Conformal Geometry on Four Manifolds

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§0. Introduction

This is the lecture notes for the author's Emmy Noether lecture at 2018, ICM, Rio de Janeiro, Brazil. It is a great honor for the author to be invited to give the lecture.

In the lecture notes, the author will survey the development of conformal geometry on four dimensional manifolds. The topic she chooses is one on which she has been involved in the past twenty or more years: the study of the integral conformal invariants on 4-manifolds and geometric applications. The development was heavily influenced by many earlier pioneer works; recent progress in conformal geometry has also been made in many different directions, here we will only present some slices of the development.

The notes is organized as follows.

In section 1, we briefly describe the prescribing Gaussian curvature problem on compact surfaces and the Yamabe problem on n-manifolds for $n \geq 3$; in both cases some second order PDE have played important roles.

In section 2, we introduce the quadratic curvature polynomial σ_2 on compact closed 4-manifolds, which appears as part of the integrand of the Gauss-Bonnet-Chern formula. We discuss its algebraic structure, its connection to the 4-th order Paneitz operator P_4 and its associated 4-th order Q curvature. We also discuss some variational approach to study the curvature and as a geometric application, results to characterize the diffeomorphism type of (S^4, g_c) and $(\mathbb{CP}^2, g_{FS})$ in terms of the size of the conformally invariant quantity: the integral of σ_2 over the manifold.

In section 3, we extend our discussion to compact 4-manifolds with boundary and introduce a third order pseudo-differential operator P_3 and 3-order curvature T on the boundary of the manifolds.

In section 4, we shift our attention to the class of conformally compact Einstein (abbreviated as CCE) four-manifolds. We survey some recent research on the problem of "filling in" a given 3-dimensional manifold as the conformal infinity of a CCE manifold. We relate the concept of "renormalized" volume in this setting again to the integral of σ_2.

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In section 5, we discuss some partial results on a compactness problem on CCE manifolds. We believe the compactness results are the key steps toward an existence theory for CCE manifolds.

The author is fortunate to have many long-term close collaborators, who have greatly contributed to the development of the research described in this article – some more than the author. Among them Matthew Gursky, Jie Qing, Paul Yang and more recently Yuxin Ge. She would like to take the chance to express her deep gratitude toward them, for the fruitful collaborations and for the friendships.

§1. Prescribing Gaussian curvature on compact surfaces and the Yamabe problem

In this section we will describe some second order elliptic equations which have played important roles in conformal geometry.

On a compact surface (M,g) with a Riemannian metric g, a natural curvature invariant associated with the Laplace operator $\Delta = \Delta_g$ is the Gaussian curvature $K = K_g$. Under the conformal change of metric $g_w = e^{2w}g$, we have

$$-\Delta w + K = K_{g_w} e^{2w} \ on\ M. \tag{1.1}$$

The classical uniformization theorem to classify compact closed surfaces can be viewed as finding solution of equation (1.1) with $K_{g_w} \equiv -1$, 0, or 1 according to the sign of $\int K_g dv_g$. Recall that the Gauss-Bonnet theorem states

$$2\pi\, \chi(M) = \int_M K_{g_w}\, dv_{g_w}, \tag{1.2}$$

where $\chi(M)$ is the Euler characteristic of M, a topological invariant. The variational functional with (1.1) as Euler equation for $K_{g_w} =$ constant is thus given by Moser's functional ([72], [73])

$$J_g[w] = \int_M |\nabla w|^2 dv_g + 2\int_M K w dv_g - \left(\int_M K dv_g\right) \log \frac{\int_M dv_{g_w}}{\int_M dv_g}. \tag{1.3}$$

There is another geometric meaning of the functional J which influences the later development of the field, that is the formula of Polyakov [80]

$$J_g[w] = 12\pi \ \log \left(\frac{\det\ (-\Delta)_g}{\det\ (-\Delta)_{g_w}} \right) \tag{1.4}$$

for metrics g_w with the same volume as g; where the determinant of the Laplacian $\det\ \Delta_g$ is defined by Ray-Singer via the "regularized" zeta function.

In [74], (see also Hong [64]), Onofri established the sharp inequality that on the 2-sphere $J[w] \geq 0$ and $J[w] = 0$ precisely for conformal factors w of the form $e^{2w}g_0 = T^* g_0$ where T is a Mobius transformation of the 2-sphere. Later Osgood-Phillips-Sarnak ([75], [76]) arrived at the same sharp inequality in their study of heights of the Laplacian. This inequality also plays an important role in their proof of the C^∞ compactness of isospectral metrics on compact surfaces.

On manifolds (M^n, g) for n greater than two, the conformal Laplacian L_g is defined as $L_g = -\Delta_g + c_n R_g$ where $c_n = \frac{n-2}{4(n-1)}$, and R_g denotes the scalar curvature of the metric g. An analogue of equation (1.1) is the equation, commonly referred to as the Yamabe equation (1.5), which relates the scalar curvature under conformal change of metric to the background metric. In this case, it is convenient to denote the conformal metric as $\hat{g} = u^{\frac{4}{n-2}} g$ for some positive function u, then the equation becomes

$$L_g u \ = c_n \ \hat{R} \ u^{\frac{n+2}{n-2}}. \tag{1.5}$$

The famous Yamabe problem to solve (1.5) with $\hat{R}$ a constant has been settled by Yamabe [87], Trudinger [85], Aubin [4] and Schoen [82]. The corresponding problem to prescribe scalar curvature has been intensively studied in the past decades by different groups of mathematicians, we will not be able to survey all the results here. We will only point out that in this case the study of $\hat{R} = c$, where c is a constant, over class of metrics $\hat{g}$ in the conformal class $[g]$ with the same volume as g, is a variational problem with respect to the functional $F_g[u] = \int_{M^n} R_{\hat{g}} dv_{\hat{g}}$ for any $n \geq 3$. Again the sign of the constant c agrees with the sign of the Yamabe invariant

$$Y(M, g) := \inf_{\hat{g} \in [g]} \frac{\int_M R_{\hat{g}} dv_{\hat{g}}}{(vol\ \hat{g})^{\frac{n-2}{n}}}. \tag{1.6}$$

§2. σ_2 curvature on 4-manifold

§2a definition and structure of σ_2.

We now introduce an integral conformal invariant which plays a crucial role in this paper, namely the integral of σ_2 curvature on four-manifolds.

To do so, we first recall the Gauss-Bonnet-Chern formula on closed compact manifold (M, g) of dimension four:

$$8\pi^2 \chi(M) = \int_M \frac{1}{4} |W_g|^2 dv_g + \int_M \frac{1}{6} (R_g^2 - 3|Ric_g|^2) dv_g, \tag{2.1}$$

where $\chi(M)$ denotes the Euler characteristic of M, W_g denotes the Weyl curvature, R_g the scalar curvature and Ric_g the Ricci curvature of the metric g.

In general, the Weyl curvature measures the obstruction to being conformally flat. More precisely, for a manifold of dimension greater or equals to four, W_g vanishes in a neighborhood of a point if an only if the metric is locally conformal to a Euclidean metric; i.e., there are local coordinates such that $g = e^{2w}|dx|^2$ for some function w. Thus for example, the standard round metric g_c on the sphere S^n has $W_{g_c} \equiv 0$.

In terms of conformal geometry, what is relevant to us is that Weyl curvature is a *pointwise* conformal invariant, in the sense that under conformal change of metric $g_w = e^{2w}g$, $|W_{g_w}| = e^{-2w}|W_g|$, thus on 4-manifold $|W_{g_w}|^2 dv_{g_w} = |W_g|^2 dv_g$; this implies in particular that the first term in the Gauss-Bonnet-Chern formula above

$$g \to \int_M |W|_g^2 dv_g$$

is conformally invariant.

For reason which will be justified later below, we denote

$$\sigma_2(g) = \frac{1}{6}(R_g^2 - 3|Ric_g|^2) \tag{2.2}$$

and draw the conclusion from the above discussion of the Gauss-Bonnet-Chern formula that

$$g \to \int_M \sigma_2(g) dv_g$$

is also an integral conformal invariant. This is the fundamental conformal invariant which will be studied in this lecture notes. We begin by justifying the name of "σ_2" curvature.

On manifolds of dimensions greater than two, the Riemannian curvature tensor Rm can be decomposed into the different components. From the perspective of conformal geometry, a natural basis is the Weyl tensor W, and the Schouten tensor, defined by

$$A_g = Ric_g - \frac{R}{2(n-1)} g.$$

The curvature tensor can be decomposed as

$$Rm_g = W_g \oplus \frac{1}{n-2} A_g \bigcirc\!\!\!\!\wedge\, g.$$

Under conformal change of metrics $g_w = e^{2w}g$, since the Weyl tensor W transforms by scaling, only the Schouten tensor depends on the derivatives of the conformal factor. It is thus natural to consider $\sigma_k(A_g)$, the k-th symmetric function of the eigenvalues of the Schouten tensor A_g, as curvature invariants of the conformal metrics.

When $k = 1$, $\sigma_1(A_g) = Tr_g\ A_g = \frac{n-2}{2(n-1)} R_g$, so the σ_1-curvature is a dimensional multiple of the scalar curvature.

When $k = 2$, $\sigma_2(A_g) = \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}(|Tr_g\ A_g|^2 - |A_g|^2)$, where the λs are the eigenvalues of the tensor A_g. For a manifold of dimension 4, we have

$$\sigma_2(g) = \sigma_2(A_g) = \frac{1}{6}(R_g^2 - 3|Ric_g|^2).$$

When $k = n$, $\sigma_n(A_g) = \textit{determinant of } A_g$, an equation of Monge-Ampère type.

In view of the Yamabe problem, it is natural to ask the question under what condition can one find a metric g_w in the conformal class of g, which solves the equation

$$\sigma_2(A_{g_w}) = \textit{constant}. \tag{2.3}$$

To do so, we first observe that as a differential invariant of the conformal factor w, $\sigma_k(A_{g_w})$ is a fully nonlinear expression involving the Hessian and the gradient of the conformal factor w. We have

$$A_{g_w} = (n-2)\{-\nabla^2 w + dw \otimes dw - \frac{|\nabla w|^2}{2}\} + A_g.$$

To illustrate that (2.3) is a fully non-linear equation, we have when $n = 4$,

$$\begin{aligned} \sigma_2(A_{g_w})e^{4w} &= \sigma_2(A_g) + 2((\Delta w)^2 - |\nabla^2 w|^2 \\ &+ \Delta w|\nabla w|^2 + (\nabla w, \nabla|\nabla w|^2)) \\ &+ \text{lower order terms.} \end{aligned} \tag{2.4}$$

where all derivative are taken with respect to the g metric.

For a symmetric $n \times n$ matrix M, we say $M \in \Gamma_k^+$ in the sense of Gårding ([40]) if $\sigma_k(M) > 0$ and M may be joined to the identity matrix by a path consisting entirely of matrices M_t such that $\sigma_k(M_t) > 0$. There is a rich literature concerning the equation

$$\sigma_k(\nabla^2 u) = f, \tag{2.5}$$

for a positive function f, which is beyond the scope of this article to cover. Here we willl only note that when $k = 2$,

$$\sigma_2(\nabla^2 u) = (\Delta u)^2 - |\nabla^2 u|^2 = f, \tag{2.6}$$

for a positive function f. We remark the leading term of equations (2.6) and (2.4) agree.

We now discuss a variational approach to study the equation (2.3) for σ_2 curvature .

Recall in section 1 we have mentioned that the functional $\mathcal{F}(g) := \int_{M^n} R_g dv_g$ is variational in the sense that when $n \geq 3$ and when one varies g in the same conformal class of metrics with fixed volume, the critical metric when attained satisfies $R_{\hat{g}} \equiv$ constant; while when $n = 2$, this is no longer true with R_g replaced by K_g and one needs to replace the functional $\mathcal{F}(g)$ by the Moser's functional J_g.

Parallel phenomenon happens when one studies the σ_2 curvature. It turns out that when $n > 2$ and $n \neq 4$, the functional $\mathcal{F}_2(g) := \int_{M^n} \sigma_2(g) dv_g$ is variational when one varies g in the same conformal class of metrics with fixed volume, while this is no longer true when $n = 4$. In section 2a below, we will describe a variational approach to study equation (2.3) in dimension 4 and the corresponding Moser's functional. Before we do so, we would like to end the discussion of this section by quoting a result of Gursky-Viaclovsky [57].

In dimension 3, one can capture all metrics with constant sectional curvature (i.e. space forms) through the study of σ_2.

Theorem 2.1. ([57]) On a compact 3-manifold, for any Riemannian metric g, denote $\mathcal{F}_2(g) = \int_M \sigma_2(A_g) dv_g$. Then a metric g with $\mathcal{F}_2(g) \geq 0$ is critical for the functional $\mathcal{F}_2$ restricted to class of metrics with volume one if and only if g has constant sectional curvature.

§2b. 4-th order Paneitz operator, Q-curvature

We now describe the rather surprising link between a 4-th order linear operator P_4, its associated curvature invariant Q_4, and the σ_2-curvature.

We first recall on (M^n, g), $n \geq 3$, the second order conformal Laplacian operator $L = -\Delta + \frac{n-2}{4(n-1)} R$ transforms under the conformal change of metric $\hat{g} = u^{\frac{4}{n-2}} g$, as

$$L_{\hat{g}}(\varphi) = u^{-\frac{n+2}{n-2}} L_g(u\,\varphi) \text{ for } \quad \text{all} \quad \varphi \in C^\infty(M^4). \tag{2.7}$$

There are many operators besides the Laplacian Δ on compact surfaces and the conformal Laplacian L on general compact manifold of dimension greater than two which have the conformal covariance property. One class of such operators of order 4 was studied by Paneitz ([79], see also [36]) defined on (M^n, g) when $n > 2$; which we call the conformal Paneitz operator:

$$P_4^n = \Delta^2 + \delta\left(a_n R g + b_n \mathrm{Ric}\right) d + \frac{n-4}{2} Q_4^n \tag{2.8}$$

and

$$Q_4^n = c_n |Ric|^2 + d_n R^2 - \frac{1}{2(n-1)} \Delta R, \tag{2.9}$$

where a_n, b_n, c_n and d_n are some dimensional constants, δ denotes the divergence, d the deRham differential.

The conformal Paneitz operator is conformally covariant. In this case, we write the conformal metric as $\hat{g} = u^{\frac{4}{n-4}} g$ for some positive function u, then for $n \neq 4$,

$$(P_4^n)_{\hat{g}}(\varphi) \;=\; u^{-\frac{n+4}{n-4}} (P_4^n)_g(u\,\varphi) \quad \text{for all } \varphi \in C^\infty(M^4). \tag{2.10}$$

Properties of (P_4^n, Q_4^n) have been intensively studied in recent years, with many surprisingly strong results. We refer the readers to the recent articles ([15], [54], [51], [60], [61] and beyond).

Notice that when n is not equal to 4, we have $P_4^n(1) = \frac{n-4}{2} Q_4^n$, while when $n = 4$, one does not read Q_4^4 from P_4^4; it was pointed out by T. Branson that nevertheless both $P := P_4^4$ and $Q := Q_4^4$ are well defined (which we named as *Branson's Q-curvature*):

$$P\varphi := \Delta^2\varphi + \delta\left(\frac{2}{3}Rg - 2\mathrm{Ric}\right)d\varphi.$$

The *Paneitz* operator P is conformally covariant of bidegree $(0,4)$ on 4-manifolds, i.e.

$$P_{g_w}(\varphi) = e^{-4\omega}P_g(\varphi) \quad \text{for all} \quad \varphi \in C^\infty(M^4)\,.$$

The Q curvature associated with P is defined as

$$2Q_g \;=\; -\frac{1}{6}\Delta R_g \;+\; \frac{1}{6}(R_g^2 - 3|Ric_g|^2)). \tag{2.11}$$

Thus the relation between Q curvature and σ_2 curvature is

$$2Q_g \;=\; -\frac{1}{6}\Delta R_g + \sigma_2(A_g). \tag{2.12}$$

The relation between P and Q curvature on manifolds of dimension four is like that of the Laplace operator $-\Delta$ and the Gaussian curvature K on compact surfaces.

$$P_g w + 2Q_g \;=\; 2Q_{g_w}e^{4w}. \tag{2.13}$$

Due to the pointwise relationship between Q and σ_2 in (2.12), notice on compact closed 4-manifolds M, $\int_M \Delta R_g dv_g = 0$, thus $\int_M Q_g dv_g = \frac{1}{2}\int \sigma_2(A_g)dv_g$ is also an integral conformal invariant. But Q curvature has the advantage of being prescribed by the linear operator P in the equation (2.13), which is easier to study. We remark the situation is different on compact 4-manifolds with boundary which we will discuss in later part of this article

Following Moser, the functional to study constant Q_{g_w} metric with $g_w \in [g]$ is given by

$$II[w] = \langle Pw, w\rangle + 4\int Qwdv - \left(\int Qdv\right)\log\frac{\int e^{4w}dv}{\int dv}.$$

In view of the relation between Q and σ_2 curvatures, if one consider the variational functional III whose Euler equation is $\Delta R = \mathit{constant}$,

$$III[w] = \frac{1}{3}\left(\int R_{g_w}^2 dv_{g_w} - \int R^2 dv\right),$$

and define

$$\mathcal{F}[w] = II[w] - \frac{1}{12}III[w],$$

we draw the conclusion:

Proposition 2.2. *([27], [19], see also [12] for an alternative approach.)*
$\mathcal{F}$ is the Lagrangian functional for σ_2 curvature.

We remark that the search for the functional $\mathcal{F}$ above was originally motivated by the study of some other variational formulas, e.g. the variation of quotients of log determinant of conformal Laplacian operators under conformal change of metrics on 4-manifolds, analogues to that of the Polyakov formula (1.4) on compact surfaces . We refer the readers to articles ([11], [10], [19], [77]) on this topic.

We also remark that on compact manifold of dimension n, there is a general class of conformal covariant operators P_{2k} of order $2k$, for all integers k with $2k \leq n$. This is the well-known class of GJMS operators [44], where P_2 coincides with the conformal Laplace operator L and where P_4 coincides with the 4-th order Paneitz operator. GJMS operators have played important roles in many recent developments in conformal geometry.

§2c Some properties of the conformal invariant $\int \sigma_2$

We now concentrate on the class of compact, closed four manifolds which allow a Riemannian metric g in the class $\mathcal{A}$, where

$$\mathcal{A} := \left\{ g|, Y(M,g) > 0, \int_M \sigma_2(A_g) dv_g > 0 \right\}.$$

Notice that for closed four manifolds, it follows from equation (2.12) that

$$2 \int_M Q_g dv_g = \int_M \sigma_2(A_g) dv_g,$$

so in the definition of $\mathcal{A}$, we could also use Q instead of σ_2.

We recall some important properties of metrics in A.

Theorem 2.3. (Chang-Gursky-Yang [26], [47], [19])

1. If $Y(M,g) > 0$, then $\int_M \sigma_2(g) dv_g \leq 16\pi^2$, equality holds if and only if (M,g) is conformally equivalent to (S^4, g_c).

2. $g \in \mathcal{A}$, then $P \geq 0$ with kernel(P) consists of constants; it follows there exists some $g_w \in [g]$ with $Q_{g_w} = constant$ and $R_{g_w} > 0$.

3. $g \in \mathcal{A}$, then there exists some $g_w \in [g]$ with $\sigma_2(A_{g_w}) > 0$ and $R_{g_w} > 0$; i. e. g_w exists in the positive two cone Γ_2^+ of A_g in the sense of Gårding [40]. We remark $g \in \Gamma_2^+$ implies $Ric_g > 0$, as a consequence the first betti number b_1 of M is zero.

4. $g \in \mathcal{A}$, then there exists some $g_w \in [g]$ with $\sigma_2(A_{g_w}) = 1$ and $R_{g_w} > 0$.

5. When (M,g) is not conformally equivalent to (S^4, g_c) and $g \in \mathcal{A}$, then for any positive smooth function f defined on M, there exists some $g_w \in [g]$ with $\sigma_2(A_{g_w}) = f$ and $R_{g_w} > 0$.

We remark that techniques for solving the σ_2 curvature equation can be modified to solve the equation $\sigma_2 = 1 + c|W|^2$ for some constant c, which is the equation we will use later in the proof of the theorems in section 2d.

As a consequence of above theorem we have

Corollary 2.1. *On (M^4, g), $g \in \mathcal{A}$ if and only if there exists some $g_w \in [g]$ with $g_w \in \Gamma_2^+$.*

A significant result in recent years is the following "uniqueness" result.

Theorem 2.4. (Gursky-Steets [55])
Suppose (M^4, g) is not conformal to (S^4, g_c) and $g \in \Gamma_2^+$, then $g_w \in [g]$ with $g_w \in \Gamma_2^+$ and with $\sigma_2(A_{g_w}) = 1$ is unique.

The result was established by constructing some norm for metrics in Γ_2^+, with respect to which the functional $\mathcal{F}$ is convex. The result is surprising in contrast with the famous example of R. Schoen [83] where he showed that on $(S^1 \times S^n, g_{prod})$, where $n \geq 2$, the class of constant scalar curvature metrics (with the same volume) is not unique.

§2d Diffeomorphism type

In terms of geometric application, this circle of ideas may be applied to characterize the diffeomorphism type of manifolds in terms of the the relative size of the conformal invariant $\int \sigma_2(A_g) dV_g$ compared with the Euler number of the underlying manifold, or equivalently the relative size of the two integral conformal invariants $\int \sigma_2(A_g) dv_g$ and $\int ||W||_g^2 dv_g$.

Note: In the following, we will view the Weyl tensor as an endomorphism of the space of two-forms: $W : \Omega^2(M) \to \Omega^2(M)$. It will therefore be natural to use the norm associated to this interpretation, which we denote by using $\|\cdot\|$. In particular,

$$|W|^2 = 4\|W\|^2.$$

Theorem 2.5. (Chang-Gursky-Yang [20])
Suppose (M, g) is a closed 4-manifold with $g \in \mathcal{A}$.
(a). If $\int_M ||W||_g^2 dv_g < \int_M \sigma_2(A_g) dv_g$ then M is diffeomorphic to either S^4 or $\mathbb{RP}^4$.
(b). If M is not diffeomorphic to S^4 or $\mathbb{RP}^4$ and $\int_M ||W||_g^2 dv_g = \int_M \sigma_2(A_g) dv_g$, then (M, g) is conformally equivalent to $(\mathbb{CP}^2, g_{FS})$.

Remark 1: The theorem above is an L^2 version of an earlier result of Margerin [70]. The first part of the theorem should also be compared to an result of Hamilton [58]; where he pioneered the method of Ricci flow and established the diffeomorphism of M^4 to the 4-sphere under the assumption when the curvature operator is positive.

Remark 2: The assumption $g \in \mathcal{A}$ excludes out the case when $(M, g) = (S^3 \times S^1, g_{prod})$, where $||W||_g = \sigma_2(A_g) \equiv 0$.

Sketch proof of Theorem 2.5

Proof. For part (a) of the theorem, we apply the existence argument to find a conformal metric g_w which satisfies the pointwise inequality

$$||W_{g_w}||^2 < \sigma_2(A_{g_w}) \quad \textit{or} \quad \sigma_2(A_{g_w}) = ||W_{g_w}||^2 + c \ \text{ for some constant } \ c > 0. \tag{2.14}$$

The diffeomorphism assertion follows from Margerin's [70] precise convergence result for the Ricci flow: such a metric will evolve under the Ricci flow to one with constant curvature. Therefore such a manifold is diffeomorphic to a quotient of the standard 4-sphere.

For part (b) of the theorem, we argue that if such a manifold is not diffeomorphic to the 4-sphere, then the conformal structure realizes the minimum of the quantity $\int |W_g|^2 dv_g$, and hence its Bach tensor vanishes; i.e.

$$B_g = \nabla^k \nabla^l W_{ikjl} + \frac{1}{2} R^{kl} W_{ikjl} = 0.$$

As we assume $g \in \mathcal{A}$, we can solve the equation

$$\sigma_2(A_{g_w}) = (1 - \epsilon)||W_{g_w}||^2 + c_\epsilon, \tag{2.15}$$

where c_ϵ is a constant which tends to zero as ϵ tends to zero. We then let ϵ tends to zero. We obtain in the limit a $C^{1,1}$ metric which satisfies the equation on the open set $\Omega = \{x | W(x) \neq 0\}$:

$$\sigma_2(A_{g_w}) = ||W_{g_w}||^2. \tag{2.16}$$

We then decompose the $Weyl$ curvature of g_w into its self dual and anti-self dual part as in the Singer-Thorpe decomposition of the full curvature tensor, apply the Bach equation to estimate the operator norm of each of these parts as endormorphism on curvature tensors, and reduce the problem to some rather sophisticated Lagrange multiplier problem. We draw the conclusion that the curvature tensor of g_w agrees with that of the Fubini-Study metric on the open set Ω. Therefore $|W_{g_w}|$ is a constant on Ω, thus W cannot vanish at all. From this, we conclude that g_w is Einstein (and under some positive orientation assumption) with $W^-_{g_w} = 0$. It follows from a result of Hitchin (see [5], chapter 13) that the limit metric g_w agrees with the Fubini-Study metric of $\mathbb{CP}^2$ □

We now discuss some of the recent joint work of M. Gursky, Siyi Zhang and myself [21] extending the theorem 2.5 above to a perturbation theorem on $\mathbb{CP}^2$.

For this purpose, for a metric $g \in \mathcal{A}$, we define the conformal invariant constant $\beta = \beta([g])$ defined as

$$\int ||W||^2_g dv_g = \beta \int_M \sigma_2(A_g) dv_g.$$

Lemma 2.1. *Given $g \in \mathcal{A}$, if $1 < \beta < 2$, then M^4 is either homemorphic to either S^4 or $\mathbb{RP}^4$ (hence $b_2^+ = b_2^- = 0$) or M^4 is homeomorphic to $\mathbb{CP}^2$ (hence $b_2^+ = 1$).*

We remark that $\beta = 2$ for the product metric on $S^2 \times S^2$.

An additional ingredient to establish the lemma above is the Signature formula :

$$12\pi^2 \tau = \int_M (||W^+||^2 - ||W^-||^2) dv,$$

where $\tau = b_2^+ - b_2^-$, $||W^+||$ is the self dual part of the Weyl curvature and $||W^-||$ the anti-self-dual part, b_2^+, b_2^- the positive and negative part of the intersection form; together with an earlier result of M. Gursky [46].

In view of the statement of Theorem 2.5 above, it is tempting to ask if one can change the "homeomorphism type" to "diffeomorphism type" in the statement of the Lemma. So far we have not been able to do so, but we have a perturbation result.

Theorem 2.6. (Chang-Gursky-Zhang [21]) There exists some $\epsilon > 0$ such that if (M, g) is a four manifold with $b_2^+ > 0$ and with a metric of positive Yamabe type satisfying with $1 < \beta([g]) < 1 + \epsilon$, then (M, g) is diffeomorphic to standard $\mathbb{CP}^2$.

Sketch proof of Theorem 2.6

Proof. A key ingredient is to apply the condition $b_2^+ > 0$ to choose a good representative metric $g_{GL} \in [g]$, which is constructed in the earlier work of Gursky [48] and used in Gursky-LeBrun ([52], [53]). To do so, they considered a generalized Yamabe curvature

$$\tilde{R}_g = R_g - 2\sqrt{6}||W^+||_g,$$

and noticed that on manifold of dimension 4, due to the conformal invariance of $||W^+||_g$, the corresponding Yamabe type functional

$$g \to \mu_g := \inf_{g_w \in [g]} \frac{\int_M \tilde{R}_{g_w} dv_{g_w}}{(vol g_w)^{\frac{1}{2}}}$$

still attains its infimum; which we denote by g_{GL}. The key observation in [48] is that $b_2^+ > 0$ implies $\mu_g < 0$ (thus $\tilde{R}_{GL} < 0$). To see this, we recall the Bochner formula satisfied by the non-trivial self dual harmonic 2-form ϕ at the extreme metric:

$$\frac{1}{2}\Delta(|\phi|^2) = |\nabla\phi|^2 - 2W^+ < \phi, \phi > + \frac{1}{3}R|\phi|^2,$$

which together with the algebraic inequality that

$$-2W^+ < \phi, \phi > + \frac{1}{3}R|\phi|^2 \geq \frac{1}{3}\tilde{R}|\phi|^2,$$

forces the sign of $\tilde{R}_{GL}$ when ϕ is non-trivial.

To continue the proof of the theorem, we notice that for a given metric g satisfying the conformal pinching condition $1 < \beta([g]) < 1 + \epsilon$ on its curvature, the corresponding g_{GL} would satisfy $G_2(g_{GL}) \le C(\epsilon)$, where for $k = 2, 3, 4$,

$$G_k(g) := \int_M \left((R - \bar{R})^k + |Ric^0|^k + ||W^-||^k + |\tilde{R}_-|^k \right) dv_g,$$

where $\bar{R}$ denotes the average of the scalar curvature R over the manifold, Ric^0 the denote the traceless part of the Ricci curvature and $\tilde{R}_-$ the negative part of $\tilde{R}$, and where $C(\epsilon)$ is a constant which tends to zero as ϵ tends to zero.

We now finish the proof of the theorem by a contradiction argument and by applying the Ricci flow method of Hamilton

$$\frac{\partial}{\partial t} g(t) = -2Ric_{g(t)}$$

to regularize the metric g_{GL}.

Suppose the statement of the theorem is not true, let $\{g_i\}$ be a sequence of metrics satisfying $1 < \beta([g_i]) < 1 + \epsilon_i$ with ϵ_i tends to zero as i tends to infinity. Choose $g_i(0) = (g_i)_{GL}$ to start the Ricci flow, we can derive the inequality

$$\frac{\partial}{\partial t} G_2(g(t)) \le aG_2(g_t) - bG_4(g(t))$$

for some positive constants a and b, also at some fixed time t_0 independent of ϵ and i for each g_i. We then apply the regularity theory of parabolic PDE to derive that some sub-sequences of $\{g_i(t_0)\}$ converges to the g_{FS} metric of $\mathbb{CP}^2$, which in turn implies the original subsequence $(M, \{g_i(0)\})$ hence a subsequence of $(M, \{g_i\})$ is diffeomorphic to $(\mathbb{CP}^2, g_{FS})$. We thus reach a contradication to our assumption. The reader is referred to the preprint [21] for details of the proof. $\square$

We end this section by pointing out there is a large class of manifolds with metrics in the class $\mathcal{A}$. By the work of Donaldson-Freedman (see [34], [41]) and Lichnerowicz vanishing theorem, the homeomorphism type of the class of simply-connected 4-manifolds which allow a metric with positive scalar curvature consists of S^4 together with $k\mathbb{CP}^2 \# l\overline{\mathbb{CP}^2}$ and $k(S^2 \times S^2)$. Apply some basic algebraic manipulations with the Gauss-Bonnet-Chern formula and the Signature formula, we can show that a manifold which admits $g \in \mathcal{A}$ satisfies $4 + 5l > k$. The round metric on S^4, the Fubini-Study metric on $\mathbb{CP}^2$, and the product metric on $S^2 \times S^2$ are clearly in the class $\mathcal{A}$. When $l = 0$, which implies $k < 4$, the class $\mathcal{A}$ also includes the metrics constructed by Lebrun-Nayatani-Nitta [66] on $k\mathbb{CP}^2$ for $k \le 2$. When $l = 1$, which implies $k < 9$, the class $\mathcal{A}$ also includes the (positive) Einstein metric constructed by D. Page [78] on $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, the (positive) Einstein metric by Chen-Lebrun-Weber [32] on $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$, and the Kähler Einstein metrics on $\mathbb{CP}^2 \# l\overline{\mathbb{CP}^2}$ for $3 \le l \le 8$ as in the work of Tian [84]. It would be an ambitious program to locate the entire class of 4-manifolds with metric in $\mathcal{A}$, and to classify their diffeomorphism types by the (relative) size of the integral conformal invariants discussed in this lecture.

§3. Compact 4-manifold with boundary, (Q, T) curvatures

To further develop the analysis of the Q-curvature equation, it is helpful to consider the associated boundary value problems. In the case of compact surface with boundary (X^2, M^1, g), where the metric g is defined on $X^2 \cup M^1$; the Gauss-Bonnet formula becomes

$$2\pi\chi(X) = \int_X K \; dv + \oint_M k \; d\sigma, \tag{3.1}$$

where k is the geodesic curvature on M. Under conformal change of metric g_w on X, the geodesic curvature changes according to the equation

$$\frac{\partial}{\partial n} w + k \;=\; k_{g_w} \, e^w \;\text{ on M.} \tag{3.2}$$

One can generalize above results to compact four manifold with boundary (X^4, M^3, g); with the role played by $(-\Delta, \frac{\partial}{\partial n})$ replaced by (P_4, P_3) and with (K, k) curvature replaced by (Q, T) curvatures; where P_4 is the Paneitz operator and Q the curvature discussed in section 2; and where P_3 is some 3rd order boundary operator constructed in Chang-Qing ([22], [23]). The key property of P_3 is that it is conformally covariant of bidegree $(0, 3)$, i.e.

$$(P_3)_{g_w} = e^{-3w} (P_3)_g$$

when operating on functions defined on the boundary of compact 4-manifolds; and under conformal change of metric $g_w = e^{2w} g$ on X^4 we have at the boundary M^3

$$P_3 w + T \;=\; T_{g_w} \, e^{3w}. \tag{3.3}$$

The precise formula of P_3 is rather complicated (see [22]). Here we will only mention that on $(B^4, S^3, |dx|^2)$, where B^4 is the unit ball in $\mathbb{R}^4$, we have

$$P_4 = (-\Delta)^2, \;\; P_3 = -\left(\frac{1}{2} \; \frac{\partial}{\partial n} \; \Delta + \tilde{\Delta} \frac{\partial}{\partial n} + \tilde{\Delta} \right) \quad \text{and} \;\; T = 2, \tag{3.4}$$

where $\tilde{\Delta}$ the intrinsic boundary Laplacian on S^3. In general the formula for T curvature is also lengthy,

$$T = \frac{1}{12} \frac{\partial}{\partial n} R + \frac{1}{6} RH - R_{\alpha n \beta n} L_{\alpha\beta} + \frac{1}{9} H^3 - \frac{1}{3} \mathrm{Tr} L^3 - \frac{1}{3} \tilde{\Delta} H,$$

where L is the second fundamental form of M in (X, g), and H the mean curvature, and n its the outside normal. In terms of these curvatures, the Gauss-Bonnet-Chern formula can be expressed as:

$$8\pi^2 \chi(X) = \int_X (||W||^2 \; + \; 2Q) \; dv \;\; + \oint_M (\mathcal{L} + 2T) \; d\sigma. \tag{3.5}$$

where $\mathcal{L}$ is a third order boundary curvature invariant that transforms by scaling under conformal change of metric, i.e. $\mathcal{L}d\sigma$ is a pointwise conformal invariant.

The property which is relevant to us is that

$$\int_X Q dv + \oint_M T d\sigma$$

is an integral conformal invariant.

It turns out for the cases which are of interest to us later in this paper, (X, g) is with totally geodesic boundary, that is, its second fundamental form vanishes. In this special case we have

$$T = \frac{1}{12}\frac{\partial}{\partial n} R. \tag{3.6}$$

Thus in view of the definitions (2.13) and (3.6) of Q and T, in this case we have

$$2(\int_X Q dv + \oint_M T d\sigma) = \int_X \sigma_2 \, dv,$$

which is the key property we will apply later to study the renormalized volume and the compactness problem of conformal compact Einstein manifolds in sections 4 and 5 below.

§4. Conformally compact Einstein manifolds

§4a. Definition and basics, some short survey

Given a manifold $(M^n, [h])$, when is it the boundary of a conformally compact Einstein manifold (X^{n+1}, g^+) with $r^2 g^+|_M = h$? This problem of finding "conformal filling in" is motivated by problems in the AdS/CFT correspondence in quantum gravity (proposed by Maldacena [69] in 1998) and from the geometric considerations to study the structure of non-compact asymptotically hyperbolic Einstein manifolds.

Here we will only briefly outline some of the progress made in this problem pertaining to the conformal invariants we are studying.

Suppose that X^{n+1} is a smooth manifold of dimension $n+1$ with smooth boundary $\partial X = M^n$. A defining function for the boundary M^n in X^{n+1} is a smooth function r on $\bar{X}^{n+1}$ such that

$$\begin{cases} r > 0 & \text{in } X; \\ r = 0 & \text{on } M; \\ dr \neq 0 & \text{on } M. \end{cases}$$

A Riemannian metric g^+ on X^{n+1} is conformally compact if $(\bar{X}^{n+1},\ r^2 g^+)$ is a compact Riemannian manifold with boundary M^n for some defining function r. We denote $h := r^2 g^+|_M$.

Conformally compact manifold (X^{n+1}, g^+) carries a well-defined conformal structure on the boundary $(M^n, [h])$ by choices of different defining functions r. We shall call $(M^n, [h])$ the conformal infinity of the conformally compact manifold (X^{n+1}, g^+).

If (X^{n+1}, g^+) is a conformally compact manifold and $\mathrm{Ric}[g^+] = -n\, g^+$, then we call (X^{n+1}, g^+) a conformally compact (Poincare) Einstein (abbreviated as CCE) manifold. We remark that on a CCE manifold X, for any given smooth metric h in the conformal infinity M, there exists a special defining function r (called the geodesic defining function) so that $r^2 g^+|_M = h,$ and $|dr|^2_{r^2 g^+} = 1$ in a neighborhood of the boundary $[0, \epsilon) \times M$, also the metric $r^2 g^+$ has totally geodesic boundary.

Some basic examples

Example 1: On $(B^{n+1}, S^n, g_{\mathbb{H}})$

$$\left(B^{n+1}, \left(\frac{2}{1 - |y|^2} \right)^2 |dy|^2 \right).$$

We can then view $(S^n, [g_c])$ as the compactification of B^{n+1} using the defining function

$$r = 2\frac{1 - |y|}{1 + |y|}$$

$$g_{\mathbb{H}} = g^+ = r^{-2} \left(dr^2 + \left(1 - \frac{r^2}{4} \right)^2 g_c \right).$$

Example 2: AdS-Schwarzchild space

$$On \ \ (R^2 \times S^2, g_m^+),$$

where

$$g_m^+ = V dt^2 + V^{-1} dr^2 + r^2 g_c,$$
$$V = 1 + r^2 - \frac{2m}{r},$$

m is any positive number, $r \in [r_h, +\infty)$, $t \in S^1(\lambda)$ and g_c the surface measure on S^2 and r_h is the positive root for $1 + r^2 - \frac{2m}{r} = 0$. We remark, it turns out that in this case, there are two different values of m so that both g_m^+ are conformal compact Einstein filling for the same boundary metric $S^1(\lambda) \times S^2$. This is the famous non-unique "filling in" example of Hawking-Page [62].

Existence and non-existence results

The most important existence result is the "Ambient Metric" construction by Fefferman-Graham ([37],[39]). As a consequence of their construction, for any given compact manifold (M^n, h) with an analytic metric h, some CCE metric exists on some tubular neighborhood $M^n \times (0, \epsilon)$ of M. This later result was recently extended to manifolds M with smooth metrics by Gursky-Székelyhidi [56].

A perturbation result of Graham-Lee [43] asserts that in a smooth neighborhood of the standard surface measure g_c on S^n, there exist a conformal compact Einstein metric on B^{n+1} with any given conformal infinity h.

There is some recent important articles by Gursky-Han and Gursky-Han-Stolz ([49], [50]), where they showed that when X is spin and of dimension $4k \geq 8$, and the Yamabe invariant $Y(M,[h]) > 0$, then there are topological obstructions to the existence of a Poincaré-Einstein g^+ defined in the interior of X with conformal infinity given by $[h]$. One application of their work is that on the round sphere S^{4k-1} with $k \geq 2$, there are infinitely many conformal classes that have no Poincaré-Einstein filling in in the ball of dimension $4k$.

The result of Gursky-Han and Gursky-Han-Stolz was based on a key fact pointed out J. Qing [81], which relies on some earlier work of J. Lee [67].

Lemma 4.1. *On a CCE manifold (X^{n+1}, M^n, g^+), assuming $Y(M,[h]) > 0$, there exists a compactification of g^+ with positive scalar curvature; hence $Y(X,[r^2 g^+]) > 0$.*

Uniqueness and non-uniqueness results

Under the assumption of positive mass theorem, J. Qing [81] has established (B^{n+1}, g_H) as the unique CCE manifold with $(S^n, [g_c])$ as its conformal infinity. The proof of this result was later refined and established without using positive mass theorem by Li-Qing-Shi [68] (see also Dutta and Javaheri [33]). Later in section 5 of this lecture notes, we will also prove the uniqueness of the CCE extension of the metrics constructed by Graham-Lee [43] for the special dimension $n = 3$.

As we have mentioned in the example 2 above, when the conformal infinity is $S^1(\lambda) \times S^2$ with product metric, Hawking-Page [62] have constructed non-unique CCE fill-ins.

4b. Renormalized volume

We will now discuss the concept of "renormalized volume" in the CCE setting, introduced by Maldacena [69] (see also the works of Witten [86], Henningson-Skenderis [63] and Graham [42]). On CCE manifolds (X^{n+1}, M^n, g^+) with geodesic defining function r,
For n even,

$$\begin{aligned}\mathrm{Vol}_{g^+}(\{r > \epsilon\}) &= c_0 \epsilon^{-n} + c_2 \epsilon^{-n+2} + \cdots \\ &\quad + c_{n-2}\epsilon^{-2} + L \log \frac{1}{\epsilon} + V + o(1).\end{aligned}$$

For n odd,

$$\begin{aligned}\mathrm{Vol}_{g^+}(\{r > \epsilon\}) &= c_0 \epsilon^{-n} + c_2 \epsilon^{-n+2} + \cdots\cdots \\ &\quad + c_{n-1}\epsilon^{-1} + V + o(1).\end{aligned}$$

We call the zero order term V the renormalized volume. It turns out for n even, L is independent of $h \in [h]$ where $h = r^2 g^+|_M$, and for n odd, V is independent of $g \in [g]$, and hence are conformal invariants.

We recall

Theorem 4.1. *(Graham-Zworski [45], Fefferman-Graham [38])*
When n is even,

$$L = c_n \oint_M Q_h dv_h,$$

where c_n is some dimensional constant.

Theorem 4.2. *(M. Anderson [1], Chang-Qing-Yang [24], [25])*
On conformal compact Einstein manifold (X^4, M^3, g^+), we have

$$V = \frac{1}{6}\int_{X^4} \sigma_2(A_g) dv_g$$

for any compactified metric g with totally geodesic boundary. Thus

$$8\pi^2\chi(X^4, M^3) = \int ||W||_g^2 dv_g + 6V.$$

Remark: There is a generalization of Theorem 4.2 above for any n odd, with X^4 replaced by X^n and with $\int_{X^4}\sigma_2$ replaced by some other suitable integral conformal invariants $\int_{X^{n+1}} v^{n+1}$ on any CCE manifold (X^{n+1}, M^n, g^+); see ([18], also [24]).

Sketch proof of Theorem 4.2 for $n = 3$

Lemma 4.2. *(Fefferman-Graham [38])*
Suppose (X^4, M^3, g^+) is conformally compact Einstein with conformal infinity $(M^3, [h])$, fix $h \in [h]$ and r its corresponding geodesic defining function. Consider

$$-\Delta_{g^+} w \;=\; 3 \;\; on \;\; X^4, \tag{4.1}$$

then w has the asymptotic behavior

$$w = log\, r + A + Br^3$$

near M, where A, B are functions even in r, $A|_M = 0$, and

$$V = \int_M B|_M.$$

Lemma 4.3. *With the same notation as in Lemma 4.2, consider the metric $g^* = g_w = e^{2w}g^+$, then g^* is totally geodesic on boundary with (1) $Q_{g^*} \equiv 0$, (2) $B|_M = \frac{1}{36}\frac{\partial}{\partial n}R_{g^*} = \frac{1}{3}T_{g^*}$.*

Proof of Lemma 4.3:

Proof. Recall we have g^+ is Einstein with $Ric_{g^+} = -3g^+$, thus

$$P_{g^+} = (-\Delta_{g^+}) \circ (-\Delta_{g^+} - 2)$$

and $2Q_{g^+} = 6$. Therefore

$$P_{g^+}w + 2Q_{g^+} = 0 = 2e^{2w}Q_{g^*}.$$

Assertion (2) follows from a straight forward computation using the scalar curvature equation and the asymptotic behavior of w. $\square$

Applying Lemmas 4.2 and 4.3, we get

$$\begin{aligned}6V &= 6\oint_{M^3} B|_M d\sigma_h = \frac{1}{6}\oint_{M^3}\frac{\partial}{\partial n}R_{g^*}d\sigma_h \\ &= 2(\int_X Q_{g^*} + \oint_M T_{g^*}) = \int_{X^4}\sigma_2(A_{g^*})dv_{g^*}.\end{aligned}$$

For any other compactified metric g with totally geodesic boundary, $\int_{X^4}\sigma_2(g)dv_g$ is a conformal invariant, and V is a conformal invariant, thus the result holds once for g^*, holds for any such g in the same conformal class, which establishes Theorem 4.2.

§5. Compactness of conformally compact Einstein manifolds on dimension 4

In this section, we will report on some joint works of Yuxin Ge and myself [16] and also Yuxin Ge, Jie Qing and myself [17].

The project we work on is to address the problem of given a sequence of CCE manifolds $(X^4, M^3, \{(g_i)^+\})$ with $M = \partial X$ and $\{g_i\} = \{r_i^2(g_i)^+\}$ a sequence of compactified metrics, denote $h_i = g_i|_M$, assume $\{h_i\}$ forms a compact family of metrics in M, is it true that some representatives $\bar{g}_i \in [g_i]$ with $\{\bar{g}_i|_M\} = \{h_i\}$ also forms a compact family of metrics in X? Let me mention the eventual goal of the study of the compactness problem is to show existence of conformal filling in for some classes of Riemannian manifolds. A plausible candidate for the problem to have a positive answer is the class of metrics (S^3, h) with the scalar curvature of h being positive. In this case by a result of Marques [71], the set of such metrics is path-connected, the non-existence argument of Gursky-Han, and Gursky-Han-Stolz ([49], [50]) also does not apply. One hopes that our compactness argument would lead via either the continuity method or degree theory to the existence of conformal filling in for this class of metrics. We remark some related program for the problem has been outlined in ([2], [3]).

The first observation is one of the difficulty of the problem is existence of some "non-local" term. To see this, we have the asymptotic behavior of the compactified metric g of CCE manifold (X^{n+1}, M^n, g^+) with conformal infinity (M^n, h) ([42], [39]) which in the special case when $n = 3$ takes the form

$$g := r^2 g^+ = h + g^{(2)}r^2 + g^{(3)}r^3 + g^{(4)}r^4 + \cdots$$

on an asymptotic neighborhood of $M \times (0, \epsilon)$, where r denotes the geodesic defining function of g. It turns out $g^{(2)} = -\frac{1}{2}A_h$ and is determined by h (we call such terms local terms), $Tr_h g^{(3)} = 0$, while

$$g^{(3)}_{\alpha,\beta} = -\frac{1}{3}\frac{\partial}{\partial n}(Ric_g)_{\alpha,\beta}$$

where α, β denote the tangential coordinate on M, is a non-local term which is not determined by the boundary metric h. We remark that h together with $g^{(3)}$ determine the asymptotic behavior of g ([39], [6]).

We now observe that different choices of the defining function r give rise to different conformal metric of $\hat{h}$ in $[h]$ on M. For convenience, In the rest of this article, we choose the representative $\hat{h} = h^Y$ be the Yamabe metric with constant scalar curvature in $[h]$ and denote it by h and its corresponding geodesic defining function by r. Similarly one might ask what is a "good' representative of $\hat{g} \in [g]$ on X? Our first attempt is to choose $\hat{g} := g^Y$, a Yamabe metric in $[g]$. The difficulty of this choice is that it is not clear how to control the boundary behavior of $g^Y|_M$ in terms of h^Y.

We also remark that in seeking the right conditions for the compactness problem, due to the nature of the problem, the natural conditions imposed should be conformally invariant.

In the statement of the results below, for a CCE manifold (X^4, M^3, g^+), and a conformal infinity $(M, [h])$ with the representative $h = h^Y \in [h]$, we solve the PDE

$$-\Delta_{g^+} w = 3 \tag{5.1}$$

and denote $g^* = e^{2w} g$ be the "Fefferman-Graham" compactification metric with $g^*|_M = h$.

We recall that $Q_{g^*} \equiv 0$, hence the renormalized volume of (X, M, g^+) is a multiple of

$$\int_X \sigma_2(A_{g*}) dv_{g^*} = 2 \oint_M T_{g*} d\sigma_h = \frac{1}{6} \oint_M \frac{\partial}{\partial n} R_{g^*} d\sigma_h.$$

Before we state our results, we recall formulas for the specific g^* metric in a model case.

Lemma 5.1. *On $(B^4, S^3, g_{\mathbb{H}})$,*

$$g^* = e^{(1-|x|^2)} |dx|^2 \ \ on \ \ B^4$$

$$Q_{g^*} \equiv 0, \quad T_{g^*} \equiv 2 \ \ on \ \ S^3$$

$$(g^*)^{(3)} \equiv 0$$

and

$$\int_{B^4} \sigma_2(A_{g*}) dv_{g^*} = 8\,\pi^2.$$

We will first state a perturbation result for the compactness problem.

Theorem 5.1. *Let $\{(B^4, S^3, \{g_i{}^+\})\}$ be a family of oriented CCE on B^4 with boundary S^3. We assume the boundary Yamabe metric h_i in conformal infinity M is of non-negative type. Let $\{g_i^*\}$ be the corresponding FG compactification. Assume*

1. *The boundary Yamabe metrics $\{h_i\}$ form a compact family in C^{k+3} norm with $k \geq 2$; and there exists some positive constant $c_1 > 0$ such that the Yamabe constant for the conformal infinity $[h_i]$ is bounded uniformly from below by c_1, that is,*

$$Y(M, [h_i]) \geq c_1;$$

2. *There exists some small positive constant $\varepsilon > 0$ such that for all i*

$$\int_{B^4} \sigma_2(A_{g_i^*}) dv_{g_i^*} \ \geq \ 8\pi^2 - \varepsilon. \tag{5.2}$$

Then the family of the g_i^* is compact in $C^{k+2,\alpha}$ norm for any $\alpha \in (0,1)$ up to a diffeomorphism fixing the boundary.

Before we sketch the proof of the theorem, we will first mention that on (B^4, S^3, g), for a compact metric g with totally geodesic boundary, Gauss-Bonnet-Chern formula takes the form:

$$8\pi^2\chi(B^4, S^3) = 8\pi^2 = \int_{B^4} (||W||_g^2 + \sigma_2(A_g))dv_g,$$

which together with the conformal invariance of the L^2 norm of the Weyl tensor, imply that the condition (5.2) in the statement of the Theorem 5.1 is equivalent to

$$\int_{B^4} ||W||^2_{g_i^+} dv_{g_i^+} \leq \varepsilon. \tag{5.3}$$

What is less obvious is that in this setting, we also have other equivalence conditions as stated in Corollary (5.2) below. This is mainly due to following result by Li-Qing-Shi ([68]).

Proposition 5.1. *Assume that (X^{n+1}, g^+) is a CCE manifold with C^3 regularity whose conformal infinity is of positive Yamabe type. Let $p \in X$ be a fixed point and $t > 0$. Then*

$$\left(\frac{Y(\partial X, [h])}{Y(\mathbb{S}^n, [g_c])}\right)^{\frac{n}{2}} \leq \frac{Vol(\partial B_{g^+}(p,t))}{Vol(\partial B_{g_{\mathbb{H}}}(p,t))} \leq \frac{Vol(B_{g^+}(p,t))}{Vol(B_{g_{\mathbb{H}}}(p,t))} \leq 1$$

where $B_{g^+}(p,t)$ and $B_{g_{\mathbb{H}}}(p,t)$ are geodesic balls.

Corollary 5.2. *Let $\{X = B^4, M = \partial X = S^3, g^+\}$ be a 4-dimensional oriented CCE on X with boundary ∂X. Assume the boundary Yamabe metric $h = h^Y$ in the conformal infinity of positive type and $Y(S^3, [h]) > c_1$ for some fixed $c_1 > 0$ and h is bounded in C^{k+3} norm with $k \geq 5$. Let g^* be the corresponding FG compactification. Then the following properties are equivalent:*

1. *There exists some small positive number $\varepsilon > 0$ such that*

$$\int_X \sigma(A_{g*})dv_{g^*} \geq 8\pi^2 - \varepsilon. \tag{5.4}$$

2. *There exists some small positive number $\varepsilon > 0$ such that*

$$\int_X ||W||^2_{g^+} dv_{g^+} \leq \varepsilon.$$

3. *There exists some small positive number $\varepsilon_1 > 0$ such that*

$$Y(S^3, [g_c]) \geq Y(S^3, [h]) > Y(S^3, [g_c]) - \varepsilon_1$$

where g_c is the standard metric on S^3.

4. *There exists some small positive number $\varepsilon_2 > 0$ such that for all metrics g^* with boundary metric h same volume as the standard metric g_c on S^3, we have*

$$T(g^*) \geq 2 - \varepsilon_2.$$

5. *There exists some small positive number $\varepsilon_3 > 0$ such that*

$$|(g^*)^{(3)}| \leq \varepsilon_3.$$

Where all the ε_i (i = 1,2,3) tends to zero when ε tends to zero and vice versa for each i.

Another consequence of Theorem 5.1 is the "uniqueness" of the Graham-Lee metrics mentioned in section 4a.

Corollary 5.3. *There exists some $\varepsilon > 0$, such that for all metrics h on S^3 with $||h - g_c||_{C^\infty} < \varepsilon$, there exists a unique CCE filling in (B^4, S^3, g^+) of h.*

Sketch proof of Theorem 5.1

We refer the readers to the articles Chang-Ge [16] and Chang-Ge-Qing [17], both will soon be posted on arXiv for details of the arguments, here we will present a brief outline.

We first state a lemma summarizing some analytic properties of the metrics g^*.

Lemma 5.2. *On a CCE manifold (X^4, M^3, g^+), where the scalar curvature of the conformal infinity (M, h) is positive. Assume h is at least C^l smooth for $l \geq 3$. Denote $g^* = e^{2w} g^+$ the FG compactification. Then*
(1) $Q(g^*) \equiv 0$,
(2) $R_{g^*} > 0$, which implies in particular $|\nabla_{g^*} w| e^w \leq 1$.
(3) g^* is Bach flat and satisfies an ϵ-regularity property, which implies in particular, once it is C^3 smooth, it is C^l smooth for $l \geq 3$.

We remark that statement (2) in the Lemma above follows from a continuity argument via some theory of scattering matrix (see Case-Chang [13], [14]), with the starting point of the argument the positive scalar curvature metric constructed by J. Lee which we have mentioned earlier in Lemma 4.1.

Sketch proof of Theorem 5.1

Proof of the theorem is built on contradiction arguments. We first note, assuming the conclusion of the theorem does not hold, then there is a sequence of $\{g_i^*\}$ which is not compact so that the L^2 norm of its Weyl tensor of the sequence tends to zero.

Our main assertion of the proof is that the C^1 norm of the curvature of the family $\{{g_i}^*\}$ remains uniformly bounded.

Assume the assertion is not true, we rescale the metric $\bar{g}_i = K_i^2 g_i{}^*$ where there exists some point $p_i \in X$ such that

$$K_i^2 = \max\{\sup_{B^4} |Rm_{g_i{}^*}|, \sqrt{\sup_{B^4} |\nabla Rm_{g_i{}^*}|}\} = |Rm_{g_i{}^*}|(p_i)(\text{ or } \sqrt{|\nabla Rm_{g_i{}^*}|(p_i)})$$

We mark the accumulation point p_i as $0 \in B^4$. Thus, we have

$$|Rm_{\bar{g}_i}|(0) = 1 \text{ or } |\nabla Rm_{\bar{g}_i}|(0) = 1 \tag{5.5}$$

We denote the corresponding defining function $\bar{w}_i$ so that $\bar{g}_i = e^{2\bar{w}_i} g_i^+$; that is $e^{2\bar{w}_i} = K_i^2 e^{2w_i}$ and denote $\bar{h}_i := \bar{g}_i|_{S^3}$. We remark that the metrics $\bar{g}_i$ also satisfy the conditions in Lemma 5.2.

As $0 \in \overline{B^4}$ is an accumulation point, depending on the location of 0, we call it either an interior or a boundary blow up point; in each case, we need a separate argument but for simplicity here we will assume we have $0 \in S^3$ is a boundary blow up point. In this case, we denote the (X_∞, g_∞) the Gromov-Hausdorff limit of the sequence $(B^4, \bar{g}_i)$, and $h_\infty := g_\infty|_{S^3}$.

Our first observation is that it follows from the assumption (1) in the statement of the theorem, we have $(\partial X_\infty, h_\infty) = (\mathbb{R}^3, |dx|^2)$.

Our second assertion is that by the estimates in Lemma 5.2, one can show $\bar{w}_i$ converges uniformly on compacta on X_∞, we call the limiting function $\bar{w}_\infty$. Hence, the corresponding metric g_∞^+ with $g_\infty = e^{2\bar{w}_\infty} g_\infty^+$ exists, satisfying $Ric_{g_\infty^+} = -3g_\infty^+$, i.e., the resulting g_∞ is again conformal to a Poincare Einstein metric g_∞^+ with $||W_{g_\infty^+}|| \equiv 0$.

Our third assertion is that (X_∞, g_∞^+) is (up to an isometry) the model space $(\mathbb{R}_+^4,\ g_{\mathbb{H}} := \frac{|dx|^2+|dy|^2}{y^2})$, where $\mathbb{R}_+^4 = \{(x,y) \in \mathbb{R}^4 | y > 0\}$. We can then apply a Liouville type PDE argument to conclude $\bar{w}_\infty = \log y$.

Thus g_∞ is in fact the flat metric $|dx|^2 + |dy|^2$, which contradicts the marking property (5.5). This contradiction establishes the main assertion.

Once the C^1 norm of the curvature of the metric $\{g_i^*\}$ is bounded, we can apply some further blow-up argument to show the diameter of the sequence of metrics is uniformly bounded, and apply a version of the Gromov-Hausdorff compactness result (see [30]) for compact manifolds with totally geodesic boundary to prove that $\{g_i^*\}$ forms a compact family in a suitable C^l norm for some $l \geq 3$. This finishes the proof of the theorem.

We end this discussion by mentioning that, from Theorem 5.1, an obvious question to ask is if we can extend the perturbation result by improving condition (5.2) to the condition $\int_{B^4} \sigma_2 > 0$. This is a direction we are working on but have not yet be able to accomplish. Below are statements of two more general theorems that we have obtained in ([16], [17]).

Theorem 5.2. *Under the assumption (1) as in Theorem 5.1, assume further the T curvature on the boundary $T_i = \frac{1}{12}\frac{\partial}{\partial n} R_{{g_i}^*}$ satisfies the following condition*

$$\liminf_{r \to 0} \inf_i \inf_{S^3} \oint_{\partial B(x,r)} T_i \geq 0. \tag{5.6}$$

Then the family of metrics $\{g_i^*\}$ is compact in $C^{k+2,\alpha}$ norm for any $\alpha \in (0,1)$ up to a diffeomorphism fixing the boundary, provided $k \geq 5$.

Theorem 5.3. *Under the assumption (1) as in Theorem 5.1, assume further that there is no concentration of $S_i := (g_i^*)^{(3)}$-tensor defined on S^3 in L^1 norm for the g_i^* metric in the following sense,*

$$\lim_{r \to 0} \sup_i \sup_x \oint_{\partial B(x,r)} |S_i| = 0. \tag{5.7}$$

Then, the family of the metrics $\{g_i^*\}$ is compact in $C^{k+2,\alpha}$ norm for any $\alpha \in (0,1)$ up to a diffeomorphism fixing the boundary, provided $k \geq 2$.

The reason we can pass the information from the T curvature in Theorem 5.2 to the S tensor in Theorem 5.3 is due to the fact that for the blow-up limiting metric g_∞, $T_{g_\infty} \equiv 0$ if and only if $S_{g_\infty} \equiv 0$.

It remains to show if there is a connection between condition (5.6) in Theorem 5.2 to the positivity of the renormalized volume, i.e. when $\int_X \sigma_2(A_g) dv_g > 0$.

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