*. What can you say about the quotient of* $\mathcal{O}$ *by one of its prime ideals?*

Question 6.19. *Do you know about Dedekind domains and class numbers?*

Question 6.20. *Talk about factorisation and primes in a polynomial ring. What is irreducibility? For what rings* R *is it true that* $R[x_1, \cdots, x_n]$ *is a unique factorisation domain? What is wrong with unique factorisation if we don't have a domain? Now, PIDs are Noetherian, but are there UFDs which are not?*

Question 6.21. *What is the radical of an ideal? What is special about elements in the nilradical?*

Question 6.22. *Define "radical". Prove it is an ideal. Prove that the ideal of all polynomials vanishing on the zero set of* I *is* $\mathrm{rad}(I)$*.*

Question 6.23. *Do you know what the radical is? Use the fact that the intersection of all prime ideals is the set of all nilpotent elements to prove that $F[x]$ has an infinite number of prime ideals, where F is a field.*

Question 6.24. *What are the radical ideals in $\mathbb{Z}$?*

Question 6.25. *Give a prime ideal in $k[x, y]$. Why is it prime? What is the variety it defines? What is the Nullstellensatz? Can you make some maximal ideals?*

Question 6.26. *State/describe Hilbert's Nullstellensatz. Sketch a proof.*

Question 6.27. *What is an irreducible variety? Give an example of a non-irreducible one.*

Question 6.28. *What are the prime ideals and maximal ideals of $\mathbb{Z}[x]$?*

Question 6.29. *Is the map $\mathbb{Z}[t]/\langle t^p - 1\rangle \to \mathbb{Z}[w]$ given by $t \mapsto w$ where $w^p = 1$ an isomorphism?*

Question 6.30. *Describe the left, right, and two-sided ideals in the ring of square matrices of a fixed size. Now identify the matrix algebra $\mathrm{M}_n(K)$ with $\mathrm{End}_K(V)$ where V is an n-dimensional K-vector space. Try to geometrically describe the simple left ideals and also the simple right ideals via that identification.*

Question 6.31. *Give examples of maximal ideals in $K = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots$, the product of countably many copies of $\mathbb{R}$. What about for a product of countably many copies of an arbitrary commutative ring R?*

Question 6.32. *Consider a commutative ring, R, and a maximal ideal, I, what can you say about the structure of R/I? What if I were prime?*

Question 6.33. *Define "Noetherian ring". Give an example.*

Question 6.34. *Prove the Hilbert basis theorem.*

Question 6.35. *What is a Noetherian ring? If I is an ideal in a Noetherian ring with a unit, what is the intersection of I^n over all positive integers n?*

Question 6.36. *What is the Jacobson radical? If R is a finitely-generated algebra over a field what can you say about it?*

Question 6.37. *Give an example of an Artinian ring.*

Question 6.38. *State the structure theorem for semisimple Artinian rings.*

Question 6.39. *What is a semisimple algebra? State the structure theorem for semisimple algebras.*

Question 6.40. *What is a matrix algebra?*

Question 6.41. *Does L^1 have a natural multiplication with which it becomes an algebra?*

Question 6.42. *Consider a translation-invariant subspace of L^1. What can you say about its relation to L^2 as a convolution algebra?*

Question 6.43. *State the structure theorem for simple rings.*

Question 6.44. *Do you know an example of a local ring? Another one? What about completions?*

Question 6.45. *Consider the space of functions from the natural numbers to $\mathbb{C}$ endowed with the usual law of addition and the following analogue of the convolution product:*

$$f * g(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right)$$

Show that this is a ring. What does this ring remind you of and what can you say about it?

Question 6.46. *Prove that any finite division ring is a field (that is, prove commutativity). Give an example of a (necessarily infinite) division ring which is NOT a field.*

Question 6.47. *Prove that all finite integral domains are fields.*

Question 6.48. *Can a polynomial over a division ring have more roots than its degree?*

Question 6.49. *Classify (finite-dimensional) division algebras over $\mathbb{R}$.*

Question 6.50. *Give an example of a $\mathbb{C}$-algebra which is not semisimple.*

Question 6.51. *What is Wedderburn's theorem? What does the group ring generated by $\mathbb{Z}/5\mathbb{Z}$ over $\mathbb{Q}$ look like? What if we take the noncyclic group of order 4 instead of $\mathbb{Z}/5\mathbb{Z}$? The quaternion group instead of $\mathbb{Z}/5\mathbb{Z}$?*

Question 6.52. *Tell me about group rings. What do you know about them?*

7 Modules

Question 7.1. *How does one prove the structure theorem for modules over PID? What is the module and what is the PID in the case of abelian groups?*

Question 7.2. *If M is free abelian, how can I put quotients of M in some standard form? What was crucial about the integers here (abelian groups being modules over $\mathbb{Z}$)? How does the procedure simplify if the ring is a Euclidean domain, not just a PID?*

Question 7.3. *Suppose D is an integral domain and the fundamental theorem holds for finitely-generated modules over D (i.e. they are all direct sums of finitely many cyclic modules). Does D have to be a PID?*

Question 7.4. *Classify finitely-generated modules over $\mathbb{Z}$, over PID, and over Dedekind rings.*

Question 7.5. *Prove a finitely-generated torsion-free abelian group is free abelian.*

Question 7.6. *What is a tensor product? What is the universal property? What do the tensors look like in the case of vector spaces?*

Question 7.7. *Now we'll take the tensor product of two abelian groups, that is, $\mathbb{Z}$-modules. Take $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/q\mathbb{Z}$, where p and q are distinct primes. What is their tensor product?*

Question 7.8. *What is a projective module?*

Question 7.9. *What is an injective module?*

Question 7.10. *Do you know an example of a flat module?*

8 Representation Theory

Question 8.1. *Define "representation" of a group. Define "irreducible representation". Why can you decompose representations of finite groups into irreducible ones? Construct an invariant inner product.*

Question 8.2. *State and prove Maschke's theorem. What can go wrong if you work over the real field? What can go wrong in characteristic p?*

Question 8.3. *Do you know what a group representation is? Do you know what the trace of a group representation is?*

Question 8.4. *State/prove/explain Schur's lemma.*

Question 8.5. *What can you say about characters? What are the orthogonality relations? How do you use characters to determine if a given irreducible representation is a subspace of another given representation?*

Question 8.6. *What's the relation between the number of conjugacy classes in a finite group and the number of irreducible representations?*

Question 8.7. *What is the character table? What field do its entries lie in?*

Question 8.8. *Why is the character table a square?*

Question 8.9. *If $\chi(g)$ is real for every character χ, what can you say about g?*

Question 8.10. *What's the regular representation?*

Question 8.11. *Give two definitions of "induced representation". Why are they equivalent?*

Question 8.12. *If you have a representation of H, a subgroup of a group G, how can you induce a representation of G?*

Question 8.13. *If you have an irreducible representation of a subgroup, is the induced representation of the whole group still irreducible?*

Question 8.14. *What can you say about the kernel of an irreducible representation? How about kernels of direct sums of irreducibles? What kind of functor is induction? Left or right exact?*

Question 8.15. *What is Frobenius reciprocity?*

Question 8.16. *Given a normal subgroup H of a finite group G, we lift all the representations of G/H to representations of G. Show that the intersection of the kernels of all these representations is precisely H. What can you say when H is the commutator subgroup of G?*

Question 8.17. *If you have two linear representations π_1 and π_2 of a finite group G such that $\pi_1(g)$ is conjugate to $\pi_2(g)$ for every g in G, is it true that the two representations are isomorphic?*

Question 8.18. *Group representations: What's special about using $\mathbb{C}$ in the definition of group algebra? Is it possible to work over other fields? What goes wrong if the characteristic of the field divides the order of the group?*

Question 8.19. *Suppose you have a finite p-group, and you have a representation of this group on a finite-dimensional vector space over a finite field of characteristic p. What can you say about it?*

Question 8.20. *Let (π, V) be a faithful finite-dimensional representation of G. Show that, given any irreducible representation of G, the n^{th} tensor power of $\mathrm{GL}(V)$ will contain it for some large enough n.*

Question 8.21. *What are the irreducible representations of finite abelian groups?*

Question 8.22. *What are the group characters of the multiplicative group of a finite field?*

Question 8.23. *Are there two nonisomorphic groups with the same representations?*

Question 8.24. *If you have a $\mathbb{Z}/5\mathbb{Z}$ action on a complex vector space, what does this action look like? What about an S_3 action? A dihedral group of any order?*

Question 8.25. *What are the representations of S_3? How do they restrict to S_2?*

Question 8.26. *Tell me about the representations of D_4. Write down the character table. What is the 2-dimensional representation? How can it be interpreted geometrically?*

Question 8.27. *How would you work out the orders of the irreducible representations of the dihedral group D_n? Why is the the sum of squares of dimensions equal to the order of the group?*

Question 8.28. *Do you know any representation theory? What about representations of A_4? Give a nontrivial one. What else is there? How many irreducible representations do we have? What are their degrees? Write the character table of A_4.*

Question 8.29. *Write the character table for S_4.*

Question 8.30. *Start constructing the character table for S_5.*

Question 8.31. *How many irreducible representations does S_n have? What classical function in mathematics does this number relate to?*

Question 8.32. *Discuss representations of $\mathbb{Z}$, the infinite cyclic group. What is the group algebra of $\mathbb{Z}$? What is the connectio*

Question 8.33. *What is a Lie group? Define a unitary representation. What is the Peter–Weyl theorem? What is the Lie algebra? The Jacobi identity? What is the adjoint representation of a Lie algebra? What is the commutator of two vector fields on a manifold?*

n with modules over PIDs? When is a representation of $\mathbb{Z}$ completely reducible? Why not always? Which are the indecomposable modules?

Question 8.34. *Talk about the representation theory of compact Lie groups. How do you know you have a finite-dimensional representation?*

Question 8.35. *How do you prove that any finite-dimensional representation of a compact Lie group is equivalent to a unitary one?*

Question 8.36. *Do you know a Lie group that has no faithful finite-dimensional representations?*

Question 8.37. *What do you know about representations of* $\mathrm{SO}(2)$*?* $\mathrm{SO}(3)$*?*

9 Categories and Functors

Question 9.1. *Which is the connection between* Hom *and tensor product? What is this called in representation theory?*

Question 9.2. *Can you get a long exact sequence from a short exact sequence of abelian groups together with another abelian group?*

Question 9.3. *Do you know what the Ext functor of an abelian group is? Do you know where it appears? What is* $\mathrm{Ext}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$*? What is* $\mathrm{Ext}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})$*?*