Some remarks on the symmetry and embedding of locally conformally flat metrics

Sun-Yung A. Chang [*]
Department of Mathematics
Fine Hall
Princeton University
Princeton, NJ 08540
chang@math.princeton.edu

Zheng-Chao Han [†]
Department of Mathematics
Rutgers University
110 Frelinghuysen Road
Piscataway, NJ 08854
zchan@math.rutgers.edu

Paul Yang [‡]
Department of Mathematics
Fine Hall
Princeton University
Princeton, NJ 08540
yang@math.princeton.edu

1 Introduction

An important question in conformal geometry is to understand under what conditions the development map of a locally conformally flat manifold into the standard sphere is an embedding. Another related question is to understand conditions on a domain $\Omega \subset \mathbb{S}^n$ under which a complete conformal metric exists on Ω with constant scalar curvature; also relevant is the uniqueness of such a metric, or possibly cataloging such metrics when uniqueness fails and Ω is some canonical domain in $\mathbb{S}^n$ such as $\mathbb{S}^n \setminus \mathbb{S}^l$ for some $0 \leq l < n$.

In [SY88] Schoen and Yau found some sufficient conditions for the development map of a locally conformally flat manifold to be an embedding. In particular they proved

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that the answer is positive if the Yamabe constant of (M, g) is non-negative. No positive result is known, as far as we are aware, in the case when the Yamabe constant of (M, g) is negative. In general the development map may not be an embedding, as shown by the elementary examples $\mathbb{S}^1_r \times \mathbb{H}^{n-1}$ where r is the radius of the circle; moreover, these examples also show that the holonomy representation of the fundamental group of M under the development map may not be discrete. However, Kulkarni and Pinkall showed in [KP86] that for a closed conformally flat n-manifold M with infinite fundamental group, its development map $d : M \mapsto \mathbb{S}^n$ is a covering map, *iff* d is not surjective.

It is natural to ask whether some kind of curvature condition in the negative Yamabe constant case would force the development map to be an embedding as well, and whether further curvature conditions would improve the estimate on the Hausdorff dimension of $\partial\Omega$?

The additional curvature conditions are often imposed in terms of the σ_k or Q curvature of a representative metric g. The σ_k curvature, denoted as $\sigma_k(A_g)$, refers to the kth elementary symmetric functions of the eigenvalues of the 1-1 tensor derived from the Weyl-Schouten tensor of the conformal metric g

$$A_g = \frac{1}{n-2}\{Ric - \frac{R}{2(n-1)}g\}.$$

Note that the σ_1 curvature is simply the scalar curvature of g, up to a dimensional constant.

The condition involving the σ_k curvature often assumes that the Weyl-Schouten tensor A_g is in the Γ_k^+ class for some $k > 1$, *i.e.*, the eigenvalues, $\lambda_1 \leq \cdots \leq \lambda_n$, of A_g at each x satisfy $\sigma_j(\lambda_1, \cdots, \lambda_n) > 0$ for all $j, 1 \leq j \leq k$. It is also natural to consider metrics whose Weyl-Schouten tensor A_g is in Γ_k^- class, namely, $(-1)^j\sigma_j(\lambda_1, \cdots, \lambda_n) > 0$ for all $j, 1 \leq j \leq k$. It is known that the operator $w \mapsto \sigma_k(A_{e^{2w}g_0})$ is elliptic when the Weyl-Schouten tensor of $g = e^{2w}g_0$ is in either Γ_k^+ or Γ_k^- class.

There are earlier results involving the Q-curvature that are relevant to the discussion here: they concern the radial symmetry and classification of solutions to constant Q-curvature equations on $\mathbb{R}^n$, where the Q-curvature of metric g is defined through

$$Q_g = c_n|Rc_g|^2 + d_n|R_g|^2 - \frac{\Delta_g R_g}{2(n-1)},$$

with c_n and d_n being some dimensional constants; and when $g = u^{\frac{4}{n-4}}|dx|^2$ on a domain in $\mathbb{R}^n$, $u>0$, $n\neq 4$,

$$(-\Delta)^2 u = \frac{n-4}{2} Q_g u^{\frac{n+4}{n-4}};$$

while on a domain in $\mathbb{R}^4$, if $g=e^{2w}|dx|^2$, then

$$(-\Delta)^2 w = 2Q_g e^{4w}.$$

In [CY97] Chang and Yang proved that any entire smooth solution $u(x)$ to

$$(-\Delta)^{\frac{n}{2}} u(x) = (n-1)! e^{nu(x)} \quad \text{on} \quad \mathbb{R}^n \tag{1}$$

with the asymptotic behavior

$$u(x) = \log \frac{2}{1+|x|^2} + w(\frac{x}{1+|x|^2}) \quad \text{for some smooth function } w \text{ defined near 0, as } x\to\infty, \tag{2}$$

must be rotationally symmetric with respect to some point $x_0\in\mathbb{R}^n$, and of the form $\log\frac{2\lambda}{1+\lambda^2|x-x_0|^2}$ for some constant $\lambda>0$.

In [L98] C.S. Lin obtained related results. For (1) in the case $n=4$, Lin obtained the same result as in [CY97] under the weaker assumption that

$$\int_{\mathbb{R}^4} e^{4u(y)} dy = \frac{8\pi^2}{3}. \tag{3}$$

Lin also obtained a related result for the positive constant Q-curvature equation on $\mathbb{R}^n$, $n>4$,

$$\begin{cases} \Delta^2 u(x) = u^{\frac{n+4}{n-4}}(x), & x\in\mathbb{R}^n; \\ u(x)>0, & x\in\mathbb{R}^n. \end{cases} \tag{4}$$

His result implies that any solution to (4) must be rotationally symmetric with respect to some point $x_0\in\mathbb{R}^n$, and of the form

$$u(x) = c_n \left(\frac{\lambda}{1+\lambda^2|x-x_0|^2}\right)^{\frac{n-4}{2}},$$

for some $\lambda > 0$ and a dimensional constant c_n.

The proofs in both [CY97] and [L98] involve the method of moving planes. In [X00] X. Xu provided a proof for the rotational symmetry of solutions to (4) using the method of moving spheres. In all the three work cited here, one crucial step is to establish that entire solutions u to (4) are superharmonic on $\mathbb{R}^n$.

Considering also that the canonical locally conformally flat metric on $\mathbb{S}^{k-1} \times \mathbb{H}^{n-k+1}$ has its scalar curvature equal to $(n-1)(2k-n-2)$ and its Q-curvature equal to $8(2k-n)(2k-n-4)/n$, the sign of the Q-curvature alone is a poor indicator of how the metric behaves. Our first result, stated below, stems from this observation that it is natural to impose some additional condition involving the scalar curvature when considering global properties of solutions to the Q-curvature equation.

Theorem 1. *Let g be a conformal, complete metric on $\Omega \subsetneq \mathbb{S}^n$ such that*

$$Q_g \equiv 1 \text{ or } 0 \quad \text{in } \Omega, \tag{5}$$

and

$$R_g \geq 0 \quad \text{in } \Omega. \tag{6}$$

Then for any ball $B \subset\subset \Omega$ in the canonical metric $g_{\mathbb{S}^n}$, the mean curvature of its boundary ∂B in metric g with respect to its inner normal is positive.

Remark 1. *Since umbilicity is invariant under a conformal change of metric, and round balls are umbilic in the canonical metric, our Theorem implies that all principal curvatures of ∂B in metric g are positive.*

In a local conformal representation for $g(x) = u^{\frac{4}{n-4}}(x)|dx|^2$ when $n \neq 4$, we have

$$(n-1)(n-4)R_g(x) = -u^{-\frac{n}{n-4}}(x)\left(\Delta u(x) + \frac{2}{n-4}\frac{|\nabla u(x)|^2}{u(x)}\right).$$

We see that the condition (6) for $n > 4$ implies that

$$\Delta u(x) \leq 0. \tag{7}$$

Condition (6) is a geometric condition and would imply (7) for any of its local representation in the form above, which is essential for our argument, while (7) itself is not a geometrically invariant condition. In Y. Li's study of entire solutions to a class of conformally invariant PDEs, and later on in his study of local behavior near isolated singularities of such solutions in [L06], he used condition (7).

A corollary of Theorem 1 is the following

Corollary (1A). *Any complete, conformal metric g on $\mathbb{S}^n \setminus \mathbb{S}^l$ for $l \leq \frac{n-2}{2}$ satisfying* (5) *and* (6) *has to be symmetric with respect to rotations of $\mathbb{S}^n$ which leave $\mathbb{S}^l$ invariant.*

A second corollary of Theorem 1 is the following

Corollary (1B). *Suppose that $g = u^{4/(n-4)} g_{\mathbb{S}^n}$ is a conformal metric on $\Omega \subsetneq \mathbb{S}^n$ such that* (5) *and* (6) *hold, and that g is complete or $u(x) \to \infty$ as $x \to \partial\Omega$, then there exists a constant $C > 0$ such that $u^{2/(n-4)}(x) \leq C\delta(x, \partial\Omega)^{-1}$, where $\delta(x, \partial\Omega)$ is the distance from x to $\partial\Omega$ in the metric $g_{\mathbb{S}^n}$.*

Remark 2. *A version of Theorem 1 for a conformal, complete metric on $\Omega \subsetneq \mathbb{S}^n$ with nonnegative constant scalar curvature, and corresponding corollaries, were first proved by Schoen in [Sc88] by using a moving spheres argument and a blow up argument. There have been many similar symmetry results on entire solutions or entire solutions with one point deleted to the constant σ_k curvature equation in the positive Γ_k class, which are generalizations of the Yamabe equation; a partial list of work in this direction includes those of Viaclovsky [V00a][V00b], Chang, Gursky and Yang [CGY02b][CGY03], Li and Li [LL03][LL05], Guan, Lin and Wang [GLW04], Li[L06].*

Remark 3. *Corollary (1B) follows from Theorem 1 as Schoen did in [Sc88] in the case of constant positive scalar curvature equation. The outline of the argument goes as follows. If the upper bound does not hold, then a sequence of rescaled solutions centered along a sequence of points approaching $\partial\Omega$ would converge to an entire solution on $\mathbb{R}^n$ to the same equation. The solutions of the latter equation are completely classified; they correspond to the round metric on $\mathbb{S}^n$, therefore their mean curvatures (and principal curvatures) along large Euclidean spheres become negative. But in the closure of any such a large Eucildean ball, this metric is the uniform limit of a sequence of metrics whose principal curvatures along its boundary sphere is positive by the version of Theorem 1. This would cause a contradiction. We will not supply a detailed proof for Corollary (1B) here.*

Remark 4. *Although Schoen's result in [Sc88] corresponding to Theorem 1 was stated and proved for a constant* positive *scalar curvature metric on a domain $\Omega \subsetneq \mathbb{S}^n$, an examination of the proof indicates that, as long as the three main ingredients for the moving plane/sphere arguments are valid, the same conclusion can be drawn, namely, the same conclusion continues to hold if the following three steps are still valid: (i). the initiation of the inequality between a solution in a half-space/ball enclosing its singular set and its reflected solution; (ii). the above inequality is a strict point wise inequality unless it becomes a point wise equality in the entire comparison domain; and (iii). the strict inequality continues to hold if the half-space/ball is moved in a small open neighborhood.*

Both (i) and (iii) involve proving that the solution in a neighborhood of its singular set stays above its reflected solution (which is a smooth solution to the equation near the singular set) by a positive amount—we did this here by using the maximum principle for superharmonic functions in a domain with a boundary component having zero Newtonian *capacity, without imposing an explicit growth condition of the solution toward its singular set.*

Both (ii) and (iii) involve using the strong maximum principle and the Hopf boundary lemma for the difference between the solution and its reflected solution, when it is assumed to be non-negative. But this part works for solutions to the constant scalar curvature equation, even if the constant is non-positive; in fact, it works even for the constant σ_k curvature equation, as long as the equation is elliptic. (i) and (iii) can be established if we assume that the conformal factor tends to ∞ uniformly upon approaching the boundary of its domain. We thus have

Theorem 2. *Let g be a conformal metric on $\Omega \subsetneqq \mathbb{S}^n$ such that (a). $\sigma_k(A_g) =$ a constant in Ω, (b). $A_g \in \Gamma_k^+$ (or Γ_k^- respectively) pointwise in Ω, and (c). if we write $g = e^{2w} g_{\mathbb{S}^n}$, when $w \to \infty$ uniformly upon approaching $\partial\Omega$, then for any ball $B \subset\subset \Omega$ in the canonical metric $g_{\mathbb{S}^n}$, the mean curvature of its boundary ∂B in metric g with respect to its inner normal is positive.*

Our next result provides a criterion for the development map of a locally conformally flat manifold to be an embedding in the negative Yamabe constant case.

Theorem 3. *Let (M, g) be a complete, locally conformally flat manifold, and $F : (M, g) \mapsto (\mathbb{S}^n, g_{\mathbb{S}^n})$ be a conformal immersion. If the Schouten tensor A_g of some metric in the conformal class of g is non-positive point wise on M, then F is an imbedding.*

Corollary (3A)**.** *If $\sigma_1(A_g) \le 0$ and*

$$\sigma_2(A_g) \ge \frac{(n-2)}{2(n-1)} (\sigma_1(A_g))^2, \tag{8}$$

then $A_g \le 0$. Thus, when $F : (M, g) \mapsto (\mathbb{S}^n, g_{\mathbb{S}^n})$ is a conformal immersion, (M, g) is a complete, locally conformally flat manifold, and satisfies $\sigma_1(A_g) \le 0$ and (8), then F is an imbedding.

Remark 5. *Theorem 3 and its corollary were obtained in the early 2000's, and were lectured by the second author in several seminar talks, including the fall 2003 CUNY Graduate Center Differential Geometry and Analysis Seminar.*

When the condition $A_g \le 0$ is not satisfied, F may not be an embedding, as shown by the canonical locally conformally flat metric on $\mathbb{S}^1_r \times \mathbb{H}^{n-1}$, whose Schouten tensor is $diag(\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2})$.

We will provide a proof for Theorem 1 in Section 2. The condition (6) implies the superharmonicity of the solution as well as its transformation under any Möbius transformation of the sphere $\mathbb{S}^n$ (when $n > 4$), which would allow the moving planes method to be carried out, as was done in [CY97], [L98], and [X00]. For completeness, we will sketch the main steps of the proof, in particular, to indicate how to handle the behavior of the solution near $\partial\Omega$.

We will provide a proof for Theorem 3 in Section 3.

2 Proof for Theorem 1

Proof. We first set up a stereographic coordinate for proving Theorem 1. Let $B \subset\subset \Omega$ be as given in Theorem 1. We can choose a stereographic coordinate such that B is mapped onto $\{x \in \mathbb{R}^n : x_1 < \lambda_0\}$. Define $\Sigma_\lambda = \{x \in \mathbb{R}^n : x_1 > \lambda\}$, and $T_\lambda = \partial(\Sigma_\lambda) = \{x \in \mathbb{R}^n : x_1 = \lambda\}$. Let Γ be the image of $\mathbb{S}^n \setminus \Omega$ under this stereographic map. Then Γ is a compact subset in Σ_{λ_0}. Define $\Sigma'_\lambda = \Sigma_\lambda \setminus \Gamma$. In this stereographic coordinate we can write

$$g(x) = u^{\frac{4}{n-4}}(x)|dx|^2 \quad \text{for } x \in \mathbb{R}^n \setminus \Gamma. \tag{9}$$

Here, we first provide the details *for the* $n > 4$ *case*; the modifications needed for the $n = 3, 4$ case will be sketched at the end.

The statement that the mean curvature of ∂B in metric g with respect to its inner normal is positive is equivalent to

$$u_{x_1}(x) > 0 \text{ for all } x \in T_{\lambda_0}. \tag{10}$$

Remark 6. *One could also represent B by a Euclidean ball $B(x_0, r)$ with x_0 as center and $r > 0$ as radius, in which case the statement that the mean curvature of ∂B in metric g with respect to its inner normal is positive is equivalent to*

$$\nabla_\theta u(x) + \frac{n-4}{2r}u(x) := \frac{\partial u(x_0 + r\theta)}{\partial r} + \frac{n-4}{2r}u(x) > 0 \quad \textit{for any } x = x_0 + r\theta, \theta \in \mathbb{S}^{n-1}, \tag{11}$$

as the mean curvature in metric g at a point x on ∂B is given by

$$\frac{2u^{\frac{2-n}{n-4}}}{n-4}\left[\nabla_\theta u(x) + \frac{n-4}{2r}u(x)\right].$$

Remark 7. *It follows from Theorem 2.7 in [SY88] that, in the situation of our Theorem 1, the Newtonian capacity $cap(\mathbb{S}^n \setminus \Omega) = 0$, which implies that $cap(\Gamma) = 0$. We will use this to deal with the behavior of $u(x)$ and that of $\Delta u(x)$ near Γ.*

Let $v(x) = -\Delta u(x)$. Then based on our set up, we have

$$\begin{cases} \Delta v(x) = -u^{\frac{n+4}{n-4}}(x) \leq 0 & \text{in } \mathbb{R}^n \setminus \Gamma, \\ v(x) \geq 0 & \text{in } \mathbb{R}^n \setminus \Gamma, \\ v(x) \quad \text{has a harmonic expansion at } x = \infty \text{ in the sense that:} \end{cases} \tag{12}$$

$$\begin{cases} v(x) = c_0|x|^{2-n} + \sum_{j=1}^{n} \frac{a_j x_j}{|x|^n} + O\left(\frac{1}{|x|^n}\right) \\ v_{x_i} = -(n-2)c_0 x_i |x|^{-n} + O\left(\frac{1}{|x|^n}\right) \\ v_{x_i x_j}(x) = O\left(\frac{1}{|x|^n}\right) \end{cases} \tag{13}$$

for some constants $c_0 > 0$ and a_j.

$u(x)$ itself has a harmonic expansion at $x = \infty$:

$$\begin{cases} u(x) = c_1|x|^{4-n} + \sum_{j=1}^{n} \frac{b_j x_j}{|x|^{n-2}} + O\left(\frac{1}{|x|^{n-2}}\right) \\ u_{x_i} = -(n-2)c_1 x_i |x|^{2-n} + O\left(\frac{1}{|x|^{n-2}}\right) \\ u_{x_i x_j}(x) = O\left(\frac{1}{|x|^{n-2}}\right) \end{cases} \tag{14}$$

for some constants $c_1 > 0$ and b_j. In fact, the harmonic expansion (13) for $v(x)$ is based on the harmonic expansion (14) of $u(x)$.

Set $x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$, which is the reflection of x with respect to T_λ, and

$$w_\lambda(x) = u(x) - u(x^\lambda) \quad \text{for } x \in \Sigma_\lambda^{'}.$$

We will prove using the moving planes method that

$$u(x) - u(x^\lambda) > 0 \text{ and } v(x) - v(x^\lambda) > 0, \text{ for all } x \in \Sigma_\lambda^{'} \text{ and } \lambda \leq \lambda_0. \tag{15}$$

It would then follow from (15) that

$$u_{x_1}(x) \geq 0 \text{ and } \partial_{x_1}(\Delta u(x)) \leq 0, \text{ for any } x \text{ with } x_1 \leq \lambda_0, \tag{16}$$

which, together with the strong maximum principle applied to $u(x) - u(x^\lambda)$ and $u(x) - u(x^\lambda)$, would conclude our proof.

In our setting it's impossible for $v(x) \equiv 0$ on $\mathbb{R}^n \setminus \Gamma$ due to (13). Then it follows from (12) and the strong maximum principle that $v(x) > 0$ in $\mathbb{R}^n \setminus \Gamma$.

We may suppose that $\Gamma \subset B(0, R_0)$ for some $R_0 > 0$. Now for any $R \geq R_0$, it follows, using $\text{cap}(\Gamma) = 0$ and (12), that there exists $\delta > 0$ depending on R such that

$$v(x) \geq \delta \quad \text{ for all } x \in B(0, R) \setminus \Gamma. \tag{17}$$

A reference for this kind of extended maximum principle is Theorem 3.4 in Chapter III of [L72].

The harmonic expansion (13) of $v(x)$ at ∞ and Lemma 2.3 in [CGS89] implies that

$$\begin{cases} \text{there exists } \lambda_1 \leq \lambda_0 \text{ and } R_1 \geq R_0 \text{ such that} \\ \\ v(x) > v(x^\lambda) \quad \text{for all } x \in \Sigma_\lambda^{'} \text{ with } |x| \geq R_1, \text{ and } \lambda \leq \lambda_1. \end{cases} \tag{18}$$

Then using (17) and the harmonic expansion (13) of $v(x)$ at ∞, we conclude that there exists $\lambda_2 \leq \lambda_1$ such that

$$v(x) > v(x^\lambda) \quad \text{for all } x \in \Sigma_\lambda^{'},\ \lambda \leq \lambda_2. \tag{19}$$

Next, $w_\lambda(x)$ satisfies

$$\Delta w_\lambda(x) = v(x^\lambda) - v(x) \leq 0 \quad \text{for all } x \in \Sigma_\lambda^{'}, \tag{20}$$

and $\lambda \leq \lambda_2$. The harmonic expansion (14) of $u(x)$ at ∞ implies that

$$w_\lambda(x) \to 0 \quad \text{ as } x \to \infty. \tag{21}$$

Using (20), (21), $w_\lambda(x) = 0$ for all $x \in T_\lambda$, and the observation that $w_\lambda(x) = u(x) - u(x^\lambda) \geq -u(x^\lambda)$ is bounded below in a neighborhood of Γ and the information that $\text{cap}(\Gamma) = 0$, we conclude that $w_\lambda(x) \geq 0$ for all $x \in \Sigma_\lambda^{'}$, $\lambda \leq \lambda_2$. The completeness assumption on g and $\Omega \neq \mathbb{S}^n$ imply that $w_\lambda(x)$ can not be $\equiv 0$, so with the strong maximum principle, we conclude that

$$w_\lambda(x) > 0 \quad \text{for all } x \in \Sigma_\lambda^{'}, \tag{22}$$

and $\lambda \le \lambda_2$.

We now define

$$\lambda_* = \sup\{\lambda \le \lambda_0 : v(x^\mu) < v(x) \quad \text{for all } x \in \Sigma_\mu^{'}, \text{ and all } \mu \le \lambda,\}$$

and proceed to prove that $\lambda_* = \lambda_0$.

By continuity (together with strong maximum principle and completeness of g), (20) and (22) continue to hold for λ_* replacing λ. We now have, using (22) for λ_* replacing λ, that

$$\Delta\left[v(x^{\lambda_*}) - v(x)\right] = u^{\frac{n+4}{n-4}}(x) - u^{\frac{n+4}{n-4}}(x^{\lambda_*}) \ge 0 \quad \text{for all } x \in \Sigma_{\lambda_*}^{'}. \tag{23}$$

$v(x^{\lambda_*}) - v(x) \le 0$ for all $x \in \Sigma_{\lambda_*}^{'}$. Now strong maximum principle, (22) and (23) imply

that $v(x^{\lambda_*}) - v(x) < 0$ for all $x \in \Sigma_{\lambda_*}^{'}$. Furthermore, using $\mathrm{cap}(\Gamma) = 0$, there exists some $\delta_* > 0$ such that

$$v(x^{\lambda_*}) - v(x) \le -\delta_* \quad \text{for } x \text{ in a neighborhood of } \Gamma.$$

This, together with (23) and Lemma 2.4 in [CGS89], implies that $\lambda_* = \lambda_0$.

We now indicate the modifications needed for the $n = 3$ case. (10) turns into

$$u_{x_1} < 0 \quad \text{for all } x \in T_{\lambda_0}; \tag{24}$$

(11) turns into

$$\nabla_\theta u(x) - \frac{u(x)}{2r} < 0; \tag{25}$$

The condition $R_g \ge 0$ turns into

$$\Delta u(x) - \frac{2|\nabla u(x)|^2}{u(x)} \ge 0; \tag{26}$$

and the 3-dimensional version of (4) for $Q = 2$ is

$$(-\Delta)^2 u = -u^{-7}, \quad x \in \mathbb{R}^3. \tag{27}$$

Setting $\tilde{v}(x) = \Delta u(x)$, we find that under (26), $\tilde{v}(x) \geq 0$; and $\eta(x) := \tilde{v}(x) - \tilde{v}(x^\lambda)$ satisfies $\eta(x) \geq -\Delta u(x^\lambda)$, as well as $\Delta\eta(x) \leq 0$ whenever $u(x) \leq u(x^\lambda)$. The version of (15) that we need to establish in 3 dimension is

$$u(x) - u(x^\lambda) < 0 \text{ and } \eta(x) = \tilde{v}(x) - \tilde{v}(x^\lambda) > 0 \quad \text{for all } x \in \Sigma_\lambda^{'} \text{ and } \lambda \leq \lambda_0. \tag{28}$$

Given (27) and the information on η above, (28) is established in almost identical way as in the $n > 4$ case. The $n = 4$ case is similar: (26) is replaced by $\Delta w(x) + |\nabla w(x)|^2 \leq 0$; (11) is replaced by $\frac{\partial w}{\partial r} + \frac{1}{r} \geq 0$.

□

Proof for Corollary 1A. It suffices to prove that, when $\mathbb{S}^l \setminus \{\infty\}$ is represented via a stereographic projection as $\mathbb{R}^l = \{x \in \mathbb{R}^n : x_{l+1} = \ldots = x_n = 0\}$, and for any $x \in \mathbb{R}^n \setminus \mathbb{R}^l$, and for any (unit) vector $e = (0, \ldots, 0, e_{l+1}, \ldots, e_n) \perp x$, we have $\nabla_e u(x) = 0$. This would also imply that, in this set up, $u = u(x_1, \ldots, x_n)$ depends on $x_{l+1}, \ldots, x_n$ only through $\sqrt{x_{l+1}^2 + \ldots + x_n^2}$.

For any $r > 0$, we see that $B(x - re, r) \subset\subset \mathbb{R}^n \setminus \mathbb{R}^l$, so the conclusion of Theorem 1 is valid on $\partial B(x - re, r)$. In particular, at $x \in \partial B(x - re, r)$, we have, by (11)

$$\nabla_e u(x) + \frac{n-4}{2r} u(x) > 0. \tag{29}$$

Since we can take $r > 0$ arbitrarily large, we conclude that $\nabla_e u(x) \geq 0$. Repeating this argument with $-e$ replacing e, we obtain $\nabla_{-e} u(x) \geq 0$, and therefore conclude that $\nabla_e u(x) = 0$. □

3 Proof for Theorem 2

We will use that the Schouten tensor A_g and the Einstein tensor E_g are related as

$$(n-2)A_g = E_g + \frac{(n-2)R_g}{2n(n-1)} g. \tag{30}$$

Proof of Theorem 3. We may assume that g itself satisfies that its Schouten tensor A_g is non-positive point wise on M. For any point $z_0 \in F(M) \subset \mathbb{S}^n$, using stereographic coordinates, there is a smooth function u on $\mathbb{S}^n$ such that $u > 0$ on $\mathbb{S}^n \setminus \{z_0\}$, $u(z_0) = 0$, and $u^{-2}g_0$ is flat. Writing $F^*(u^{-2}g_0) = v^{-2}g \stackrel{\text{def}}{=} \widehat{g}$ on $M \setminus F^{-1}(z_0)$, then $\widehat{g}$ is flat. Hence, on $M \setminus F^{-1}(z_0)$, $\widehat{E} = 0$, $\widehat{R} = 0$.

Under a (pointwise) conformal change of the metric g, $\widehat{g} = v^{-2}g$, the Einstein tensor and scalar curvature transform as follows.

$$\widehat{E} = E + \frac{n-2}{v}\{\nabla^2 v - \frac{\Delta v}{n} g\}, \tag{31}$$

$$\widehat{R} = v^2\{R + 2(n-1)\frac{\Delta v}{v} - n(n-1)\frac{|\nabla v|^2}{v^2}\}. \tag{32}$$

Thus in the situation here, we have, by (31) and (32),

$$E = -\frac{n-2}{v}\{\nabla^2 v - \frac{\Delta v}{n} g\}, \tag{33}$$

$$R = -(n-1)\{2\frac{\Delta v}{v} - n\frac{|\nabla v|^2}{v^2}\}. \tag{34}$$

It now follows that

$$A = -\frac{\nabla^2 v}{v} + \frac{|\nabla v|^2}{2v^2} g. \tag{35}$$

Under our assumption that $A \leq 0$, we therefore have

$$\nabla^2 v \geq \frac{|\nabla v|^2}{2v} g, \tag{36}$$

on $M \setminus F^{-1}(z_0)$. By a limiting argument, $v(\gamma(s))$ is a non-negative convex function along any geodesic (in metric g) $\gamma(s)$ on M.

If $P_0 \neq P_1 \in M$ are such that $F(P_0) = F(P_1)$, we set $z_0 = F(P_0) = F(P_1)$ and carry out the computations in the paragraph above. Since (M, g) is assumed to be complete, we may joint P_0 and P_1 by a geodesic (in metric g) $\gamma(s)$ parametrized over $s \in [0,1]$ with $\gamma(0) = P_0$ and $\gamma(1) = P_1$, then $v(\gamma(0)) = v(\gamma(1)) = 0$. Since $v > 0$ on $M \setminus F^{-1}(z_0)$, this would imply that $\gamma(s) \in F^{-1}(z_0)$ for all $s \in [0,1]$, using the convexity of v. But this is not possible, and this contradiction implies that F must be an imbedding. □

It follows from (30) that

$$2\sigma_2(A_g) = \frac{n-1}{n}(\sigma_1(A_g))^2 - \frac{||E_g||^2}{(n-2)^2}, \tag{37}$$

where $||E_g||$ is the metric norm of E with respect to g.

Remark 8. *Condition* (8) *is a kind of pinching condition, as it is equivalent to*

$$\frac{||E_g||^2}{(n-2)^2} \leq \frac{(\sigma_1(A_g))^2}{(n-1)n}, \tag{38}$$

using (37)*.* (8) *is also equivalent to*

$$(n-1)||A_g||^2 \leq (\sigma_1(A_g))^2 \tag{39}$$

Proof of Corollary 3A. (30) can be rewritten as

$$A_g = \frac{E_g}{n-2} + \frac{\sigma_1(A_g)}{n} g. \tag{40}$$

Since E is trace free, we have the sharp inequality

$$-\sqrt{\frac{n-1}{n}}||E_g|| \leq E \leq \sqrt{\frac{n-1}{n}}||E_g||, \tag{41}$$

Thus, when (8) holds, we have (38), as remarked earlier, which then implies that

$$\frac{||E_g||}{(n-2)} \leq \frac{|\sigma_1(A_g)|}{\sqrt{n(n-1)}}. \tag{42}$$

It now follows from (42), (40) and (41) that $A_g \leq 0$.

□

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