WORK OF CHARLES FEFFERMAN IN CONFORMAL GEOMETRY

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As colleague and collaborator, we feel privileged to have had the opportunity to work with Charlie. His contributions to conformal geometry are centered around the ambient metric construction. We begin with a discussion of some background leading up to its discovery.

The conformal ambient metric grew out of Charlie's work in several complex variables; in particular in his efforts to understand the asymptotic expansion of the Bergman kernel. As Charlie had previously shown, the restriction to the diagonal of the Bergman kernel K of a smooth, bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ can be written in the form $K(z,z) = \varphi(z)\rho(z)^{-n-1} + \psi(z)\log\rho(z)$, where ρ is a smooth defining function for $\partial\Omega$ and φ, $\psi \in C^\infty(\overline{\Omega})$. This expansion can be viewed by analogy to the asymptotic expansion of the heat kernel of a Riemannian manifold restricted to the diagonal. The heat kernel can be expanded in powers of the time variable t and the coefficients in the expansion are local scalar invariants of Riemannian metrics which can be constructed as contractions of tensor products of covariant derivatives of the curvature tensor. The Bergman kernel on the diagonal is determined locally by the boundary up to a smooth function, so it was natural to try to find an analogous expansion for φ to order $n+1$ and for ψ to infinite order. But several problems immediately arose in contemplating carrying this out. One was that Ω is not canonically a product near $\partial\Omega$, so there was no obvious analogue of t, nor was there an obvious way to formulate an expansion in such a way that the coefficients would be geometric invariants of the boundary. But even if one could surmount these difficulties, the most glaring problem was the fact that it was not known how to construct general scalar invariants of CR structures, the geometric structures induced on nondegenerate hypersurfaces by the complex structure on the background $\mathbb{C}^n$.

Charlie resolved these difficulties in his groundbreaking paper [F2]. His solution was to construct a Lorentz signature, asymptotically Kähler-Einstein metric $\widetilde{g}$ on $\mathbb{C}^* \times \Omega$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, via a formal solution of a degenerate complex Monge-Ampère equation. This metric $\widetilde{g}$ is invariant under rotations and homogeneous under dilations in $\mathbb{C}^*$ and is invariantly associated to the CR geometry on $\partial\Omega$. The formal solution to the Monge-Ampère equation plays the role of the time variable t, and scalar CR invariants can be constructed using the curvature tensor and Levi-Civita connection of $\widetilde{g}$ in a manner roughly analogous to the case of Riemannian geometry.

The conformal ambient metric, introduced in [FG1] and also denoted $\widetilde{g}$, is an analogue in a different setting: it is determined by the datum of a conformal class $(M,[g])$ of Riemannian metrics on a manifold M of dimension $n \geq 3$. Metrics in the conformal class are sections of a ray subbundle $\mathcal{G}$ of the bundle of symmetric 2-tensors on M, and $\widetilde{g}$ is a Lorentz signature metric on the ambient space $\widetilde{\mathcal{G}} = \mathcal{G} \times \mathbb{R}$ determined asymptotically along $\mathcal{G} \cong \mathcal{G} \times \{0\}$. The model is the sphere S^n, whose group of conformal motions is the

Lorentz group $O(n+1,1)$ of linear transformations of $\mathbb{R}^{n+2}$ preserving a quadratic form of signature $(n+1,1)$. The ray bundle $\mathcal{G}$ can be identified with the forward pointing half of the null cone of the quadratic form. The ambient metric for the sphere is just the Minkowski metric on $\mathbb{R}^{n+2}$, which is clearly preserved by the conformal motions in $O(n+1,1)$ in their linear action on $\mathbb{R}^{n+2}$. For a general conformal class of metrics $(M,[g])$, the ambient metric is a Lorentz signature metric on $\widetilde{\mathcal{G}}$ determined asymptotically along $\mathcal{G}\times\{0\}$ by the following three conditions:

(1) $\widetilde{g}$ is homogeneous of degree two with respect to natural dilations on $\widetilde{\mathcal{G}}$
(2) $\iota^*\widetilde{g} = \mathbf{g_0}$
(3) $\mathrm{Ric}(\widetilde{g})$ vanishes asymptotically at $\mathcal{G}\times\{0\}$.

Here $\mathbf{g_0}$ is a tautological symmetric 2-tensor on $\mathcal{G}$ determined by the conformal class, and $\iota : \mathcal{G}\to\mathcal{G}\times\{0\}\subset\mathcal{G}\times\mathbb{R}$ is the inclusion. In (3), the asymptotic order of vanishing is infinite if n is odd, and is $n/2-1$ if n is even. The ambient metric $\widetilde{g}$ is uniquely determined up to diffeomorphism by these conditions, to infinite order if n is odd, and to order $n/2$ if n is even.

The conformal ambient metric was inspired by Charlie's construction in several complex variables, but the relationship is closer than mere analogy: the Kähler-Lorentz metric in the several complex variables construction can be viewed as a special case of a conformal ambient metric. One takes the conformal class to be the so-called "Fefferman metric" (there are too many of these!) on $\partial\Omega\times S^1$, which Charlie had constructed earlier in [F1]. Although one usually thinks of conformal geometry as simpler than CR geometry, by this construction the class of CR structures induced on non-degenerate boundaries of domains in $\mathbb{C}^n$ can be viewed as a subclass of the class of conformal structures on even-dimensional manifolds.

There is a second, equivalent formulation of ambient metrics; namely, as Poincaré metrics. The model here is hyperbolic space. Recall that S^n can be viewed as the boundary at infinity of hyperbolic space, which arises as the restriction of the Minkowski metric on $\mathbb{R}^{n+2}$ to one sheet of the hyperboloid arising as the -1-level set of the Lorentz signature quadratic form. This construction generalizes to the case of a general conformal manifold as the "conformal infinity"; the ambient metric can be restricted to a natural hypersurface in $\widetilde{\mathcal{G}}$ and the resulting Poincaré metric has asymptotically constant negative Ricci curvature.

These constructions have been enormously influential; the ambient and Poincaré metrics are now viewed as fundamental in conformal geometry and beyond. As described above, one of Charlie's original motivations for the construction in CR geometry was to construct and characterize scalar invariants of CR structures to describe the asymptotic expansion of the Bergman kernel. The ambient metric enables construction of scalar conformal invariants as "Weyl invariants", constructed as linear combinations of complete contractions of covariant derivatives of the curvature tensor of the ambient metric. Determining the extent to which all invariants arise by this construction involved developing a new "parabolic invariant theory"; this was carried out in [F2], [BEG], and [FG3].

The ambient metric opened up new arenas for study in geometric analysis. Conformally invariant powers of the Laplacian were constructed in terms of the ambient metric in [GJMS], leading to Branson's construction of Q-curvature in [B]. Q-curvature is a higher-dimensional version of scalar curvature in dimension 2. Branson's original definition proceeded by analytic continuation in the dimension and Q curvature was originally regarded as rather mysterious. Charlie's work in in [FG2], [FH] helped illuminate its nature. Q curvature enters into the Gauss-Bonnet integrand in higher dimensions (albeit in a more complicated way than the scalar curvature in dimension 2). In dimension 4 and modulo the part which is pointwise conformally invarant, the Gauss-Bonnet integrand has fully non-linear structure under conformal change of metric. The analytic study of such partial differential equations has become a central topic in conformal geometry; see for example [CGY].

The fundamental idea underlying the ambient/Poincaré metric is to study geometry in dimension n by passing to a different but essentially equivalent description in dimension $n+1$ or $n+2$. The AdS/CFT correspondence in physics, a major development since its introduction by Maldacena in 1997, is based on the same idea. In fact, the Poincaré metric construction amounts to the geometry underlying the AdS/CFT duality between conformal field theories on a boundary at infinity and supergravity in the bulk. This synergy between geometry and physics has stimulated both fields, and continues to be a source of exciting developments today.

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