REMEMBERING DONALD SARASON (1933-2017)

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Donald Erik Sarason was a leading figure at the intersection of complex analysis and operator theory for several generations. He earned his PhD in 1963 under the direction of Paul Halmos and worked at U.C. Berkeley from 1964 until his retirement in 2012. Since a complete account of Don's work and his mathematical legacy would run to many pages, we focus here on a few career highlights and personal reminiscences from three of his former students.

Among Don's earlier papers was the 1967 masterpiece "Generalized interpolation in H^∞" [1], which laid the groundwork for several decades of research by hundreds of mathematicians worldwide. This remarkable paper brought new techniques to the study of holomorphic interpolation problems and function algebras, while also marking the birth of commutant-lifting theory. We briefly examine the impact and ramifications of these ground-shaking ideas below.

Let $z_1, z_2, \ldots, z_n$ be distinct points in the open unit disk $\mathbb{D}$ in the complex plane and let $w_1, w_2 \ldots, w_n \in \mathbb{D}$ be arbitrary. The *Nevanlinna–Pick interpolation problem* asks whether there exists a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ so that $f(z_i) = w_i$ for $i = 1, 2, \ldots, n$. A necessary and sufficient condition for the existence of a solution is that the $n \times n$ matrix

$$\left[\frac{1-\overline{w_j}w_i}{1-\overline{z_j}z_i}\right]_{i,j=1}^{n}$$

is positive semidefinite. Don realized that this, and the Carathéodory interpolation problem (which asks for a holomorphic $f : \mathbb{D} \to \mathbb{D}$ whose first n Taylor coefficients are prescribed), can be recast in terms of reproducing kernel Hilbert spaces [1]. This perspective permitted a unified treatment of a variety of holomorphic interpolation problems, including contemporary multivariate and noncommutative analogues.

Don's work paved the way for deep work on function algebras, which reached new heights in the 1970s. His *Bulletin of the AMS* paper [2] drew attention to the outstanding open problems of the day. Let H^∞ denote the algebra of bounded analytic functions on $\mathbb{D}$, identified with a subspace of $L^\infty := L^\infty(\mathbb{T})$ by associating each $f \in H^\infty$ with its almost-everywhere defined boundary function on the unit circle $\mathbb{T}$. Of particular interest are the closed algebras that lie between L^∞ and H^∞. R. G. Douglas conjectured that each such algebra is generated by H^∞ and a set of reciprocals of inner functions (defined below). This became known as the *Douglas problem*. Don made several huge contributions to this area, one of which was the observation that $H^\infty + C$, in which $C = C(\mathbb{T})$ denotes the algebra of continuous complex-valued functions on $\mathbb{T}$, is a Douglas algebra. In fact, $H^\infty + C$ is the smallest norm-closed subalgebra of L^∞ that properly contains H^∞. Don discovered deep connections between the Douglas problem and L. Carleson's work on the corona problem. This ultimately culminated in the complete description of Douglas algebras by Don's student S.-Y. A. Chang and by D. Marshall in 1976.

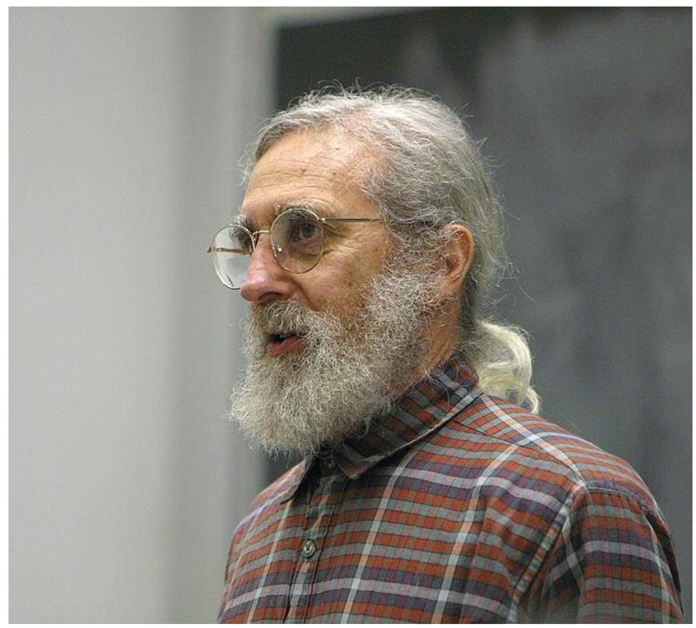

FIGURE 1. Don Sarason in 2003, at his 70th birthday conference. Photo courtesy of Jonathan Shapiro.

The most obvious infinite-dimensional generalization of a Jordan block, the unilateral shift $(a_0, a_1, a_2, \ldots) \mapsto (0, a_0, a_1, \ldots)$ on the sequence space $\ell^2(\mathbb{N})$, is of fundamental importance. It is unitarily equivalent to the linear operator $Sf \mapsto zf$ on the Hardy–Hilbert space H^2 of all holomorphic $f(z) = \sum_{n=0}^\infty a_n z^n$ on $\mathbb{D}$ for which $\|f\|^2 = \sum_{n=0}^\infty |a_n|^2 < \infty$. A fundamental result of A. Beurling provides an elegant function-theoretic characterization of the nontrivial, closed S-invariant subspaces of H^2. They are of the form $uH^2 = \{uh : h \in H^2\}$, in which u is an *inner function*; that is,

$$u(z) = \alpha z^N \Big(\prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z_n} z} \Big) \exp\Big[-\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t) \Big],$$

in which $\alpha \in \mathbb{T}$, $N \in \{0, 1, 2, \ldots\}$, (z_n) is a possibly finite sequence of points in $\mathbb{D}$ that satisfy $\sum_{n=1}^\infty (1 - |z_n|) < \infty$, and μ is a nonnegative, singular Borel measure on $\mathbb{T}$. Inner functions are precisely the elements of H^∞ whose boundary functions have modulus one almost everywhere on $\mathbb{T}$.

The so-called *model spaces* $\mathcal{K}_u = H^2 \ominus uH^2$ are of great importance in function-related operator theory. Don observed that if B commutes with the compression of S to $\mathcal{K}_u$, then there is a $\phi \in H^\infty$ so that $\|B\| = \|\phi\|_\infty$ and $B = T_\phi|_{\mathcal{K}_u}$ is the restriction of the *Toeplitz operator* $T_\phi f = P(\phi f)$ to $\mathcal{K}_u$; here P denotes the Riesz projection, the orthogonal projection from $L^2 := L^2(\mathbb{T})$ onto H^2 [1]. Since $ST_\phi = T_\phi S$, this result was referred to as *commutant lifting*. Don's theorem ultimately led to the general tools developed by Sz.-Nagy and Foiaş. "[T]he story of how the author had the good fortune to be able to prove the primordial version of the commutant lifting theorem" is recounted in Don's article [8], written in memory of his advisor, Paul Halmos.

The year 1975 saw the publication of Don's landmark paper "Functions of vanishing mean oscillation" [3]. The space BMO of functions of bounded mean oscillation had appeared in seminal papers of John–Nirenberg and Fefferman–Stein. A complex-valued function defined on the unit circle has *vanishing mean oscillation* if the average of the absolute value of its difference from its average over an interval tends to zero as the length of the interval shrinks to zero. Thus, VMO is a subspace of BMO. Don proved that functions in VMO can be decomposed as the sum of a functions in $H^\infty + C$ and the harmonic conjugate of a function in $H^\infty + C$. The concept of VMO has since found wide applications in many areas of operator theory, harmonic analysis, and partial differential equations.

In 1978, Don gave a series of ten lectures at Virginia Polytechnic Institute and State University (now Virginia Tech). These lectures tied together classical function theory, contemporary functional analysis, and exciting new discoveries in harmonic analysis, such as Fefferman's duality theorem. His lecture notes, entitled *Function Theory on the Unit Circle* [4], were widely circulated and extremely influential. Despite never being formally published, these lecture notes are Sarason's sixth most-cited work, according to MathSciNet.

In the 1990s, Don pioneered the abstract treatment of contractive containment of Hilbert spaces and established a fruitful connection between de Branges–Rovnyak spaces and the ranges of certain Toeplitz operators [5]. Using reproducing kernel Hilbert space techniques, he gave elegant new proofs of the Julia–Carathéodory and the Denjoy–Wolf theorems from complex dynamics.

2007 saw the republication, by the AMS, of Don's book *Complex Function Theory* [7], an undergraduate-level textbook known for its rigorous yet approachable development of the subject.

During the period 2007-2013, Don focused on the development of truncated Toeplitz operators and their relatives. The seminal paper "Algebraic properties of truncated Toeplitz operators" [6], published in 2007, echoed the title of the famed Brown–Halmos paper from 1963 that largely initiated the study of Toeplitz operators. He observed many new phenomena that spurred further research and provided operator theorists with a host of new problems and challenges that continue to guide the field.

Although he published many influential papers, perhaps Don's most enduring contributions are his many students. Don had 40 PhD students, of whom many are now successful mathematicians; see Table 1 and Figure 4. Although most of his students remained in operator theory or complex analysis, a number of them branched out and became prominent in other areas. We shall hear from three of them: Sun-Yung Alice Chang, a differential geometer; David Cruz-Uribe, a harmonic analyst; and John Doyle, an engineer.

Lee, Matthew	1969	Laroco, Jr., Leonardo	1988
Nash, David	1970	Lotto, Benjamin	1988
Hively, Gregory	1972	Izzo, Alexander	1989
Chang, Sun-Yung	1974	LeBlanc, Emile	1989
Axler, Sheldon	1975	Li, Kin	1989
Gerver, Joseph	1975	McCarthy, John	1989
Cowen, Carl	1976	Crofoot, Robert	1991
Rosenthal, Erik	1977	Davis, Benjamin	1991
Guillory, Carroll	1978	King-Smith, Oliver	1991
Hayashi, Eric	1978	Cruz-Uribe, David	1993
Hoffman, Michael	1979	Sand, Michael	1994
Wolff, Thomas	1979	Silva, Jorge-Nuno	1994
Bravo, Jaime	1980	Marchant, Simon	1995
Budde, Paul	1982	Shapiro, Jonathan	1995
McMillen, William	1983	Chartrand, Rick	1999
Smith, Wayne	1983	Grinshpan, Anatolii	2001
Doyle, John	1984	Serra, Antonio	2002
Hochwald, Scott	1984	Garcia, Stephan	2003
Walsh, Shelley	1984	Neumann, Genevra	2003
Nguyen, Hung	1985	Judson, Zachary	2009

TABLE 1. Students of Donald Sarason. Source: `http://www.genealogy.ams.org/id.php?id=5021`.

Sun-Yung Alice Chang. I first met Don Sarason at my preliminary examination in 1971, when I was a second-year graduate student at U.C. Berkeley. I had arrived in the U.S. from Taiwan the year before and barely spoke English. I was admitted only to the masters degree program at Berkeley; my financial support and admission into the PhD program depended on how well I did on the examination. I was very nervous about it.

The examination was oral, with topics in algebra, analysis, and geometry, each for one hour. Students were questioned on each topic by two faculty members. Partially due to my nervousness, I did rather poorly in the first topic in algebra and I felt enormous pressure to do better in the second topic, which was analysis – my strong suit.

Don Sarason was one of my examiners and posed his questions slowly. He asked me some very basic questions at the beginning, which calmed me down. Later in the exam, he not only asked me to answer the questions but also encouraged me to display more of the in-depth knowledge I had about the topics. After the examination, while I was waiting in the corridor for the outcome, Don walked up to me and told me I did exceptionally well. This gave me the confidence to finish the rest of my examinations.

FIGURE 2. Don Sarason with Sun-Yung Alice Chang, one of his first PhD students, in 1979. Photo courtesy of Sun-Yung Alice Chang.

This was the pattern of the interaction between Don Sarason and me. He later became my PhD advisor. I learned later that Don was an extremely shy person, who did not talk much. But during the many occasions that I sat in his office asking questions, he provided me with a lot of guidance. Sometimes he even followed up and passed to me hand-written notes after our meetings. He was always warm and tremendously supportive.

At the time, Don had a large number of graduate students, as he did throughout his career. We formed learning seminars and studied together. Despite his quiet personality, Don made a conscious effort to provide us with a stimulating environment. He often organized dinners after seminar talks and held frequent Sunday buffets at his home, where we lingered the whole afternoon. Long after I graduated from Berkeley, I continued to attend such gatherings when I visited him. I remember vividly that once I brought my young son with me to one of the lunch buffets, my son became the center of attention at the party, and we had a great time. I also remember the happy time when after a conference I attended a country music concert with Don and his wife Mary in Nashville.

During my early career, partially due to the two-body problem (my husband is also a mathematician), I found myself facing a strong constraint of choices in the job market. Once I was very discouraged and talked to Don on the phone. As usual, Don did not speak much, but later he wrote me a two-page letter, telling me, "The basic challenge is the same wherever you are…, the most important thing is what you do, not where you are." I was deeply moved by the letter. Later in my career people sometimes asked me if as a woman and a minority I felt discriminated against. I must say that sometimes I am sensitive about the issue; but I always remember that there are some great people and outstanding mathematicians like Don and others who have gone out of their ways to help and encourage me. Don was a great advisor, mentor, and friend whom I could lean on. I am sure this is the same role he has played to many of his other students and young colleagues. We all truly miss him.

David Cruz-Uribe. I first met Donald Sarason in the fall of 1989, when I was a brand-new graduate student at Berkeley. Don was teaching complex analysis. Although I intended to study algebraic number theory, I took his course. He started the semester by introducing himself and then saying, "Please call me Don." It took me years to say it to his face. Over the course of the semester I grew disillusioned by number theory and increasingly attracted by this soft spoken but incredibly engaging lecturer. Paul Halmos said: "He is one of the smoothest and clearest lecturers I know and he has an extraordinary sense of time. He knows almost to the minute how long it will take him to explain something." In the spring I took classical harmonic analysis from Don, and late in my first year I asked if I could be his student.

This was towards the end of a period in which Don produced a quarter of his 40 PhD students; see Table 1. Nevertheless, Don seemed happy to take me on as his student. He started me off by having me read his VPI Lecture Notes [4], based on a series of lectures he gave in 1978 at what is now Virginia Tech. Though never published they were widely circulated and well known to operator theorists of that era. I still have my photocopy, heavily marked up. I would go to Don's office every week and he would patiently answer my questions.

At every level, from introductory calculus to graduate courses and interactions with his students, Don was an amazing teacher. Though a small thing, I was impressed by his ability to draw a perfect, three-foot diameter circle on the chalkboard with a single stroke. His explanations were always lucid, and his choice of material took students to the heart of the subject. The one exception to this that I can remember was in complex analysis, when we spent the last few weeks grinding through the theory of elliptic functions. After one particularly technical lecture about the intricacies of theta functions, I asked him why he included this material. He laconically replied, "Because I always wanted to learn about them." As his advisor, Paul Halmos, put it, "Don never used eight words when seven would do."

Don was a very careful writer and editor. Every draft of my thesis and of the one small paper I wrote came back covered with red ink. My mathematical reasoning, of course, was scrutinized, but he also corrected my exposition with an exactitude I had not seen since my high school composition classes. He wanted every sentence to be as clear and as concise as possible. Once, he sent me scrambling to find a mysterious paper after he wrote in the margin, "See Fowler." This was not some unknown mathematician, but rather H. W. Fowler's *A Dictionary of Modern English Usage*, and the reference was to correct a subtle grammatical error.

Don had a very dry sense of humor. When teaching a large calculus course, he would stop in the middle of each lecture and tell a joke. They were usually corny, but his deadpan delivery always evoked laughter. One winter I met him in Sproul Plaza just before the start of classes. We were chatting and I asked if he had done anything interesting over break. He simply responded, "I got married – I guess you know to who." As a more extended joke, whenever he lectured, he always wore a tie. The ties, however, were ugly and only occasionally went with his outfit. I later learned that he had a huge collection in his office. As he was leaving to go teach he would select one at random and put it on. When I asked him why, he explained that he had gotten a student evaluation that complained his dress was too casual for a professor, so he had gone to a thrift store and bought a bag of ties for a few dollars.

Every student of Don's that I have spoken to agrees that Don was an excellent advisor. Though I complained at the time about his lack of direction, I only realized many years later the confidence that Don had in my ability, and the support he gave me to develop as an independent research mathematician. The thesis problem he gave later became known as the Sarason conjecture on the product of unbounded Toeplitz operators. I suspect that Don had in mind a partial result via classical operator theoretic methods. However, I soon became interested in a real-variable approach to the problem and showed that it was equivalent to an open question in the study of two-weight norm inequalities. On my own I decided that this was the correct way to approach the problem; or, as Ben Lotto, another of Don's students put it, I sold out to the dark side of analysis.

FIGURE 3. Don Sarason (middle) with S.R. Garcia (left) and G. Karaali (right) in Berkeley, May 2004. Photo courtesy Stephan Ramon Garcia.

I plunged ahead, unfazed (and indeed, unaware) that my advisor knew very little about this area and so could provide almost no guidance. I remember one meeting, shortly before I resolved the major sticking point in my thesis, where I outlined what I was trying to prove and all of my failed attempts. In response to my pleas for help, he only responded, "Those are very interesting questions, David." But he carefully read my thesis and his many marginal comments significantly improved the final results.

My real reward came a decade later, in 2003, at a conference organized in honor of his 70th birthday. In the decade since getting my PhD, I had made some progress in the study of two-weight norm inequalities, and in my talk I sketched out the partial results as they applied to the Sarason conjecture. As a joke I described my work as the result of an "intellectual falling out" that Don and I had when I was a graduate student. At the end of my talk, Don got up, said, "David, you were right," and then sat down again.

John Doyle. For a pure mathematician, Don had a surprisingly large impact on control engineering. This connection started in 1978 when I began graduate work at Berkeley. Initially my favorite subject was differential geometry, but when I took functional analysis from Don he definitely became my favorite instructor. He was truly compassionate, and took everything very seriously except himself. This showed in his humor and politics, both of which I enjoyed. He was also an avid runner, though our routines were very different, so we didn't train together.

At the time, the research frontiers in the new subject of robust control theory began to coincide with pure math in Don's area of operator theory, and particularly optimization in H^∞. This was particularly relevant to me, as I was commuting between Berkeley and Minneapolis, where Honeywell was the main client of my aerospace consulting business. I planned to focus on pure math when in Berkeley and I talked to Don briefly and somewhat superficially about some potential research directions. I had some questions about what was known at the time and about what I might study to catch up. Don said he wasn't sure about my questions and told me to come back in a week after he'd had time to think about it.

When I came back, Don gave me a personalized lecture on his assessment of existing results plus a set of notes with key references that were great starting points. This amazed me but I learned that this sort of generosity was typical. I don't recall when he became my official advisor but he immediately helped me get started in important directions.

The pure math side progressed rapidly, building on work by Don and others. My research quickly added a focus on overcoming the apparent computational intractability of the otherwise promising theoretical solutions. This departed somewhat from Don's interest and our interactions were thus less frequent but always fun.

Don was not thrilled that many initial applications of this work, by others, would likely be to military aerospace systems, but he was always very tolerant of my admittedly hypocritical compartmentalization. It helped that everything was open and unclassified, and ultimately the applications were overwhelmingly academic and commercial, though he was quite reasonably not interested in any of these distinctions. My thesis ended up being more applied and computationally oriented than either of us expected, and it even received the Friedman applied math thesis award. It also formed one of the foundations for what became the Matlab Robust Control Toolbox, which is widely used in academia and industry.

Although Don was a pure mathematician at heart, four of my advisees from Caltech's Control and Dynamical Systems program – Don's academic grandchildren – ultimately became professors in various parts of Berkeley Engineering.

FIGURE 4. Don Sarason, with students and colleagues, at his 70th birthday conference in 2003. S.-Y. Alice Chang is in the lower right and S.R. Garcia is in the upper left. Photo courtesy of Jonathan Shapiro.

FIGURE 5. Annotated version of the above. Photo courtesy of Jonathan Shapiro.

SELECTED PAPERS OF DONALD SARASON

[1] Generalized interpolation in H^∞. *Trans. Amer. Math. Soc.*, 127:179–203, 1967.

[2] Algebras of functions on the unit circle. *Bull. Amer. Math. Soc.*, 79:286–299, 1973.

[3] Functions of vanishing mean oscillation. *Trans. Amer. Math. Soc.*, 207:391–405, 1975.

[4] *Function theory on the unit circle*. Virginia Polytechnic Institute and State University, Department of Mathematics, Blacksburg, Va., 1978. Notes for lectures given at a Conference at Virginia Polytechnic Institute and State University, Blacksburg, Va., June 19–23, 1978.

[5] *Sub-Hardy Hilbert spaces in the unit disk*, volume 10 of *University of Arkansas Lecture Notes in the Mathematical Sciences*. John Wiley & Sons, Inc., New York, 1994. A Wiley-Interscience Publication.

[6] Algebraic properties of truncated Toeplitz operators. *Oper. Matrices*, 1(4):491–526, 2007.

[7] *Complex function theory*. American Mathematical Society, Providence, RI, second edition, 2007.
[8] Commutant lifting. In *A glimpse at Hilbert space operators*, volume 207 of *Oper. Theory Adv. Appl.*, pages 351–357. Birkhäuser Verlag, Basel, 2010.

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FIGURE 6. Pamela Gorkin (left), student of Sheldon Axler (second from left), student of Donald Sarason (second from right), student of Paul Halmos (right) in Lancaster, 1984. Halmos, once wrote, "Don never used eight words when seven would do." Photo rights held by MAA.