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CR Geometry in 3-D*

Paul C. YANG[1]

(*Dedicated to Professor Haim Brezis with admiration*)

Abstract CR geometry studies the boundary of pseudo-convex manifolds. By concentrating on a choice of a contact form, the local geometry bears strong resemblence to conformal geometry. This paper deals with the role conformally invariant operators such as the Paneitz operator plays in the CR geometry in dimension three. While the sign of this operator is important in the embedding problem, the kernel of this operator is also closely connected with the stability of CR structures. The positivity of the CR-mass under the natural sign conditions of the Paneitz operator and the CR Yamabe operator is discussed. The CR positive mass theorem has a consequence for the existence of minimizer of the CR Yamabe problem. The pseudo-Einstein condition studied by Lee has a natural analogue in this dimension, and it is closely connected with the pluriharmonic functions. The author discusses the introduction of new conformally covariant operator P-prime and its associated Q-prime curvature and gives another natural way to find a canonical contact form among the class of pseudo-Einstein contact forms. Finally, an isoperimetric constant determined by the Q-prime curvature integral is discussed.

Keywords Paneitz operator, Embedding problem, Yamabe equation, Mass, P-prime, Q-prime curvature
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1 Introduction

A CR manifold is a pair (M^{2n+1}, J) of a smooth oriented (real) $(2n+1)$-dimensional manifold together with a formally integrable complex structure $J\colon H \to H$ on a maximally nonintegrable codimension one subbundle $H \subset TM$. In particular, the bundle $E = H^{\perp} \subset T^*M$ is orientable and any nonvanishing section θ of E is a contact form, i.e., $\theta \wedge (\mathrm{d}\theta)^n$ is nonvanishing. We assume further that (M^{2n+1}, J) is strictly pseudo-convex, meaning that the symmetric tensor $\mathrm{d}\theta(\cdot, J\cdot)$ on $H^* \otimes H^*$ is positive definite; since E is one-dimensional, this is independent of the choice of contact form θ.

Given a CR manifold (M^{2n+1}, J), we can define the subbundle $T^{1,0}$ of the complexified tangent bundle as the +i-eigenspace of J, and $T^{0,1}$ as its conjugate. We likewise denote by $\Lambda^{1,0}$ the space of $(1,0)$-forms (that is, the subbundle of $T^*_{\mathbb{C}}M$ which annihilates $T^{0,1}$) and by $\Lambda^{0,1}$ its conjugate. The CR structure is said to be integrable if $T^{0,1}$ is closed under the Lie bracket, a condition that is vacuous when $n = 1$. The canonical bundle K is the complex line-bundle $K = \Lambda^{n+1}(\Lambda^{1,0})$.

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[1]Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton NJ 08544, USA. E-mail: yang@math.princeton.edu
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A pseudohermitian manifold is a triple (M^{2n+1}, J, θ) of a CR manifold (M^{2n+1}, J) together with a choice of contact form θ. The assumption that $\mathrm{d}\theta(\cdot, J\cdot)$ is positive definite implies that the Levi form $L_\theta(U \wedge \overline{V}) = -2\mathrm{i}\mathrm{d}\theta(U \wedge \overline{V})$ defined on $T^{1,0}$ is a positive-definite Hermitian form. Since another choice of contact form $\widehat{\theta}$ is equivalent to a choice of (real-valued) function $\sigma \in C^\infty(M)$ such that $\widehat{\theta} = \mathrm{e}^\sigma\theta$, and the Levi forms of $\widehat{\theta}$ and θ are related by $L_{\widehat{\theta}} = \mathrm{e}^\sigma L_\theta$, we see that the analogy between CR geometry and conformal geometry begins through the similarity of choosing a contact form or a metric in a conformal class (see [17]).

Given a pseudohermitian manifold (M^{2n+1}, J, θ), the Reeb vector field T is the unique vector field such that $\theta(T) = 1$ and $\mathrm{d}\theta(\cdot, T) = 0$. An admissible coframe is a set of $(1,0)$-forms $\{\theta^\alpha\}_{\alpha=1}^n$ whose restriction to $T^{1,0}$ forms a basis for $(T^{1,0})^*$ and such that $\theta^\alpha(T) = 0$ for all α. Denote by $\theta^{\overline{\alpha}} = \overline{\theta^\alpha}$ the conjugate of θ^α. Then $\mathrm{d}\theta = \mathrm{i}h_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}}$ for some positive definite Hermitian matrix $h_{\alpha\overline{\beta}}$. Denote by $\{T, Z_\alpha, Z_{\overline{\alpha}}\}$ the frame for $T_{\mathbb{C}}M$ dual to $\{\theta, \theta^\alpha, \theta^{\overline{\alpha}}\}$, so that the Levi form is

$$L_\theta(U^\alpha Z_\alpha, V^{\overline{\alpha}} Z_{\overline{\alpha}}) = h_{\alpha\overline{\beta}} U^\alpha V^{\overline{\beta}}.$$

Tanaka [35] and Webster [38] defined a canonical connection on a pseudohermitian manifold (M^{2n+1}, J, θ) as follows: Given an admissible coframe $\{\theta^\alpha\}$, define the connection forms ${\omega_\alpha}^\beta$ and the torsion form $\tau_\alpha = A_{\alpha\beta}\theta^\beta$ by the relations

$$\begin{aligned} \mathrm{d}\theta^\beta &= \theta^\alpha \wedge {\omega_\alpha}^\beta + \theta \wedge \tau^\beta, \\ \omega_{\alpha\overline{\beta}} + \omega_{\overline{\beta}\alpha} &= \mathrm{d}h_{\alpha\overline{\beta}}, \\ A_{\alpha\beta} &= A_{\beta\alpha}, \end{aligned}$$

where we use the metric $h_{\alpha\overline{\beta}}$ to raise and lower indices, e.g., $\omega_{\alpha\overline{\beta}} = h_{\gamma\overline{\beta}}{\omega_\alpha}^\gamma$. In particular, the connection forms are pure imaginary. The connection forms define the pseudohermitian connection on $T^{1,0}$ by $\nabla Z_\alpha = {\omega_\alpha}^\beta \otimes Z_\beta$, which is the unique connection preserving $T^{1,0}$, T, and the Levi form.

The curvature form ${\Pi_\alpha}^\beta := \mathrm{d}{\omega_\alpha}^\beta - {\omega_\alpha}^\gamma \wedge {\omega_\gamma}^\beta$ can be written as

$${\Pi_\alpha}^\beta = {R_\alpha}^\beta{}_{\gamma\overline{\delta}}\theta^\gamma \wedge \theta^{\overline{\delta}} \mod \theta,$$

defining the curvature of M. The pseudohermitian Ricci tensor is the contraction $R_{\alpha\overline{\beta}} := {R_\gamma}^\gamma{}_{\alpha\overline{\beta}}$ and the pseudohermitian scalar curvature is the contraction $R := {R_\alpha}^\alpha$. As shown by Webster [38], the contraction ${\Pi_\gamma}^\gamma$ is given by

$${\Pi_\gamma}^\gamma = \mathrm{d}{\omega_\gamma}^\gamma = R_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}} + \nabla^\beta A_{\alpha\beta}\theta^\alpha \wedge \theta - \nabla^{\overline{\beta}} A_{\overline{\alpha}\overline{\beta}}\theta^{\overline{\alpha}} \wedge \theta. \tag{1.1}$$

For computational and notational efficiency, it is usually more useful to work with the pseudohermitian Schouten tensor

$$P_{\alpha\overline{\beta}} := \frac{1}{n+2}\Big(R_{\alpha\overline{\beta}} - \frac{1}{2(n+1)}Rh_{\alpha\overline{\beta}}\Big)$$

and its trace $P := {P_\alpha}^\alpha = \frac{R}{2(n+1)}$. The following higher order derivatives:

$$\begin{aligned} T_\alpha &= \frac{1}{n+2}(\nabla_\alpha P - \mathrm{i}\nabla^\beta A_{\alpha\beta}), \\ S &= -\frac{1}{n}(\nabla^\alpha T_\alpha + \nabla^{\overline{\alpha}} T_{\overline{\alpha}} + P_{\alpha\overline{\beta}}P^{\alpha\overline{\beta}} - A_{\alpha\beta}A^{\alpha\beta}) \end{aligned}$$

also appear frequently (see [19, 30]).

In performing computations, we usually use abstract index notation, so for example τ_α denotes a $(1,0)$-form and $\nabla_\alpha\nabla_\beta f$ denotes the $(2,0)$-part of the Hessian of a function. Of course, given an admissible coframe, these expressions give the components of the equivalent tensor. The following commutator formulas are useful.

Lemma 1.1

$$\nabla_\alpha\nabla_\beta f - \nabla_\beta\nabla_\alpha f = 0, \qquad \nabla_{\overline{\beta}}\nabla_\alpha f - \nabla_\alpha\nabla_{\overline{\beta}} f = \mathrm{i}h_{\alpha\overline{\beta}}\nabla_0 f,$$
$$\nabla_\alpha\nabla_0 f - \nabla_0\nabla_\alpha f = A_{\alpha\gamma}\nabla^\gamma f, \quad \nabla^\beta\nabla_0\tau_\alpha - \nabla_0\nabla^\beta\tau_\alpha = A^{\gamma\beta}\nabla_\gamma\tau_\alpha + \tau_\gamma\nabla_\alpha A^{\gamma\beta},$$

where ∇_0 denotes the derivative in the direction T.

The following consequences of the Bianchi identities are also useful.

Lemma 1.2

$$\nabla^\alpha P_{\alpha\overline{\beta}} = \nabla_{\overline{\beta}} P + (n-1)T_{\overline{\beta}}, \tag{1.2}$$
$$\nabla_0 R = \nabla^\alpha\nabla^\beta A_{\alpha\beta} + \nabla_\alpha\nabla_\beta A^{\alpha\beta}. \tag{1.3}$$

In particular,

$$-\Delta_b R - 2n\mathrm{Im}\nabla^\alpha\nabla^\beta A_{\alpha\beta} = -2\nabla^\alpha(\nabla_\alpha R - \mathrm{i}n\nabla^\beta A_{\alpha\beta}). \tag{1.4}$$

An important operator in the study of pseudohermitian manifolds is the sublaplacian

$$\Delta_b := (\nabla^\alpha\nabla_\alpha + \nabla_\alpha\nabla^\alpha).$$

Defining the subgradient $\nabla_b u$ as the projection of $\mathrm{d}u$ onto $H^*\otimes\mathbb{C}$ (that is, $\nabla_b f = \nabla_\alpha f + \nabla_{\overline{\alpha}} f$), it is easy to show that

$$-\int_M u\Delta_b v\,\theta\wedge \mathrm{d}\theta^n = \int_M \langle\nabla_b u, \nabla_b v\rangle\theta\wedge\mathrm{d}\theta^n$$

for any $u, v \in C^\infty(M)$, at least one of which is compactly supported, and where $\langle\cdot,\cdot\rangle$ denotes the Levi form.

One important consequence of the Bianchi identity is that the operator P has the following two equivalent forms:

$$\begin{aligned} Pf &:= \Delta_b^2 f + n^2\nabla_0^2 f - 2\mathrm{i}n\nabla_\beta(A^{\alpha\beta}\nabla_\alpha f) + 2\mathrm{i}n\nabla^\beta(A_{\alpha\beta}\nabla^\alpha f) \\ &= 4\nabla^\alpha(\nabla_\alpha\nabla_\beta\nabla^\beta f + \mathrm{i}nA_{\alpha\beta}\nabla^\beta f). \end{aligned} \tag{1.5}$$

In dimension $n = 1$, the operator P is the compatibility operator found by Graham and Lee [22]. Hirachi [24] later observed that in this dimension P is a CR covariant operator, in the sense that it satisfies a particularly simple transformation formula under a change of contact form. Thus, in this dimension P is the CR Paneitz operator P_4.

1.1 CR pluriharmonic functions

Given a CR manifold (M^{2n+1}, J), a CR pluriharmonic function is a function $u \in C^\infty(M)$ which is locally the real part of a CR function $v \in C^\infty(M;\mathbb{C})$, i.e., $u = \mathrm{Re}(v)$ for v satisfying $\nabla_{\overline{\alpha}} v = 0$. We denote by $\mathcal{P}$ the space of pluriharmonic functions on M, which is usually an infinite-dimensional vector space. When additionally a choice of contact form θ is given, Lee [31] proved the following alternative characterization of CR pluriharmonic functions which does not require solving for v.

Proposition 1.1 *Let (M^{2n+1}, J, θ) be a pseudohermitian manifold. A function $u \in C^\infty(M)$ is CR pluriharmonic if and only if*

$$B_{\alpha\overline{\beta}} u := \nabla_{\overline{\beta}} \nabla_\alpha u - \frac{1}{n} \nabla^\gamma \nabla_\gamma u \, h_{\alpha\overline{\beta}} = 0, \quad \text{if } n \geq 2,$$
$$P_\alpha u := \nabla_\alpha \nabla_\beta \nabla^\beta u + in A_{\alpha\beta} \nabla^\beta u = 0, \quad \text{if } n = 1.$$

It is straightforward to check that (see [22])

$$\nabla^{\overline{\beta}}(B_{\alpha\overline{\beta}} u) = \frac{n-1}{n} P_\alpha u. \tag{1.6}$$

In particular, we see that the vanishing of $B_{\alpha\overline{\beta}} u$ implies the vanishing of $P_\alpha u$ when $n > 1$. Moreover, the condition $B_{\alpha\overline{\beta}} u = 0$ is vacuous when $n = 1$, and by (1.6), we can consider the condition $P_\alpha u = 0$ from Proposition 1.1 as the "residue" of the condition $B_{\alpha\overline{\beta}} u = 0$.

Note also that, using the second expression in (1.5), we have that $P = 4\nabla^\alpha P_\alpha$. In particular, it follows that $\mathcal{P} \subset \ker P_4$ for three-dimensional CR manifolds (M^3, J). It is easy to see that this is an equality when (M^3, J) admits a torsion-free contact form (see [22]), the general case is a question of interest that will be addressed in Section 3.

The space of CR pluriharmonic functions is stable for the one-parameter family (M^3, J^t, θ) of pseudohermitian manifold if for every $\varphi \in \mathcal{P}^t$ and every $\varepsilon > 0$, there is a $\delta > 0$ such that for each s satisfying $|t - s| < \delta$, there is a CR pluriharmonic function $f_s \in \mathcal{P}^s$ such that

$$\|\varphi - f_s\|_2 < \varepsilon.$$

We also discuss the question of stability in Section 3.

2 The Embedding Problem

The typical examples of CR structure are the smooth boundaries of strictly pseudo-convex complex manifolds. Let Ω be a smooth strictly pseudo-convex domain in a Stein manifold given by a defining function $u < 0$ which satisfies the nondegeneracy condition at the boundary $\Sigma = \{u = 0\}$, and u is strictly plurisubharmonic near Σ. Then one easily verifies that $\theta = \mathrm{Im}(\overline{\partial} u)$ restricts to a contact form on Σ, and $\ker \theta = \xi$ inherits the ambient almost complex structure J. The converse question whether every CR structure arises this way is known as the embedding problem. In dimensions $2n + 1 \geq 5$, this was answered affirmatively by the work of Boutet de Monvel [1]. The special case $n = 1$ has received a good deal of attention since the work of Rossi [34] and is the focus of this section.

The example of Rossi is a small perturbation of the standard CR 3-sphere: Let $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 | |z_1|^2 + |z_2|^2 = 1\}$ be the boundary of the unit ball in $\mathbb{C}^2$. The standard CR structure is given by the vector field $Z = \overline{z}_2(\frac{\partial}{\partial z_1}) - \overline{z}_1(\frac{\partial}{\partial z_2})$. Consider for each t the new CR structure given by $Z_t = Z + t\overline{Z}$. For $0 < |t| < 1$, it is shown by Burns [4] that the CR holomorphic functions must be even functions, hence this CR structure cannot be realized in $\mathbb{C}^N$ for any N. In this same paper it is shown that if for a given CR structure in 3-D, the CR holomorphic functions separate points and the Szego projection is continuous in the C^∞ topology, then the structure may be realized in some $\mathbb{C}^N$. In [29], Kohn showed that these conditions are indeed satisfied if the $\Box_b$ operator on functions has closed range. Subsequently, Lempert [32] showed that when the torsion vanishes a CR structure in 3-D may be realised in $\mathbb{C}^n$. We note the vanishing torsion condition means that the Reeb vector field generates a one parameter family of biholomorphic transformations of the CR structure.

In order to relax the condition of vanishing torsion, one realizes that the Paneitz operator plays an important role in the embedding question.

Theorem 2.1 (see [12]) *If (M^3, θ, J) satisfy the condition $P_4 \geq 0$ and the Webster scalar curvature $R \geq c > 0$, then the nonzero eigenvalues λ of the operator $\Box_b$ satisfy the lower bound $\lambda \geq \min R$. As a consequence, the $\Box_b$ operator has closed range.*

It is helpful to remark that the vanishing torsion condition implies the condition $P_4 \geq 0$: This follows from the two identities involving the Paneitz operator and the $\Box_b$ operator:

$$4P_4\phi = \Box_b\overline{\Box}_b - 2\mathrm{i}(A^{11}\phi_1)_{,1} = \overline{\Box}_b\Box_b + 2\mathrm{i}(A^{\overline{11}}\phi_{\overline{1}})_{,\overline{1}}.$$

Thus when torsion vanishes, $\Box_b$ commutes with $\overline{\Box}_b$, and hence $P_4 \geq 0$.

As a consequence of this eigenvalue bound, it is possible to verify the stability of embedding for a family of CR structures $\{J_t \mid |t| < \epsilon\}$ on a given (M^3, ξ) satisfying the same assumptions as in Theorem 2.1. If the CR structure J_0 is embeddable in $\mathbb{C}^N$, then for t sufficiently small, there is an embedding of J_t which is close to that for J_0.

In the next section, we discuss the natural questions of positivity of Paneitz operator which is closely related to the stability question.

3 When is P_4 Non-negative?

To find criteria to verify when P_4 is non-negative, one would like to know if CR manifolds embedded in $\mathbb{C}^2$ with some additional nice properties satisfy these nonnegativity conditions assumed in Theorem 2.1. Working in this direction, [13] showed that these nonnegative conditions hold for small deformations of a strictly pseudo-convex hypersurface with vanishing torsion in $\mathbb{C}^2$.

Another closely related question concerning the CR Paneitz operator is the identification of its kernel. It follows from its definition that, on a three-dimensional CR manifold, the space of CR pluriharmonic functions is contained in the kernel of the CR Paneitz operator. Moreover, Graham and Lee showed [22] that if a three-dimensional CR manifold admits a torsion-free contact form, then the kernel of the CR Paneitz operator consists solely of the CR pluriharmonic functions. One would like to characterize CR manifolds for which this equality holds. Since there are known non-embedded examples for which the equality does not hold, we restrict our attention to embedded CR manifolds. Motivated by this problem, Hsiao [26] showed that for embedded CR manifolds, there is a finite-dimensional vector space W, such that the kernel of the CR Paneitz operator P_4 splits into a direct sum:

$$\ker P_4 = \mathcal{P} \oplus W. \tag{3.1}$$

There is an elementary proof of this fact in [8].

Theorem 3.1 (see [8]) *Let (M^3, J^t, θ) be a family of embedded CR manifolds for $t \in [-1,1]$ with the following properties:*

(1) *J^t is real analytic in the deformation parameter t.*

(2) *The Szegö projectors $S^t\colon F^{2,0} \to (\ker \overline{\partial}_b^t \subset F^{2,0})$ vary continuously in the deformation parameter t ($F^{2,0}$ denotes the L^2 sections of the canonical bundle).*

(3) *For the structure J^0 we have $P_4^0 \geq 0$ and $\ker P_4^0 = \mathcal{P}^0$, the space of CR pluriharmonic functions with respect to J^0.*

(4) *There is a uniform constant $c > 0$ such that*

$$\inf_{t\in[-1,1]} \min_M R^t \geq c > 0. \tag{3.2}$$

(5) *The CR pluriharmonic functions are stable for the family (M^3, J^t, θ). Then $P_4^t \geq 0$ and $\ker P_4^t = \mathcal{P}^t$ for all $t \in [-1,1]$.*

Remark 3.1 The assumption (3.2) can be replaced by the assumption that the CR Yamabe constants $Y[J^t]$ are uniformly positive. Since the assumptions on the CR Paneitz operator are CR invariant (see [24]), this allows us to recast Theorem 3.1 in a CR invariant way.

As an application, consider the family of ellipsoids in $\mathbb{C}^2$ as deformations of the standard CR three-sphere. The formula established in [28, Theorem 1] expressing the Szegö kernel in terms of the defining function implies that condition (2) holds. Since the standard contact form on the CR three-sphere is torsion-free, its CR Paneitz operator is nonnegative and has kernel consisting only of the CR pluriharmonic functions (see [22]). An elementary calculation (see [13]) shows that the ellipsoids have positive Webster scalar curvature thus verifying condition (4) of the theorem. The condition (5) then follows from the stability result of [12]. Stability of CR functions for strictly pseudo-convex domains in $\mathbb{C}^2$ first appeared in the work of Lempert [33].

Corollary 3.1 *The ellipsoids in $\mathbb{C}^2$ are such that the Paneitz operator is nonnegative and has kernel consisting only of the CR pluriharmonic functions.*

Theorem 3.1 shows that the stability of the CR pluriharmonic functions plays a role in preventing the existence of the supplementary space. If one wishes to use deformations to exhibit examples of CR manifolds for which the supplementary space exists, one should thus look at unstable families. Indeed there are conditions which guarantee the existence of the supplementary space.

First, using the Baire Category theorem, one can show that generically the supplementary space exists.

Theorem 3.2 (see [9]) *Let (M^3, J^t, θ) be a family of embedded CR manifolds with $t \in [-1,1]$. Assume that the Szegö projector $S^t\colon F^{2,0} \to (\ker \overline{\partial}_b^t \subset F^{2,0})$ varies real analytically in the deformation variable t. Then*

(1) $n_0 := \sup_{t\in[-1,1]} \dim W^t < \infty$.

(2) *The set*

$$F := \{t \in [-1,1]\colon\ \dim W^t \le n_0 - 1\}$$

is a closed set with no accumulation points. In particular, F has no interior points and hence F is of the first category.

(3) *The set*

$$E := \{t \in [-1,1]\colon\ \dim W^t = n_0\}$$

has nonempty interior.

Remark 3.2 The theorem above states that for generic values t of the deformation parameter, $\dim W^t = n_0$. Since $n_0 > 0$ if there exists a $t_0 \in [-1,1]$ with $W^{t_0} \neq \{0\}$, the supplementary space exists for a generic value of t if it exists for some t_0. Moreover, if $n_0 > 0$, then $\dim W^t = 0$ for a thin set $F \subset [-1,1]$ of the first category.

Under an assumption on the rate of vanishing of the first non-zero eigenvalue of the CR Paneitz operator for a family of CR structures, the loss of stability of the CR pluriharmonic functions implies the existence of the supplementary space. To make this precise, we list our assumptions.

Let $(M^3, J^t, \theta) =: M^t$ be a family of embedded CR manifolds for which J^t is C^6 in the deformation parameter t for some interval $|t - t_0| < \mu$ with $\mu > 0$. Suppose that there is a constant $c > 0$ independent of t, such that the following assertions hold.

(1) For any $t \neq t_0$ and any $f \perp \ker P_4^t$, it holds that

$$|t - t_0|\eta(|t - t_0|)\|f\|_2 \le \|P_4^t f\|_2, \tag{3.3a}$$

where $\eta(s) \to \infty$ as s tends to zero.

(2) For any $f \perp \ker P_4^{t_0}$, it holds that

$$c\|f\|_2 \le \|P_4^{t_0} f\|_2. \tag{3.3b}$$

Together, the assumptions (3.3a) and (3.3b) imply that the lowest nonzero absolute value of the eigenvalues of the CR Paneitz operator P_4^t jumps up as $t \to t_0$.

Next, we assume that there is a family of diffeomorphisms $\Phi^t\colon M^t \to M^0 := M^{t_0}$ which is C^6 in the deformation parameter t, such that Φ^0 is the identity map.

Finally, we assume that the CR pluriharmonic functions are unstable at t_0. More precisely, we assume that there is a CR pluriharmonic function $f_0 \in C^5$ for the structure M^0 and constants $\varepsilon > 0$ and $0 < \delta < \mu$, such that for any t with $|t - t_0| < \delta$ and any CR pluriharmonic function $\psi \in \mathcal{P}^t$, it holds that

$$\|f_0 - \psi\|_2 \ge \varepsilon. \tag{3.4}$$

Theorem 3.3 (see [9]) *Assume that (M^3, J^t, θ) is as described above. Then for all $t \neq t_0$ with $|t - t_0| < \delta$ and δ sufficiently small, the supplementary space W^t exists.*

4 The Positivity of the CR Mass

An important application of the embedding theorem is the solvability of the $\Box_b$ equation, a key fact in the CR positive mass theorem. The situation is quite different from the positivity of mass in Riemannian geometry.

We consider a compact three-dimensional pseudohermitian manifold (M, J, θ) (with no boundary) of positive Tanaka-Webster class. This means that the first eigenvalue of the conformal sublaplacian

$$L_b := -4\Delta_b + R$$

is strictly positive. Here Δ_b stands for the sublaplacian of M and R for the Tanaka-Webster curvature. The conformal sublaplacian has the following covariance property under a conformal change of contact form:

$$\widehat{L}_b(\phi) = u^{-\frac{Q+2}{Q-2}} L_b(u\phi), \quad \widehat{\theta} = u^2\theta,$$

where $Q = 4$ is the homogeneous dimension of the manifold. The conformal sublaplacian rules the change of the Tanaka-Webster curvature under the above conformal deformation, through the following formula:

$$-4\Delta_b u + Ru = \widehat{R}u^{\frac{Q+2}{Q-2}},$$

where $\widehat{R}$ is the Tanaka-Webster curvature corresponding to the pseudohermitian structure $(J, \widehat{\theta})$. The positivity of the Tanaka-Webster class is equivalent to the condition

$$\mathcal{Y}(J) := \inf_{\widehat{\theta}} \frac{\int_M R_{J,\widehat{\theta}} \widehat{\theta} \wedge \mathrm{d}\widehat{\theta}}{\left(\int_M \widehat{\theta} \wedge \mathrm{d}\widehat{\theta}\right)^{\frac{1}{2}}} > 0, \tag{4.1}$$

where $\widehat{\theta}$ is any contact form which annihilates ξ. Under the assumption $\mathcal{Y}(J) > 0$, we have that L_b is invertible, so for any $p \in M$, there exists a Green's function G_p for which

$$(-4\Delta_b + R)G_p = 16\delta_p.$$

One can show that in CR normal coordinates (z, t), the Green's function G_p admits the following expansion:

$$G_p = \frac{1}{2\pi}\rho^{-2} + A + O(\rho),$$

where A is some real constant and we have set $\rho^4(z, t) = |z|^4 + t^2,\ z \in \mathbb{C},\ t \in \mathbb{R}$. Analogous to the Riemannian construction for the blow-up of a compact manifold, we consider the new pseudohermitian manifold with a blow-up of contact form

$$N = (M \setminus \{p\}, J, \theta = G_p^2\widehat{\theta}). \tag{4.2}$$

With an inversion of coordinates, we then obtain a pseudohermitian manifold which has asymptotically the geometry of the Heisenberg group. Starting from this model, we give a definition of asymptotically flat pseudohermitian manifold and we introduce its pseudohermitian mass (p-mass) by the formula

$$m(J, \theta) := \lim_{\Lambda \to +\infty} \mathrm{i} \oint_{S_\Lambda} \omega_1^1 \wedge \theta,$$

where we set $S_\Lambda = \{\rho = \Lambda\}$, $\rho^4 = |z|^4 + t^2$, and ω_1^1 stands for the connection form of the structure. The above quantity is indeed a natural candidate, since it satisfies the same property as the ADM mass in Riemannian geometry, and moreover it coincides with the zero-th order term in the expansion of the Green's function for L_b. In the case that N arise as the blowup of M^3, m is a positive multiple of the constant A.

[14] gave some general conditions which ensure the nonnegativity of the p-mass, characterizing also the zero case as (CR equivalent to) the standard three-dimensional CR sphere.

Theorem 4.1 *Let M be a smooth, strictly pseudo-convex three dimensional compact CR manifold. Suppose $\mathcal{Y}(J) > 0$, and that the CR Paneitz operator is nonnegative. Let $p \in M$ and let θ be a blow-up of contact form as in* (4.2)*. Then*

(a) $m(J, \theta) \geq 0$;

(b) *if $m(J, \theta) = 0$, M is CR equivalent* (*or isomorphic as pseudohermitian manifold*) *to S^3, endowed with its standard CR structure.*

The assumptions here are conformally invariant, and are needed to ensure the positivity of the right-hand side in (4.3). The proof is patterned after the spinorial argument of Witten. By the embeddability result, the conditions on $\mathcal{Y}(J)$ and P imply the embeddability of M: We use this property to find a solution of $\Box_b \beta = 0$ with the correct asymptotics (to make the first term in the right-hand side of (4.3) vanish).

$$\frac{2}{3} m(J, \theta) = \int_N \Big\{ -|\Box_b \beta|^2 + 2|\beta_{,\overline{11}}|^2 + 2R|\beta_{,\overline{1}}|^2 + \frac{1}{2}\overline{\beta} P \beta \Big\}\, \theta \wedge \mathrm{d}\theta. \tag{4.3}$$

Here $\beta : N \to \mathbb{C}$ is a function satisfying

$$\beta = \overline{z} + \beta_{-1} + O(\rho^{-2+\epsilon}) \quad \text{near } \infty, \qquad \Box_b \beta = O(\rho^{-4}),$$

with β_{-1} a suitable function with homogeneity -1 in ρ.

The full solvability of $\Box_b \beta = 0$ then reduces to a mapping theorem in weighted spaces worked out by Hsiao and Yung [27].

It is important to point out that the assumption $P_4 \geq 0$ is necessary, since there are small perturbations of the CR structure on the standard 3-sphere for which the CR mass is actually negative. Indeed it appears likely that this holds for generic perturbations of the standard 3-sphere.

5 Pseudo-Einstein Contact Forms

For strictly pseudo-convex domains in $\mathbb{C}^{n+1}$, Fefferman [16] introduced the following complex Monge-Ampere equation:

$$J[u] = (-1)^{n+1} \det \begin{pmatrix} u & u_{\overline{j}} \\ u_i & u_{i\overline{j}} \end{pmatrix} = 1. \tag{5.1}$$

An iterative computational procedure is given to find approximate solutions of the equation (1.1) that is accurate to order $n+1$ near the boundary: Let ψ be any smooth defining function of the boundary:

$$u_1 = \frac{\psi}{J(\psi)^{\frac{1}{3}}},$$
$$u_s = u_{s-1}\Big(1 + \frac{1 - J[u_{s-1}]}{(n+1-s)s}\Big) \quad \text{for } 2 \le s \le n+1.$$

The resulting contact form $\theta = \mathrm{Im}(\overline{\partial} u)$ satisfies the following pseudo-Einstein condition:

$$R_{\alpha\overline{\beta}} - \Big(\frac{1}{n}\Big) R h_{\alpha\overline{\beta}} = 0, \quad \text{if } n \ge 2,$$
$$\nabla_\alpha R - i\nabla^\beta A_{\alpha\overline{\beta}} = 0, \quad \text{if } n = 1.$$

Jack Lee [30] showed that the pseudo-Einstein condition is equivalent to the statement that at each point $p \in M$ there exists a neighborhood of p in which there is a closed section ω of the canonical bundle with respect to which θ is volume normalized, i.e.,

$$\theta \wedge (\mathrm{d}\theta)^n = \mathrm{i}^{n^2} n! \theta \wedge (T \lrcorner \omega) \wedge (T \lrcorner \overline{\omega}).$$

The special condition when $n = 1$ is motivated by the observation that the Bianchi identity gives

$$\nabla^{\overline{\beta}}\Big(R_{\alpha\overline{\beta}} - \Big(\frac{1}{n}\Big) R h_{\alpha\overline{\beta}}\Big) = \frac{n-1}{n}(\nabla_\alpha R - \mathrm{i}n\nabla^\beta A_{\alpha\beta}).$$

There is a close connection of pseudo-Einstein contact forms with the pluriharmonic functions. Let (M^3, J, θ) be a pseudohermitian manifold and define the $(1,0)$-form W_α by

$$W_\alpha := \nabla_\alpha R - \mathrm{i}\nabla^\beta A_{\alpha\beta}.$$

Observe that W_α vanishes if and only if θ is pseudo-Einstein. As first, observed by Hirachi [24], W_α satisfies a simple transformation formula; given another contact form $\widehat{\theta} = \mathrm{e}^\sigma \theta$, a straightforward computation shows that

$$\widehat{W_\alpha} = W_\alpha - 3P_\alpha \sigma. \tag{5.2}$$

An immediate consequence of (5.2) is the following correspondence between pseudo-Einstein contact forms and CR pluriharmonic functions.

Proposition 5.1 *Let (M^3, J, θ) be a pseudo-Einstein three-manifold. Then the set of pseudo-Einstein contact forms on (M^3, J) is given by*

$$\{\mathrm{e}^u \theta \colon u \text{ is a CR pluriharmonic function}\}.$$

Since there is a large supply of pseudo-Einstein contact form once there exists one, it is of interest to find a canonical one. Although the positive mass theorem provide a contact form that minimizes the CR Yamabe quotient, there is no assurance that it is pseudo-Einstein. In the next section, we introduce a new operator that is relevant to this question.

6 The P-Prime Operator

Using spectral methods, Branson, Fontana and Morpurgo [2] have recently identified a new operator P_4' on the standard CR three-sphere (S^3, J, θ_0), such that P_4' is of the form Δ_b^2 plus lower-order terms, P_4' is invariant under the action of the CR automorphism group of S^3, and P_4' appears in an analogue of Q-curvature equation in which the exponential term is present. However, the operator P_4' acts only on the space $\mathcal{P}$ of CR pluriharmonic functions on S^3, namely those functions which are the boundary values of pluriharmonic functions in the ball $\{(z, w)\colon |z|^2+|w|^2 < 1\} \subset \mathbb{C}^2$. The space of CR pluriharmonic functions on S^3 is itself invariant under the action of the CR automorphism group, so it makes sense to discuss the invariance of P_4'. Using this operator, Branson, Fontana and Morpurgo [2] showed that

$$\int_{S^3} u\, P_4' u + 2\int_{S^3} Q_4' u - \Big(\int_{S^3} Q_4'\Big) \log\Big(\int_{S^3} \mathrm{e}^{2u}\Big) \geq 0 \tag{6.1}$$

for all $u \in \mathcal{P}$, where $Q_4' = 1$ and equality holds in (6.1) if and only if $\mathrm{e}^{u}\theta_0$ is a standard contact form. In [7], we extended the definition of P' to more general CR structures in dimension $n = 1$.

To describe our results, let us begin by discussing in more detail the ideas which give rise to the definitions of P_4' and Q_4'. To define P_4', we follow the same strategy of Branson, Fontana, and Morpurgo [2]. First, Gover and Graham [19] showed that on a general CR manifold (M^{2n+1}, J), one can associate to each choice of contact form θ a formally-self adjoint real fourth-order operator $P_{4,n}$ which has leading order term $\Delta_b^2 + T^2$, and that this operator is CR covariant. On three-dimensional CR manifolds, this reduces to the well-known operator

$$P_4 := P_{4,1} = \Delta_b^2 + T^2 - 4\mathrm{Im}\nabla^\alpha(A_{\alpha\beta}\nabla^\beta)$$

which, through the work of Graham and Lee [22] and Hirachi [24], is known to serve as a good analogue of the Paneitz operator of a four-dimensional conformal manifold. As pointed out by Graham and Lee [22], the kernel of P_4 (as an operator on a three-dimensional CR manifold) contains the space $\mathcal{P}$ of CR pluriharmonic functions, and thus one can ask whether the operator

$$P_4' := \lim_{n\to 1} \frac{2}{n-1} P_{4,n}|_{\mathcal{P}}$$

is well-defined. This is the case. It then follows from standard arguments (see [3]) that if $\widehat{\theta} = \mathrm{e}^{\sigma}\theta$ is any other choice of contact form, then the corresponding operator $\widehat{P_4'}$ is related to P_4' by

$$\mathrm{e}^{2\sigma}\widehat{P_4'}(f) = P_4'(f) + P_4(\sigma f) \tag{6.2}$$

for any $f \in \mathcal{P}$. Thus the relation between P_4' and P_4 is analogous to the relation between the Q-curvature and the Paneitz operator. More precisely, the P'-operator can be regarded as a Q-curvature operator in the sense of Branson and Gover [3]. Moreover, since the Paneitz operator is self-adjoint and kills pluriharmonic functions, the transformation formula (6.2) implies that

$$\mathrm{e}^{2\sigma}\widehat{P_4'}(f) = P_4'(f) \mod \mathcal{P}^\perp$$

for any $f \in \mathcal{P}$, returning P_4' to the status of a Paneitz-type operator. This is the sense in which the P'-operator is CR invariant, and is the way that it is studied in (6.1).

From its construction, one easily sees that $P_4'(1)$ is exactly Hirachi's Q-curvature. Thus, unlike the Paneitz operator, the P'-operator does not necessarily kill constants. However, there is a large and natural class of contact forms for which the P'-operator does kill constants, namely the pseudo-Einstein contact forms. In this setting, it is natural to ask whether there is a scalar invariant Q_4' such that $P_4'(1) = \frac{n-1}{2}Q_4'$. Indeed, if (M^3, J, θ) is a pseudo-Einstein manifold, then the scalar invariant

$$Q_4' := \lim_{n\to 1} \frac{4}{(n-1)^2} P_{4,n}(1)$$

is well-defined. As a consequence, if $\widehat{\theta} = \mathrm{e}^{\sigma}\theta$ is another pseudo-Einstein contact form (in particular, $\sigma \in \mathcal{P}$), then

$$\mathrm{e}^{2\sigma}\widehat{Q_4'} = Q_4' + P_4'(\sigma) + \frac{1}{2}P_4(\sigma^2). \tag{6.3}$$

Taking the point of view that P_4' is a Paneitz-type operator, we may also write

$$\mathrm{e}^{2\sigma}\widehat{Q_4'} = Q_4' + P_4'(\sigma) \quad \mathrm{mod}\ \mathcal{P}^{\perp}. \tag{6.4}$$

The upshot is that, on the standard CR three-sphere, $Q_4' = 1$, so that this indeed recovers the interpretation of the Beckner-Onofri-type inequality (6.1) of Branson-Fontana-Morpurgo [2] as an estimate involving a Paneitz-type operator and Q-type curvature. Additionally, we also see from (6.3) that the integral of Q_4' is a CR invariant. More precisely, if (M^3, J) is a compact CR three-manifold and $\theta, \widehat{\theta}$ are two pseudo-Einstein contact forms, then

$$\int_M \widehat{Q_4'}\, \widehat{\theta} \wedge \mathrm{d}\widehat{\theta} = \int_M Q_4'\, \theta \wedge \mathrm{d}\theta.$$

In conformal geometry, the total Q-curvature plays an important role in controlling the topology of the underlying manifold. For instance, the total Q-curvature can be used to prove sphere theorems (see, e.g., [23, Theorem B] and [11, Theorem A]). We have the following CR analogue of Gursky's theorem [23, Theorem B].

Theorem 6.1 *Let (M^3, J, θ) be a compact three-dimensional pseudo-Einstein manifold with nonnegative Paneitz operator and nonnegative CR Yamabe constant. Then*

$$\int_M Q_4'\, \theta \wedge \mathrm{d}\theta \le \int_{S^3} Q_0'\, \theta_0 \wedge \mathrm{d}\theta_0,$$

with equality if and only if (M^3, J) is CR equivalent to the standard CR three sphere.

The proof of Theorem 6.1 relies upon the existence of a CR Yamabe contact form, that is, the existence of a smooth unit-volume contact form with constant Webster scalar curvature equal to the CR Yamabe constant (see [14]). In particular, it relies on the CR Positive Mass theorem (see [14]). One complication which does not arise in the conformal case (see [23]) is the possibility that the CR Yamabe contact form may not be pseudo-Einstein. We overcome this difficulty by computing how the local formula for Q_4' transforms with a general change of contact form, i.e., without imposing the pseudo-Einstein assumption.

In conformal geometry, the total Q-curvature also arises when considering the Euler characteristic of the underlying manifold. Burns and Epstein [5] showed that there is a biholomorphic invariant, now known as the Burns-Epstein invariant, of the boundary of a strictly pseudo-convex domain which is related to the Euler characteristic of the domain in a similar way. It turns out that the Burns-Epstein invariant is a constant multiple of the total Q'-curvature, and thus there is a nice relationship between the total Q'-curvature and the Euler characteristic.

Theorem 6.2 *Let (M^3, J) be a compact CR manifold which admits a pseudo-Einstein contact form θ, and denote by $\mu(M)$ the Burns-Epstein invariant of (M^3, J). Then*

$$\mu(M) = -16\pi^2 \int_M Q' \, \theta \wedge \mathrm{d}\theta.$$

In particular, if (M^3, J) is the boundary of a strictly pseudo-convex domain X, then

$$\int_X \Big(c_2 - \frac{1}{3} c_1^2 \Big) = \chi(X) - \frac{1}{16\pi^2} \int_M Q' \, \theta \wedge \mathrm{d}\theta,$$

where c_1 and c_2 are the first and second Chern forms of the Kähler-Einstein metric in X obtained by solving Fefferman's equation, and $\chi(X)$ is the Euler characteristic of X.

In general dimensions, the existence of P' operators is given by Hirachi [25] and described in terms of tractor calculus by Case-Gover in a forthcoming publication.

7 An Extremal Pseudo-Einstein Contact Form

In pursuing the analogy with the pair (P_4, Q_4) in conformal geometry, it is more natural to focus on the operator $\overline{P_4'}$ which is given by $\tau \circ P'$ where τ is the projection to the pluriharmonics. Similarly, it is more natural to consider the scalar invariant $\overline{Q_4'} := \tau Q_4' \in \mathcal{P}$. The transformation property of Q_4' implies that if $\widehat{\theta} = \mathrm{e}^w \theta$ are both pseudo-Einstein, then equation (6.4) showed that $(\overline{P_4'}, \overline{Q_4'})$ have the same formal properties as (P_4, Q_4). Note that if θ is pseudo-Einstein, then $\widehat{\theta}$ is pseudo-Einstein if and only if $w \in \mathcal{P}$ [24], so that (6.2) makes sense.

We will construct contact forms for which the Q'-curvature $\overline{Q_4'}$ is constant by constructing minimizers of the II-functional given by

$$\mathrm{II}(w) = \int_M w \, \overline{P_4' w} + 2 \int_M \overline{Q_4'} w - \Big(\int_M \overline{Q_4'} \Big) \log \Big(\int_M \mathrm{e}^{2w} \Big) \tag{7.1}$$

on a pseudo-Einstein three-manifold $(M^3, T^{1,0}M, \theta)$. Note that, since II is only defined on w, the projections in (7.1) can be removed, i.e., we can equivalently define the II-functional in terms of P_4' and Q_4'. In general, the II-functional is not bounded below. However, under natural positivity conditions, it is bounded below and coercive, in which case we can construct the desired minimizers.

Theorem 7.1 (see [6]) *Let $(M^3, T^{1,0}M, \theta)$ be a compact pseudo-Einstein three-manifold, such that the P'-operator P_4' is nonnegative and* $\ker P_4' = \mathbb{R}$. *Suppose additionally that*

$$\int_M \overline{Q_4'} \theta \wedge \mathrm{d}\theta < 16\pi^2. \tag{7.2}$$

Then there exists a function $w \in \mathcal{P}$ which minimizes the II*-functional* (7.1)*. Moreover, the contact form $\widehat{\theta} := \mathrm{e}^w \theta$ is such that $\overline{\widehat{Q}_4'}$ is constant.*

Some comments on the statement of Theorem 7.1 are in order. First, [7] provided some sufficient condition for the operator P_4' to be non-negative. Second, the assumptions of the theorem are all CR invariant: The assumptions on $\overline{P_4'}$ are independent of the choice of contact form and the assumption (7.2) is independent of the choice of pseudo-Einstein contact form. In particular, if one is interested only in boundaries of domains in $\mathbb{C}^2$, the assumptions are biholomorphic invariants. Third, the assumption that P' is nonnegative with trivial kernel (i.e., $\ker \overline{P_4'} = \mathbb{R}$) automatically holds if there exists a pseudo-Einstein contact form with nonnegative scalar curvature (see [7]). Fourth, the assumption (7.2) holds if one assumes instead that $(M^3, T^{1,0}M, \theta)$ has nonnegative CR Paneitz operator and nonnegative CR Yamabe constant (see [7]). Note here that $16\pi^2$ is the total Q'-curvature of the standard CR three-sphere. Fifth, the conclusion that $\overline{\widehat{Q_4'}}$ is constant is the best that one can hope for: Though it is tempting to speculate that minimizers of the II-functional actually give rise to contact forms for which $\widehat{Q}_4'$ is constant, as happens on the standard CR three-sphere, the natural structure on $S^1 \times S^2$ is an example where there is a unique, up to homothety, contact form for which $\widehat{Q}_4'$ is constant but Q_4' is nonconstant.

The proof of Theorem 7.1 is analogous to the corresponding result for the Q-curvature in four-dimensional conformal geometry (see [10]), though there are many new difficulties, we must overcome. Since we are minimizing within $\mathcal{P}$, there is a Lagrange multiplier in the Euler equation for the II-functional which lives in the orthogonal complement $\mathcal{P}^{\perp}$ to $\mathcal{P}$. Rather than obtain estimates on the Lagrange multiplier, we establish regularity of minimizers by studying the Green's function of $\overline{P}_4'$. The greater difficulty lies in showing that minimizers in $W^{2,2} \cap \mathcal{P}$ for the II-functional exist under the hypotheses of Theorem 7.1. The basic idea here is to use the positivity of P_4' and (7.2) to show that the II-functional is coercive. However, doing so requires showing that $\overline{P_4'}$ satisfies a Moser-Trudinger inequality with the same constant as on the CR three-sphere. We do this by appealing to the general results of Fontana and Morpurgo [18], which depends upon having a fairly detailed understanding of the properties of $(\overline{P_4'})^{\frac{1}{2}}$. This is technically the most involved part of the argument.

8 An Isoperimetric Inequality

The Q' curvature integral has in addition to its topological meaning, an analytic consequence as an invariant that controls the isoperimetric constant for a class of pseudo-Einstein contact forms.

Consider the Heisenberg group H^1 which may be realized as $\{(x, y, t) \in \mathbb{R}^3\}$ with the contact form $\theta_0 = \mathrm{d}t + x\mathrm{d}y - y\mathrm{d}x$ and the CR structure given by $X = \partial_x + y\partial_t,\ JX = Y = \partial_y - x\partial_t$. Both curvature and torsion vanish hence it is pseudo-Einstein. Consider $\widehat{\theta} = \mathrm{e}^{2u}\theta$ where $\sigma \in \mathcal{P}$. The P' operator is simply given by $P' = (\Delta_b)^2$, and one checks easily that P' maps $\mathcal{P}$ to itself, so that there is no need to project back to $\mathcal{P}$.

Given a contact form (M, θ), there is a natural distance function $d(p, q)$ defined as the infimum over the length of contact curves joining p to q, where the length of a contact curve $\gamma : [a, b] \to M$ is defined by $\int_a^b \sqrt{\mathrm{d}\theta(\gamma', J\gamma')}\mathrm{d}t$. Associated to the contact form, there is also a natural notion of area of a surface in M (see, e.g., [15]). It is of interest to study the isoperimetric inequality for such geometry: Give $V > 0$ to bound the area of all domains $\Omega \subset M$ with $\mathrm{vol}(\Omega) = V$.

Theorem 8.1 (see [37]) *Let* $\theta = \mathrm{e}^{2u}\theta_0$ *be a pseudo-Einstein complete contact form on* H^1*. Suppose that the Webster curvature is non-negative near infinity, suppose also* $Q' \geq 0$ *and* $\int Q'\theta \wedge \mathrm{d}\theta < c_1$*. Then there is an isoperimetric constant* $\mathrm{I} = \mathrm{I}(\int Q'\theta \wedge \mathrm{d}\theta)$ *so that* $\mathrm{Area}(\partial\Omega) \geq \mathrm{I}V^{\frac{3}{4}}$.

We make three remarks concerning this bound. First, this result is a weak analogue of the stronger result in conformal geometry (see [36]). Second, the isoperimetric constant only depends on the size of the Q'-curvature integral. Third, the constant c_1 is the constant appearing in the following integral representation formula to be established for such u:

$$u(x) = \frac{1}{c_n}\int_{\mathbb{H}^1} \log\frac{\rho(y)}{\rho(y^{-1}x)}Q(y)\mathrm{e}^{4u(y)}\mathrm{d}y + C. \tag{8.1}$$

Naturally it is expected that the condition $Q' \geq 0$ should be replaced by $\int Q'_+\theta\mathrm{d}\theta < c_1$, and the constant $\mathrm{I} = \mathrm{I}(\int Q', \int Q'_-)$.

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