

# SOME HIGHER ORDER ISOPERIMETRIC INEQUALITIES VIA THE METHOD OF OPTIMAL TRANSPORT

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ABSTRACT. In this paper, we establish some sharp inequalities between the volume and the integral of the  $k$ -th mean curvature for  $k+1$ -convex domains in the Euclidean space. The results generalize the classical Alexandrov-Fenchel inequalities for convex domains. Our proof utilizes the method of optimal transportation.

## 1. INTRODUCTION

Classical isoperimetric inequality for domains in Euclidean space was rigorously established by H. A. Schwartz [Sch84](1884) using the method of symmetrization. Later De Giorgi [DG58] gave a simpler proof based on an early argument of Steiner. In [Gro85], Gromov established the inequality by constructing a map from the domain to the unit ball and applying the divergence theorem. Inspired by Gromov's idea, it became well-known that optimal transport method is effective in establishing various sharp geometric inequalities. See for example [McC97], [CENV04], [FMP10] etc..

In this paper, we apply optimal transport method to establish some sharp higher order isoperimetric inequalities of Alexandrov-Fenchel type for a class of domains which includes the convex domains.

Suppose  $\Omega \subset \mathbb{R}^m$  is a bounded convex set. Consider the set

$$\Omega + tB := \{x + ty | x \in \Omega, y \in B\}$$

where  $B$  is the unit ball and  $t > 0$ . By a theorem of Minkowski [Min11], the volume of the set is a polynomial of degree  $m$ , whose expansion is given by

$$\text{Vol}(\Omega + tB) = \sum_{k=0}^m \binom{m}{k} W_k(\Omega) t^k.$$

Here  $W_k(\Omega)$ ,  $k = 0, \dots, m$  are coefficients determined by the set  $\Omega$ , and  $\binom{m}{k} = \frac{m!}{k!(m-k)!}$ . The  $k$ -th quermassintegral  $V_k$  is defined as a multiple of the coefficient  $W_{m-k}(\Omega)$ .

$$V_k(\Omega) := \frac{\omega_k}{\omega_m} W_{m-k}(\Omega). \quad (1.1)$$

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Here  $\omega_k$  denotes the volume of the unit  $k$ -ball; for an arbitrary domain  $\Omega$ ,  $V_m(\Omega) = \text{vol}(\Omega)$  denotes the volume of  $\Omega$ .

If the boundary  $\partial\Omega$  is smooth, the quermassintegrals can also be represented as the integrals of invariants of the second fundamental form: Let  $L_{\alpha\beta}$  be the second fundamental form on  $\partial\Omega$ , and let  $\sigma_l(L)$  with  $l = 0, \dots, m-1$  be the  $l$ -th elementary symmetric function of the eigenvalues of  $L$ . (Define  $\sigma_0(L) = 1$ .) Then

$$V_{m-k}(\Omega) := \frac{(m-k)!(k-1)! \omega_{m-k}}{m! \omega_m} \int_{\partial\Omega} \sigma_{k-1}(L) d\mu, \quad (1.2)$$

where  $d\mu$  is the surface area of  $\partial\Omega$ . From the above definition, one can see that  $V_0(\Omega) = 1$ , and  $V_{m-1}(\Omega) = \frac{\omega_{m-1}}{m\omega_m} \text{Area}(\partial\Omega)$ . As a consequence of the Alexandrov-Fenchel inequalities [Ale37], one obtains the following family of inequalities: if  $\Omega$  is a convex domain in  $\mathbb{R}^m$  with smooth boundary, then for  $0 \leq l \leq m-1$ ,

$$\left( \frac{V_{l+1}(\Omega)}{V_{l+1}(B)} \right)^{\frac{1}{l+1}} \leq \left( \frac{V_l(\Omega)}{V_l(B)} \right)^{\frac{1}{l}}. \quad (1.3)$$

For  $0 \leq l \leq m-2$ , (1.3) is equivalent to, due to the identity (1.2),

$$\left( \int_{\partial\Omega} \sigma_{k-1}(L) d\mu \right)^{\frac{1}{m-k}} \leq \bar{C} \left( \int_{\partial\Omega} \sigma_k(L) d\mu \right)^{\frac{1}{m-1-k}}, \quad (1.4)$$

where  $k = m-1-l$ . Here  $\bar{C} = C(k, m)$  denotes the (sharp) constant which is obtained only when  $\Omega$  is a ball in  $\mathbb{R}^m$ . When  $l = m-1$ , (1.3) is the well-known isoperimetric inequality

$$\text{vol}(\Omega)^{\frac{m-1}{m}} \leq \frac{1}{m\omega_m^{\frac{1}{m}}} \text{Area}(\partial\Omega). \quad (1.5)$$

An open question in the field is if (1.4) holds when the domain is only  $k$ -convex in the sense that  $\sigma_l(L)(x) > 0$  for all  $l \leq k$  and all  $x \in \partial\Omega$ . (In the following, we denote the condition of  $k$ -convexity by  $L \in \Gamma_k^+$ .) This is indeed the case under the additional assumption that  $\Omega$  is star-shaped, as established by Guan-Li [GL09]. In the work of Huisken [Hui], he has proved that (1.4) holds for  $k = 1$  when one assumes in addition that the domain is outward minimizing. Inequalities of the type (1.4) were also discussed by Trudinger [Tru94]. In [Cas10], Castillon has applied the method of optimal transport to give a new proof of the Michael-Simon inequality, which in particular implies (1.4) for  $k = 1$  with some constant  $C(m)$ . In [CW11, CW], we have established inequalities of the type (1.4) for all  $l \leq k$  with some (non-sharp) constant  $C(k, m)$  when  $\Omega$  is  $(k+1)$ -convex.

In this note, we apply the method of optimal transport to establish the following ‘‘end version’’ of the sharp inequalities in (1.4).

**Theorem 1.1.** *Let  $\Omega$  be a domain in  $\mathbb{R}^m$  with smooth boundary. Suppose  $\Omega$  is 2-convex, i.e.  $L \in \Gamma_2^+$ . Then*

$$\text{vol}(\Omega)^{\frac{m-2}{m}} \leq \left(\frac{1}{\omega_m}\right)^{\frac{2}{m}} \frac{1}{m(m-1)} \int_{\partial\Omega} H d\mu, \quad (1.6)$$

where  $H = \sigma_1(L)$  is the mean curvature of  $\partial\Omega$  and  $d\mu$  is the surface area of  $\partial\Omega$ . The constant in the inequality is sharp and equality holds only when  $\Omega$  is a ball in  $\mathbb{R}^m$ .

We also prove the inequality between the volume of  $\Omega$  and the integral of  $\sigma_2(L)$  with the sharp constant.

**Theorem 1.2.** *Let  $\Omega$  be a domain in  $\mathbb{R}^m$  with smooth boundary. Suppose  $\Omega$  is 3-convex, i.e.  $L \in \Gamma_3^+$ . Then*

$$\text{vol}(\Omega)^{\frac{m-3}{m}} \leq \left(\frac{1}{\omega_m}\right)^{\frac{3}{m}} \frac{1}{3\binom{m}{3}} \int_{\partial\Omega} \sigma_2(L) d\mu. \quad (1.7)$$

The constant in the inequality is sharp and equality holds only when  $\Omega$  is a ball in  $\mathbb{R}^m$ .

We remark that our proof indicates that the sharp version of these inequalities for  $(k+1)$ -convex domains are likely to be derived by optimal transport method for all  $k$ ; but the proof would be algebraically challenging. It also remains an open question if the additional one level of convexity assumption (that is we assume  $\Omega$  is  $(k+1)$ -convex instead of  $k$ -convex) in our proof is necessary—but the proof we present here heavily depends on this assumption.

The main difference of Theorem 1.1, and 1.2 from our previous work in [CW] is that we have applied the optimal transport map from a measure on  $\Omega$  to the unit  $m$ -ball, instead of applying the optimal transport map from the projections of the measure on  $\partial\Omega$  to the unit  $(m-1)$ -balls on hyperplanes of  $\mathbb{R}^m$ . Also, in order to obtain the sharp constants, we study the delicate interplays between terms involving the tangential and normal directions of the optimal transport map. We apply the Taylor expansion and derive recursive inequalities in order to estimate each individual term in the expansion.

Finally we would like to clarify that although an optimal transport map  $\bar{\nabla}\phi$  from  $\Omega$  to the unit ball in  $\mathbb{R}^m$  is smooth up to the boundary if  $\Omega$  is strictly convex by the results of Caffarelli ([Caf92a, Caf96]), the proof of our main theorems still work if we only assume  $\Omega$  is 2 or 3-convex respectively by an approximation argument. This type of approximation process is quite standard and has been explained in detail for example in [CW] and in many other articles in this research field. We will not emphasize this point later in our argument and will take for granted that we can integrate by parts for derivatives of  $\phi$  on  $\partial\Omega$ .

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## 2. PRELIMINARIES

**2.1. Optimal Transport.** We begin with a similar set-up that Gromov has used in establishing the classical isoperimetric inequality.

By the result of Brenier [Bre91] on optimal transportation, given a probability measure  $f(x)dx$  on  $\Omega$ , there exists a convex potential function  $\phi : \mathbb{R}^m \rightarrow \mathbb{R}$ , such that  $\bar{\nabla}\phi$  is the unique optimal transport map from  $\Omega$  to  $B(0, 1)$  (the unit  $m$ -ball centered at the origin) which pushes forward the probability measure  $f(x)dx$ , to the probability measure  $g(y)dy = \frac{1}{\omega_m}dy$  on  $B(0, 1)$ . We adopt the convention to denote by  $\bar{\nabla}\phi$ ,  $\bar{\nabla}_{ij}^2\phi$  the gradient and the Hessian of  $\phi$  with respect to the ambient Euclidean metric, in order to distinguish them from  $\nabla\phi$ ,  $\nabla_{\alpha\beta}^2\phi$  (or  $\phi_{\alpha\beta}$ )—the gradient and the Hessian of  $\phi$  with respect to the metric of  $\partial\Omega$ . We remark that in our notation, both the coordinate of the form and the coordinate the vector are denoted using the lower index  $\phi_\alpha$  (or  $\phi_\beta$  etc.). For simplicity, we denote the boundary  $\partial\Omega$  by  $M$  from now on.

Since  $\bar{\nabla}\phi$  preserves the measure, we have the equation

$$\det(\bar{\nabla}^2\phi)(x) = \frac{f(x)}{g(\bar{\nabla}\phi(x))} = \omega_m f(x). \quad (2.1)$$

The function  $f(x)$  will be specified later. Also  $\bar{\nabla}\phi$  is the optimal transport map from  $\Omega$  to  $B(0, 1)$ . Therefore  $|\bar{\nabla}\phi| \leq 1$ . Thus

$$|\nabla\phi|^2 + \phi_n^2 \leq 1. \quad (2.2)$$

Here  $n$  denotes the outward unit normal to  $M$ , and  $\phi_n$  denotes the directional derivative of  $\phi$  in this direction. This is a fact that will be frequently used later in the argument.

Now the convexity of  $\phi$  implies  $\bar{\nabla}^2\phi$  is nonnegative definite. Therefore by the geometric-arithmetic inequality  $(\det(\bar{\nabla}^2\phi))^{\frac{k+1}{m}} \leq \frac{1}{\binom{m}{k+1}}\sigma_{k+1}(\bar{\nabla}^2\phi)$ . Thus we obtain, by integrating over  $\Omega$ , that

$$\begin{aligned} \int_{\Omega} (\omega_m f(x))^{\frac{k+1}{m}} dx &= \int_{\Omega} (\det(\bar{\nabla}^2\phi))^{\frac{k+1}{m}} dx \\ &\leq \int_{\Omega} \frac{1}{\binom{m}{k+1}} \sigma_{k+1}(\bar{\nabla}^2\phi) dx \\ &= \int_{\Omega} \frac{1}{(k+1)\binom{m}{k+1}} \bar{\nabla}_{ij}^2\phi [T_k]_{ij}(\bar{\nabla}^2\phi) dx. \end{aligned} \quad (2.3)$$

Here  $[T_k]_{ij}$  is the Newton transformation tensor, defined by  $[T_k]_{ij}(A) := [T_k]_{ij}(\overbrace{A, \dots, A}^k)$  and

$$[T_k]_{ij}(A_1, \dots, A_k) := \frac{1}{k!} \delta_{j,j_1, \dots, j_k}^{i,i_1, \dots, i_k} (A_1)_{i_1 j_1} \cdots (A_k)_{i_k j_k}. \quad (2.4)$$

For example

$$[T_1]_{ij}(A) = Tr(A)\delta_{ij} - A_{ij},$$

where  $Tr(A)$  is the trace of  $A$ . We used in the last line of (2.3) that  $\sigma_{k+1}(A) = \frac{1}{k+1} A_{ij} [T_k]_{ij}(A)$ . For more properties of  $[T_k]_{ij}$ , we refer the readers to [CW11, section 5.1]. Using the fact  $\partial_j(\overline{\nabla}_{ij}^2 \phi) = \partial_i(\overline{\nabla}_{jj}^2 \phi)$ , one can easily show that  $\partial_j([T_k]_{ij})(\overline{\nabla}^2 \phi) = 0$ . Here  $\partial_j$  is the coordinate derivative on  $\mathbb{R}^m$ . Hence we have by the divergence theorem that

$$\begin{aligned} & \int_{\Omega} \frac{1}{(k+1)\binom{m}{k+1}} \overline{\nabla}_{ij}^2 \phi [T_k]_{ij}(\overline{\nabla}^2 \phi) dx \\ &= \frac{1}{(k+1)\binom{m}{k+1}} \int_{\Omega} \partial_j \left( \phi_i [T_k]_{ij}(\overline{\nabla}^2 \phi) \right) dx \\ &= \frac{1}{(k+1)\binom{m}{k+1}} \int_M [T_k]_{ij}(\overline{\nabla}^2 \phi) \phi_i n_j d\mu, \end{aligned} \quad (2.5)$$

where  $n_j$  is the coordinate of the outward unit normal on  $M$ . If we combine (2.3) and (2.5) and specify the probability measure  $f(x)dx := \frac{1}{vol(\Omega)} dx$  on  $\Omega$ , we then get

$$vol(\Omega)^{1-\frac{k+1}{m}} \leq \left(\frac{1}{\omega_m}\right)^{\frac{k+1}{m}} \frac{1}{(k+1)\binom{m}{k+1}} \int_M [T_k]_{ij}(\overline{\nabla}^2 \phi) \phi_i n_j d\mu. \quad (2.6)$$

Comparing the constants in (2.6) (for  $k = 1$  and  $2$  respectively) with the constants in (1.6) and (1.7) of Theorem 1.1 and 1.2 respectively, we notice that to prove these theorems it suffices to establish inequalities (2.7) and (2.8) below.

$$\int_M [T_1]_{ij}(\overline{\nabla}^2 \phi) \phi_i n_j d\mu \leq \int_M H d\mu, \quad (2.7)$$

$$\int_M [T_2]_{ij}(\overline{\nabla}^2 \phi) \phi_i n_j d\mu \leq \int_M \sigma_2(L) d\mu. \quad (2.8)$$

In the following, we will in fact prove two inequalities which are slightly stronger than (2.7) and (2.8).

**Theorem 2.1.** *Let  $\phi$  be a smooth convex function on  $\Omega$ , and  $|\nabla \phi| \leq 1$ . Suppose  $M = \partial\Omega$  is smooth and the second fundamental form  $L \in \Gamma_2^+$ . Then*

$$\int_M [T_1]_{ij}(\overline{\nabla}^2 \phi) \phi_i n_j d\mu \leq \int_M H - \frac{1}{3} [T_1]_{\alpha\beta}(L) \phi_\alpha \phi_\beta d\mu. \quad (2.9)$$

**Theorem 2.2.** *Let  $\phi$  be a smooth convex function on  $\Omega$ , and  $|\nabla\phi| \leq 1$ . Suppose  $M = \partial\Omega$  is smooth and the second fundamental form  $L \in \Gamma_3^+$ . Then*

$$\int_M [T_2]_{ij}(\bar{\nabla}^2\phi)\phi_i n_j d\mu \leq \int_M \sigma_2(L) - \frac{1}{4} \int_M [T_2]_{\alpha\beta}(L)\phi_\alpha\phi_\beta d\mu. \quad (2.10)$$

Note for  $L \in \Gamma_2^+$ ,  $[T_1]_{\alpha\beta}(L) \geq 0$ ; and for  $L \in \Gamma_3^+$ ,  $[T_2]_{\alpha\beta}(L) \geq 0$ . In section 3 and 4, we will prove these two theorems.

We remark that although by our optimal transport method with approximation argument,  $\phi$  is smooth on  $M$ , it is enough to assume  $\phi$  is  $C^3$ , since in the proof we will only take covariant derivatives of  $\phi$  up to the third order. Also, the assumption that  $M$  is smooth can be much weaker. In fact, one only needs to assume  $M$  is  $C^2$  (so that the second fundamental form is well defined) and apply an approximation argument. In this paper, however, we do not want to focus on the problem of looking for the weakest regularity assumption, thus we simply assume  $\phi$  and  $M$  are smooth.

*Proof of Theorem 1.1 and 1.2:* It is clear that inequality (2.9) implies (2.7), which in turn implies (1.6); similarly, inequality (2.10) implies (2.8), which in turn implies (1.7). Now we will show that equalities in (1.6) and (1.7) hold only when  $\Omega$  is a ball. In fact we first claim (1.6) attains the equality only if

$$\bar{\nabla}^2\phi(x) = \lambda(x)Id. \quad (2.11)$$

The proof of this claim is given in the next paragraph. The same argument applies when (1.7) becomes an equality.

We can approximate  $\Omega$  by a sequence of subsets  $\{\Omega_l\}_{l=1}^\infty$  such that  $\Omega_l \Subset \Omega$  and  $\partial\Omega_l \rightarrow \partial\Omega$  in  $C^\infty$  norm. Since  $\Omega_l$  is contained in the interior of  $\Omega$ , and thus  $\phi$  on  $\bar{\Omega}_l$  is a smooth function up to the boundary [Caf92b], we can apply Theorem 2.1 on each  $\Omega_l$  and derive

$$\begin{aligned} \omega_m^{2/m} \text{vol}(\Omega_l) \text{vol}(\Omega)^{-2/m} &= \int_{\Omega_l} \det^{2/m}(\bar{\nabla}^2\phi) dx \\ &\leq \frac{2}{m(m-1)} \int_{\Omega_l} \sigma_2(\bar{\nabla}^2\phi) dx \\ &\leq \frac{1}{m(m-1)} \int_{\partial\Omega_l} H_{\partial\Omega_l} d\mu_l. \end{aligned} \quad (2.12)$$

Here  $H_{\partial\Omega_l}$  denotes the mean curvature of boundary of  $\Omega_l$ ; and  $\mu_l$  denotes the area form of the boundary of  $\Omega_l$ . We then take  $l \rightarrow \infty$ . By dominating convergence theorem, we derive the same inequalities for  $\Omega$ . Namely,

$$\begin{aligned} \omega_m^{2/m} \text{vol}(\Omega)^{1-2/m} &= \int_{\Omega} \det^{2/m}(\bar{\nabla}^2\phi) dx \\ &\leq \frac{2}{m(m-1)} \int_{\Omega} \sigma_2(\bar{\nabla}^2\phi) dx \\ &\leq \frac{1}{m(m-1)} \int_{\partial\Omega} H d\mu. \end{aligned} \quad (2.13)$$

Hence equality is attained only if  $\det^{2/m}(\overline{\nabla}^2\phi) = \frac{2}{m(m-1)}\sigma_2(\overline{\nabla}^2\phi)$ , and thus  $\overline{\nabla}^2\phi(x) = \lambda(x)Id$ . This gives the proof of (2.11).

Now, by our choice  $f(x) = \frac{1}{\text{vol}(\Omega)}$  as well as the equation (2.1), we see that  $\lambda(x)$  must be a constant  $\lambda$  on each connected component of  $\Omega$ , possibly with different values of  $\lambda$  on different components. Thus the unique solution  $\overline{\nabla}\phi$  to the optimal transport problem is a dilation map  $\overline{\nabla}\phi(x) = \lambda x$  on each connected component of  $\Omega$ . Hence each connected component of  $\Omega$  is a ball. However, if  $\Omega$  is the union of two or more balls, we can compute directly that it does not attain the equality in (2.9). Therefore  $\Omega$  is a ball.

**2.2. Elementary facts.** By Taylor expansion

$$(1 - s^2)^{1/2} = 1 - \sum_{k=1}^{\infty} C_k s^{2k}, \quad (2.14)$$

where  $C_k = \binom{2k}{k} \frac{1}{2^{2k}(2k-1)} = \frac{(2k-3)!!}{k!2^k} \geq 0$ . In the following, we always take  $s = |\nabla\phi|$ , and  $\psi := \sqrt{1 - |\nabla\phi|^2}$ . By (2.2),  $0 \leq s \leq 1$ . Formally, the convergence holds uniformly only if  $|s| < 1$ . In the proof of Proposition 3.3, we will assume  $0 \leq s < 1$  and apply the Taylor expansion. Later in the proof of Theorem 2.1, we will use an approximation argument to derive the theorem for general  $0 \leq s \leq 1$ . For  $0 \leq s < 1$ , we have the following facts.

**Fact (a):**

$$\psi = 1 - \sum_{k=1}^{\infty} C_k |\nabla\phi|^{2k}. \quad (2.15)$$

**Fact (b):**

$$\sum_{k=1}^{\infty} 2k C_k |\nabla\phi|^{2(k-1)} \psi \equiv 1. \quad (2.16)$$

*Proof.* If we take derivative on both sides of (2.14), then

$$\frac{-s}{(1 - s^2)^{1/2}} = - \sum_{k=1}^{\infty} 2k C_k s^{2k-1}. \quad (2.17)$$

Let  $s = |\nabla\phi|$ . It then deduces (2.16).  $\square$

Define

$$F(x) := \sum_{k=1}^{\infty} \frac{k}{k+1} C_k |\nabla\phi(x)|^{2(k-1)},$$

and

$$G(x) := \frac{1}{2} - \sum_{k=1}^{\infty} \frac{1}{2(k+1)} C_k |\nabla\phi(x)|^{2k}.$$

$F$  and  $G$  will appear in the proof of Theorem 2.2. If we multiply  $s$  on both sides of (2.14) and integrate over  $[0, s]$ , then we derive

**Fact (c):**  $3G|\nabla\phi|^2 = 1 - \psi^3$ .

By a simple calculation, we also have

**Fact (d):**  $2G = \psi + F|\nabla\phi|^2$ .

**Fact (e):**  $F\psi \leq \frac{1}{4}$ .

*Proof of Fact (e):* Since  $\frac{1}{2(k+1)} \leq \frac{1}{4}$  for  $k \geq 1$ ,

$$\begin{aligned} F\psi &= \left( \sum_{k=1}^{\infty} \frac{k}{k+1} C_k |\nabla\phi|^{2(k-1)} \right) \psi \\ &\leq \frac{1}{4} \sum_{k=1}^{\infty} 2k C_k |\nabla\phi|^{2(k-1)} \psi = \frac{1}{4}, \end{aligned} \tag{2.18}$$

where the last equality uses Fact (b).

### 3. PROOF OF THEOREM 2.1

Consider the isometric embedding  $i : M \rightarrow \mathbb{R}^m$ , where  $M := \partial\Omega$ . For  $x \in M$ , one can write the Hessian of  $\phi$  in coordinates of tangential derivatives and normal derivatives of  $T_x M$ . Let indices  $\alpha, \beta$  with  $\alpha, \beta = 1, \dots, m-1$  be the tangential directions,  $n$  be the outward unit normal direction on  $M$ , and let  $i, j$  with  $i, j = 1, \dots, m$  be the coordinates of  $\mathbb{R}^m$ . It is well known that

$$\bar{\nabla}_{\alpha\beta}^2 \phi = \phi_{\alpha\beta} + L_{\alpha\beta} \phi_n,$$

and

$$\bar{\nabla}_{\alpha n}^2 \phi = \nabla_{\alpha}(\phi_n) - L_{\alpha\beta} \phi_{\beta}.$$

Thus we can decompose  $\bar{\nabla}^2 \phi = A + B$ , where

$$A = \left( \begin{array}{ccc|c} \cdots & \cdots & \cdots & \vdots \\ \cdots & \phi_{\alpha\beta} & \cdots & \nabla_{\alpha}(\phi_n) \\ \cdots & \cdots & \cdots & \vdots \\ \cdots & \nabla_{\alpha}(\phi_n) & \cdots & \bar{\nabla}_{nn}^2 \phi \end{array} \right), \tag{3.1}$$

and

$$B = \left( \begin{array}{ccc|c} \cdots & \cdots & \cdots & \vdots \\ \cdots & L_{\alpha\beta} \phi_n & \cdots & -L_{\alpha\beta} \phi_{\beta} \\ \cdots & \cdots & \cdots & \vdots \\ \cdots & -L_{\alpha\beta} \phi_{\beta} & \cdots & 0 \end{array} \right). \tag{3.2}$$

Define

$$L_k := \int_M [T_{k-1}]_{ij} (A + B) \phi_i n_j d\mu. \tag{3.3}$$



The inequality in Theorem 2.1 is equivalent to

$$L_2 \leq \int_M H - \frac{1}{3} [T_1]_{\alpha\beta}(L) \phi_\alpha \phi_\beta d\mu. \quad (3.4)$$

To prove this, we write  $L_2 = L_{2,1} + L_{2,2}$ , where

$$L_{2,1} := \int_M [T_1]_{ij}(A) \phi_i n_j d\mu,$$

and

$$L_{2,2} := \int_M [T_1]_{ij}(B) \phi_i n_j d\mu.$$

**Proposition 3.1.**

$$L_{2,1} = 2 \int_M \Delta \phi \phi_n d\mu. \quad (3.5)$$

$$L_{2,2} = \int_M H \phi_n^2 + L_{\alpha\beta} \phi_\alpha \phi_\beta d\mu. \quad (3.6)$$

*Proof.* Recall

$$[T_1]_{ij}(A) = \text{Tr}(A) \delta_{ij} - A_{ij},$$

where  $\text{Tr}(A)$  denotes the trace of  $A$ .

$$\begin{aligned} L_{2,1} &= \int_M (\Delta \phi + \phi_{nn}) \phi_n - (\nabla_\alpha(\phi_n) \phi_\alpha + \phi_{nn} \phi_n) d\mu \\ &= \int_M \Delta \phi \phi_n - \nabla_\alpha(\phi_n) \phi_\alpha d\mu \\ &= \int_M 2\Delta \phi \phi_n d\mu. \end{aligned} \quad (3.7)$$

$$\begin{aligned} L_{2,2} &= \int_M (\text{Tr}(B) \delta_{in} - B_{in}) \phi_i d\mu \\ &= \int_M (\text{Tr}(B) \delta_{\alpha n} - B_{\alpha n}) \phi_\alpha d\mu + \int_M (\text{Tr}(B) \delta_{nn} - B_{nn}) \phi_n d\mu \\ &= \int_M H \phi_n^2 + L_{\alpha\beta} \phi_\alpha \phi_\beta d\mu. \end{aligned} \quad (3.8)$$

□

We now define  $M_{2,1}$ , and  $M_{2,2}$  the analogous expressions of  $L_{2,1}$ ,  $L_{2,2}$  in (3.5) and (3.6), with  $\phi_n$  replaced by  $\psi$ :

**Definition 3.2.** *Define*

$$M_{2,1} := 2 \int_M \Delta \phi \psi d\mu, \quad (3.9)$$

and

$$M_{2,2} := \int_M H \psi^2 + L_{\alpha\beta} \phi_\alpha \phi_\beta d\mu. \quad (3.10)$$

**Proposition 3.3.** *Suppose  $\phi$  in addition satisfies  $|\nabla\phi| < 1$ . Then*

$$\begin{aligned} M_{2,1} &\leq \frac{2}{3} \int_M (H\delta_{\alpha\beta} - L_{\alpha\beta})\phi_\alpha\phi_\beta d\mu \\ &= \frac{2}{3} \int_M [T_1]_{\alpha\beta}(L)\phi_\alpha\phi_\beta d\mu. \end{aligned} \quad (3.11)$$

$$M_{2,2} = \int_M H - [T_1]_{\alpha\beta}(L)\phi_\alpha\phi_\beta d\mu. \quad (3.12)$$

We now assert the inequality (2.9) in Theorem 2.1 follows the inequalities in the above proposition. To see this, we have

$$\begin{aligned} L_2 &= L_{2,1} + L_{2,2} \\ &= M_{2,1} + M_{2,2} + \int_M 2\Delta\phi(\phi_n - \psi) + H(\phi_n^2 - \psi^2) d\mu \end{aligned} \quad (3.13)$$

We claim that

$$\int_M 2\Delta\phi(\phi_n - \psi) + H(\phi_n^2 - \psi^2) d\mu \leq 0. \quad (3.14)$$

To see (3.14), we re-write  $\Delta\phi$  as  $\Delta\phi = \bar{\Delta}|_{T_x M}\phi - H\phi_n$ , where  $\bar{\Delta}|_{T_x M}\phi$  denotes the trace of  $\bar{\nabla}^2\phi$  restricted on  $T_x M$ . Since  $\bar{\nabla}^2\phi$  is nonnegative,  $\bar{\Delta}|_{T_x M}\phi \geq 0$ . Therefore, we obtain

$$\begin{aligned} &\int_M 2\Delta\phi(\phi_n - \psi) + H(\phi_n^2 - \psi^2) d\mu \\ &= \int_M 2\bar{\Delta}|_{T_x M}\phi(\phi_n - \psi) - 2H\phi_n(\phi_n - \psi) + H(\phi_n^2 - \psi^2) d\mu \\ &= \int_M 2\bar{\Delta}|_{T_x M}\phi(\phi_n - \psi) - H(\phi_n - \psi)^2 d\mu. \end{aligned} \quad (3.15)$$

Using (2.2),  $\phi_n \leq \psi = \sqrt{1 - |\nabla\phi|^2}$ ; we also have  $\bar{\Delta}|_{T_x M}\phi \geq 0$  and  $H \geq 0$ . Thus (3.14) holds.

We now deduce from (3.13), (3.14) and Proposition 3.3 that for  $\phi$  satisfying  $|\nabla\phi| < 1$ ,

$$\begin{aligned} L_2 &\leq M_{2,1} + M_{2,2} \\ &\leq \int_M H - \frac{1}{3}[T_1]_{\alpha\beta}(L)\phi_\alpha\phi_\beta d\mu. \end{aligned} \quad (3.16)$$

This finishes the proof of Theorem 2.1 for  $\phi$  satisfying  $|\nabla\phi| < 1$ .

If  $|\nabla\phi| \leq 1$ , then we can apply the above argument to the function  $\phi_\delta := (1 - \delta) \cdot \phi$ . It is obvious that  $\phi_\delta$  is a smooth convex function on  $\Omega$ , and  $|\nabla\phi_\delta| < 1$ . Therefore, Theorem 2.1 holds for each individual  $\phi_\delta$ . Namely

$$\int_M [T_1]_{ij}(\bar{\nabla}^2\phi_\delta)(\phi_\delta)_i n_j d\mu \leq \int_M H - \frac{1}{3}[T_1]_{\alpha\beta}(L)(\phi_\delta)_\alpha(\phi_\delta)_\beta d\mu.$$

When  $\delta \rightarrow 0$ ,  $\phi_\delta$  converges to  $\phi$  in  $C^\infty$  on  $M$ . Thus the above inequality is also valid for  $\phi$ . This finishes the proof of Theorem 2.1 for general  $|\nabla\phi| \leq 1$ .

We now begin the proof of Proposition 3.3. The strategy we will apply is to expand  $\psi$  by the Taylor series and to derive a recursive inequality for each individual term  $\int_M \Delta\phi |\nabla\phi|^{2k} d\mu$  in the Taylor series. Let us begin with the following lemma.

**Lemma 3.4.** *Define*

$$J_k := \int_M \Delta\phi |\nabla\phi|^{2k} d\mu.$$

Then

$$J_k = \frac{2k}{2k+1} \int_M [T_1]_{\alpha\beta} (\nabla^2\phi) \phi_\alpha \phi_\beta |\nabla\phi|^{2(k-1)} d\mu. \quad (3.17)$$

*Proof.* By integration by parts

$$\begin{aligned} J_k &:= \int_M \Delta\phi |\nabla\phi|^{2k} d\mu \\ &= \int_M -2k \phi_\alpha \phi_\beta \phi_{\alpha\beta} |\nabla\phi|^{2(k-1)} d\mu. \end{aligned} \quad (3.18)$$

By the definition of  $[T_1]$ ,  $\phi_{\alpha\beta} = \Delta\phi \delta_{\alpha\beta} - [T_1]_{\alpha\beta} (\nabla^2\phi)$ . Then

$$\begin{aligned} J_k &= \int_M -2k \Delta\phi |\nabla\phi|^{2k} d\mu \\ &\quad + \int_M 2k [T_1]_{\alpha\beta} (\nabla^2\phi) \phi_\alpha \phi_\beta |\nabla\phi|^{2(k-1)} d\mu \\ &= -2kJ_k + \int_M 2k [T_1]_{\alpha\beta} (\nabla^2\phi) \phi_\alpha \phi_\beta |\nabla\phi|^{2(k-1)} d\mu. \end{aligned} \quad (3.19)$$

Thus

$$J_k = \frac{2k}{2k+1} \int_M [T_1]_{\alpha\beta} (\nabla^2\phi) \phi_\alpha \phi_\beta |\nabla\phi|^{2(k-1)} d\mu. \quad (3.20)$$

This finishes the proof of Lemma 3.4.  $\square$

*Proof of Proposition 3.3:* First, by Taylor expansion

$$\begin{aligned} M_{2,1} &:= \int_M 2\Delta\phi\psi d\mu \\ &= 2 \int_M \Delta\phi \left(1 - \sum_{k=1}^{\infty} C_k |\nabla\phi|^{2k}\right) d\mu. \end{aligned} \quad (3.21)$$

Notice that  $\int_M \Delta\phi d\mu = 0$  and for  $|\nabla\phi| < 1$  the convergence in the Taylor expansion is uniform. Thus

$$M_{2,1} = -2 \sum_{k=1}^{\infty} C_k J_k,$$

where  $J_k := \int_M \Delta\phi |\nabla\phi|^{2k} d\mu$ . Using Lemma 3.4, we get

$$M_{2,1} = -2 \sum_{k=1}^{\infty} \frac{2k}{2k+1} C_k \int_M [T_1]_{\alpha\beta} (\nabla^2\phi) \phi_\alpha \phi_\beta |\nabla\phi|^{2(k-1)} d\mu.$$

Since  $\nabla_{\alpha\beta}^2\phi = \bar{\nabla}_{\alpha\beta}^2\phi - L_{\alpha\beta}\phi_n$ ,

$$\begin{aligned} M_{2,1} &= - \sum_{k=1}^{\infty} \frac{4k}{2k+1} C_k \int_M [T_1]_{\alpha\beta}(\bar{\nabla}^2\phi) \phi_{\alpha}\phi_{\beta} |\nabla\phi|^{2(k-1)} d\mu \\ &\quad + \sum_{k=1}^{\infty} \frac{4k}{2k+1} C_k \int_M [T_1]_{\alpha\beta}(L)\phi_n\phi_{\alpha}\phi_{\beta} |\nabla\phi|^{2(k-1)} d\mu. \end{aligned} \quad (3.22)$$

Since  $[T_1]_{\alpha\beta}(\bar{\nabla}^2\phi) \geq 0$ , the first sum in (3.22) is non-positive. Also, in the second sum, we notice  $\frac{2}{2k+1} \leq \frac{2}{3}$  for any  $k \geq 1$ . Therefore

$$M_{2,1} \leq \frac{2}{3} \int_M [T_1]_{\alpha\beta}(L)\phi_{\alpha}\phi_{\beta} \left( \sum_{k=1}^{\infty} 2kC_k |\nabla\phi|^{2(k-1)} \right) \phi_n d\mu. \quad (3.23)$$

Now by Fact (b),

$$\left( \sum_{k=1}^{\infty} 2kC_k |\nabla\phi|^{2(k-1)} \right) \phi_n \leq \left( \sum_{k=1}^{\infty} 2kC_k |\nabla\phi|^{2(k-1)} \right) \psi = 1.$$

Also we have  $[T_1]_{\alpha\beta}(L)\phi_{\alpha}\phi_{\beta} \geq 0$ . Therefore

$$\frac{2}{3} \int_M [T_1]_{\alpha\beta}(L)\phi_{\alpha}\phi_{\beta} \left( \sum_{k=1}^{\infty} 2kC_k |\nabla\phi|^{2(k-1)} \right) \phi_n d\mu \leq \frac{2}{3} \int_M [T_1]_{\alpha\beta}(L)\phi_{\alpha}\phi_{\beta} d\mu.$$

This completes the proof for  $M_{2,1}$ .

For the term  $M_{2,2}$ , it is straightforward to see that

$$\begin{aligned} M_{2,2} &:= \int_M H\psi^2 + L_{\alpha\beta}\phi_{\alpha}\phi_{\beta} d\mu \\ &= \int_M H - H|\nabla\phi|^2 + L_{\alpha\beta}\phi_{\alpha}\phi_{\beta} d\mu \\ &= \int_M H - [T_1]_{\alpha\beta}(L)\phi_{\alpha}\phi_{\beta} d\mu. \end{aligned} \quad (3.24)$$

This finishes the proof of Proposition 3.3.

#### 4. PROOF OF THEOREM 2.2

In this section, we will prove

$$L_3 := \int_M [T_2]_{ij}(\bar{\nabla}^2\phi) \phi_i n_j d\mu \leq \int_{\partial\Omega} \sigma_2(L) - \frac{1}{4} \int_M [T_2]_{\alpha\beta}(L)\phi_{\alpha}\phi_{\beta} d\mu.$$

To prove this, we first decompose  $L_3$ , using the multi-linearity of  $[T_2]_{ij}(\cdot)$ , into  $L_3 = L_{3,1} + L_{3,2} + L_{3,3}$ , where

$$\begin{aligned} L_{3,1} &:= \int_M [T_2]_{ij}(A, A)\phi_i n_j d\mu, \\ L_{3,2} &:= 2 \int_M [T_2]_{ij}(A, B)\phi_i n_j d\mu, \end{aligned}$$

and

$$L_{3,3} := \int_M [T_2]_{ij}(B, B) \phi_i n_j d\mu.$$

$A$  and  $B$  are as defined in (3.1) and (3.2).

**Proposition 4.1.**

$$L_{3,1} = 3 \int_M \sigma_2(\nabla^2 \phi) \phi_n d\mu - \int_M Ric_{\alpha\beta} \phi_\alpha \phi_\beta \phi_n d\mu. \quad (4.1)$$

$$L_{3,2} = \frac{3}{2} \int_M \Sigma_2(\nabla^2 \phi, L) \phi_n^2 d\mu + \int_M [T_1]_{\alpha\beta}(\nabla^2 \phi) L_{\beta\gamma} \phi_\alpha \phi_\gamma d\mu. \quad (4.2)$$

And

$$L_{3,3} = \int_M \sigma_2(L) \phi_n^3 d\mu + \int_M [T_1]_{\alpha\beta}(L) L_{\beta\gamma} \phi_\alpha \phi_\gamma \phi_n d\mu. \quad (4.3)$$

Here  $\Sigma_2(A, B) := Tr(A)Tr(B) - Tr(AB)$  for any two tensors  $A$  and  $B$ . It is the linear polarization of  $\sigma_2(\cdot)$  (up to a multiplicative constant) in the sense that  $\Sigma_2(A, A) = 2\sigma_2(A)$ , and it is symmetric and linear with respect to  $A$  and  $B$ . For polarization of  $\sigma_k$ , we refer to [CW11, CW]. The proof of Proposition 4.1 is by direct computation. Thus we omit it here. Now as what we did in the previous section, we define  $M_{3,1}$ ,  $M_{3,2}$ ,  $M_{3,3}$ , simply by substituting  $\phi_n$  by  $\psi$  in the formulas of Proposition 4.1.

**Definition 4.2.**

$$M_{3,1} = 3 \int_M \sigma_2(\nabla^2 \phi) \psi d\mu - \int_M Ric_{\alpha\beta} \phi_\alpha \phi_\beta \psi d\mu. \quad (4.4)$$

$$M_{3,2} = \frac{3}{2} \int_M \Sigma_2(\nabla^2 \phi, L) \psi^2 d\mu + \int_M [T_1]_{\alpha\beta}(\nabla^2 \phi) L_{\beta\gamma} \phi_\alpha \phi_\gamma d\mu. \quad (4.5)$$

And

$$M_{3,3} = \int_M \sigma_2(L) \psi^3 d\mu + \int_M [T_1]_{\alpha\beta}(L) L_{\beta\gamma} \phi_\alpha \phi_\gamma \psi d\mu. \quad (4.6)$$

We first simplify the formula of  $M_{3,2}$ .

**Proposition 4.3.**

$$M_{3,2} = -2 \int_M [T_2]_{\alpha\beta}(\nabla^2 \phi, L) \phi_\alpha \phi_\beta d\mu. \quad (4.7)$$

*Proof.* Since  $\psi^2 = 1 - |\nabla \phi|^2$ , applying integration by parts we have

$$\begin{aligned} & \int_M \Sigma_2(\nabla^2 \phi, L) \psi^2 d\mu \\ &= \int_M (\Delta \phi H - \phi_{\alpha\beta} L_{\alpha\beta}) (1 - |\nabla \phi|^2) d\mu \\ &= - \int_M \phi_\alpha H_\alpha (1 - |\nabla \phi|^2) + \phi_\alpha H (-2\phi_\gamma \phi_{\gamma\alpha}) d\mu \\ & \quad + \int_M \phi_\alpha L_{\alpha\beta, \beta} (1 - |\nabla \phi|^2) + \phi_\alpha L_{\alpha\beta} (-2\phi_\gamma \phi_{\gamma\beta}) d\mu \end{aligned} \quad (4.8)$$

By Codazzi equation  $H_\alpha = L_{\alpha\beta,\beta}$ . Hence the first term and the third term in the last equality above are canceled. So we have

$$\begin{aligned} & \int_M \Sigma_2(\nabla^2\phi, L)\psi^2 d\mu \\ &= 2 \int_M (\phi_\alpha H\phi_\gamma\phi_{\gamma\alpha} - \phi_\alpha L_{\alpha\beta}\phi_\gamma\phi_{\gamma\beta})d\mu \\ &= 2 \int_M \phi_\alpha\phi_\gamma[T_1]_{\alpha\beta}(L)\phi_{\beta\gamma}d\mu. \end{aligned} \quad (4.9)$$

Substituting  $\frac{1}{2} \int_M \Sigma_2(\nabla^2\phi, L)\psi^2 d\mu$  by formula (4.9) in  $M_{3,2}$ , we have

$$\begin{aligned} M_{3,2} &= (1 + \frac{1}{2}) \int_M \Sigma_2(\nabla^2\phi, L)\psi^2 d\mu + \int_M [T_1]_{\alpha\beta}(\nabla^2\phi)L_{\beta\gamma}\phi_\alpha\phi_\gamma d\mu \\ &= \int_M \Sigma_2(\nabla^2\phi, L)\psi^2 + \phi_\alpha\phi_\gamma[T_1]_{\alpha\beta}(L)\phi_{\beta\gamma} + [T_1]_{\alpha\beta}(\nabla^2\phi)L_{\beta\gamma}\phi_\alpha\phi_\gamma d\mu \\ &= \int_M \Sigma_2(\nabla^2\phi, L)(1 - |\nabla\phi|^2) + \phi_\alpha\phi_\gamma[T_1]_{\alpha\beta}(L)\phi_{\beta\gamma} \\ &\quad + [T_1]_{\alpha\beta}(\nabla^2\phi)L_{\beta\gamma}\phi_\alpha\phi_\gamma d\mu. \end{aligned} \quad (4.10)$$

By the Codazzi equation again,

$$\begin{aligned} \int_M \Sigma_2(\nabla^2\phi, L)d\mu &= \int_M \phi_\alpha H_\alpha - \phi_\alpha L_{\alpha\beta,\beta}d\mu \\ &= 0, \end{aligned} \quad (4.11)$$

Therefore (4.10) implies

$$\begin{aligned} M_{3,2} &= \int_M \Sigma_2(\nabla^2\phi, L)(-|\nabla\phi|^2) + \phi_\alpha\phi_\gamma[T_1]_{\alpha\beta}(L)\phi_{\beta\gamma} \\ &\quad + [T_1]_{\alpha\beta}(\nabla^2\phi)L_{\beta\gamma}\phi_\alpha\phi_\gamma d\mu. \end{aligned} \quad (4.12)$$

Using the fact that for tensors  $A_{\alpha\beta}, B_{\alpha\beta}$

$$2[T_2]_{\alpha\gamma}(A, B) = \Sigma_2(A, B)\delta_{\alpha\gamma} - [T_1]_{\alpha\beta}(A)B_{\beta\gamma} - [T_1]_{\alpha\beta}(B)A_{\beta\gamma}, \quad (4.13)$$

we have

$$M_{3,2} = -2 \int_M [T_2]_{\alpha\beta}(\nabla^2\phi, L)\phi_\alpha\phi_\beta d\mu. \quad (4.14)$$

□

**Proposition 4.4.** *Define*

$$A_k := \int_M \sigma_2(\nabla^2\phi)|\nabla\phi|^{2k} d\mu.$$

Then

$$\begin{aligned} A_k &= \frac{k}{k+1} \int_M [T_2]_{\alpha\beta}(\nabla^2\phi, \nabla^2\phi)\phi_\alpha\phi_\beta|\nabla\phi|^{2(k-1)}d\mu \\ &\quad + \frac{1}{2(k+1)} \int_M Ric_{\alpha\beta}\phi_\alpha\phi_\beta|\nabla\phi|^{2k}d\mu. \end{aligned} \quad (4.15)$$

*Proof.* Applying properties of  $T_1$  and the integration by parts, we have

$$\begin{aligned} A_k &:= \int_M \sigma_2(\nabla^2\phi)|\nabla\phi|^{2k}d\mu \\ &= \int_M \frac{1}{2}\phi_{\alpha\beta}[T_1]_{\alpha\beta}(\nabla^2\phi)|\nabla\phi|^{2k}d\mu \\ &= \int_M -\frac{1}{2}\phi_\alpha([T_1]_{\alpha\beta}(\nabla^2\phi))_\beta|\nabla\phi|^{2k} - \frac{1}{2}\phi_\alpha[T_1]_{\alpha\beta}(\nabla^2\phi)(|\nabla\phi|^{2k})_\beta d\mu. \end{aligned} \quad (4.16)$$

$$([T_1]_{\alpha\beta}(\nabla^2\phi))_\beta = (\Delta\phi\delta_{\alpha\beta} - \phi_{\alpha\beta})_\beta = -Ric_{\alpha\beta}\phi_\beta, \quad (4.17)$$

where  $Ric_{\alpha\beta}$  denotes the Ricci curvature of  $M$ . Therefore (4.16) becomes

$$A_k = \int_M \frac{1}{2}\phi_\alpha\phi_\beta Ric_{\alpha\beta}|\nabla\phi|^{2k} - k\phi_\alpha[T_1]_{\alpha\beta}(\nabla^2\phi)\phi_\gamma\phi_{\gamma\beta}|\nabla\phi|^{2(k-1)}d\mu. \quad (4.18)$$

We now notice that  $[T_1]_{\alpha\beta}(\nabla^2\phi)\phi_{\gamma\beta} = \sigma_2(\nabla^2\phi)\delta_{\alpha\gamma} - [T_2]_{\alpha\gamma}(\nabla^2\phi, \nabla^2\phi)$ . Thus

$$\begin{aligned} A_k &= \int_M \left[ \frac{1}{2}\phi_\alpha\phi_\beta Ric_{\alpha\beta}|\nabla\phi|^{2k} \right. \\ &\quad \left. - k\phi_\alpha(\sigma_2(\nabla^2\phi)\delta_{\alpha\gamma} - [T_2]_{\alpha\gamma}(\nabla^2\phi, \nabla^2\phi))\phi_\gamma|\nabla\phi|^{2(k-1)} \right] d\mu \\ &= \int_M \frac{1}{2}\phi_\alpha\phi_\beta Ric_{\alpha\beta}|\nabla\phi|^{2k} - kA_k + k\phi_\alpha[T_2]_{\alpha\gamma}(\nabla^2\phi, \nabla^2\phi)\phi_\gamma|\nabla\phi|^{2(k-1)}d\mu. \end{aligned} \quad (4.19)$$

Therefore

$$\begin{aligned} A_k &= \frac{k}{k+1} \int_M [T_2]_{\alpha\beta}(\nabla^2\phi, \nabla^2\phi)\phi_\alpha\phi_\beta|\nabla\phi|^{2(k-1)}d\mu \\ &\quad + \frac{1}{2(k+1)} \int_M Ric_{\alpha\beta}\phi_\alpha\phi_\beta|\nabla\phi|^{2k}d\mu. \end{aligned} \quad (4.20)$$

□

We will now assume  $|\nabla\phi| < 1$  as in the previous section.

**Corollary 4.5.** *Suppose  $\phi$  in addition satisfies  $|\nabla\phi| < 1$ . Then*

$$\begin{aligned} M_{3,1} &= 3 \int_M \sigma_2(\nabla^2\phi)\psi d\mu - \int_M Ric_{\alpha\beta}\phi_\alpha\phi_\beta\psi d\mu \\ &= -3 \int_M [T_2]_{\alpha\beta}(\nabla^2\phi, \nabla^2\phi)\phi_\alpha\phi_\beta F d\mu \\ &\quad + 3 \int_M Ric_{\alpha\beta}\phi_\alpha\phi_\beta G d\mu - \int_M Ric_{\alpha\beta}\phi_\alpha\phi_\beta\psi d\mu, \end{aligned} \quad (4.21)$$

where

$$F(x) := \sum_{k=1}^{\infty} \frac{k}{k+1} C_k |\nabla \phi|^{2(k-1)},$$

$$G(x) := \frac{1}{2} - \sum_{k=1}^{\infty} \frac{1}{2(k+1)} C_k |\nabla \phi|^{2k}.$$

*Proof.* By Fact (a),

$$\begin{aligned} M_{3,1} &= 3 \int_M \sigma_2(\nabla^2 \phi) \psi d\mu - \int_M Ric_{\alpha\beta} \phi_\alpha \phi_\beta \psi d\mu \\ &= 3 \int_M \sigma_2(\nabla^2 \phi) \left(1 - \sum_{k=1}^{\infty} C_k |\nabla \phi|^{2k}\right) d\mu - \int_M Ric_{\alpha\beta} \phi_\alpha \phi_\beta \psi d\mu \quad (4.22) \\ &= 3 \int_M \sigma_2(\nabla^2 \phi) d\mu - 3 \sum_{k=1}^{\infty} C_k A_k - \int_M Ric_{\alpha\beta} \phi_\alpha \phi_\beta \psi d\mu. \end{aligned}$$

Substituting the formula of  $A_k$  in Proposition 4.4 and the equality that

$$\begin{aligned} \int_M \sigma_2(\nabla^2 \phi) d\mu &= \frac{1}{2} \int_M \phi_{\alpha\beta} (\Delta \phi \delta_{\alpha\beta} - \phi_{\alpha\beta}) d\mu \\ &= \frac{1}{2} \int_M -\phi_\alpha ((\Delta \phi)_\alpha - \phi_{\alpha\beta,\beta}) d\mu \quad (4.23) \\ &= \frac{1}{2} \int_M Ric_{\alpha\beta} \phi_\alpha \phi_\beta d\mu, \end{aligned}$$

in (4.22), we derive (4.21).  $\square$

Applying Proposition 4.3 and Corollary 4.5, we get

$$\begin{aligned} &M_{3,1} + M_{3,2} + M_{3,3} \\ &= -3 \int_M [T_2]_{\alpha\beta} (\nabla^2 \phi, \nabla^2 \phi) \phi_\alpha \phi_\beta F d\mu + 3 \int_M Ric_{\alpha\beta} \phi_\alpha \phi_\beta G d\mu \\ &\quad - 2 \int_M [T_2]_{\alpha\beta} (\nabla^2 \phi, L) \phi_\alpha \phi_\beta d\mu + \int_M \sigma_2(L) \psi^3 d\mu \quad (4.24) \\ &\quad + \int_M Ric_{\alpha\beta} \phi_\alpha \phi_\beta \psi d\mu - \int_M [T_1]_{\alpha\beta}(L) L_{\beta\gamma} \phi_\alpha \phi_\gamma \psi d\mu. \end{aligned}$$

By the Gauss equation,

$$Ric_{\alpha\beta} = [T_1]_{\alpha\gamma}(L) L_{\gamma\beta}. \quad (4.25)$$

Thus the last two terms in (4.24) are canceled. Therefore

$$\begin{aligned} &M_{3,1} + M_{3,2} + M_{3,3} \\ &= -3 \int_M [T_2]_{\alpha\beta} (\nabla^2 \phi, \nabla^2 \phi) \phi_\alpha \phi_\beta F d\mu + 3 \int_M Ric_{\alpha\beta} \phi_\alpha \phi_\beta G d\mu \quad (4.26) \\ &\quad - 2 \int_M [T_2]_{\alpha\beta} (\nabla^2 \phi, L) \phi_\alpha \phi_\beta d\mu + \int_M \sigma_2(L) \psi^3 d\mu. \end{aligned}$$



We now apply Fact (c):  $3G|\nabla\phi|^2 = 1 - \psi^3$  to combine the second and the last term in (4.26)

$$\begin{aligned} & 3 \int_M Ric_{\alpha\beta} \phi_\alpha \phi_\beta G d\mu + \int_M \sigma_2(L) \psi^3 d\mu \\ &= 3 \int_M Ric_{\alpha\beta} \phi_\alpha \phi_\beta G d\mu + \int_M \sigma_2(L) (1 - 3G|\nabla\phi|^2) d\mu \end{aligned} \quad (4.27)$$

By (4.25) and the fact that

$$[T_1]_{\alpha\gamma}(L) L_{\gamma\beta} - \sigma_2(L) \delta_{\alpha\beta} = [T_2]_{\alpha\beta}(L, L), \quad (4.28)$$

(4.27) is equal to

$$\begin{aligned} & 3 \int_M [T_1]_{\alpha\gamma}(L) L_{\gamma\beta} \phi_\alpha \phi_\beta G d\mu + \int_M \sigma_2(L) (1 - 3G|\nabla\phi|^2) d\mu \\ &= \int_M \sigma_2(L) d\mu - 3 \int_M [T_2]_{\alpha\beta}(L, L) \phi_\alpha \phi_\beta G d\mu. \end{aligned} \quad (4.29)$$

In conclusion (4.26) is deduced to

$$M_{3,1} + M_{3,2} + M_{3,3} = \int_M \sigma_2(L) d\mu + E_1 + E_2 + E_3, \quad (4.30)$$

where

$$\begin{aligned} E_1 &= -3 \int_M [T_2]_{\alpha\beta}(\nabla^2\phi, \nabla^2\phi) \phi_\alpha \phi_\beta F d\mu \\ E_2 &= -2 \int_M [T_2]_{\alpha\beta}(\nabla^2\phi, L) \phi_\alpha \phi_\beta d\mu \\ E_3 &= -3 \int_M [T_2]_{\alpha\beta}(L, L) \phi_\alpha \phi_\beta G d\mu. \end{aligned} \quad (4.31)$$

In the following we will prove

**Proposition 4.6.** *If  $\phi$  in addition satisfies  $|\nabla\phi| < 1$ , then*

$$E_1 + E_2 + E_3 \leq -\frac{1}{4} \int_M [T_2]_{\alpha\beta}(L) \phi_\alpha \phi_\beta d\mu. \quad (4.32)$$

*From this, it is obvious that for  $\phi$  satisfying  $|\nabla\phi| < 1$ ,*

$$M_{3,1} + M_{3,2} + M_{3,3} \leq \int_M \sigma_2(L) d\mu - \frac{1}{4} \int_M [T_2]_{\alpha\beta}(L) \phi_\alpha \phi_\beta d\mu. \quad (4.33)$$

*Proof.* By  $\nabla_{\alpha\beta}^2\phi = \bar{\nabla}_{\alpha\beta}^2\phi - L_{\alpha\beta}\phi_n$ , we obtain

$$\begin{aligned}
E_1 &= -3 \int_M [T_2]_{\alpha\beta}(\bar{\nabla}^2\phi - L\phi_n, \bar{\nabla}^2\phi - L\phi_n)\phi_\alpha\phi_\beta F d\mu \\
&= -3 \int_M [T_2]_{\alpha\beta}(\bar{\nabla}^2\phi, \bar{\nabla}^2\phi)\phi_\alpha\phi_\beta F d\mu \\
&\quad + 6 \int_M [T_2]_{\alpha\beta}(\bar{\nabla}^2\phi, L)\phi_n\phi_\alpha\phi_\beta F d\mu \\
&\quad - 3 \int_M [T_2]_{\alpha\beta}(L, L)\phi_n^2\phi_\alpha\phi_\beta F d\mu \\
&=: E_{1,1} + E_{1,2} + E_{1,3}
\end{aligned} \tag{4.34}$$

Since  $\phi$  is a convex function,  $\bar{\nabla}^2\phi$  is nonnegative. Therefore

$$[T_2]_{\alpha\beta}(\bar{\nabla}^2\phi, \bar{\nabla}^2\phi) \geq 0. \tag{4.35}$$

This together with  $F \geq 0$  implies that  $E_{1,1} \leq 0$ . So  $E_1 \leq E_{1,2} + E_{1,3}$ .

For  $E_2$ , using  $\nabla_{\alpha\beta}^2\phi = \bar{\nabla}_{\alpha\beta}^2\phi - L_{\alpha\beta}\phi_n$ , we get

$$\begin{aligned}
E_2 &= -2 \int_M [T_2]_{\alpha\beta}(\bar{\nabla}^2\phi, L)\phi_\alpha\phi_\beta d\mu \\
&\quad + 2 \int_M [T_2]_{\alpha\beta}(L, L)\phi_n\phi_\alpha\phi_\beta d\mu \\
&:= E_{2,1} + E_{2,2}.
\end{aligned} \tag{4.36}$$

We next observe that

**Lemma 4.7.**

$$E_{1,2} + E_{2,1} \leq -\frac{1}{2} \int_M [T_2]_{\alpha\beta}(\bar{\nabla}^2\phi, L)\phi_\alpha\phi_\beta d\mu \leq 0. \tag{4.37}$$

*Proof.*

$$E_{1,2} + E_{2,1} = \int_M [T_2]_{\alpha\beta}(\bar{\nabla}^2\phi, L)\phi_\alpha\phi_\beta(6F\phi_n - 2) d\mu. \tag{4.38}$$

On the one hand,  $6F\phi_n - 2 \leq -\frac{1}{2}$  because  $F\phi_n \leq F\psi$ , and  $F\psi \leq \frac{1}{4}$  (Fact (e)); on the other hand, since  $L_{\alpha\beta} \in \Gamma_3^+$  and  $\bar{\nabla}^2\phi \geq 0$ , we have

$$[T_2]_{\alpha\beta}(\bar{\nabla}^2\phi, L) \geq 0. \tag{4.39}$$

Therefore

$$E_{1,2} + E_{2,1} \leq -\frac{1}{2} \int_M [T_2]_{\alpha\beta}(\bar{\nabla}^2\phi, L)\phi_\alpha\phi_\beta d\mu \leq 0. \tag{4.40}$$

□

We continue the proof of Proposition 4.6. Applying Lemma 4.7, we have

$$\begin{aligned}
E_1 + E_2 + E_3 &\leq E_{1,3} + E_{2,2} + E_3 \\
&= \int_M [T_2]_{\alpha\beta}(L)\phi_\alpha\phi_\beta P(x) d\mu,
\end{aligned} \tag{4.41}$$

where  $P(x) := -3F\phi_n^2 + 2\phi_n - 3G$ . By Fact (d),  $2G = \psi + F|\nabla\phi|^2$ . Thus

$$P = -3F\phi_n^2 + 2\phi_n - \frac{3}{2}\psi - \frac{3}{2}|\nabla\phi|^2.$$

It is not hard to show  $P \leq -\frac{1}{4}$ . In fact,

$$\begin{aligned} P &= -3F\phi_n^2 + 2\phi_n - \frac{3}{2}\psi - \frac{3}{2}F|\nabla\phi|^2 \\ &= -3\sum_{k=1}^{\infty} \frac{k}{k+1} C_k |\nabla\phi|^{2(k-1)} \phi_n^2 + 2\phi_n - \frac{3}{2}\psi - \frac{3}{2}\sum_{k=1}^{\infty} \frac{k}{k+1} C_k |\nabla\phi|^{2k}. \end{aligned} \quad (4.42)$$

To estimate  $-3\sum_{k=1}^{\infty} \frac{k}{k+1} C_k |\nabla\phi|^{2(k-1)} \phi_n^2$  in (4.42), we observe  $C_k \geq 0$ , and  $C_1 = \frac{1}{2}$ . So we drop the sum over  $k \geq 2$  and get

$$-3\sum_{k=1}^{\infty} \frac{k}{k+1} C_k |\nabla\phi|^{2(k-1)} \phi_n^2 \leq -\frac{3}{4}\phi_n^2. \quad (4.43)$$

And to estimate  $-\frac{3}{2}\sum_{k=1}^{\infty} \frac{k}{k+1} C_k |\nabla\phi|^{2k}$  in (4.42), we use the fact that  $\frac{k}{k+1} \geq \frac{1}{2}$  for all  $k \geq 1$ , and Fact (a):  $\psi = 1 - \sum_{k=1}^{\infty} C_k |\nabla\phi|^{2k}$  to get

$$-\frac{3}{2}\sum_{k=1}^{\infty} \frac{k}{k+1} C_k |\nabla\phi|^{2k} \leq -\frac{3}{4}\sum_{k=1}^{\infty} C_k |\nabla\phi|^{2k} = -\frac{3}{4}(1 - \psi). \quad (4.44)$$

Applying (4.43) and (4.44) to (4.42), we have

$$\begin{aligned} P &\leq -\frac{3}{4}\phi_n^2 + 2\phi_n - \frac{3}{2}\psi - \frac{3}{4}(1 - \psi) \\ &\leq -\frac{3}{4}\phi_n^2 + 2\phi_n - \frac{3}{4}\phi_n - \frac{3}{4}, \end{aligned} \quad (4.45)$$

due to the fact that  $\phi_n \leq \psi$ . Now

$$-\frac{3}{4}\phi_n^2 + 2\phi_n - \frac{3}{4}\phi_n - \frac{3}{4} = -\frac{1}{4}(3\phi_n - 2)(\phi_n - 1) - \frac{1}{4} \leq -\frac{1}{4}, \quad (4.46)$$

for  $\phi_n \leq 1$ . Thus we obtain  $P \leq -\frac{1}{4}$ . By (4.41), it concludes that

$$E_1 + E_2 + E_3 \leq -\frac{1}{4} \int_M [T_2]_{\alpha\beta}(L) \phi_\alpha \phi_\beta d\mu \leq 0. \quad (4.47)$$

This completes the proof of Proposition 4.6 □

Finally, we are ready to give the proof of Theorem 2.2 using Proposition 4.6.

*Proof.* By Proposition 4.1 and Definition 4.2

$$\begin{aligned} L_3 &= L_{3,1} + L_{3,2} + L_{3,3} \\ &= M_{3,1} + M_{3,2} + M_{3,3} + \int_M 3\sigma_2(\nabla^2\phi)(\phi_n - \psi) \\ &\quad + \frac{3}{2}\Sigma_2(\nabla^2\phi, L)(\phi_n^2 - \psi^2) + \sigma_2(L)(\phi_n^3 - \psi^3) d\mu. \end{aligned} \quad (4.48)$$

We will prove

$$\int_M 3\sigma_2(\nabla^2\phi)(\phi_n - \psi) + \frac{3}{2}\Sigma_2(\nabla^2\phi, L)(\phi_n^2 - \psi^2) + \sigma_2(L)(\phi_n^3 - \psi^3)d\mu \leq 0. \quad (4.49)$$

Recall that  $\Sigma_2(A, B) := \text{Tr}(A)\text{Tr}(B) - \text{Tr}(AB)$ . This immediately implies that

$$\sigma_2(A) = \frac{1}{2}\Sigma_2(A, A).$$

Using  $\nabla_{\alpha\beta}^2\phi = \bar{\nabla}_{\alpha\beta}^2\phi - L_{\alpha\beta}\phi_n$  and the linearity of  $\Sigma_2(\cdot, \cdot)$ , first of all

$$\begin{aligned} & 3 \int_M \sigma_2(\nabla^2\phi)(\phi_n - \psi)d\mu \\ &= \frac{3}{2} \int_M \Sigma_2(\bar{\nabla}_{\alpha\beta}^2\phi - L_{\alpha\beta}\phi_n, \bar{\nabla}_{\alpha\beta}^2\phi - L_{\alpha\beta}\phi_n)(\phi_n - \psi)d\mu \\ &= \int_M 3\sigma_2(\bar{\nabla}^2\phi)(\phi_n - \psi) - 3\Sigma_2(\bar{\nabla}^2\phi, L)\phi_n(\phi_n - \psi) + 3\sigma_2(L)\phi_n^2(\phi_n - \psi)d\mu. \end{aligned} \quad (4.50)$$

Since  $\sigma_2(\bar{\nabla}^2\phi) \geq 0$  and  $\phi_n \leq \psi$ , the first term in the above line is non-positive. Thus

$$\begin{aligned} & 3 \int_M \sigma_2(\nabla^2\phi)(\phi_n - \psi)d\mu \\ & \leq \int_M -3\Sigma_2(\bar{\nabla}^2\phi, L)\phi_n(\phi_n - \psi) + 3\sigma_2(L)\phi_n^2(\phi_n - \psi)d\mu, \end{aligned} \quad (4.51)$$

Secondly,

$$\begin{aligned} & \frac{3}{2}\Sigma_2(\nabla^2\phi, L)(\phi_n^2 - \psi^2) \\ &= \frac{3}{2}\Sigma_2(\bar{\nabla}^2\phi, L)(\phi_n^2 - \psi^2) - 3\sigma_2(L)\phi_n(\phi_n^2 - \psi^2). \end{aligned} \quad (4.52)$$

Using (4.51) and (4.52), we have

$$\begin{aligned} & \int_M 3\sigma_2(\nabla^2\phi)(\phi_n - \psi) + \frac{3}{2}\Sigma_2(\nabla^2\phi, L)(\phi_n^2 - \psi^2) + \sigma_2(L)(\phi_n^3 - \psi^3)d\mu \\ & \leq -3 \int_M \Sigma_2(\bar{\nabla}^2\phi, L) \left( \phi_n(\phi_n - \psi) - \frac{1}{2}(\phi_n^2 - \psi^2) \right) d\mu \\ & \quad + \int_M \sigma_2(L) (3\phi_n^2(\phi_n - \psi) - 3\phi_n(\phi_n^2 - \psi^2) + (\phi_n^3 - \psi^3)) d\mu \\ & = -\frac{3}{2} \int_M \Sigma_2(\bar{\nabla}^2\phi, L)(\phi_n - \psi)^2 d\mu \\ & \quad + \int_M \sigma_2(L)(\phi_n - \psi)^3 d\mu. \end{aligned} \quad (4.53)$$

This is less or equal than 0, due to the fact that  $\Sigma_2(\bar{\nabla}^2\phi, L) \geq 0$ ,  $\sigma_2(L) \geq 0$  and  $\phi_n \leq \psi$ . Thus (4.49) is proved.

Plugging (4.49) into (4.48) and applying Proposition 4.6, we conclude that for  $\phi$  satisfying  $|\nabla\phi| < 1$ ,

$$\begin{aligned} L_3 &\leq M_{3,1} + M_{3,2} + M_{3,3} \\ &\leq \int_M \sigma_2(L) d\mu - \frac{1}{4} \int_M [T_2]_{\alpha\beta}(L) \phi_\alpha \phi_\beta d\mu. \end{aligned} \quad (4.54)$$

As in the previous section, if  $|\nabla\phi| \leq 1$ , then we can apply the above argument to  $\phi_\delta := (1 - \delta) \cdot \phi$  and take  $\delta \rightarrow 0$ . This completes the proof of Theorem 2.2.  $\square$

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