SHRINKWRAPPING AND THE TAMING OF HYPERBOLIC 3–MANIFOLDS

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0. Introduction

During the period 1960–1980, Ahlfors, Bers, Kra, Marden, Maskit, Sullivan, Thurston and many others developed the theory of *geometrically finite ends* of hyperbolic 3–manifolds. It remained to understand those ends which are not geometrically finite; such ends are called *geometrically infinite*.

Around 1978 William Thurston gave a conjectural description of geometrically infinite ends of complete hyperbolic 3–manifolds. An example of a geometrically infinite end is given by an infinite cyclic covering space of a closed hyperbolic 3-manifold which fibers over the circle. Such an end has cross sections of uniformly bounded area. By contrast, the area of sections of geometrically finite ends grow exponentially in the distance from the convex core.

For the sake of clarity we will assume throughout this introduction that $N = \mathbb{H}^3/\Gamma$ where Γ is parabolic free. Precise statements of the parabolic case will be given in §7.

Thurston's idea was formalized by Bonahon [Bo] and Canary [Ca] with the following.

Definition 0.1. The end $\mathcal{E}$ of a hyperbolic 3-manifold N is *simply degenerate* if it is topologically of the form $S \times [0, \infty)$ where S is a closed surface, and there exists a sequence $\{S_i\}$ of CAT(-1) surfaces exiting $\mathcal{E}$ which are homotopic to $S \times 0$ in $\mathcal{E}$. This means that there exists a sequence of maps $f_i : S \to N$ such that the induced path metrics induce CAT(-1) structures on the S_i's, $f(S_i) \subset S \times [i, \infty)$ and f is homotopic to a homeomorphism onto $S \times 0$ via a homotopy supported in $S \times [0, \infty)$.

Here by CAT(-1), we mean as usual a geodesic metric space for which geodesic triangles are "thinner" than comparison triangles in hyperbolic space. If the metrics pulled back by the f_i are smooth, this is equivalent to the condition that the Riemannian curvature is bounded above by -1. See [BH] for a reference. Note that by Gauss–Bonnet, the area of a CAT(-1) surface can be estimated from its Euler characteristic; it follows that a simply degenerate end has cross sections of uniformly bounded area, just like the end of a cyclic cover of a manifold fibering over the circle.

Francis Bonahon [Bo] observed that geometrically infinite ends are exactly those ends possessing an exiting sequence of closed geodesics. This will be our working definition of such ends throughout this paper.

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The following is our main result.

Theorem 0.2. *The end $\mathcal{E}$ of a complete hyperbolic 3-manifold N with finitely generated fundamental group is simply degenerate if there exists a sequence of closed geodesics exiting $\mathcal{E}$.*

Consequently we have,

Theorem 0.3. *Let N be a complete hyperbolic 3-manifold with finitely generated fundamental group, then every end of N is geometrically tame, i.e. it is either geometrically finite or simply degenerate.*

In 1974 Marden [Ma] showed that a geometrically finite hyperbolic 3-manifold is *topologically tame*, i.e. is the interior of a compact 3-manifold. He asked whether all complete hyperbolic 3-manifolds with finitely generated fundamental group are topologically tame. This question is now known as the *Tame Ends Conjecture* or *Marden Conjecture*.

Theorem 0.4. *If N is a complete hyperbolic 3-manifold with finitely generated fundamental group, then N is topologically tame.*

There have been many important steps towards Theorem 0.2. The seminal result was obtained by Thurston [T], Theorem 9.2, who proved Theorems 0.3 and 0.4 for algebraic limits of Fuchsian groups. Bonahon [Bo] established Theorems 0.2 and 0.4 when $\pi_1(N)$ is freely indecomposible and Canary [Ca] proved that topological tameness implies geometrical tameness. Results in the direction of 0.4 were also obtained by Canary-Minsky [CaM], Kleineidam–Souto [KS], Evans [Ev], Brock–Bromberg–Evans–Souto [BBES], Brock–Souto [BS] and Souto [So]. Actually [So] plays a crucial role, for it is used in obtaining Theorem 0.4 from our main technical result described later in the introduction. Independently, and very recently, Ian Agol [Ag] has announced a proof of 0.4.

Thurston first discovered how to obtain analytic conclusions from the existence of exiting sequences of CAT(-1) surfaces. Thurston's work as generalized by Bonahon [Bo] and Canary [Ca] combined with Theorem 0.2 yields a positive proof of the Ahlfors' Measure Conjecture [A2].

Theorem 0.5. *If Γ is a finitely generated Kleinian group, then the Lebesgue measure of its limit set is either full or zero. If $L_\Gamma = S^2_\infty$, then Γ acts ergodically on S^2_∞.*

Theorem 0.5 is one of the many analytical consequences of our main result. Indeed Theorem 0.2 implies that a complete hyperbolic 3-manifold N with finitely generated fundamental group is *analytically tame* as defined by Canary [Ca]. It follows from Canary that the various results of §9 [Ca] hold for N.

Our main result is the last step needed to prove the following monumental result, the other parts being established by Alhfors, Bers, Kra, Marden, Masket, Mostow, Prasad, Sullivan, Thurston, Minsky, Masur–Minsky and Brock–Canary–Minsky. See [Mi] and [BCM].

Theorem 0.6 (Classification Theorem)**.** *If N is a complete hyperbolic 3-manifold with finitely generated fundamental group, then N is determined up to isometry by its topological type, the conformal boundary of its geometrically finite ends and the ending laminations of its geometrically infinite ends.*

The following result was conjectured by Bers, Sullivan and Thurston. Theorem 0.4 is one of many results, many of them recent, needed to build a proof. See [BCM] or [BB] for a more detailed discussion.

Theorem 0.7 (Density Theorem)**.** *If $N = \mathbb{H}^3/\Gamma$ is a complete finitely generated 3-manifold with finitely generated fundamental group, then Γ is the algebraic limit of geometrically finite Kleinian groups.*

The main technical innovation of this paper is a new technique called *shrinkwrapping* for producing CAT(-1) surfaces in hyperbolic 3-manifolds. Historically, such surfaces have been immensely important in the study of hyperbolic 3-manifolds, e.g. see [T], [Bo], [Ca] and [CaM].

Given a locally finite set Δ of pairwise disjoint simple closed curves in the 3-manifold N, we say that the embedded surface $S \subset N$ is *2-incompressible* rel. Δ if every compressing disc for S meets Δ at least twice. Here is a sample theorem.

Theorem 0.8 (Existence of shrinkwrapped surface)**.** *Let M be a complete, orientable, parabolic free hyperbolic 3–manifold, and let Γ be a finite collection of pairwise disjoint simple closed geodesics in M. Further, let $S \subset M\backslash\Gamma$ be a closed embedded 2–incompressible surface rel. Γ which is either nonseparating in M or separates some component of Γ from another. Then S is homotopic to a CAT(-1) surface T via a homotopy*

$$F : S \times [0, 1] \to M$$

such that

(1) $F(S \times 0) = S$
(2) $F(S \times t) = S_t$ *is an embedding disjoint from* Γ *for* $0 \leq t < 1$
(3) $F(S \times 1) = T$
(4) *If T' is any other surface with these properties, then area$(T) \leq$ area(T')*

We say that T is obtained from S by shrinkwrapping rel. Γ*, or if Γ is understood, T is obtained from S by* shrinkwrapping.

In fact, we prove the stronger result that T is Γ–*minimal* (to be defined in §1) which implies in particular that it is intrinsically CAT(-1)

Here is the main technical result of this paper.

Theorem 0.9. *Let $\mathcal{E}$ be an end of the complete orientable hyperbolic 3-manifold N with finitely generated fundamental group. Let C be a 3-dimensional compact core of N, $\partial_\mathcal{E}$ the component of ∂C facing $\mathcal{E}$ and $g = genus(\partial_\mathcal{E})$. If there exists a sequence of closed geodesics exiting $\mathcal{E}$, then there exists a sequence $\{S_i\}$ of CAT(-1) surfaces of genus g exiting $\mathcal{E}$ such that each $\{S_i\}$ is homologically separating in $\mathcal{E}$. That is, each S_i homologically separates $\partial_\mathcal{E}$ from $\mathcal{E}$.*

Theorem 0.4 now follows directly from Souto [So] or can be derived from the pleated surface interpolation technique introduced by Thurston [T] and developed by [Ca2], [CaM] and [So]. See §6.

The proof of Theorem 0.9 blends elementary aspects of minimal surface theory, hyperbolic geometry, and 3-manifold topology. The method will be demonstrated in §4 where we give a proof of Canary's theorem. The first time reader is urged to begin with that section.

This paper is organized as follows. In §1 and §2 we establish the shrinkwrapping technique for finding CAT(-1) surfaces in hyperbolic 3-manifolds. In §3 we prove the existence of ϵ-separated simple geodesics exiting the end of parabolic free manifolds. In §4 we prove Canary's theorem. This proof will model the proof of the general case. The general strategy will be outlined at the end of that section. In §5 we develop the topological theory of end reductions in 3-manifolds. In §6 we give the proofs of our main results. In §7 we give the necessary embellishments of our methods to state and prove our results in the case of manifolds with parabolic elements.

Notation 0.10. If $X \subset Y$, then $N(X)$ denotes a regular neighborhood of X in Y and int(X) denotes the interior of X. If X is a topological space, then $|X|$ denotes the number of components of X.

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1. Shrinkwrapping

In this section, we introduce a new technical tool for finding CAT(-1) surfaces in hyperbolic 3–manifolds, called *shrinkwrapping*. Roughly speaking, given a collection of simple closed geodesics Γ in a hyperbolic 3–manifold M and an embedded surface $S \subset M \backslash \Gamma$, a surface $T \subset M$ is obtained from S by *shrinkwrapping S rel.* Γ if it homotopic to S, can be approximated by an isotopy from S supported in $M \backslash \Gamma$, and is least area subject to these constraints.

Given mild topological conditions on M, Γ, S (namely 2–incompressibility, to be defined below) the shrinkwrapped surface exists, and is CAT(-1) with respect to the path metric induced by the Riemannian metric on M.

We use some basic analytical tools throughout this section, including the Gauss–Bonnet formula, the coarea formula, and the Arzela–Ascoli theorem. At a number of points we must invoke results from the literature to establish existence of minimal surfaces ([MSY]), existence of limits with area and curvature control ([CiSc]), and regularity of the shrinkwrapped surfaces along Γ ([Ri]). General references are [CM], [Js] [Fed] and [B].

1.1. **Geometry of surfaces.**

For convenience, we state some elementary but fundamental lemmas concerning curvature of surfaces.

The following lemma is formula 5.6 on page 100 of [CM].

Lemma 1.1 (Monotonicity of curvature)**.** *Let Σ be a minimal surface in a Riemannian manifold M. Let K_Σ denote the curvature of Σ, and K_M the sectional curvature of M. Then restricted to the tangent space $T\Sigma$,*

$$K_\Sigma = K_M - \frac{1}{2}|A|^2$$

where A denotes the second fundamental form of Σ.

In particular, if the Riemannian curvature on M is bounded from above by some constant K, then the curvature of a minimal surface Σ in M is also bounded above by K.

The following lemma is just the usual Gauss–Bonnet formula:

Lemma 1.2 (Gauss–Bonnet formula)**.** *Let Σ be a C^3 Riemannian surface with (possibly empty) C^3 boundary $\partial\Sigma$. Let K_Σ denote the Riemann curvature of Σ, and κ the geodesic curvature along $\partial\Sigma$. Then*

$$\int_\Sigma K_\Sigma = 2\pi\chi(\Sigma) - \int_{\partial\Sigma} \kappa \, dl$$

Many simple proofs exist in the literature. For example, see [Js].

If $\partial\Sigma$ is merely piecewise C^3, with finitely many corners p_i and external angles α_i, the Gauss–Bonnet formula must be modified as follows:

Lemma 1.3 (Gauss–Bonnet with corners)**.** *Let Σ be a C^3 Riemannian surface with boundary $\partial\Sigma$ which is piecewise C^3, and has external angles α_i at finitely many points p_i. Let K_Σ and κ be as above. Then*

$$\int_\Sigma K_\Sigma = 2\pi\chi(\Sigma) - \int_{\partial\Sigma} \kappa \, dl - \sum_i \alpha_i$$

Observe for abc a geodesic triangle with external angles $\alpha_1, \alpha_2, \alpha_3$ that Lemma 1.3 implies

$$\int_{abc} K = 2\pi - \sum_i \alpha_i$$

Notice that the geodesic curvature κ vanishes precisely when $\partial\Sigma$ is a geodesic, that is, a critical point for the length functional. More generally, let ν be the normal bundle of $\partial\Sigma$ in Σ, oriented so that the inward unit normal is a positive section. The exponential map restricted to ν defines a map

$$\phi : \partial\Sigma \times [0, \epsilon] \to \Sigma$$

for small ϵ, where $\phi(\cdot, 0) = \mathrm{Id}|_{\partial\Sigma}$, and $\phi(\partial\Sigma, t)$ for small t is the boundary in Σ of the tubular t neighborhood of $\partial\Sigma$. Then

$$\int_{\partial\Sigma} \kappa \, dl = -\frac{d}{dt}\Big|_{t=0} \mathrm{length}(\phi_t(\partial\Sigma))$$

Note that if Σ is a surface with sectional curvature bounded above by -1, then by integrating this formula we see that the ball $B_t(p)$ of radius t in Σ about a point $p \in \Sigma$ satisfies

$$\mathrm{area}(B_t(p)) \geq 2\pi(\cosh(t) - 1) > \pi t^2$$

for small $t > 0$.

1.2. **Comparison geometry.** For basic elements of the theory of comparison geometry, see [BH].

Definition 1.4 (Comparison triangle)**.** Let $a_1a_2a_3$ be a geodesic triangle in a geodesic metric space X. Let $\kappa \in \mathbb{R}$ be given. A *κ–comparison triangle* is a geodesic triangle $\overline{a_1a_2a_3}$ in the complete simply–connected Riemannian 2–manifold of constant sectional curvature κ, where the edges a_ia_j and $\overline{a_ia_j}$ satisfy

$$\mathrm{length}(a_ia_j) = \mathrm{length}(\overline{a_ia_j})$$

Given a point $x \in a_1a_2$ on one of the edges of $a_1a_2a_3$, there is a corresponding point $\overline{x} \in \overline{a_1a_2}$ on one of the edges of the comparison triangle, satisfying

$$\text{length}(a_1x) = \text{length}(\overline{a_1x})$$

and

$$\text{length}(xa_2) = \text{length}(\overline{xa_2})$$

Remark 1.5. Note that if $\kappa > 0$, the comparison triangle might not exist if the edge lengths are too big, but if $\kappa \leq 0$ the comparison triangle always exists, and is unique up to isometry.

There is a slight issue of terminology to be aware of here. In a *surface*, a triangle is a polygonal disk with 3 geodesic edges. In a *path metric space*, a triangle is just a union of 3 geodesic segments with common endpoints.

Definition 1.6 (CAT(κ))**.** Let S be a closed surface with a path metric g. Let $\tilde{S}$ denote the universal cover of $\tilde{S}$, with path metric induced by the pullback of the path metric g. Let $\kappa \in \mathbb{R}$ be given. S is said to be *CAT*(κ) if for every geodesic triangle abc in $\tilde{S}$, and every point z on the edge bc, the distance in $\tilde{S}$ from a to z is no more than the distance from $\overline{a}$ to $\overline{z}$ in a κ–comparison triangle.

By Lemma 1.3 applied to geodesic triangles, one sees that a C^3 surface Σ with sectional curvature K_Σ satisfying $K_\Sigma \leq \kappa$ everywhere is CAT(κ) with respect to the Riemannian path metric.

More generally, if Σ is a surface which is C^3 outside a subset $X \subset \Sigma$ without interior, if $K_\Sigma \leq \kappa$ in $\Sigma \backslash X$, and if the formula from Lemma 1.3 holds for every geodesic triangle with vertices in $\Sigma \backslash X$ then Σ is CAT(κ).

This is an easy exercise in comparison geometry; see [B] for details.

Definition 1.7 (Γ–minimal surfaces)**.** Let $\kappa \in \mathbb{R}$ be given. Let M be a complete Riemannian 3–manifold with sectional curvature bounded above by κ, and let Γ be an embedded collection of simple closed geodesics in M. An immersion

$$\psi : S \to M$$

is Γ*–minimal* if it is smooth with mean curvature 0 on $S \backslash \Gamma$, and is metrically CAT(κ) with respect to the path metric induced by ψ from the Riemannian metric on M.

Notice by Lemma 1.1 that a smooth surface S with mean curvature 0 in M is CAT(κ), so a minimal surface (in the usual sense) is an example of a Γ–minimal surface.

1.3. **Statement of shrinkwrapping theorem.**

Definition 1.8 (2–incompressibility)**.** An embedded surface S in a 3–manifold M disjoint from a collection Γ of simple closed curves is said to be 2*–incompressible rel.* Γ if any essential compressing disk for S must intersect Γ in at least two points. If Γ is understood, we say S is 2*–incompressible*.

Theorem 1.9 (Existence of shrinkwrapped surface)**.** *Let M be a complete, orientable, parabolic free hyperbolic 3–manifold, and let Γ be a finite collection of pairwise disjoint simple closed geodesics in M. Further, let $S \subset M \backslash \Gamma$ be a closed embedded 2–incompressible surface rel. Γ which is either nonseparating in M or separates some component of Γ from another. Then S is homotopic to a Γ–minimal surface T via a homotopy*

$$F : S \times [0,1] \to M$$

such that

(1) $F(S \times 0) = S$
(2) $F(S \times t) = S_t$ *is an embedding disjoint from* Γ *for* $0 \leq t < 1$
(3) $F(S \times 1) = T$
(4) *If* T' *is any other surface with these properties, then area*$(T) \leq$ *area*(T')

We say that T *is obtained from* S *by* shrinkwrapping rel. Γ*, or if* Γ *is understood,* T *is obtained from* S *by* shrinkwrapping*.*

The remainder of this section will be taken up with the proof of Theorem 1.9.

Remark 1.10. In fact, for our applications, the property we want to use of our surface T is that we can estimate its diameter (rel. the thin part of M) from its Euler characteristic. This follows from Gauss–Bonnet and the bounded diameter lemma (Lemma 1.14, to be proved below). In fact, our argument will show directly that the surface T satisfies Gauss–Bonnet; the fact that it is CAT(-1) is logically superfluous for the purposes of this paper.

1.4. **Deforming metrics along geodesics.**

Definition 1.11 (δ–separation)**.** Let Γ be a collection of disjoint simple geodesics in a Riemannian manifold M. The collection Γ is *δ–separated* if any path $\alpha : I \to M$ with endpoints on Γ and satisfying

$$\text{length}(\alpha(I)) \leq \delta$$

is homotopic rel. endpoints into Γ. The supremum of such δ is called the *separation constant* of Γ. The collection Γ is *weakly δ–separated* if

$$\text{dist}(\gamma, \gamma') > \delta$$

whenever γ, γ' are distinct components of Γ. The supremum of such δ is called the *weak separation constant* of Γ.

Definition 1.12 (Neighborhood and tube neighborhood)**.** Let $r > 0$ be given. For a point $x \in M$, we let $N_r(x)$ denote the closed ball of radius r about x, and let $N_{<r}(x), \partial N_r(x)$ denote respectively the interior and the boundary of $N_r(x)$. For a closed geodesic γ in M, we let $N_r(\gamma)$ denote the closed tube of radius r about γ, and let $N_{<r}(\gamma), \partial N_r(\gamma)$ denote respectively the interior and the boundary of $N_r(\gamma)$. If Γ denotes a union of geodesics γ_i, then we use the shorthand notation

$$N_r(\Gamma) = \bigcup_{\gamma_i} N_r(\gamma_i)$$

Remark 1.13. Topologically, $\partial N_r(x)$ is a sphere and $\partial N_r(\gamma)$ is a torus, for sufficiently small r. Similarly, $N_r(x)$ is a closed ball, and $N_r(\gamma)$ is a closed solid torus. If Γ is δ–separated, then $N_{\delta/2}(\Gamma)$ is a union of solid tori.

Lemma 1.14 (Bounded Diameter Lemma)**.** *Let* M *be a complete hyperbolic* 3*–manifold. Let* Γ *be a disjoint collection of* δ*–separated embedded geodesics. Let* $\epsilon > 0$ *be a Margulis constant for dimension* 3*, and let* $M_{\leq \epsilon}$ *denote the subset of* M *where the injectivity radius is at most* ϵ*. If* $S \subset M \backslash \Gamma$ *is a* 2*–incompressible* Γ*–minimal surface, then there is a constant* $C = C(\chi(S), \epsilon, \delta) \in \mathbb{R}$ *and* $n = n(\chi(S), \epsilon, \delta) \in \mathbb{Z}$ *such that for each component* S_i *of* $S \cap (M \backslash M_{\leq \epsilon})$*, we have*

$$diam(S_i) \leq C$$

Furthermore, S *can only intersect at most* n *components of* $M_{\leq \epsilon}$*.*

Proof. Since S is 2–incompressible, any point $x \in S$ either lies in $M_{\leq\epsilon}$, or is the center of an embedded m–disk in S, where

$$m = \min(\epsilon/2, \delta/2)$$

Since S is CAT(-1), Gauss–Bonnet implies that the area of an embedded m–disk in S has area at least $2\pi(\cosh(m) - 1) > \pi m^2$.

This implies that if $x \in S \cap M \backslash M_{\leq\epsilon}$ then

$$\text{area}(S \cap N_m(x)) \geq \pi m^2$$

The proof now follows by a standard covering argument. □

A surface S satisfying the conclusion of the Bounded Diameter Lemma is sometimes said to have diameter bounded by C modulo $M_{\leq\epsilon}$.

Remark 1.15. Note that if ϵ is a Margulis constant, then $M_{\leq\epsilon}$ consists of Margulis tubes and cusps. Note that the same argument shows that, away from the thin part of M and an ϵ neighborhood of Γ, the diameter of S can be bounded by a constant depending only on $\chi(S)$ and ϵ.

Definition 1.16 (Deforming metrics)**.** Let $\delta > 0$ be such that Γ is δ–separated. Choose some small r with $r < \delta/2$. For $t \in [0, 1)$ we define a family of Riemannian metrics g_t on M in the following manner. The metrics g_t agree with the hyperbolic metric away from some fixed tubular neighborhood $N_r(\Gamma)$.

Let

$$h : N_{r(1-t)}(\Gamma) \to [0, r(1-t)]$$

be the function whose value at a point p is the hyperbolic distance from p to Γ. We define a metric g_t on M which agrees with the hyperbolic metric outside $N_{r(1-t)}(\Gamma)$, and on $N_{r(1-t)}(\Gamma)$ is conformally equivalent to the hyperbolic metric, with

$$\frac{g_t \text{ length element}}{\text{hyperbolic length element}} = 1 + 2\sin^2\left(\frac{\pi\, h(p)}{r(1-t)}\right)$$

We are really only interested in the behaviour of the metrics g_t as $t \to 1$. As such, the choice of r is irrelevant. However, for convenience, we will fix some small r throughout the remainder of §1.

The deformed metrics g_t have the following properties:

Lemma 1.17 (Metric properties)**.** *The g_t metric satisfies the following properties:*

1. *For each t there is an $f(t)$ satisfying $r(1-t)/3 < f(t) < 2r(1-t)/3$ such that the union of tori $\partial N_{f(t)}(\Gamma)$ are totally geodesic for the g_t metric*
2. *For each component γ_i and each t, the metric g_t restricted to $N_r(\gamma_i)$ admits a family of isometries which preserve γ_i and acts transitively on the unit normal bundle (in M) to γ_i*
3. *The area of a disk cross–section on $N_{r(1-t)}$ is $O(1-t)$.*
4. *The metric g_t dominates the hyperbolic metric on 2–planes. That is, for all 2–vectors ν, the g_t area of ν is at least as large as the hyperbolic area of ν*

Proof. These properties can be verified by a straightforward computation. □

Notation 1.18. We denote length of an arc $\alpha : I \to M$ with respect to the g_t metric as $\text{length}_t(\alpha(I))$, and area of a surface $\psi : R \to M$ with respect to the g_t metric as $\text{area}_t(\psi(R))$.

1.5. **Constructing the homotopy.**

Lemma 1.19 (Minimal surface exists)**.** *Let M, Γ, S be as in the statement of Theorem 1.9. For each t, there exists an embedded surface S_t isotopic in $M \backslash \Gamma$ to S, which is g_t–least area among all such surfaces.*

Proof. Note that with respect to the g_t metrics, the surfaces $\partial N_{f(t)}(\Gamma)$ described in Lemma 1.17 are totally geodesic, and therefore act as barrier surfaces. If there exists a lower bound on the injectivity radius in $M \backslash \Gamma$ with respect to the g_t metric, then the main theorem of [MSY] implies that either such a minimal surfaces S_t can be found, or S is the boundary of a twisted I–bundle over a closed surface, or else S can be homotoped off every compact set in $M \backslash \Gamma$.

First we show that these other two possibilities cannot occur. If S is nonseparating in M, then it intersects some essential loop β with algebraic intersection number 1. It follows that S cannot be homotoped off β, and does not bound an I–bundle. Similarly, if γ_1, γ_2 are distinct geodesics of Γ separated from each other by S, then the γ_i's can be joined by an arc α which has algebraic intersection number 1 with the surface S. The same is true of any S' homotopic to S; it follows that S cannot be homotoped off the arc α, nor does it bound an I–bundle disjoint from Γ.

Now suppose that the injectivity radius on M is not bounded below. We use the following trick. Let g'_t be obtained from the metric g_t by perturbing it on the complement of some enormous compact region E so that it has a flaring end there, and there is a barrier minimal surface close to ∂E. Then by [MSY] there is a minimal surface S'_t for the g'_t metric. Since S'_t must either intersect β or α, by the Bounded Diameter Lemma 1.14, unless the hyperbolic area of $S'_t \cap E$ is very large, the diameter of S'_t in E is much smaller than the distance from α or β to ∂E. Since by hypothesis, S'_t is least area for the g'_t metric, its restriction to E has hyperbolic area less than the hyperbolic area of S, and therefore there is an *a priori* upper bound on its diameter in E. By choosing E big enough, we see that S'_t is contained in the interior of E, where g_t and g'_t agree. Thus S'_t is minimal for the g_t metric, and therefore $S_t = S'_t$ exists for any t. □

Lemma 1.20. *Fix small r as in Definition 1.16. Suppose for some t, the surface S_t does not intersect $N_{r(1-t)}(\Gamma)$. Then S_t is a minimal surface in M, and we can choose $S_u = S_t$ for $u > t$.*

Proof. Since g_t agrees with the hyperbolic metric outside $N_{r(1-t)}(\Gamma)$, a g_t–minimal surface S_t which does not intersect $N_{r(1-t)}$ is actually a minimal surface for the hyperbolic metric. Since the g_u metric dominates the hyperbolic metric for each u, such an S_t is also g_u minimal for all $u > t$. □

By hypothesis, S_t is isotopic to S in $M \backslash \Gamma$. A minimal surface in the complement of Γ is obviously Γ–minimal. Suppose there is some other Γ–minimal surface R which can be perturbed to an embedded surface $R' \subset M \backslash \Gamma$ isotopic to S in $M \backslash \Gamma$. If there is an inequality $\mathrm{area}(R) < \mathrm{area}(S_t)$ then for some such perturbation R' we have $\mathrm{area}(R') < \mathrm{area}(S_t)$. On the other hand,

$$R' \subset M \backslash N_{r(1-u)}(\Gamma)$$

for some $u \geq t$, contrary to the definition of S_u and the choice of $S_u = S_t$.

It follows that if S_t is disjoint from $N_{r(1-t)}$ for some t, then the proof of Theorem 1.9 follows, by taking $T = S_u = S_t$ for all $u \geq t$.

For the remainder of the proof of Theorem 1.9 therefore, we assume that there is an infinite sequence $t_i \to 1$ such that S_{t_i} intersects $N_{r(1-t_i)}$.

To extract good limits of sequences of minimal surfaces, one generally needs *a priori* bounds on the area and the total curvature of the limiting surfaces. Here for a surface S, the *total curvature* of S is just the integral of the absolute value of the (Gauss) curvature over S. For minimal surfaces of a fixed topological type in a manifold with sectional curvature bounded above, a curvature bound follows from an area bound by Gauss–Bonnet. However, our surfaces S_t are minimal with respect to the g_t metrics, which have no uniform upper bound on their sectional curvature, so we must work slightly harder to show that the the S_t have uniformly bounded total curvature. More precisely, we show that their restrictions to the complement of any fixed tubular neighborhood $N_\epsilon(\Gamma)$ have uniformly bounded total curvature.

Lemma 1.21 (Finite total curvature)**.** *Let S_t be the surfaces constructed in Lemma 1.19. Fix some small, positive ϵ. Then the subsurfaces*

$$S_t' = S_t \cap M \backslash N_\epsilon(\Gamma)$$

have uniformly bounded total curvature.

Proof. Having chosen ϵ, we choose t large enough so that $r(1-t) < \epsilon/2$.

Observe firstly that each S_t has g_t area less that the g_t area of S, and therefore hyperbolic area less that the hyperbolic area of S for sufficiently large t.

Let $\tau_{t,s} = S_t \cap \partial N_s(\Gamma)$ for small s. By the coarea formula (see [Fed],[CM]) we can estimate

$$\text{area}(S_t \cap (N_{\epsilon(\Gamma)} \backslash N_{\epsilon/2}(\Gamma))) \geq \int_{\epsilon/2}^{\epsilon} \text{length}(\tau_{t,s})\, ds$$

If the integral of geodesic curvature along a component σ of $\tau_{t,\epsilon}$ is large, then the length of the curves obtained by isotoping σ into $S_t \cap N_\epsilon(\Gamma)$ grows very rapidly.

Since there is an *a priori* bound on the hyperbolic area of S_t, it follows that for each constant $C_1 > 0$ there is a constant $C_2 > 0$, such that for each component σ of $\tau_{t,\epsilon}$ which has length $\geq C_1$ there is a loop

$$\sigma' \subset S_t \cap (N_\epsilon(\Gamma) \backslash N_{\epsilon/2}(\Gamma))$$

isotopic to σ by a short isotopy, satisfying

$$\int_\sigma \kappa\, dl \leq C_2$$

For otherwise, the hyperbolic area of an $\epsilon/2$ tubular neighborhood of such a σ in S_t would be bigger than the hyperbolic area of S, which is absurd.

On the other hand, since S_t is g_t minimal, for t sufficiently large, there is a constant $C_1 > 0$ such that each component σ of $\tau_{t,\epsilon}$ which has length $\leq C_1$ bounds a subdisk of S_t contained in $M \backslash N_{r(1-t)}$.

By the coarea formula above, we can choose ϵ so that $\text{length}(\tau_{t,s})$ is *a priori* bounded. It follows that if S_t'' is the subsurface of S_t bounded by the components of $\tau_{t,s}$ of length $> C_1$ then we have *a priori* upper bounds on the area of S_t'', on $\int_{\partial S_t''} \kappa\, dl$, and on $-\chi(S_t'')$. Moreover, S_t'' is contained in $M \backslash N_{r(1-t)}$ where the metric g_t agrees with the hyperbolic metric, so the curvature K of S_t'' is bounded above by -1 pointwise, by Lemma 1.1. By the Gauss–Bonnet formula, this gives an *a priori* upper bound on the total curvature of S_t'', and therefore on $S_t' \subset S_t''$. $\square$

Remark 1.22. A more highbrow proof of Lemma 1.21 follows from theorem 1 of [S], using the fact that the surfaces S_t are globally least area.

Lemma 1.23 (Limit exists)**.** *Let S_t be the surfaces constructed in Lemma 1.19. Then there is an increasing sequence*

$$0 < t_1 < t_2 < \cdots$$

such that $\lim_{i\to\infty} t_i = 1$, and the S_{t_i} converge on compact subsets of $M\backslash\Gamma$ to some $T' \subset M\backslash\Gamma$ with closure T in M.

Proof. By definition, the surfaces S_t have g_t area bounded above by the g_t area of S. Moreover, since S is disjoint from Γ, for sufficiently large t, the g_t area of S is equal to the hyperbolic area of S. Since the g_t area dominates the hyperbolic area, it follows that the S_t have hyperbolic area bounded above, and by Lemma 1.21, for any ϵ, the restrictions of S_t to $M\backslash N_\epsilon(\Gamma)$ have uniformly bounded finite total curvature. Moreover, by assumption, each S_t intersects $N_{r(1-t)}$, so by the bounded diameter Lemma 1.14 they are all contained in some fixed compact subset E of M. By standard compactness theorems (see e.g. [CiSc]) any infinite sequence S_{t_i} contains a subsequence which converges on compact subsets of $E\backslash\Gamma$, away from finitely many points where some subsurface with nontrivial topology might collapse.

But S is 2–incompressible rel. Γ, so in particular it is incompressible in $M\backslash S$, and no such collapse can take place, and the limit T' exists (compare [MSY]). Since each S_t is a minimal surface with respect to the g_t metric, it is a minimal surface with respect to the hyperbolic metric on $M\backslash N_{r(1-t)}(\Gamma)$. It follows that T' is real analytic of mean curvature 0, properly embedded in $M\backslash\Gamma$, and we can define T to be the closure in M. □

Lemma 1.24 (Interpolating isometry)**.** *Let $\{t_i\}$ be the sequence as in Lemma 1.23. Then after possibly passing to a subsequence, there is an isotopy $F : S \times [0,1) \to M\backslash\Gamma$ such that*

$$F(S, t_i) = S_{t_i}$$

and such that for each $p \in S$ the track of the isotopy $F(p, [0,1))$ either converges to some well–defined limit $F(p,1) \in M\backslash\Gamma$ or else it is eventually contained in $N_\epsilon(\Gamma)$ for any $\epsilon > 0$.

Proof. Fix some small ϵ. Outside $N_\epsilon(\Gamma)$, the surface S_{t_i} converge uniformly and smoothly to T'. It follows for i sufficiently large, that S_{t_i} and $S_{t_{i+1}}$ are both sections of the exponentiated unit normal bundle of $T'\backslash N_\epsilon(\Gamma)$, and therefore we can isotope S_{t_i} to $S_{t_{i+1}}$ along the fibers of the normal bundle.

Let Z be obtained from $N_\epsilon(\Gamma)$ by isotoping it slightly into $M\backslash N_\epsilon(\Gamma)$ so that it is transverse to T, and therefore also to S_{t_i} for i sufficiently large. For each i, we consider the intersection

$$\tau_i = S_{t_i} \cap \partial Z$$

and observe that the limit satisfies

$$\lim_{i\to\infty} \tau_i = \tau = T \cap \partial Z$$

Let σ be a component of τ which is inessential in ∂Z. Then for large i, σ can be approximated by $\sigma_i \subset \tau_i$ which are inessential in ∂Z. Since the S_{t_i} are 2–incompressible rel. Γ, the loops σ_i must bound subdisks D_i of S_{t_i}. Since ∂Z is a convex surface with respect to the hyperbolic metric, and the g_t metric agrees with the hyperbolic metric for large t, it follows that the disks D_i are actually contained in $Z\backslash\Gamma$ for large i. It follows that D_i and D_{i+1} are isotopic by an isotopy supported in $Z\backslash\Gamma$, which restricts to a very small isotopy of σ_i to σ_{i+1} in ∂Z.

Let σ be a component of τ which is essential in ∂Z. Then so is σ_i for large i. Again, since S, and therefore S_{t_i} is 2–incompressible rel. Γ, it follows that σ_i cannot be a meridian of ∂Z, and must actually be a longitude. It follows that there is another essential curve σ_i' in each τ_i, such that the essential curves σ_i' and σ_i cobound a subsurface A_i in $S_{t_i} \cap Z\backslash\Gamma$. After passing to a diagonal subsequence, we can assume that the σ_i' converge to some component σ' of τ.

By 2–incompressibility, the surfaces A_i are annuli. Note that there are *two* relative isotopy classes of such annuli. By passing to a further diagonal subsequence, we can assume A_i and A_{i+1} are isotopic in $Z\backslash\Gamma$ by an isotopy which restricts to a very small isotopy of $\sigma_i \cup \sigma_i'$ to $\sigma_{i+1} \cup \sigma_{i+1}'$ in ∂Z.

We have shown that for any small ϵ and any sequence S_{t_i}, there is an arbitrarily large index i and infinitely many indices j with $i < j$ so that the surfaces S_{t_i} and S_{t_j} are isotopic, and the isotopy can be chosen to have the following properties:

(1) The isotopy takes $N_\epsilon(\Gamma) \cap S_{t_i}$ to $N_\epsilon(\Gamma) \cap S_{t_j}$ by an isotopy supported in $N_\epsilon(\Gamma)$.
(2) Outside $N_\epsilon(\Gamma)$, the tracks of the isotopy are contained in fibers of the exponentiated normal bundle of $T'\backslash N_\epsilon(\Gamma)$.

Choose a sequence $\epsilon_i \to 0$, and pick a subsequence of the S_{t_i}'s and relabel so that $S_{t_i}, S_{t_{i+1}}$ satisfy the properties above with respect to $N_{\epsilon_i}(\Gamma)$. Then the composition of this infinite sequence of isotopies is F. □

Remark 1.25. The reason for the circumlocutions in the statement of Lemma 1.24 is that we have not yet proved that T is a limit of the S_t as maps from S to M. This will follow in §1.6, where we analyse the structure of T near a point $p \in \Gamma$, and show it has a well–defined tangent cone.

1.6. **Existence of tangent cone.**

Lemma 1.26 (Tangent cone)**.** *Let T', T be as constructed in Lemma 1.23. Let $p \in T \cap \Gamma$. Then near p, T' is a surface with a well–defined tangent cone.*

Proof. We use what is essentially a curve–shortening argument. For each small s, define

$$T_s = \partial N_s(p) \cap T$$

For each point $q \in T\backslash\Gamma$, we define $\alpha(q)$ to be the angle between the tangent space to T at q, and the radial geodesic through q emanating from p. By the coarea formula, we can calculate

$$\text{area}(T \cap N_s(p)) = \int_0^s \int_{T_t} \frac{1}{\cos(\alpha)} dl\, dt \geq \int_0^s \text{length}(T_t) dt$$

where dl denotes the length element in each T_t. Note that this estimate implies that T_t is rectifiable for a.e. t. We choose s to be such a rectifiable value.

Now, each component τ of T_s is a limit of components τ_i of $S_{t_i} \cap \partial N_s(p)$ for large i. By 2–incompressibility of the S_{t_i}, each τ_i is a loop bounding a subdisk D_i of S_{t_i} for large i. $\partial N_s(p)$ is convex in the hyperbolic metric. By cutting out the disks $\partial N_s(p) \cap N_{r(1-t)}(\Gamma)$ and replacing them with disks orthogonal to Γ which are totally geodesic in both the g_t and the hyperbolic metrics, we can approximate

$\partial N_s(p)$ by a surface ∂B bounding a ball $B \subset N_s(p)$ which is convex in the g_t metric. Let τ_i' be the component of $S_{t_i} \cap \partial B$ approximating τ_i, and let D_i' be the subdisk of S_{t_i} which it bounds. Then the disk D_i' must be contained in B, by convexity of ∂B in the g_t metric. The disks D_i' converge to the component $D \subset T'$ bounded by τ, and the hyperbolic areas of the D_i' converge to the hyperbolic area of D. Now, let D' be the cone on τ to the point p. D' can be perturbed an arbitrarily small amount to an embedded disk D'', and therefore by comparing D'' with the D_i', we see that the hyperbolic area of D' must be at least as large as that of D.

Since this is true for each component τ of T_s, by abuse of notation we can replace T by the component of $T \cap N_s(p)$ bounded by a single component τ. We use this notational convention for the remainder of the proof of the lemma. Note that the inequality above still holds. It follows that we must have

$$\text{area}(T \cap N_s(p)) \leq \int_0^s \text{length}(T_s)\frac{\sinh(t)}{\sinh(s)}dt = \text{area}(\text{cone on } T_s)$$

For each $t \in (0, s]$, define $\|T_t\|$ by the formula

$$\|T_t\| = \text{length}(T_t)\frac{\sinh(s)}{\sinh(t)}$$

Geometrically, if $\phi : N_s(p)\backslash p \to \partial N_s(p)$ is the projection map along radial geodesics, then $\phi(T_t) \subset \partial N_s(p)$, and

$$\|T_t\| = \text{length}(\phi(T_t))$$

It follows from the inequalities above that for some intermediate s' we must have

$$\|T_{s'}\| \leq \|T_s\|$$

with equality iff $T \cap N_s(p)$ is equal to the cone on T_s.

Now, the cone on T_s is not minimal in $N_s(p)\backslash\Gamma$ unless T_s is a great circle or geodesic bigon in $\partial N_s(p)$ (with endpoints on $\partial N_s(p) \cap \Gamma$), in which case the lemma is proved. So we may suppose that for any s there is some $s' < s$ such that $\|T_{s'}\| \leq \|T_s\|$. Therefore we choose a sequence of values s_i with $s_i \to 0$ such that $\|T_{s_i}\| > \|T_{s_{i+1}}\|$, such that $\|T_{s_i}\|$ converges to the infimal value of $\|T_t\|$ with $t \in (0, s]$, and such that $\|T_{s_i}\|$ is the minimal value of $\|T_t\|$ on the interval $t \in [s_i, 1]$. Note that for any small t, the cone on T_t has area $\frac{t}{2}\,\text{length}(T_t) + O(t^3)$.

The set of loops in the sphere with length bounded above by some constant, parameterized by arclength, is *compact*, by the Arzela–Ascoli theorem (see [Fed]). Therefore we may choose $i < j$ such that $\psi(T_{s_i})$ and $\psi(T_{s_j})$ are C^0 close in $\partial N_s(p)$. Now, the cone on T_{s_i} can be perturbed slightly to interpolate a surface $F_{s_j}^{s_i}$ between T_{s_i} and T_{s_j} satisfying

$$\text{area}(F_{s_j}^{s_i}) \leq \frac{\epsilon s_i^2}{2} + \int_{s_j}^{s_i} \text{length}(T_t)dt$$

where ϵ can be chosen as small as we like, for large i, j.

Suppose the curve $\phi(T_{s_i})$ is not very close (in the C^0 topology) to being a great circle or geodesic bigon in $\partial N_s(p)$. Then by pushing in regions where the (geodesic) curvature of T_{s_i} is positive, we can push $F_{s_j}^{s_i}$ rel. boundary to a new surface $G_{s_j}^{s_i}$ in $N_s(p)\backslash\Gamma$, where

$$\text{area}(F_{s_j}^{s_i}) \geq \frac{\delta s_i^2}{2} + \text{area}(G_{s_j}^{s_i})$$

and where δ depends only on the Hausdorff distance from $\phi(T_{s_i})$ to a geodesic bigon or great circle in $\partial N_s(p)$.

But then $G_{s_j}^{s_i}$ has area strictly less than the cone on T_{s_i}, which as we have pointed out, is absurd. It follows that the curves $\phi(T_{s_i})$ converge geometrically to a great circle or a geodesic bigon in $\partial N_s(p)$.

Let $C \subset \partial N_s(p)$ be this geometric limit. Then inside an ϵ neighborhood of C in $\partial N_s(p)$, we can find a pair of curves $C^\pm$, where C^+ is convex, and C^- is convex except for two acute angles on $\Gamma \cap \partial N_s(p)$. The cone on $C^\pm$ is a pair of barrier surfaces in $N_s(p)$. In particular, once $\phi(T_{s_i})$ and $\phi(T_{s_{i+1}})$ are both trapped between C^+ and C^-, the same is true of $\phi(T_r)$ for all $r \in [s_{i+1}, s_i]$. This is enough to establish the existence of the tangent cone. □

Notice that Lemma 1.26 actually implies that T is a rectifiable surface in M, which is a local (topological) embedding. In particular, this shows that the isotopy $F : S \times [0,1) \to M$ constructed in Lemma 1.24 can be chosen to limit to a homotopy $F : S \times [0,1] \to M$ such that $F(S,1) = T$.

1.7. **The thin obstacle problem.** From the proof of Lemma 1.26, we see that T exists as a C^0 map which by abuse of notation we denote $u : T \to M$. Moreover, the formula

$$\mathrm{area}(T \cap N_s(p)) = \int_0^s \int_{T_t} \frac{1}{\cos(\alpha)} dl\, dt$$

implies that the derivative du is in L^1, and therefore u is in the Sobolev space $H^{1,1}$. See [Mor] for a definition.

Actually, the fact that u is a limit of maps $F(\cdot, t_i) : S \to M$ which are minimal for the g_t metric, and therefore L^2 energy minimizers for the metric structure on S pulled back by $F(\cdot, t_i)$, easily implies that du is L^2.

We need to establish the regularity of du along Γ in the following sense. Let $L = T \cap \Gamma$, and call this the *coincidence set*. For each local sheet of T, we want the derivative du to be well–defined along the interior of L from either side, and tangentially also at a non–interior point of L.

Now, if $I \subset L$ is an interval, then the reflection principle (see [Oss]) implies that each local sheet T^+ of T with $\partial T^+ = I$ can be analytically continued to a minimal surface across I, by taking another copy of T^+, rotating it through angle π along the axis I and gluing it to the original T^+ along I. It follows that du is real analytic from either side along the interior of L. Note that if the tangent cone at a point p is not literally a tangent *plane*, then an easy comparison argument implies that p is an interior point of the coincidence set. See [N] page 90 for a fuller discussion.

Non–interior points of L are more difficult to deal with, and we actually want to conclude that du is continuous at such points. Fortunately, this is a well–known problem in the theory of variational problems, known as the *Signorini problem*, or the (two dimensional) *thin obstacle problem*.

In the literature, this problem is usually formulated in the following terms:

Thin Obstacle Problem. Let Ω be a bounded open subset of $\mathbb{R}^2$, and A an oriented line contained in Ω. Let $\psi : A \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ be given, with $g \geq \psi$ on $\partial\Omega \cap A$. Define

$$\mathbb{K} = \{v \in g + H_0^{1,p} \mid v \geq \psi \text{ on } A\}$$

Minimize

$$J(u) = \int F(x, u, \nabla u) dx$$

over $u \in \mathbb{K}$.

Here $H^{1,p}$ denotes the usual Sobolev space over Ω for the L^p norm, with zero boundary conditions.

The main conditions typically imposed on F are sufficient regularity of F and its partial derivatives (Lipschitz is usually enough) and ellipticity (see [Fre] page 281 for details). For example, if we want the graph of u to be a (Euclidean) minimal surface away from $\psi(A)$, then the formula for F is $F = (1 + |\nabla u|^2)^{1/2}$ which is certainly real analytic and elliptic. The definition of F for a nonparametric minimal surface in exponential co–ordinates on hyperbolic space is more complicated, but certainly F is real analytic and elliptic in the sense of Frehse.

See Figure 1 for an example of the graph of a function solving the Dirichlet thin obstacle problem, where $\psi|_A$ is constant. This surface is visually indistinguishable from the graph of the function solving the unparameterized minimal surface thin obstacle problem with the same boundary and obstacle data, but for computer implementation, the Dirichlet problem is less computationally costly.

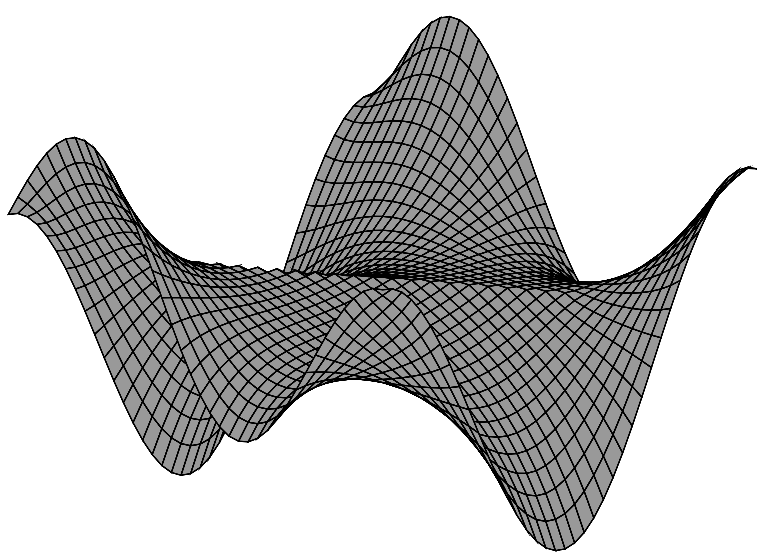

FIGURE 1. The graph of a function solving the thin obstacle problem

The next theorem establishes not only the desired continuity of ∂u, but actually gives an estimate for the modulus of continuity.

The following is a restatement of theorem 1.3 on page 26 of [Ri] in our context:

Theorem 1.27 (Richardson [Ri] Regularity of thin obstacle)**.** *Let u be a solution to the thin obstacle problem for F elliptic in the sense of Frehse and $p \in [1, \infty]$, and suppose that $\partial\Omega, \psi, g$ are smooth. Then ∂u is continuous along A in the tangent direction, one–sided* *continuous in the normal direction on either side, and continuous in the normal direction at a non–interior point. Furthermore, ∂u is Hölder continuous, with exponent $1/2$; i.e. the modulus of continuity of ∂u is $O(t^{1/2})$.*

We apply this theorem to our context:

Lemma 1.28 (Regularity along coincidence set)**.** *For $u : T \to M$ defined as above, the derivative du along local sheets of T is continuous from each side along the coincidence set L, and continuous at non–interior points.*

Proof. If p is an interior point of L, this follows by the reflection principle. Otherwise, by Lemma 1.26 and the discussion above, the tangent cone is a plane π in the tangent space at p.

We show how to choose local co–ordinates in a ball B near each point $p \in L$ such that $B \cap \Gamma$ is the x–axis, each local sheet of T is the graph of a function $u : \Omega \to \mathbb{R}$, and u is non–negative along the x–axis. Let $\gamma = B \cap \Gamma$, and let γ' be another geodesic through p orthogonal to γ and tangent to π. Let $\mathcal{F}$ and $\mathcal{G}$ be foliations of B by totally geodesic planes orthogonal to γ and γ' respectively. Then each leaf of $\mathcal{F}$ is totally geodesic for both the hyperbolic and the g_t metric for all t, and each leaf of $\mathcal{G}$ is totally geodesic for the g_t metric for sufficiently large t. It follows that T has no source or sink singularities with respect to either foliation. Since $T \cap B$ is a (topological) disk, by reasons of Euler characteristic it can have no saddle singularities either, and therefore no singularities at all. We let $\mathcal{F}$ and $\mathcal{G}$ be level sets of two co–ordinate functions on B. Define a third co–ordinate function to be (signed) hyperbolic distance to the plane containing γ and γ', and observe that u is a graph in these co–ordinates.

It follows that u solves an instance of the thin obstacle problem, and by Theorem 1.27 the desired regularity of du follows. $\square$

Remark 1.29. For our applications, the fact that u is $C^{1+1/2}$ is more than necessary. In fact, all we use is that u is C^1. This is proved (with a logarithmic modulus of continuity for du) by [Fre], and (with a Hölder modulus of continuity for du) in arbitrary dimension by [K].

Remark 1.30. The structure of the coincidence set is important to understand, and it has been studied by various authors. Hans Lewy [Lew] showed that for J the Dirichlet integral and ψ analytic, the coincidence set is a finite union of points and intervals. Athanasopoulos [Ath] proved the same result for the minimal surface question, for symmetric domain Ω and obstacle A, but his (very short and elegant) proof relies fundamentally on the symmetry of the problem, and we do not see how it applies in our context.

Note that if the Hausdorff dimension of the coincidence set is strictly < 1, then since T is $C^{1+1/2}$ (and therefore Lipschitz) along this coincidence set, the theory of removable singularities implies that T is actually real analytic along Γ. It follows in this case that the coincidence set consists of a finite union of isolated points, and that T is actually a minimal surface. See e.g. [Car] for details.

Remark 1.31. Existence results for the thin obstacle problem for minimal surfaces with analytic obstacles (see e.g. [Ri], [K], [N]) gives an alternative proof of the existence of the limit T. Given S, we can shrinkwrap S near Γ in small balls by using existence for the thin obstacle problem, and away from Γ by replacing small disks with least area embedded disks with the same boundary. The argument of [HS] implies that for S 2–incompressible, this converges to a surface T.

1.8. **CAT(-1) property.** We have shown that T satisfies all the properties of the conclusion of Theorem 1.9, except that we have not yet shown that it is intrinsically CAT(-1). In this subsection we show that T is CAT(-1) with respect to the path metric induced from M, after possibly replacing it by a new surface T with the same properties.

Lemma 1.32 (CAT(-1) property)**.** *After possibly replacing T by a new immersed surface with the same properties, T is CAT(-1) with respect to the path metric induced from M.*

Proof. To show that T is CAT(-1) we will show that there is no distributional positive curvature concentrated along the coincidence set L. Since $T\backslash\Gamma$ is a minimal surface, the curvature of T is bounded above by -1 on this subset. It will follow by Gauss–Bonnet that T is CAT(-1).

We first treat a simpler problem in Euclidean 3–space, which we denote $\mathbb{R}^3$. Let Σ be an embedded surface in $\mathbb{R}^3$ which is C^3 outside a subset X which is contained in a geodesic γ in $\mathbb{R}^3$, and which is C^1 along X from either side along the interior of X, and C^1 at non–interior points of X. Then we claim, for each subsurface $R \subset \Sigma$ with C^3 boundary $\partial R \subset \Sigma\backslash X$, that

$$\int_{R\backslash X} K_\Sigma = 2\pi\chi(R) - \int_{\partial R} \kappa\, dl$$

Compare Lemma 1.2.

In other words, we want to show that X is a "removable singularity" for R, at least with respect to the Gauss–Bonnet formula.

Let $\phi : R\backslash X \to S^2$ denote the Gauss map, which takes each point $p \in R$ to its unit normal, in the unit sphere of S^2. Then K_Σ is the pullback of the area form by ϕ. Let $\overline{R}$ denote the completion of $R\backslash X$ with respect to the path metric. Then $\overline{R}$ is obtained from R by cutting it open along each interval in $R \cap X$, and sewing in two copies of the interval thereby removed. Notice that there is a natural forgetful map $\overline{R} \to R$.

By the assumptions about the regularity of R, the Gauss map ϕ actually extends to a *continuous* map $\phi : \overline{R} \to S^2$. Moreover, since X is contained in a geodesic γ of $\mathbb{R}^3$, the image $\phi(\overline{R}\backslash R)$ is contained in a great circle C in S^2.

For each boundary component τ of $\overline{R}$, we claim that the map $\phi : \tau \to C$ has degree zero. For, otherwise, by a degree argument, there are points $p^\pm \in \overline{R}$ which map to the same point in p, for which $\phi(p^+) = -\phi(p^-)$, and the graphs of $\phi(p^+)$ and $-\phi(p^-)$ locally have a nonzero algebraic intersection number. It follows that the local sheets of R from either side must actually intersect along p, contrary to the fact that Σ is embedded. It follows that we can sew in a disk to $\overline{R}$ along each boundary component to get a surface $\overline{R}'$ homeomorphic to R, with $\partial\overline{R}' = \partial R$, and extend ϕ to $\phi : \overline{R}' \to S^2$ by mapping each such disk into C.

Now, the surface $\overline{R}'$ can be perturbed slightly in a neighborhood of X to a new surface $\overline{R}''$ which is C^3 in S^2, in such a way that the Gauss map of $\overline{R}''$ is a perturbation of ϕ. So the usual Gauss–Bonnet formula (Lemma 1.2) shows that

$$\int_{\overline{R}''} K = 2\pi\chi(R) - \int_{\partial R} \kappa\, dl$$

But $\int_{\overline{R}''} K$ is just the integral of the area form on S^2 pulled back by the Gauss map; it follows that

$$\int_{\overline{R}''} K = \int_{S^2} \text{degree}(\phi(\overline{R}'))$$

and

$$\int_{S^2} \text{degree}(\phi(\overline{R}'')) = \int_{S^2} \text{degree}(\phi(\overline{R}'))$$

since one map is obtained from the other by a small perturbation supported away from the boundary. Since the measure of C is zero, this last integral is just equal to

$$\int_{S^2 \backslash C} \text{degree}(\phi(\overline{R}')) = \int_{R \backslash X} K$$

and the claim is proved.

Now we show how to apply this to our shrinkwrapped surface T. We use the following trick. Let j_t with $t \in [0,1)$ be a family of metrics on M, conformally equivalent to the hyperbolic metric, which agree with the hyperbolic metric outside $N_{r(1-t)}$, which are Euclidean on $N_{r(1-t)/2}$, and which have curvature pinched between -1 and 0, and are rotationally and translationally symmetric along the core geodesic. Then we let T_t be the surface obtained by shrinkwrapping S *with respect to the* j_t *metric*. I.e. we let $g_{s,t}$ be a family of metrics as in Definition 1.16 which agree with the j_t metric outside $N_{r(1-t)(1-s)}$, construct minimal surfaces $S_{s,t}$ as in Lemma 1.19, and so on, limiting to the immersed surface T_t which is minimal for the j_t metric on $M \backslash \Gamma$, and $C^{1+1/2}$ along $T_t \cap \Gamma$. Arguing locally as above, we see that small subsurfaces of T_t contained in the Euclidean tubes $N_{r(1-t)/2}$ satisfy Gauss–Bonnet in the complement of the coincidence set. By Lemma 1.1, the surfaces T_t all have curvature bounded above by 0, and bounded above by -1 outside $N_{r(1-t)}$. By Gauss–Bonnet for geodesic triangles, T_t is CAT(0), and actually CAT(-1) outside $N_{r(1-t)}$.

Now take the limit as $t \to 1$. Some subsequence of the surfaces T_t converges to a limit T. Again, by Gauss–Bonnet for geodesic triangles, the limit is actually CAT(-1), and the lemma is proved. $\square$

This completes the proof of Theorem 1.9.

Problem 1.33. *Develop a simplicial or PL theory of shrinkwrapping.*

2. The Main Construction Lemma

The purpose of this section is to state the main construction Lemma 2.1 and show how it follows easily from Theorem 1.9.

2.1. **The main construction lemma.** We now state and prove the main construction lemma.

Lemma 2.1 (Main construction lemma)**.** *Let $\mathcal{E}$ be an end of the complete open orientable parabolic free hyperbolic 3–manifold N with finitely generated fundamental group. Let $W \subset N$ be a submanifold such that $\partial W \cap int(N)$ separates W from $\mathcal{E}$. Let $\Delta_1 \subset N \backslash \partial W$ be a finite collection of simple closed geodesics with $\Delta = int(W) \cap \Delta_1$ a non–empty proper subset of Δ_1. Suppose further that ∂W is 2–incompressible rel. Δ.*

Let G be a finitely generated subgroup of $\pi_1(W)$, and let X be the covering space of W corresponding to G. Let Σ be the preimage of Δ in X, and $\hat{\Delta} \subset \Sigma$ a subset which maps *homeomorphically onto Δ under the covering projection, and let $B \subset \hat{\Delta}$ be a nonempty union of geodesics. Suppose there exists an embedded closed surface $S \subset X \backslash B$ that is 2–incompressible rel. B in X, which separates every component of B from ∂X.*

Then ∂W can be homotoped to a Δ_1–minimal surface which, by abuse of notation, we call $\partial W'$, and the map of S into N given by the covering projection is homotopic to a map whose image T' is Δ_1–minimal. Also, $\partial W'$ (resp. T') can be perturbed by an arbitrarily small perturbation to be an embedded (resp. smoothly immersed) surface ∂W_t (resp. T_t) bounding W_t with the following properties:

(1) *There exists an isotopy from ∂W to ∂W_t which never crosses Δ_1, and which induces an isotopy from W to W_t, and a corresponding deformation of hyperbolic manifolds X to X_t which fixes Σ pointwise.*
(2) *There exists an isotopy from S to $S_t \subset X_t$ which never crosses B, such that T_t is the projection of S_t to N.*

Proof. The proof is reasonably straightforward, given the work in §1. First, we obtain $\partial W'$ from ∂W by shrinkwrapping rel. Δ_1. Since $\Delta = \text{int}(W) \cap \Delta_1$ is a nonempty and proper subset of Δ_1, ∂W satisfies the hypotheses of Theorem 1.9, and therefore $\partial W'$ exists, and satisfies the desired properties.

For each $t = t_i$ in our approximating sequence, the metric g_t on W_t lifts to a metric on X_t which, by abuse of notation, we also call g_t. Let $\hat{g}_t$ be the metric on X_t which agrees with g_t near the components B of Σ, near (hyperbolically) concave subsurfaces of the boundary ∂X_t covering ∂W_t, and agrees with the hyperbolic metric elsewhere.

This requires a small amount of explanation: the metric on N is deformed along tubular neighborhoods of some geodesics in $\Delta_1 \backslash W$ in such a way that the tubular neighborhoods intersect the interior of W, near ∂W. If γ is a component in $\Delta_1 \backslash W$ (i.e. an "exterior geodesic"), we let $\hat{g}_t$ agree with the deformed metric g_t on X_t near ∂W, even though the "core" of the corresponding geodesics do not exist in X_t. On the other hand, if γ is a component of $(\Delta_1 \cap W) \backslash B$ (i.e. an "interior geodesic") such that the surface ∂X_t intersects small tubular neighborhoods of some lift γ' to X_t, then by Lemma 1.26, the normal direction to ∂X_t into X_t points into the tube $N_{r(1-t)}(\Delta_1)$, and therefore ∂X_t is mean convex in the hyperbolic metric near γ' and acts as locally as a barrier surface for the hyperbolic metric. So we let $\hat{g}_t$ agree with the hyperbolic metric in tubular neighborhoods of such "interior geodesics". The point of this construction is that with respect to the $\hat{g}_t$ metric, ∂X_t acts as a barrier surface, and we can find a $\hat{g}_t$ least area surface $S_t \subset \text{int}(X_t)$ by [MSY].

There are two alternative approaches which deserve mention. Firstly, the limiting surface $\partial X'$ itself, covering $\partial W'$ acts as a barrier surface for the hyperbolic metric near interior geodesics, by monotonicity properties of the thin obstacle problem (see e.g. [N]). Secondly, it is straightforward to develop a theory of shrinkwrapping for surfaces which might be transverse at finitely many points to some of the geodesics about which it is shrinkwrapped; in the limit, such "transverse obstacles" become invisible, and the limiting surfaces are actually minimal (with respect to the hyperbolic metric) near such transverse obstacles. This should be familiar to the reader who has experimented by pushing needles through soap bubbles, and observing that this operation does not deform their geometry.

In any case, we obtain our $\hat{g}_t$ least area surface S_t. The immersed surfaces $T_t \subset N$ are obtained by mapping S_t to W_t by the covering projection. After passing to a further subsequence of values $t = t_i$, the limit of the T_t exists as a map from S to N, with image T', by the argument of Lemma 1.23 applied locally. The regularity of T' locally along Δ_1 follows from the argument of Lemma 1.28, since that argument is completely local. It follows that T' is Δ_1–minimal. Notice that some local sheets of T' are actually *minimal* (in the usual sense) near geodesics in Δ_1, corresponding to subsets of S_t in X_t crossing components of $\Sigma \backslash B$ where the $\hat{g}_t$ metric agrees with the hyperbolic metric. In any case, T' is intrinsically CAT(-1), and the theorem is proved. □

2.2. **Nonsimple geodesics.** When we come to consider hyperbolic manifolds with parabolics, we need to treat the case that the geodesics Δ_1 might not be simple. But there is a standard trick to reduce this case to the simple case, at the cost of slightly perturbing the hyperbolic metric.

Explicitly, suppose $\Gamma \subset M$ is as in the statement of Theorem 1.9 except that some of the components are possibly not simple. Then for every $\epsilon > 0$ there exists a perturbation g of the hyperbolic metric on M in a neighborhood of Γ with the following properties:

(1) The new metric g agrees with the hyperbolic metric outside $N_\epsilon(\Gamma)$
(2) With respect to the metric g, the curves in Γ are homotopic to a collection of simple geodesics Γ'
(3) The metric g is hyperbolic (i.e. has constant curvature -1) on $N_{\epsilon/2}(\Gamma')$
(4) The metric g is $1+\epsilon$–bilipschitz equivalent to the hyperbolic metric, and the sectional curvature of the g metric is pinched between $-1-\epsilon$ and $-1+\epsilon$.

The existence of such a metric g follows from lemma 5.5 of [Ca]. It is clear that the methods of §1 apply equally well to the metric g, and therefore shrinkwrapping can be done with respect to the metric g, producing a surface which is intrinsically CAT$(-1+\epsilon)$.

In fact, since such a metric exists for each ϵ, we can take a sequence of such metrics g_ϵ for each small $\epsilon > 0$, produce a shrinkwrapped surface T_ϵ for each such ϵ, and take a limit T as $\epsilon \to 0$ which is intrinsically CAT(-1), and which can be approximated by embedded surfaces, isotopic to S, in the complement of $\Gamma \backslash C$ where C is a finite subset of geodesics whose cardinality can be *a priori* bounded above in terms of the genus of S. We will not be using this stronger fact in the sequel, since the existence of a CAT$(-1+\epsilon)$ surface is quite enough for our purposes.

3. Asymptotic Tube Radius and Length

By [Bo] an end of a complete hyperbolic 3–manifold N is geometrically infinite if and only if there exists an exiting sequence of closed geodesics. In this chapter we show that if $\pi_1(N)$ is parabolic free, then the geodesics can be chosen to be η–separated; in particular, all are simple.

Definition 3.1. Let N be a complete hyperbolic 3-manifold with geometrically infinite end $\mathcal{E}$. Define the *$\mathcal{E}$-asymptotic tube radius* to be the supremum over all sequences $\{\gamma_i\}$ of closed geodesics exiting $\mathcal{E}$, of

$$\limsup_{i\to\infty} \text{ tube radius}(\gamma_i)$$

Similarly define the *$\mathcal{E}$-asymptotic length* to be infimum over all sequences $\{\gamma_i\}$ as before of

$$\liminf_{i\to\infty} \text{ length}(\gamma_i)$$

We will drop the prefix $\mathcal{E}$ when the end in question is understood.

Proposition 3.2. *If $\mathcal{E}$ is a geometrically infinite end of the complete hyperbolic 3-manifold N without parabolics, then asymptotic tube radius $> 1/4$ asymptotic length. If asymptotic length = 0, then asymptotic tube radius = ∞. There exists a uniform lower bound η to the asymptotic tube radius of a geometrically infinite end of a complete parabolic free hyperbolic 3–manifold.*

Proof. The second statement follows from the fact [Me] that tube radius goes to infinity as length goes to 0. The third statement is an immediate consequence of the first statement and [Me]. (Actually, Proposition 3.3 will show that $\log(3)/2$ is a lower bound.)

Now suppose that asymptotic length = $L \notin \{0, \infty\}$. Then there exists a sequence $\{\gamma_i\}$ exiting $\mathcal{E}$ such that length(γ_i) $\to L$. As in [G2] if the tube radius $(\gamma_t) \leq \frac{1}{4}\operatorname{length}(\gamma_i)$, then there exists a geodesic β_i homotopic to a curve which is a union of a segment of γ_i and an orthogonal arc from γ_i to itself, and each of these segments has length $\leq \operatorname{length}(\gamma_i)/2$. By rounding corners and applying the above result from [Me], this implies that $\operatorname{length}(\beta_i) < \operatorname{length}(\gamma_i) - C$ for some constant $C > 0$. Thus if $\limsup$ tube radius $(\gamma_i) < L/4$, there exists a sequence $\{\beta_i\}$ such that $\liminf \operatorname{length}(\beta_i) \leq L - C$ where β_i is as above. Since $\infty > L$, $\{\beta_i\}$ must exit the same end as $\{\gamma_i\}$, which is a contradiction.

Now suppose that asymptotic length is infinite and $\{\gamma_i\}$ is an exiting sequence such that length$(\gamma_i) \to \infty$. Given $R \geq 10$ we produce a new exiting sequence $\{\sigma_i\}$ with tube radius $(\sigma_i) > R$ for all i. If possible let α_i be a smallest segment of γ_i such that there is a geodesic path β_i connecting $\partial(\alpha_i)$, $\operatorname{length}(\beta_i) \leq 10R$ and β_i is not homotopic to α_i rel endpoints. If α_i does not exist, then tube radius$(\gamma_i) \geq 5R$. So let us assume that for all i, α_i exists. Note that $\operatorname{length}(\alpha_i) \to \infty$ or else we can find an exiting sequence of bounded length. Therefore for i sufficiently large we can assume that $\operatorname{length}(\alpha_i) > 10R$, $\operatorname{length}(\beta) = 10R$, and both of the angles between β_i and α_i are at least $\pi/2$. The geodesic σ_i homotopic to the curve obtained by concatenating α_i and β_i lies within distance 2 of $\alpha_i \cup \beta_i$ and for the most part lies extremely close. If tube radius$(\sigma_i) \leq R$, then there would be an arc τ_i connecting points of σ_i such that $\operatorname{length}(\tau_i) \leq 2R$ and τ_i cannot be homotoped rel endpoints into σ_i. If for infinitely many i, both endpoints of τ_i uniformly close to β_i, then asymptotic length is bounded. Otherwise for i sufficiently large one finds new essential geodesic paths β_i' of length $\leq 10R$ with endpoints in $\alpha_i - \partial\alpha_i$. This contradicts the minimality property of α_i. ☐

Proposition 3.3. *If N is a complete, orientible, hyperbolic 3-manifold and $\pi_1(N)$ has no parabolic elements, then asymptotic tube radius$> log(3)/2$.*

Remark 3.4. We will not be using Proposition 3.3 in this paper.

Proof. If not, there exists an exiting sequence γ_i such that

$$\lim_i \text{length}(\gamma_i) = t = \text{asymptotic length}$$

and

$$\lim_i \text{tube radius}(\gamma_i) = r = \text{asymptotic tube radius}$$

where $r \leq log(3)/2$. By passing to a subsequence we can find sequences $\{\alpha_i\}, \{\beta_i\}$ where α_i is a lift of γ_i to $\mathbb{H}^3$ and β_i is a nearest $\pi_1(N)$ translate of α_i. Furthermore we can assume that the associated triple (L_i, D_i, R_i) (as defined in [GMT]) converges to (L, D, R). Here $\mathrm{Re}(D) = 2r$ and $\mathrm{Re}(L) = t$. As in [GMT], (L, D, R) gives rise to a marked 2-generator group $\langle f, w\rangle$. Since $\mathrm{Re}(D) < \log(3)$, (L, D, R) must lie in the parameter space $\mathcal{P}$. It cannot lie in one of the 6 exceptional regions, or else by [GMT], [JR], [Li] and [CLLM], it and (L_i, D_i, R_i) correspond to a closed hyperbolic 3-manifold for i sufficiently large. Therefore some word $u(f, w)$ in f and w either gives rise to an element of shorter length or a translate A' of $\mathrm{Axis}(f) := A$ at distance less than $Re(D)$ from A. In either case the reduction of length or distance is bounded below by some constant ϵ. If (L, D, R) gives rise to a length reducing killer word, then so does (L_i, D_i, R_i) for i sufficiently large. Since $\pi_1(N)$ has no parabolics, this word corresponds to a hyperbolic element and hence a geodesic $\sigma_i \subset N$. If $u(f, w)$ is loxodromic, then the corresponding geodesic σ is of bounded distance from α, the geodesic associated to f. Therefore the geodesics $\{\sigma_i\}$ are at bounded distance from $\{\gamma_i\}$ and hence exit the same end. If $u(f, w)$ is parabolic, then $\mathrm{length}(\sigma_i) \to 0$, hence $\{\sigma_i\}$ is exiting and must exit the same end as $\{\gamma_i\}$. Indeed, in $\mathbb{H}^3, u(f, v)$ takes a point x to y where $d(x, y) < t/4$. Then for i sufficiently large there are essential closed curves of length $< t/2$ at distance at most $2d(x, A)$ from α_i. Similarly (L, D, R) does not give rise to an ortholength reducing killer word, or else for i sufficiently large we would obtain a contradiction to the fact that r is the asymptotic tube radius. □

Question 3.5. *What is the maximal lower bound for the asymptotic tube radius of a geometrically infinite end $\mathcal{E}$ of a complete, orientable, hyperbolic manifold with finitely generated fundamental group, both in the cases that $\mathcal{E}$ is parabolic free or not?*

Question 3.6. *What is the upper bound for asymptotic length of a geometrically infinite end $\mathcal{E}$? It follows from Theorem 0.9 that there is an upper bound which is a function of rank$(\pi_1(\mathcal{E}))$.*

4. CANARY'S THEOREM

In this section we give a proof of Canary's theorem (Theorem 4.1) when N is parabolic free. Our proof of Theorem 0.9 will closely parallel this argument.

Theorem 4.1 (Canary). *If $\mathcal{E}$ is a topologically tame end of the complete, orientable, hyperbolic 3-manifold $N = \mathbb{H}^3/\Gamma$, where Γ has no parabolic elements, then there exists a sequence of CAT(-1) surfaces exiting the end. If $\mathcal{E}$ is parametrized by $S \times [0, \infty)$, then these surfaces are homotopic to surfaces of the form $S \times t$, via a homotopy supported in $S \times [0, \infty)$.*

Proof. It suffices to consider the case that $\mathcal{E}$ is geometrically infinite. By Lemma 3.2 there exists a sequence of pairwise disjoint η-separated simple closed geodesics $\Delta = \{\delta_i\}$ exiting $\mathcal{E}$. Assume that Δ and the parametrization of $\mathcal{E}$ are chosen so that for all $i \in \mathbb{N}$, $\delta_i \subset S \times (i - 1, i)$. Let g=genus(S), $\Delta_i = \{\delta_1, \cdots, \delta_i\}$ and $\{\alpha_i\}$ a locally finite collection of embedded proper rays in $\mathcal{E}$ such that $\partial\alpha_i \in \delta_i$.

An idea used repeatedly, in various guises, throughout this paper is the following. If R is a closed oriented surface and T is obtained by shrinkwrapping R rel the geodesics Δ_R, then R is homotopic to T via a homotopy which does not meet Δ_R, except possibly at the last instant. Therefore, if $\delta_i \subset \Delta_R$ and $\langle R, \alpha_i\rangle = 1$, then $T \cap \alpha_i \neq \emptyset$ and if $T \cap \delta_i = \emptyset$, then $\langle T, \alpha_i\rangle = 1$. Here $\langle\cdot, \cdot\rangle$ denotes algebraic intersection number.

Case 1. Each $S \times i$ is incompressible in $N \backslash \Delta$. (E.g. $N = S \times \mathbb{R}$)

Proof of Case 1. Apply Lemma 2.1 to shrinkwrap $S \times i$ rel. Δ_{i+1} to a CAT(-1) surface S_i. Since $\langle S \times i, \alpha_i \rangle = 1$, $S_i \cap \alpha_i \neq \emptyset$. Since $\{\alpha_i\}$ is locally finite, the Bounded Diameter Lemma implies that the S_i's must exit $\mathcal{E}$. Therefore for i sufficiently large $S_i \subset \mathcal{E}$ and $\langle S_i, \alpha_1 \rangle = 1$; and hence the projection of S_i into $S \times 0$, (given by the product structure on $\mathcal{E}$) is a degree-1 map between surfaces of the same genus. Since such maps are homotopic to homeomorphisms, we see that S_i can be homotoped within $\mathcal{E}$ to a homeomorphism onto $S \times 0$. See Figure 2 for a schematic view. $\square$

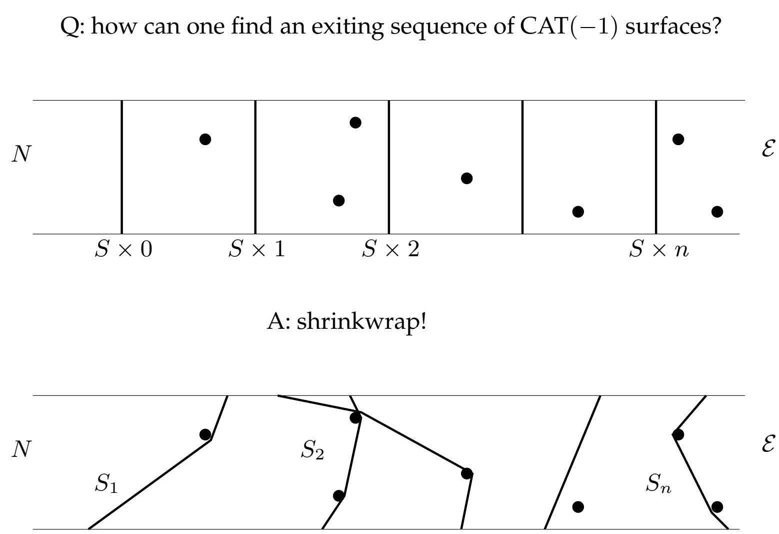

FIGURE 2. A schematic depiction of shrinkwrapping in action

Case 2. General Case. (E.g. N is a homotopy handlebody.)

Proof of Case 2. Without loss of generality we can assume that every closed orientable surface separates N, (see Lemma 5.1 and Lemma 5.6). We first use a purely combinatorial/topological argument to find a particular sequence of surfaces. We then shrinkwrap these surfaces and show that they have the desired escaping and homological properties.

Fix i. If possible, compress $S \times i$, via a compression which either misses Δ or crosses Δ once say at $\delta_{i_1} \subset \Delta_i$. If possible, compress again via a compression meeting $\Delta \backslash \delta_{i_1}$ at most once say at $\delta_{i_2} \subset \Delta_i$. After at most $n \leq 2g-2$ such operations we obtain embedded connected surfaces $S^i_1, \cdots, S^i_{i_r}$, none of which is a 2-sphere and each is 2-incompressible rel $\Delta_{i+1} \backslash \{\delta_{i_1} \cup \cdots \cup \delta_{i_n}\}$. With at most $2g-2$ exceptions, each $\delta_j, j \leq i$, is separated from $\mathcal{E}$ by exactly one surface S^i_k. Call *Bag*^i_k the region separated from $\mathcal{E}$ by S^i_k.

Since each $i_r \leq g$, we can find a $p \in \mathbb{N}$ and a reordering of the S^i_j's (and their bags) so that for infinitely many $i \geq p$, $\delta_p \in \mathrm{Bag}^i_1$; furthermore, for i such that $\delta_p \in \mathrm{Bag}^i_1$, if $p(i)$ denotes the maximal index such that $\delta_{p(i)} \in \mathrm{Bag}^i_1$ then the set $\{p(i)\}$ is unbounded. By Lemma 2.1, S^i_1 is homotopic rel $\Delta_{i+1} \backslash \{\delta_{i_1}, \cdots, \delta_{i_n}\}$ to a

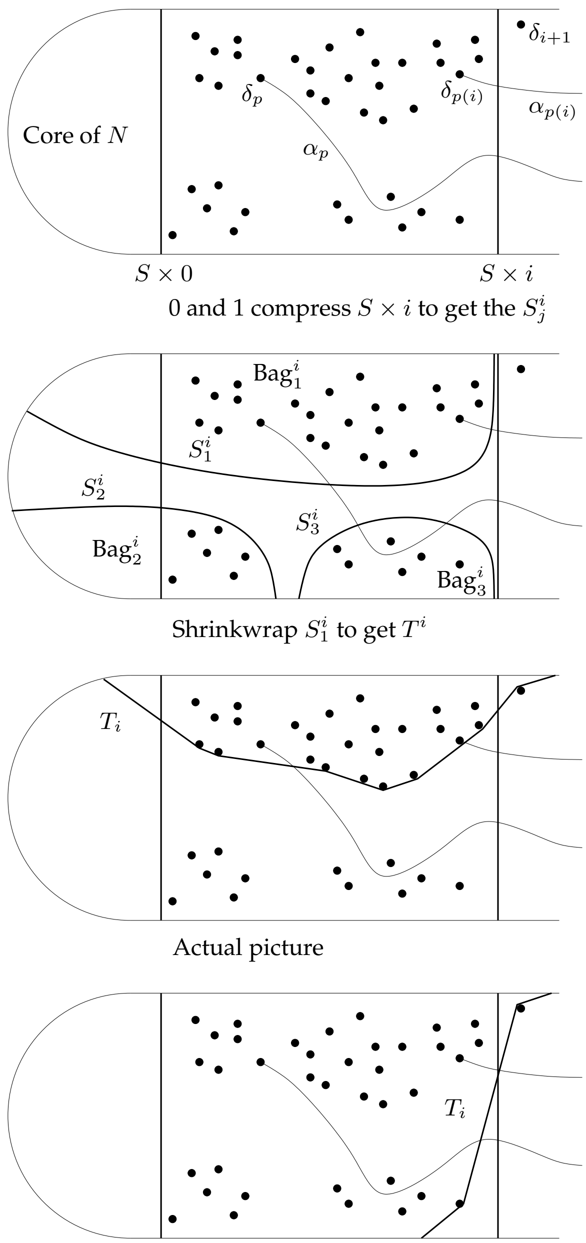

FIGURE 3. The Bounded Diameter Lemma and the intersection number argument show that $S \times i$ undergoes no compression, and T_i actually separates all δ_i from $\mathcal{E}$.

CAT(-1) surface S_i. Since the collection $\{\alpha_{p(i)}\}$ is infinite and locally finite, the Bounded Diameter Lemma implies that a subsequence of these S^i_1's must exit $\mathcal{E}$. Call this subsequence $T_1, T_2, \cdots$. Therefore, for i sufficiently large, T_i must lie in $S \times (p, \infty)$ and hence $\langle T_i, \alpha_p \rangle = 1$. Therefore, projection of T_i to $S \times p$ is degree

1. This in turn implies that genus $T_i = g$ and T_i can be homotoped within $\mathcal{E}$ to a homeomorphism onto $S \times 0$. See Figure 3 for another schematic view. $\square$

Remark 4.2. This argument shows that for i sufficiently large, $S\times i$ is 2-incompressible in $N\setminus\Delta_i$. Also, given any η-separated collection of exiting geodesics a sufficiently large finite subset is 2-disc busting. Actually using the technology of the last chapter, this statement holds for any sequence of exiting closed geodesics.

The proof of Theorem 0.9 follows a similar strategy. Here is the outline in the case that N has a single end $\mathcal{E}$ and no parabolics. Given a sequence of η-separated exiting simple closed geodesics $\Delta = \{\delta_i\}$ we pass to subsequence (and possibly choose δ_1 to have finitely many components) and find a sequence of connected embedded surfaces denoted $\{\partial W_i\}$ such that for each $i,\ \partial W_i$ separates $\Delta_i = \delta_1 \cup \delta_2 \cup \cdots \cup \delta_i$ from $\Delta - \Delta_i$ and is 2-incompressible rel Δ_i. It is *a priori* possible that the ∂W_i's do not exit $\mathcal{E}$. If W_i denotes the compact region split off by ∂W_i, then after possibly deleting an initial finite set of W_i's (and adding the associated δ_i's to δ_1) we find a compact 3-manifold $D \subset W_1$ which is a core for $\mathcal{W} = \cup W_i$.

We next find an immersed genus $\leq g$ surface T_i, which in a twisted sense, separates off a subset B_i of Δ_i from $\mathcal{E}$. For infinitely many i, B_i includes a fixed δ_p and for these i's the set $\{p(i)\}$ is unbounded, where $p(i)$ is the largest index of a $\delta_k \subset B_i$. The surface T_i separates B_i from the rest in the sense that T_i lifts to an embedded surface $\hat{T}_i$ in the $\pi_1(D)$-cover $\hat{W}_i$ of W_i and in that cover $\hat{T}_i$ separates a lift $\hat{B}_i$ from $\partial\hat{W}_i$, the preimage of ∂W_i. The argument to this point is purely topological and applies to any 3-manifold with finitely generated fundamental group. In the general case, $\{\partial W_i\}$ will not be an exiting sequence.

Next we shrinkwrap ∂W_i rel Δ_{i+1} to a CAT(-1) surface which we continue to call ∂W_i. Then we homotope $\hat{T}_i$ rel $\hat{\Delta}_i$ to a CAT(-1) surface in the induced $\hat{W}_i$ and let T_i now denote the projected surface in N. (The point of shrinkwrapping ∂W_i is that $\partial\hat{W}_i$ is now a barrier which prevents $\hat{T}_i$ from popping out of $\hat{W}_i$ during the shrinkwrapping process.) We use the $\delta_{p(i)}$'s to show that, after passing to subsequence, the T_i's exit $\mathcal{E}$. We use δ_p to show that for i sufficiently large, T_i homologically separates $\mathcal{E}$ from a Scott core of N.

We have outlined the strategy. For purposes of exposition, the construction of the T_i's in §6 is slightly different from the sketch above.

In §7 we make the necessary embellishments to handle the parabolic case.

The next chapter develops the theory of end reductions which enables us to define the submanifolds W_i.

5. End Manifolds and End Reductions

In this section, we prove a structure theorem for the topology of an end of a 3–manifold with finitely generated fundamental group.

The first step is to replace our original manifold with a 1–ended manifold M with the homotopy type of a bouquet of circles and closed orientable surfaces. We then prove Theorem 5.25, the *infinite end engulfing theorem*, which says that given an exiting seqence of homotopically non trivial simple closed curves we can pass to a subsequence Γ and find a submanifold $\mathcal{W}$, with finitely generated fundamental group, containing Γ which has the following properties:

(1) $\mathcal{W}$ can be exhausted by codimension-0 compact submanifolds W_i whose boundaries are 2–incompressible rel $\Gamma \cap W_i$.
(2) $\mathcal{W}$ has a core which lies in W_1.

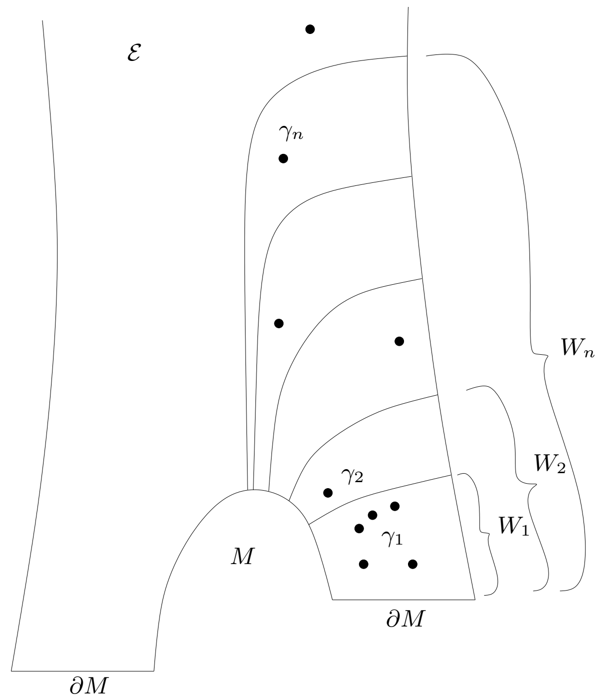

FIGURE 4. A schematic view of $\{W_i\}$, $\{\gamma_i\}$ and M.

This completes the preliminary step in the proof of Theorem 0.9, as explained at the end of §4. The proof of Theorem 0.9 itself is in §6.

In what follows we will assume that all 3-manifolds are orientable and irreducible.

Lemma 5.1. *If $\mathcal{E}$ is the end of an open Riemannian 3-manifold M' with finitely generated fundamental group, then $\mathcal{E}$ is isometric to the end of a 1-ended 3-manifold M whose (possibly empty) boundary is a finite union of closed orientable surfaces. A core of M is obtained by attaching 1-handles to the components of ∂M, unless $\partial M = \emptyset$, in which case a core is a 1-complex and M=M'.*

Proof. A thickened Scott core C [Sc] of M' is a union of 1-handles (possibly empty) attached to a compact 3-manifold X with incompressible boundary. Split M' along all the boundary components of X and let M be the component which contains $\mathcal{E}$. □

Remark 5.2. M is a submanifold of M'. M is isometric to a submanifold $\hat{M}$ of the covering of M' corresponding to the inclusion $\pi_1(M) \to \pi_1(M')$, and the inclusion $M \to \hat{M}$ is a homotopy equivalence.

Definition 5.3. Call a finitely generated group a *free/surface group* if it is a free product of orientable surface groups and a free group. Call a 1-ended, irreducible, orientable, 3-manifold M an *end–manifold* if it has a compact (possibly empty) boundary and a compact core of the form $\partial M \times I \cup 1$-handles if $\partial M \neq \emptyset$ or a handlebody if $\partial M = \emptyset$.

Note that $\pi_1(M)$ is a free/surface group for M an end–manifold.

Lemma 5.4. *If G is a subgroup of a free/surface group, then its π_1-rank equals its H_1-rank, both in $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$-coeficients.* □

Proof. A finitely generated subgroup of a free/surface group is a free/surface group, and equality holds in that case. An infinitely generated subgroup of a free/surface group contains an infinitely generated free summand. Consequently, both π_1 and H_1 rank are infinite for such subgroups. □

Lemma 5.5. *An H_1-injective subgroup G of a free/surface group K is finitely generated.*

Proof. Rank $\pi_1(G) = \mathrm{rank}(G/[G,G]) \leq \mathrm{rank}(K/[K,K]) = \mathrm{rank}\,\pi_1(K) < \infty$. □

Lemma 5.6. *A 1-ended, orientable, irreducible 3-manifold M with compact boundary is an end-manifold if and only if $\pi_1(M)$ is a free/surface group, $H_2(M, \partial M) = 0$ and ∂M is π_1-injective.*

Every closed embedded π_1-injective surface in an end-manifold is boundary parallel.

Proof. Let M be an end-manifold with core C of the form $\partial M \times I \cup 1$-handles or handlebody if $\partial M = \emptyset$. Since the inclusion $C \to M$ is a homotopy equivalence, ∂M is incompressible and $\pi_1(M)$ is a free/surface group. If $T \subset M$ is a compact properly embedded π_1-injective surface, then T can be homotoped rel ∂T into C. The cocores D_i of the 1-handles are properly embedded disks whose boundary misses ∂T. Since T is homotopically essential, it follows that each intersection $T \cap D_i$ is homotopically inessential in T, and therefore T can be homotoped off the cocores of the 1–handles. Once this is done, T can be further homotoped rel. boundary into ∂M, since C deformation retracts to ∂M in the complement of the cocores of the 1–handles. This implies that $H_2(M, \partial M) = 0$.

Conversely, since $H_2(M, \partial M) = 0$ and M has incompressible boundary, a connected, closed orientable incompressible surface R must separate off a connected, compact Haken manifold X with incompressible boundary. Since $\pi_1(M)$ is a free/surface group, $\pi_1(X)$ is a closed orientable surface group and using [St] we conclude that $X = N(T)$ for some component T of ∂M, so R is boundary parallel.

If $\partial M = \emptyset$, then any core is a handlebody. Conversely, if $\partial M \neq \emptyset$, by [Mc] M has a core C which contains ∂M. If C' is obtained by maximally compressing C, then by the second pararaph, each component of $\partial C'$ is boundary parallel and hence $C = \partial M \times I \cup 1$-handles. □

Corollary 5.7. *If $\mathcal{W}$ is a 1-ended, π_1-injective submanifold of the end-manifold M such that $\pi_1(\mathcal{W})$ is finitely generated and $\partial\mathcal{W}$ is a union of components of ∂M, then $\mathcal{W}$ is an end-manifold.*

Definition 5.8. Given a connected compact subset J of an open irreducible 3-manifold M, the *end reduction* of J to M is to first approximation the smallest open submanifold of M which can engulf, up to isotopy, any closed surface in

$M\backslash J$ which is incompressible in $M\backslash J$. End reductions were introduced by Brin-Thickstun [BT1, BT2]. Their basic properties were developed by Brin-Thickstun [BT1, BT2] and Myers [My]. In particular [BT1] show that W_J can be created via the following procedure. If $V_1 \subset V_2 \subset \cdots$ is an exhaustion of M by compact connected codimension-0 submanifolds such that $J \subset V_1$, then one inductively obtains an exhaustion $W_1 \subset W_2 \subset \cdots$ of W_J as follows. Transform V_1 to W_1 through a maximal series of intermediate manifolds $U_1 = V_1, U_2, \cdots, U_n = W_1$ where U_{k+1} is obtained from U_k by one of the following 3 operations.

(1) Compress along a disc disjoint from J.
(2) Attach a 2-handle to U_k which lies in $M\backslash\text{int}(U_k)$, and whose attaching core circle is essential in U_k.
(3) Delete a component of U_k disjoint from J.

Having constructed W_i, pass to a subsequence of the V_j's and reorder so that $W_i \subset \text{int}(V_{i+1})$. Finally pass from V_{i+1} to W_{i+1} via a maximal sequence of the above operations. Since ∂W_i is incompressible in $M - J$, an essential compression of U_k can be isotoped rel boundary to one missing W_i. Therefore, we will assume such operations miss W_i and hence $W_i \subset \text{int}(W_{i+1})$. Brin and Thickstun [BT1] show that W_J is up to isotopy independent of all choices.

We will say that $\{W_i\}$ is a *standard exhaustion* of W_J if it is an exhaustion of W_J which arises, as above, i.e. via the three end-reduction operations from an exhaustion of M by compact sets.

Remark 5.9. Note that operations (1) and (2) reduce the sum of the ranks of π_1 of the components. It follows that the transition from V_i to W_i is obtained by a *finite* sequence of operations.

Remark 5.10. (Historical Note) Brin and Thickstun [BT1], [BT2] study end reductions to develop a necessary and a sufficient condition, *end 1-movability*, for taming an end of a 3-manifold. More recently, Myers [My] has promoted the use of end reductions to address both the $\mathbb{R}^3$-covering space conjecture and the Marden conjecture.

Lemma 5.11. *The inclusion $i_J : \mathcal{W}_J \to N$ induces π_1 and H_1-injections. The latter in both $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ homology.*

Proof. The π_1-injectivity was first proven in [BT2] and rediscovered in [My]. Our proof of H_1-injectivity mimics the proof of π_1-injectivity in [My]. Suppose $C \subset W_i$ is a union of oriented simple closed curves bounding the surface S. Note that by elementary 3–manifold topology, we can assume S is embedded.

By choosing n sufficiently large we can assume that $W_i \cup S \subset V_n$. If V_n^1 is obtained by adding a 2-handle to V_n, then $S \subset V_n^1$. If V_n^1 is obtained by compressing V_1, via a compression missing J, then by modifying S near the compressing disc we obtain a surface S_1 spanning C (orientably, if need be) with $S_1 \subset V_n^1$. If V_n^1 is obtained by deleting components of V_1 which miss C, then $S_1 = S \cap V_n^1$ still spans C. Since W_n is obtained from V_n by a finite sequence of such operations it follows that C bounds in W_n and hence in $\mathcal{W}_J$. □

H_1-injectivity of $\mathcal{W}_J$ in N gives us the following crucial corollary:

Corollary 5.12. *An end-reduction in an end-manifold has finitely generated fundamental group.*

Proof. Combine Lemma 5.11 with Lemma 5.5. □

Definition 5.13. If W_J is an end reduction of the codimension-0 submanifold J in N, then we say that W_J is *trivial* if W_J is isotopic to an open regular neighborhood of J or equivalently W_J is isotopic to $\text{int}(J)$. W_J is *eventually trivial* if it has an exhaustion $W_1 \subset W_2 \subset \cdots$ such that ∂W_i is parallel to ∂W_j for all i, j.

We now study end reductions of disconnected spaces J. While the following technology and definitions can be given for more general objects we restrict our attention to a finite unions of pairwise disjoint closed (possibly non simple) curves none of which lie in a 3-cell. Ultimately we will address end reductions of infinite sequences of exiting curves.

Definition 5.14. If J is a finite union of pairwise disjoint closed curves in an open irreducible 3-manifold M, we say that that J is *end nonseparable* if there is a compact connected submanifold H such that $J \subset \text{int}(H)$ and ∂H is incompressible in $M \backslash J$. Such an H is called a *house* of J. If J is end nonseparable, then define W_J to be an end-reduction of H, and call W_J the *end reduction of* J.

Lemma 5.15. *The end-reduction $\mathcal{W}_J$ of an end nonseparable union J of simple closed curves is well defined up to isotopy.*

Proof. Let H and H' be two houses for J. We want to show that if $\mathcal{W}_H$ is an end reduction of H, then there is an isotopy of H' to H'_1 fixing J, so that the end reduction $\mathcal{W}_{H'_1}$ of H'_1 is equal to $\mathcal{W}_H$. By the definition of a house for J, both H and H'_1 satisfy the property that they are connected submanifolds of M whose boundaries are incompressible in $M \backslash J$.

Let $\{W_i\}$ be a standard exhaustion of $\mathcal{W}_H$ arising from the exhaustion $\{V_i\}$ of M. By passing to a subsequence we can assume that $H' \cup H \subset V_2$. By considering the passage of V_2 to W_2, we observe that H' can be isotoped to H'_1 rel J to lie in $\text{int}(W_2)$ and that ∂W_2 is incompressible in $M \backslash H'_1$. Thus $\mathcal{W}_H$ is also an end-reduction of H'. Since end reductions are unique up to isotopy the result follows, and we may unambiguously denote $\mathcal{W}_H$ by $\mathcal{W}_J$. □

Lemma 5.16. *Let A be a finite union of pairwise disjoint closed curves in the open irreducible 3-manifold M. Then A canonically decomposes into finitely many maximal pairwise disjoint end non separable subsets $A_1, \cdots, A_n$. Indeed, if B is a maximal end non separable subset of A, then $B = A_i$ for some i.*

Proof. It suffices to show that if B and C are end non separable subsets of A, then either $C \cup B$ is end non separable or $C \cap B = \emptyset$. Let H_B and H_C be houses for B and C respectively. Let $V \subset N$ be a compact submanifold containing $H_B \cup H_C$. By considering the passage of V to W by a maximal sequence of compressions, 2-handle additions, and deletions which are taken with respect to $B \cup C$, one sees that H_B (resp. H_C) can be isotoped to lie in W via an isotopy fixing B (resp. C). If $B \cap C \neq \emptyset$, then W is connected and hence is a house for $B \cup C$. □

Lemma 5.17. *If $A_1, \cdots A_n$ are the maximal end non separable components of a finite set A of pairwise disjoint closed curves in an open irreducible 3-manifold M, then they have pairwise disjoint end reductions. In particular they have pairwise disjoint houses.*

Proof. Let $A_1, A_2, \cdots, A_n$ be the maximal end separable subsets of A. Let $\{V_k\}$ be an exhaustion of M with $A \subset V_1$. Consider a sequence $V_1 = U_1, \cdots, U_n = W_1$

where the passage from one to the next is isotopy, compression, 2-handle addition or deletion, where the compressions or deletions are taken with respect to A. By passing to a subsequence of the exhaustion we can assume that $W_1 \subset V_2$, and in the above manner pass from V_2 to W_2. In like manner construct $W_3, W_4, \cdots$. By deleting finitely many of the first W_i's from the sequence and reindexing, we can assume that all the W_i's have the same number of components.

It suffices to show that if W is a component of W_k, then W contains a unique A_i and that ∂W is incompressible in $M \backslash A_i$. Indeed, it suffices to prove incompressibility of ∂W in $M \backslash (W \cap A)$, for then W is a house and can only contain one A_i by maximality. If ∂W is compressible in $M \backslash (W \cap A)$ it must compress to the outside via some compressing disc D. Consider a term V_n in the exhausting sequence with $W_k \cup D \subset V_n$. By considering the passage of V_n to W_n we can rechoose the disc spanning ∂D to obtain a new compressing disc $E \subset W_n$. Since ∂W is incompressible in $M \backslash \text{int}(W_k)$, it follows that E must hit a component of W_k distinct from W. This imples that W_n contains fewer components than W_k, which is a contradiction. □

Lemma 5.18. *If $A_1, A_2, \cdots, A_n$ are as in Lemma 5.17, with pairwise disjoint end reductions $W_{A_1}, W_{A_2}, \cdots, W_{A_n}$, then $W_{A_1} \cup W_{A_2} \cup \cdots \cup W_{A_n}$ is H_1-injective in M, in both $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ coefficients.*

Proof. Repeat the proof of Lemma 5.11. □

Corollary 5.19. *Let A be a union of finitely many pairwise disjoint simple closed curves in the end-manifold M. If each component of A is homotopically nontrivial, then A breaks up into at most rank$(\pi_1(M))$ maximal non separable subsets.*

Proof. If A partitions into maximal non separable subsets $A_1, \cdots, A_n$, then the H_1-rank of W_{A_i} is non trivial, since $\pi_1(W_{A_i})$ is a nontrivial subgroup of a free/surface group. Now apply the previous lemma. □

Lemma 5.20. *If $J \subset J'$ are finite, end nonseparable unions of homotopically essential, pairwise disjoint, simple closed curves with end-reductions $\mathcal{W}$ and $\mathcal{W}'$, then $\mathcal{W}$ is isotopic rel J to $\mathcal{W}_1$, where $\mathcal{W}_1 \subset \mathcal{W}'$.*

Proof. Let $W_1 \subset W_2 \subset \cdots$ be a standard exhaustion of $\mathcal{W}$. Let $Z_1 \subset Z_2 \subset \cdots$ be a standard exhaustion of $\mathcal{W}'$ arising from the exhaustion $\{V_i\}$ of M. By passing to subsequence we can assume that $W_1 \subset V_1$. By considering the passage of V_1 to Z_1 we can isotope W_1 rel J to lie in Z_1. Proceeding by induction and passing to subsequence, we can assume that $W_k \subset V_k$ and $W_{k-1} \subset Z_{k-1}$. By considering the passage of V_k to Z_k (which fixes Z_{k-1}) we can isotope W_k rel W_{k-1} to lie in Z_k. The isotoped W_i's give rise to an isotopy of $\mathcal{W}$ to $\mathcal{W}_1$ with $\mathcal{W}_1 \subset \mathcal{W}'$. □

Lemma 5.21. *Let $\gamma_1, \gamma_2, \cdots$ be a sequence of homotopically nontrivial, pairwise disjoint simple closed curves in the end-manifold M. Then we can group together finitely many of the curves into γ_1, and pass to a subsequence so that*

(1) *$\Gamma_i = \{\gamma_1, \cdots, \gamma_i\}$ is end nonseparable.*
(2) *For each $\gamma \in \Gamma_i$, there exists a $\gamma' \in \Gamma_1 - \gamma$ such that $[\gamma] = [\gamma'] \in H_1(M, \mathbb{Z}/2\mathbb{Z})$.*

Proof. By passing to subsequence we can assume that each γ_i represents the same element of $H_1(M, \mathbb{Z}/2\mathbb{Z})$. By Lemma 5.16, if T is a finite subset of Γ, then T canonically partitions into finitely many end nonseparable subsets $S_1, \cdots, S_n$ with corresponding pairwise disjoint end reductions $\mathcal{W}_1, \cdots, \mathcal{W}_n$. Define

$$C(T) = \sum_{i=1}^{n} \mathrm{rank}(H_1(\mathcal{W}_i, \mathbb{Z}/2\mathbb{Z})) = \mathrm{rank}(H_1(\cup_{i=1}^{n} \mathcal{W}_i, \mathbb{Z}/2\mathbb{Z})) \leq \mathrm{rank}(H_1(M, \mathbb{Z}/2\mathbb{Z}))$$

where the last inequality follows from Lemma 5.18. Define

$$C(\Gamma) = \max\{C(T) \,|\, T \text{ is a finite subset of } \Gamma\}$$

Now pass to an infinite subset of Γ with $C(\Gamma)$ is minimal. By Lemma 5.18, if $T \subset \Gamma$ with $C(T) = C(\Gamma)$, then adding a new element to the T does not increase the number of end non separable subsets in its canonical partition. Since $C(\Gamma)$ is minimal, we can enlarge T by adding finitely many elements so that the enlarged T, which by abuse of notation we still call T, is end nonseparable. Again by maximality of $C(T)$, T together with any finite subset of Γ is still end non separable. Now express Γ as $\cup\gamma_i$ with $\gamma_1 = T$. □

Lemma 5.22. *Let $\{\gamma_i\}$ be a set of pairwise disjoint homotopically non trivial simple closed curves in the end-manifold M, with γ_1 possibly having multiple components. Let $\Gamma_i = \cup_{j=1}^{i}\gamma_j$. Fix $x \in \gamma_1$. Let $\mathcal{W}_i$ denote an end-reduction to Γ_i and $G_i = in_*(\pi_1(\mathcal{W}_i, x))$, where $in : \mathcal{W}_i \to M$ is inclusion. Then*

1. *$G_1 \subset G_2 \subset G_3 \cdots$.*
2. *If $G = \cup G_i$, then the natural map $G \to \pi_1(M)$ is H_1-injective in $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ coefficients, i.e. $G/[G, G] \to \pi_1(M, x)/[\pi_1(M, x), \pi_1(M, x)]$ is injective (also after reducing mod 2 in the $\mathbb{Z}/2\mathbb{Z}$ case).*
3. *G is finitely generated.*

Proof. Conclusion (1) follows directly from Lemma 5.20. By Lemma 5.11, each G_i is H_1–injective in M. Since $G = \cup G_i$, (2) follows. Conclusion (3) follows from Lemma 5.5. □

Lemma 5.23. *If $\gamma_1, \gamma_2, \cdots$ is a sequence of pairwise disjoint, homotopically non trivial, simple closed curves in the end-manifold M, then after passing to subsequence, allowing γ_1 to have multiple components and fixing a base point $x \in \gamma_1$, there exists a compact set $D_1 \subset M$ and compact submanifolds $\{W_i\}$ of M such that*

1. *$\partial W_i \cap \partial M$ is a union of components of ∂M and $\partial W_i - \partial M$ is connected.*
2. *If $\Gamma_i = \cup_{j=1}^{i}\gamma_j$, then $\Gamma_i \subset W_i$, and D_1 can be isotoped rel x to $D_i \subset W_i$ such that Γ_i can be homotoped into D_i via a homotopy supported in W_i.*
3. *∂W_i is 2-incompressible rel Γ_i.*

Proof. By Lemma 5.21 we can assume that if $\Gamma_i = \cup_{j=1}^{i}\gamma_i$, then Γ_i is end nonseparable, each γ_i represents the same element of $H_1(M, \mathbb{Z}/2\mathbb{Z})$ and $|\Gamma_1| \geq 2$. By Lemma 5.22, after possibly enlarging Γ_1, we can further assume that if G_i denotes $in_*(\pi_1(\mathcal{W}_i, x)) \subset \pi_1(M, x)$, then $G_i = G_j$ for all i, j. For each i, fix an end-reduction $\mathcal{W}_i$ of Γ_i, with $W_i^1 \subset W_i^2 \subset W_i^3 \subset \cdots$ a standard exhaustion arising from the exhaustion $\{V_j\}$ of M.

Since each component of ∂M is incompressible, W_i^j is connected and every closed surface in M separates, it follows that for j sufficiently large, $W_i^j \cap \partial M$ is a union of components of ∂M and $\partial W_i^j - \partial M$ is connected. This proves (1).

Let D_1 be a Scott core of $\mathcal{W}_1$ with $x \in D_1$. To each $\gamma \in \Gamma_1$ pick a $\emptyset \neq \gamma' \in \Gamma_1 - \gamma$ (though one can use the $\emptyset$ if $0 = [\gamma] \in H_1(M, \mathbb{Z}/2\mathbb{Z})$). By construction there exists a $\mathbb{Z}/2\mathbb{Z}$-homology between γ and γ' and a homotopy from γ into D_1 supported in $\mathcal{W}_1$. Choose W_1^i sufficiently large to contain these homotopies, homologies as well as D_1. Let $W_1 = W_1^i$. By construction W_1 satisfies the conclusions of 2) and ∂W_1 is incompressible in $M - \Gamma_1$. To show 2-incompressibility, note that if W_1 had an essential compressing disc E transverse to $\gamma \in \Gamma_1$, then $|E \cap \gamma| \geq 2$ if $0 = [\gamma] \in H_1(M, \mathbb{Z}/2\mathbb{Z})$ and $E \cap \gamma' \neq \emptyset$ otherwise, for some $\gamma' \in \Gamma_1 - \gamma$.

Now fix $i > 1$. By Lemma 5.20, D_1 can be isotoped rel x into $\mathcal{W}_i$. Let D_i be an isotoped image. Since $G_1 = G_i$, the inclusion $D_i \to \mathcal{W}_i$ is a homotopy equivalence. Choose j sufficiently large, so that W_i^j engulfs D_i, homotopies of Γ_i into D_i and homologies between elements of Γ_i and Γ_1. As in the previous paragraph if $W_i = W_i^j$, then W_i satisfies the conclusions (2) and (3). □

Remark 5.24. By Corollary 5.7, $\mathcal{W}_1$ is an end-manifold, hence $D_1 \subset \mathcal{W}_1$ can be taken to be of the form $N(\partial \mathcal{W}_1) \times I \cup$ 1-handles, if $\partial \mathcal{W}_1 \neq \emptyset$ and a handlebody otherwise.

Theorem 5.25 (Infinite end engulfing theorem)**.** *If $\gamma_1, \gamma_2, \cdots$ is a locally finite sequence of pairwise disjoint, homotopically non trivial, simple closed curves in the end-manifold M, then after passing to subsequence, allowing γ_1 to have multiple components and fixing a base point $x \in \gamma_1$, there exist compact submanifolds $D \subset W_1 \subset W_2 \subset \cdots$ of M such that*

1. *$\partial W_i \cap \partial M$ is a union of components of ∂M and $\partial W_i - \partial M$ is connected.*
2. *If $\Gamma_i = \cup_{j=1}^i \gamma_j$, then $\Gamma_i \subset W_i$, and Γ_i can be homotoped into D via a homotopy supported in W_i.*
3. *∂W_i is 2-incompressible rel Γ_i.*
4. *If $\mathcal{W} = \cup W_i$, then $\mathcal{W}$ is π_1 and H_1–injective in both $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ coeficients.*
5. *D is a core of $\mathcal{W}$ and is of the form $\partial \mathcal{W} \times I \cup$ 1-handles.*

The conclusion of this theorem is schematically depicted in Figure 4.

Proof. Let Γ_i' (resp. $\{\gamma_i'\} = \Gamma'$) denote the subset of Γ_i (resp. $\{\gamma_i\} = \Gamma$) produced in lemma 5.23. For each i, let $\mathcal{W}_i'$ be an end-reduction of Γ_i'. Let D_1, W_1 and G_i be be as in the proof of that lemma. In particular we can assume that for all i, $G_1 = G_i$. Define $\Gamma_1 = \gamma_1 = \Gamma_1'$ and $D = D_1$. By Remark 5.24 we can assume that D is of the form $\partial \mathcal{W}_1 \times I \cup$ 1-handles.

As discussed in the proof of Lemma 5.21, if Σ is a finite subset of Γ' which contains γ_1, then Σ is end nonseparable. Therefore, if $\Sigma \subset \Gamma_i'$, then $G_1 \subset G_\Sigma \subset G_i = G_1$ and hence $G_1 = G_\Sigma$, where $G_\Sigma = in_*(\pi_1(\mathcal{W}_\Sigma, x))$ and $\mathcal{W}_\Sigma$ is an end-reduction of Σ.

Define Γ_2 to be $\Gamma_1 \cup \gamma_{i_2}'$ where i_2 is the smallest integer such that $\gamma_{i_2}' \cap W_1 = \emptyset$. Define $\gamma_2 = \gamma_{i_2}'$. Since $\gamma_2 \cap W_1 = \emptyset$ it follows, by considering the three end-reduction operations, that there exists an end-reduction $\mathcal{W}_2$ of Γ_2 such that $W_1 \subset \mathcal{W}_2$. Since $G_1 = G_{\Gamma_2}$ the inclusion $D \to \mathcal{W}_2$ is a homotopy equivalence. Let W_2 be a sufficiently large term in a standard exhaustion of $\mathcal{W}_2$ to contain $W_1 \cup \gamma_2$ together with a homotopy of γ_2 into D and a homology of γ_2 to an element of Γ_1. By construction, W_2 satisfies conditions (2). As in the proof of Lemma 5.23 it satisfies (3).

Having inductively constructed Γ_k and W_k define $\Gamma_{k+1} = \Gamma_k \cup \gamma'_{i_{k+1}}$ where i_{k+1} is the smallest integer such that $\gamma'_{i_{k+1}} \cap W_k = \emptyset$. Now construct W_{k+1} as above and complete the proof by induction.

The π_1 and H_1-injectivity follows as in the proof of lemma 5.11 after noting that W_k is obtained by applying the usual end-reduction operations to $V_{j_k} \operatorname{rel} \Gamma_k$, where $\{V_j\}$ is an exhaustion of M and $j_k \to \infty$. □

Definition 5.26. Call the $\mathcal{W}$ constructed in theorem 5.25 an *end-engulfing* of Γ.

The following remarks and results are not used in the proof of Theorem 0.9

Remark 5.27. Given $J \subset J'$ with end-reductions $\mathcal{W}$ and $\mathcal{W}'$ one can isotope $\mathcal{W} \operatorname{rel} J$ to $\mathcal{W}_1$ so that $\mathcal{W}_1 \subset \mathcal{W}'$ (lemma 5.20). On the other hand one cannot in general isotope $\mathcal{W}'$ to contain $\mathcal{W}$. One need only look at the case of $J \subset J'$ being nested balls in the Whitehead manifold to find examples. Such considerations make it challenging to find nested end–reductions $\mathcal{W}_1 \subset \mathcal{W}_2 \subset \mathcal{W}_3 \subset \cdots$.

Theorem 5.28 (Finite end reduction theorem)**.** *Let M be an end-manifold. If $\Gamma = \{\gamma_i\}$ is a end non separable union of finitely many homotopically essential, pairwise disjoint, closed curves, then an end-reduction $\mathcal{W}_\Gamma$ of Γ has finitely generated fundamental group and given a standard exhaustion $\{W_i\}$, then by passing to subsequence, for all $i, j < k$,*

$$in_*(\pi_1(W_i)) = in_*(\pi_1(W_j)) \subset \pi_1(W_k)$$

and the map $in_* : \pi_1(W_k) \to \pi_1(\mathcal{W}_\Gamma)$ restricted to $in_*(\pi_1(W_i))$ induces an isomorphism onto $\pi_1(\mathcal{W}_\Gamma)$. Here in_* denotes the map induced by inclusion.

We first prove a topological lemma.

Lemma 5.29. *If M is an end-manifold, then M has an exhaustion by compact manifolds $V_1 \subset V_2 \subset \cdots$, such that for each $i > 1$ either V_i is a handlebody, in which case $\partial M = \emptyset$, or V_i is obtained by attaching 1-handles to a $N(\partial M)$.*

Proof. If $\partial M = \emptyset$, then $\pi_1(M)$ is free and this result follows directly from [FF]. If $\partial M \neq \emptyset$, it suffices to show that if X is any compact submanifold of M, then $X \subset V$ where V is obtained by thickening ∂M and attaching 1-handles. We use the standard argument, e.g. see [BF], [BT2] or [FF]. Using the loop theorem we can pass from X to a submanifold Y, with incompressible boundary via a sequence of compressions and external 2-handle additions. By appropriately enlarging X to X_1, so as to contain these 2-handles, we can pass from X_1 to Y by only compressions. By enlarging Y, and hence X_1, we can assume that $\partial M \subset Y$ and no component of $M \setminus \operatorname{int}(Y)$ is compact. By Lemma 5.6 each component of ∂Y is boundary parallel and hence Y is of the form $N(\partial M) \cup$ 1-handles. □

Proof of theorem 5.28. Let $V_1 \subset V_2 \subset \cdots$ be an exhaustion of M as in Lemma 5.29 so that $\Gamma \subset V_1$. Let $W_1 \subset W_2 \subset \cdots$ be a standard exhaustion of $\mathcal{W}_\Gamma$ arising from the exhaustion $\{V_i\}$ of M.

By Definition 5.14 and Lemma 5.12, $\pi_1(\mathcal{W}_\Gamma)$ is finitely generated, so we can pass to subsequence and assume that the induced map $\pi_1(W_1) \to \pi_1(\mathcal{W}_\Gamma)$ is surjective.

Let $H_i = in_*(\pi_1(W_1))$ where in : $W_1 \to W_i$ is inclusion. We now show that after passing to a subsequence of the $W_i' s, i \geq 2$, the induced maps

$$H_2 \to H_3 \to \cdots \to \pi_1(\mathcal{W}_\Gamma)$$

are all isomorphisms.

For $j \geq 1$, let $G_j = \alpha^j_*(\pi_1(W_1))$, where $\alpha^j : W_1 \to V_j$ is inclusion. Each G_j is a finitely generated subgroup of $\pi_1(V_j)$ and hence is a free product of finitely many closed orientable surface groups and a finitely generated free group. Since for all j, $\text{rank}(G_j) \leq \text{rank}(G_1)$, there are only finitely many possibilities for such groups and hence by passing to a subsequence we can assume that for $j, k > 1$, the groups G_j and G_k are abstractly isomorphic. Free/surface groups are obviously linear, hence residually finite by Malcev [Mv]. Further, Malcev [Mv] went on to show that finitely generated residually finite groups are Hopfian, i.e. surjective self maps are isomorphisms. This implies that the induced maps $G_2 \to G_3 \to G_4 \to \cdots$ are all isomorphisms. If

$$K = \ker(\text{in}_* : \pi_1(W_1) \to \pi_1(\mathcal{W}_\Gamma))$$

then

$$K = \ker(\pi_1(W_1) \to \pi_1(M)) = \ker(\pi_1(W_1) \to G_2)$$

We now show that $K = \ker(\pi_1(W_1) \to \pi_1(W_2))$. One readily checks that if $W_1 \subset V$ and $K = \ker(\pi_1(W_1) \to \pi_1(V))$, and V' is obtained from V by compression, 2-handle addition or deletion where the operation is done avoiding W_1, then $K \subset \ker(\pi_1(W_1) \to \pi_1(V')$. This implies that if $K_2 = \ker(\pi_1(W_1) \to \pi_1(W_2))$, then $K \subset K_2$. On the other hand $K_2 \subset K$ since $K = \ker(\pi_1(W_1) \to \pi_1(\mathcal{W}_\Gamma))$. Therefore, the induced maps $H_2 \to H_3 \to \pi_1(\mathcal{W}_\Gamma)$ are isomorphisms.

Apply the argument of the previous paragraph to obtain a subsequence of $\{W_i\}$ which starts with W_1 and W_2 such that the π_1-image of W_2 in $W_j, j > 2$, maps isomorphically to $\pi_1(W_\Gamma)$, via the map induced by inclusion. Continue in this manner to construct $W_3, W_4, \cdots$. □

Addendum to theorem 5.25 *We can obtain the following additional property. If $i, j < k$, then*

$$in_*(\pi_1(W_i)) = in_*(\pi_1(W_j)) \subset \pi_1(W_k)$$

where in denotes inclusion. The map $in_* : \pi_1(W_k) \to \pi_1(\mathcal{W})$ restricted to $in_*(\pi_1(W_i))$ induces an isomorphism onto $\pi_1(\mathcal{W})$.

Proof. Apply Theorem 5.25 to produce the space D as well as the sets $\{\gamma_i\}, \{\Gamma_i\}$, $\{W_i\}$ which we now redefine as $\{\gamma'_i\}, \{\Gamma'_i\}, \{W'_i\}$. Define $\gamma_1 = \Gamma_1 = \gamma'_1$, $W_1 = W'_1$, $\gamma_2 = \gamma'_2$ and $\Gamma_2 = \Gamma'_2$. Let $W'_2 = W^1_2 \subset W^2_2 \subset W^3_2 \subset \cdots$ be a standard exhaustion of an end reduction $\mathcal{W}_2$ of Γ_2, which we can assume satisfies the conclusions of Theorem 5.28. Defining $W_2 = W^2_2$, we see that the restriction of in:$_*\pi_1(W_2) \to \pi_1(\mathcal{W}_2) = \pi_1(D)$ to $\text{in}_*(\pi_1(W_1)) \subset \pi_1(W_2)$ is an isomorphism. Choose $\gamma_3 = \gamma'_{i_3} \in \{\gamma'_i\}$ so that $\gamma_3 \cap W_2 = \emptyset$ and define $\Gamma_3 = \Gamma_2 \cup \gamma_3$. Let $W^1_3 \subset W^2_3 \subset W^3_3 \subset \cdots$ be a standard exhaustion of an end-reduction $\mathcal{W}_3$ of Γ_3 which satisfies the conclusions of Theorem 5.28 and has $W_2 \subset W^1_3$. Defining $W_3 = W^2_3$, we see that the restriction of in:$_*\pi_1(W_3) \to \pi_1(\mathcal{W}_3) = \pi_1(D)$ to $\text{in}_*(\pi_1(W_2)) \subset \pi_1(W_3)$ is an isomorphism. Now define $\gamma_3 = \gamma'_{i_3}$ and $\Gamma_2 = \Gamma_1 \cup \gamma_{i_3}$. In a similar manner construct $\gamma_4, \gamma_5 \cdots, \Gamma_4, \Gamma_5, \cdots, W_4, W_5, \cdots$ and finally define $\mathcal{W} = \cup W_i$. □

Remarks 5.30. If one allows each γ_i to be a finite set of elements, then we can obtain the conclusion (in Theorem 5.25 and its addendum) that each γ_i is $\mathbb{Z}/2\mathbb{Z}$-homologically trivial.

Question 5.31. *Let M be a connected, compact, orientable, irreducible 3-manifold such that $\chi(M) \neq 0$ and let G be a subgroup of $\pi_1(M)$. If the induced map $G/[G,G] \to H_1(M)$ is injective, then is G finitely generated?*

Question 5.32. *Let Γ be a locally finite collection of pairwise disjoint homotopically essential simple closed curves such that $C(\Gamma) = C(\Gamma')$ for any infinite subset Γ' of Γ. Is it true, that given $n \in \mathbb{N}$, there exists an end-engulfing of $\mathcal{W} = \cup W_i$ of Γ such that for all i, $|E \cap \Gamma_i| \geq n$ for all essential compressing discs E of W_i.*

6. Proof of Theorems 0.9, 0.4 and 0.2: Parabolic Free Case

Proof of Theorem 0.9. By Lemma 5.1 and Remark 5.2 it suffices to consider the case that $\mathcal{E}$ is the end of an end-manifold $M \subset N$ such that the inclusion $M \to N$ is a homotopy equivalence. By Lemma 3.2 there exists an η-separated collection $\Delta = \{\delta_i\}$ of closed geodesics which exit $\mathcal{E}$. We let Δ_i denote the union $\Delta_i = \cup_{j \leq i} \delta_j$. Apply Theorem 5.25 to Δ and M to pass to a subsequence also called Δ where we allow δ_1 to have finitely many components. Theorem 5.25 also produces a manifold $\mathcal{W}$ open in M and exhausted by compact manifolds $\{W_i\}$ having the following properties.

1. $\mathcal{W}$ is π_1 and H_1 injective (in $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ coeficients) in M and hence $\pi_1(\mathcal{W})$ is a free/surface group.
2. $\partial W_i \backslash \partial M$ is a closed connected surface which separates Δ_i from $\mathcal{E}$ and is 2-incompressible in N rel. Δ_i.
3. There exists a compact submanifold core $D \subset W_1$ of $\mathcal{W}$ such that for each i, δ_i can be homotoped into D via a homotopy supported in W_i. D is either of the form $\partial\mathcal{W} \times I$ with 1-handles attached to the 1-side, if $\partial\mathcal{W} \neq \emptyset$ or a handlebody, otherwise.

Let G_i denote $\text{in}_*(\pi_1(D)) \subset \pi_1(W_i)$. Let $\partial_e D$ denote $D \cap \partial M = \partial\mathcal{W}$. Let X_i denote the covering space of W_i with group G_i and let $\hat{D}$ denote the lift of D. Pick a homotopy of Δ_i into D supported in W_i. This homotopy lifts to a homotopy of $\hat{\Delta}_i$ into $\hat{D}$, thereby picking out the closed preimages $\hat{\Delta}_i$ of Δ_i which are in 1-1 correspondence with Δ_i. Let $\{\hat{\delta}_1, \cdots, \hat{\delta}_i\}$ denote these elements.

Since W_i is an atoroidal Haken manifold with non empty boundary, it follows from Thurston (see Proposition 3.2 [Ca] or [Mo]) that $\text{int}(X_i)$ is topologically tame. Let $\bar{X}_i$ denote its manifold compactification. Since the lift $\hat{D}$ is a core of X_i, there is a canonical identification of $\partial_e D$ with some set of components of $\partial\bar{X}_i$. Let $\partial_e\hat{D}$ denote these components. Having the same homotopy type as D, it follows by the usual group theoretic reasons that $\bar{X}_i$ either compresses down to a 3-ball, or to a possibly disconnected (closed orientable surface)$\times I$. In the former case $\bar{X}_i$ is a handlebody which, for reasons of Euler characteristic, is of the same genus as D. In the latter case, since $\partial_e\hat{D}$ is an incompressible surface, $\bar{X}_i$ is topologically $\partial_e\hat{D} \times I$ with 1-handles attached to the 1-side. Let $\bar{S}_i$ denote $\partial\bar{X}_i - \partial_e\hat{D}$. Again by reason of Euler characteristic, $\text{genus}(\bar{S}_i) = \text{genus}(\partial_{\mathcal{E}} D)$, where $\partial_{\mathcal{E}} D = \partial D - \partial_e D$.

Define $g' = \text{genus}(\bar{S}_i) = \text{genus}(\partial_{\mathcal{E}} D)$. We show that $g' \leq g = \text{genus}(\partial_{\mathcal{E}} C)$, where C is the original core of M. By construction $D \cap \partial M$ is a union of components of $C \cap \partial M$, therefore it suffices to show that the number of 1-handles attached to $N(D \cap \partial M)$ is not more than the number of 1-handles attached to $N(C \cap \partial M)$ in the constructions of D and C respectively. If $D \cap \partial M = C \cap \partial M$, then this follows

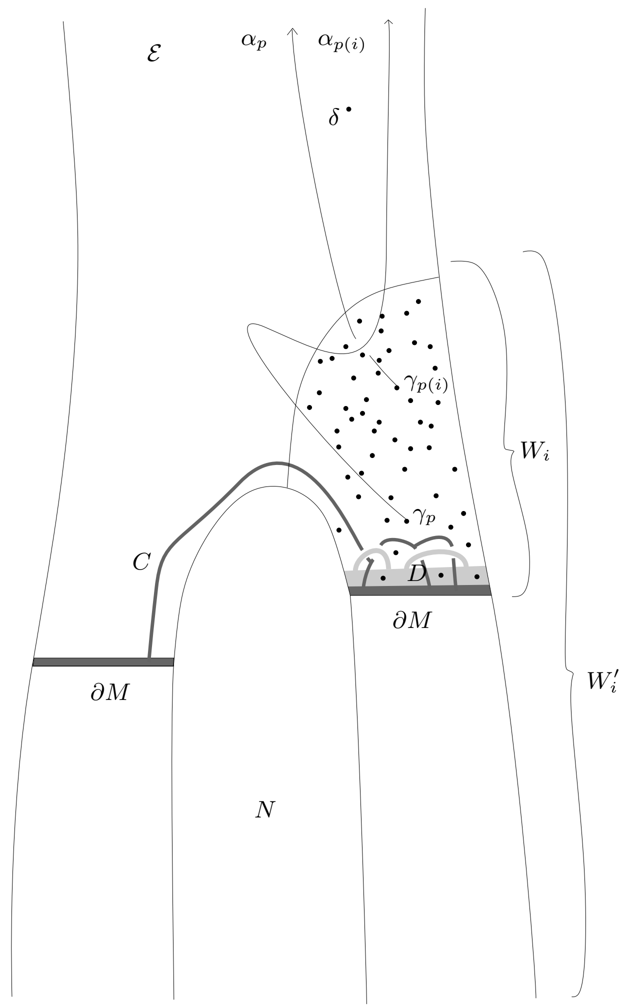

FIGURE 5.

immediately from the fact that D and C are cores respectively of $\mathcal{W}$ and M and the H_1-injectivity of $\mathcal{W}$ in M. Let $E = (C \cap \partial M)\backslash(D \cap \partial M)$. The H_1-injectivity of C implies that the inclusion $H_1(C \cap \partial M) \to H_1(M)$ is injective. If the kernel of $in_* : H_1(D \cup E) \to H_1(M)$ is nontrivial, then a non trivial homology between D and E would lie in some V_j, where $j > 1$ and V_j is a term in the exhausting sequence of M used for constructing $\{W_i\}$. But in this case, $E \cap W_j \neq \emptyset$. This contradicts the choice of W_1. Therefore

$$(*)\quad \text{number of 1-handles of } D \leq \operatorname{rank} H_1(M) - (\operatorname{rank} H_1(D \cap \partial M) + \operatorname{rank} H_1(E))$$
$$= \operatorname{rank} H_1(M) - \operatorname{rank}(H_1(\partial C \cap M))$$
$$= \text{number of 1-handles of } C$$

Isotope $\bar{S}_i \subset \bar{X}_i$ to an embedded surface $\hat{S}^i \subset X_i$ via an isotopy which does not cross $\hat{\Delta}_i$. Next, if possible, compress $\hat{S}^i$ via a compression either disjoint from $\hat{\Delta}_i$ or crossing $\hat{\Delta}_i$ once, say at $\hat{\delta}_{i_1}$. If possible, compress the resulting surface via a compression crossing $\hat{\Delta}_i \backslash \hat{\delta}_{i_1}$ at most once and so on. Note that by the calculations of the previous paragraph, the genus of the $\hat{S}^i$ are uniformly bounded, so there is an *a priori* upper bound on the number of compressions we need to do. In the end we obtain connected surfaces $\hat{S}^i_1, \cdots, \hat{S}^i_n$ in X_i which are 2-incompressible rel $\hat{\Delta}\backslash\{\hat{\delta}_{i_1}, \cdots, \hat{\delta}_{i_m}\}$ where $m < 2g' - 1$ and n and $\mathrm{genus}(\hat{S}^i_j) \leq g'$. Since $X_i \backslash \hat{\Delta}_i$ is irreducible, we can assume that no $\hat{S}^i_j$ is a 2-sphere. These $\hat{S}^i_j$'s create a partition $B^i_1, \cdots B^i_n$ of $\hat{\Delta}_i\backslash\{\hat{\delta}_{i_1}, \cdots, \hat{\delta}_{i_m}\}$, where B^i_j is the subset of $\hat{\Delta}_i\backslash\{\hat{\delta}_{i_1}, \cdots, \hat{\delta}_{i_m}\}$ separated from $\bar{S}_i$ by $\hat{S}^i_j$. Each $\hat{S}^i_j$ is incompressible to the $\bar{S}_i$ side, since the component of $\bar{X}_i$ split along $\hat{S}^i_1 \cup \cdots \cup \hat{S}^i_n$ which contains $\bar{S}_i$ is homeomorphic to $N(\hat{S}^i_1 \cup \cdots \cup \hat{S}^i_n) \cup$ 1-handles. Therefore, each $\hat{S}^i_j$ is 2-incompressible rel B^i_j.

As in the proof of Canary's theorem, after appropriately reordering the B^i_j's we can find a $p \in \mathbb{N}$ and a sequence $k_1 < k_2 < \cdots$ such that $\hat{\delta}_p \subset B^{k_i}_1$ and if $p(i)$ denotes the largest index of a $\hat{\delta}_j \in B^{k_i}_1$, then $\lim_{i\to\infty} p(i) = \infty$. In general reorder the $\hat{S}^i_j$'s so that, if possible, $\hat{\delta}_p \subset B^i_1$.

Fix i. Let W'_i be the union of W_i together with the components of $N\backslash\mathrm{int}(M)$ which non trivially intersect ∂W_1. Let Y_i denote the covering of W'_i with fundamental group G_i. View $X_i, \hat{\Delta}_i$, and the $\hat{S}^i_j$'s etc. as sitting naturally in Y_i. Let $\delta \in \Delta$ be disjoint from W_i. Apply Lemma 2.1 to $W_i,\ \delta \cup \Delta_i$ and S^i_1.

We have the following dictionary between terms appearing in our current setup (on the left) and the terms appearing in the hypothesis of Lemma 2.1 (on the right):

$$\begin{array}{rcl}
\text{the geodesics } \delta \cup \Delta_i & \longleftrightarrow & \text{the geodesics } \Delta_1 \\
\text{the manifold } W'_i & \longleftrightarrow & \text{the manifold } W \\
W'_i \cap (\delta \cup \Delta_i) = \Delta_i & \longleftrightarrow & W \cap \Delta_1 = \Delta \\
\text{the subgroup } G_i \text{ of } \pi_1(W'_i) & \longleftrightarrow & \text{the subgroup } G \text{ of } \pi_1(W) \\
\text{the cover } Y_i \text{ with } \pi_1(Y_i) = G_i & \longleftrightarrow & \text{the cover } X \text{ with } \pi_1(X) = G \\
\text{the lifted geodesics } B^i_1 & \longleftrightarrow & \text{the lifted geodesics } B \\
\text{the surface } \hat{S}^i_1 & \longleftrightarrow & \text{the surface } S
\end{array}$$

Then Lemma 2.1 constructs surfaces T_i and P_i where the correspondence is

$$\begin{array}{rcl}
\text{the shrinkwrapped surface } T^i & \longleftrightarrow & \text{the shrinkwrapped surface } T \\
\text{the approximating surface } P^i & \longleftrightarrow & \text{the approximating surface } T_t
\end{array}$$

In more detail: W'_i is isotopic to a manifold W^{new}_i, via an isotopy fixing $\Delta_i \cup \delta$ pointwise. This isotopy induces a homotopy of the covering projection $\pi : Y_i \to W'_i \subset N$ to a covering projection $\pi^{\text{new}} : Y_i \to W^{\text{new}}_i \subset N$. Our $\hat{S}^i_1$ is isotopic to a surface $\hat{P}^i$ via an isotopy avoiding B^i_1 and the projection of $\hat{P}^i$ into N is a surface P^i which is homotopic to a CAT(-1) surface T^i. Furthermore, P^i and T^i are at

Hausdorff distance ≤ 1 and the homotopy from P^i to T^i is supported within the 1-neighborhood of P^i.

We relabel superscripts, and by abuse of notation we let the sequence $\{T^i\}$ stand for the old subsequence $\{T^{k_i}\}$, with $\delta_{p(k_i)}$ being denoted by $\delta_{p(i)}$, etc. We also drop the superscript *new* so in particular the projection $\pi : Y_i \to W_i'$ now refers to $\pi^{\text{new}} : Y_i \to W_i^{\text{new}}$.

We use the $\delta_{p(i)}$'s to show that $\{T^i\}$ exits $\mathcal{E}$. Let $\{\alpha_i\}$ be a locally finite collection of properly embedded rays from $\{\delta_i\}$ to $\mathcal{E}$. For each i, $\hat{S}_1^i$ intersects some component ω of $\pi^{-1}(\alpha_{p(i)})$ with algebraic intersection number 1, so $\hat{P}^i \cap \pi^{-1}(\alpha_{p(i)}) \neq \emptyset$. Therefore for all i, we have an inequality $\text{dist}(T^i, \alpha_{p(i)}) \leq 1$. Our assertion now follows from the Bounded Diameter Lemma.

Lemma 6.1. *Let $\mathcal{E}$ be an end of M an orientable 3-manifold with finitely generated fundamental group. If C is a 3-manifold compact core of M and Z is the component of $M \setminus C$ which contains $\mathcal{E}$, then $[\partial_{\mathcal{E}} C]$ generates $H_2(Z)$ and is Thurston norm minimizing. Here $\partial_{\mathcal{E}} C$ is the component of ∂C which faces $\mathcal{E}$.*

Proof. First, $\partial_{\mathcal{E}} C$ is connected, or else there exists a closed curve κ in M intersecting a component of $\partial_{\mathcal{E}} C$ once, hence κ is not homologous to a cycle in C, contradicting the fact that C is a core. That $[\partial_{\mathcal{E}} C]$ generates $H_2(Z)$ follows from the fact that any 2-cycle w in Z is homologous to one in C, so the restriction of that homology to Z gives a homology of w to $n[\partial_{\mathcal{E}} C]$ for some n. Equivalently, observe that the inclusion $C \to M$ is a homotopy equivalence, and use excision for homology.

Let $Q \subset \text{int}(Z)$ be a Thurston norm minimizing surface representing $[\partial_{\mathcal{E}} C]$. We can choose Q to be connected since $H_2(Z) = \mathbb{Z}$. Let $V \subset Z$ be the submanifold between $\partial_{\mathcal{E}} C$ and Q. If $\text{genus}(\partial_{\mathcal{E}} C) > \text{genus}(Q)$, then there exists a nonzero z in the kernel of $in_* : H_1(\partial_{\mathcal{E}} C) \to H_1(V)$. This follows from the well known fact that for any compact orientable 3-manifold V, the rank of the kernel of the map $in_* : H_1(\partial V) \to H_1(V)$ is $\frac{1}{2}\,\text{rank}(H_1(\partial V))$. Since C is a core, z is in the kernel of the map $in_* : H_1(\partial_{\mathcal{E}} C) \to H_1(C)$. This gives rise to a class $w' \in H_2(M)$ and dual class $z' \in H_1(\partial_{\mathcal{E}} C)$ with $\langle z', w' \rangle \neq 0$, which is again a contradiction. □

Using this lemma, we now complete the proof of Theorem 0.9.

Let Z denote the component of N split open along $\partial_{\mathcal{E}} C$ which contains $\mathcal{E}$. By Lemma 6.1, $\partial_{\mathcal{E}} C$ generates $H_2(Z)$. Next, observe that if β is any ray in Z from $\partial_{\mathcal{E}} C$ to $\mathcal{E}$ and R is any immersed closed orientable surface in Z, then $[R] = n[\partial_{\mathcal{E}} C] \in H_2(Z)$ where $\langle R, \beta \rangle = n$. To see this, note that $\langle n[\partial_{\mathcal{E}} C], \beta \rangle = n$ by considering n copies of $\partial_{\mathcal{E}} C$ slightly pushed into Z.

We now use δ_p to show that for i sufficiently large $[T^i]$ is homologous in Z to $[\partial_{\mathcal{E}} C] \in H_2(Z) \cong \mathbb{Z}$. Let β be the ray $\sigma * \alpha_p$ where $\sigma \subset Z$ is a path from $\partial_{\mathcal{E}} C$ to $\partial \alpha_p$. For what follows assume that i is sufficiently large so that

$$N_2(P^i) \cap (\sigma \cup \delta_p \cup C) = \emptyset$$

where $N_2(P^i)$ denotes the 2–neighborhood of P^i, and hence

$$\langle T^i, \beta \rangle = \langle P^i, \alpha_p \rangle$$

We now compute this value. By perturbing P^i, if necessary, we can assume that P^i is transverse to α_p and no intersections occur at double points of P^i. There is a 1–1 correspondence of sets

$$\{\alpha_p \cap P^i\} \longleftrightarrow \{\pi^{-1}(\alpha_p) \cap \hat{P}^i\}$$

Let Bag_i denote the component of Y_i split along $\hat{P}^i$ which is disjoint from $\bar{S}_i$. Note that $\pi^{-1}(\delta_p) \cap \partial \mathrm{Bag}_i = \emptyset$. If κ is a component of $\pi^{-1}(\alpha_p)$, then $\langle \kappa, \hat{P}^i \rangle = 0$ if no endpoints lie in Bag_i while $\langle \kappa, \hat{P}^i \rangle = 1$ if exactly one endpoint lies in Bag_i. To see this, orient α_p so that the positive end escapes to $\mathcal{E}$. Then the positive end of each lift κ is in ∂Y_i, which is outside the submanifold bounded by $\hat{P}^i$. It follows that if p is an endpoint of κ in Bag_i, then p is the negative end of κ, and $\langle \kappa, \hat{P}^i \rangle = 1$. Since $\hat{\delta}_p$ lies in Bag_i, there is at least 1 component κ of $\pi^{-1}(\alpha_p)$ with such an endpoint in Bag_i, and therefore

$$\langle \pi^{-1}(\alpha_p), \hat{P}^i \rangle \geq 1$$

and hence

$$[T^i] = n[\partial_{\mathcal{E}} C] \in H_2(Z)$$

for some $n \geq 1$.

Therefore

$$|\chi(\partial_{\mathcal{E}} C)| \geq |\chi(T_i)| \geq x_s(n[\partial_{\mathcal{E}} C]) = x(n[\partial_{\mathcal{E}} C]) = nx([\partial_{\mathcal{E}} C]) = n|\chi(\partial_{\mathcal{E}} C)|$$

and hence $n = 1$ and $\mathrm{genus}(T^i) = \mathrm{genus}(\partial_{\mathcal{E}} C)$. Here x (resp. x_s) denotes the Thurston (resp. singular Thurston) norm on $H_2(Z)$. The first inequality follows by construction, the second by definition, the third since $x_s = x$ [G1], the fourth since x is linear on rays [T2] and the fifth by Lemma 6.1. This completes the proof of Theorem 0.9. □

Remark 6.2. Since for i sufficiently large, genus$(T^i) = g$, it follows that for such i, no compressions occur in the passage from $\bar{S}^i$ to $\hat{S}^i_1$. This mirrors the similar phenomenon seen in the proof of Canary's theorem.

If the shrinkwrapped $\partial W_i'$ is actually a Δ_i-minimal surface disjoint from Δ_i, then ∂X_i is a least area minimal surface for the hyperbolic metric, and we can pass directly from $\hat{S}^i_1$ to a $\hat{\Delta}_i$-minimal surface $\hat{T}^i$ by shrinkwrapping in X_i. Our T^i is then the projection of $\hat{T}^i$ to N.

If the shrinkwrapped $\partial W_i'$ touches Δ_i, then we can still shrinkwrap $\hat{S}^i_1$ in X_i. In this case X_i is bent and possibly squeezed along parts of $\hat{\Delta}_i$ and it is cumbersome to discuss the geometry and topology of X_i. Therefore we chose for the purposes of exposition to express T^i as a limit of surfaces. These surfaces are projections of g_{t_k}–minimal surfaces in the smooth Riemannian manifolds X_i with Riemannian metrics g_{t_k}. As metric spaces, the (X_i, g_{t_k}) converge to the bent and squeezed hyperbolic "metric" on X_i.

Tameness Criteria Let $\mathcal{E}$ be an end of the complete hyperbolic 3-manifold N with finitely generated fundamental group and compact core C. Let Z be the component of $N \setminus \mathrm{int}(C)$ containing $\mathcal{E}$ with $\partial_{\mathcal{E}} C$ denoting ∂Z. Let $T_1, T_2, \cdots$ be a sequence of surfaces mapped into N. Consider the following properties.

(1) $\mathrm{genus}(T_i) = \mathrm{genus}(\partial_{\mathcal{E}} C)$.
(2) $T_i \subset Z$ and exit $\mathcal{E}$.
(3) Each T_i homologically separates C from $\mathcal{E}$ (i.e. $[T_i] = [\partial_{\mathcal{E}} C] \in H_2(Z)$).
(4) Each T_i is CAT(-1).

Theorem 6.3 (Souto [So]). *If $T_1, T_2, \cdots$ is a sequence of mapped surfaces in the complete hyperbolic 3-manifold N with core C and end $\mathcal{E}$ which satisfies criteria (1), (2) and (3), then $\mathcal{E}$ is topologically tame.* □

Theorem 6.3 follows directly from the proof of Theorem 2, [So]. That proof makes essential use of the work of Bonahon [Bo] and Canary [Ca]. We now show how criterion (4) enables us to establish tameness without invoking the impressive technology of [Bo] and [Ca]. This argument combines the heart of Souto's proof with basic 3-manifold topology.

A topological argument that criteria (1) - (4) imply tameness. It suffices to consider the case that $\mathcal{E}$ is the end of an end-manifold M which includes by a homotopy equivalence into N, and that C is of the form $N(\partial M) \cup$ 1-handles.

Using standard arguments, we can replace the T_i's by CAT(-1) simplicial hyperbolic surfaces. The idea of how to do this is simple: the CAT(-1) property implies that each T_i has an essential simple closed curve κ_i of length uniformly bounded above. If κ_i^* denotes the geodesic in N homotopic to κ_i, then either the κ_i^* have length bounded below by some constant, and are therefore contained within a bounded neighborhood of κ_i, or else the lengths of the κ_i^* get arbitrarily short, and therefore they escape to infinity. In either case, the sequence $\kappa_1^*, \kappa_2^*, \cdots$ exits $\mathcal{E}$. Then we can triangulate T_i by a 1–vertex triangulation with a vertex on κ_i^*, and pull the simplices tight to geodesic triangles. This produces a pleated surface, homotopic to T_i, which is contained in a bounded neighborhood of κ_i^* rel. the thin part of N, and therefore these surfaces also exit $\mathcal{E}$.

Note that either $\partial_{\mathcal{E}} C$ is incompressible in N or each T_i is compressible, i.e. there exists an essential simple closed curve in T_i that is homotopically trivial in N. Indeed, using the π_1-surjectivity of C and the irreducibility of N, T_i can be homotoped into C. If T_i were incompressible, then T_i could be homotoped off the 1-handles and then homotoped into a component of ∂M. For homological reasons, the degree of this map would be one, which implies that T is homeomorphic to a component of ∂M, which in turn implies that $\text{genus}(T_i) < \text{genus}(\partial_{\mathcal{E}} C)$, contradicting Lemma 6.1.

Thurston [T] showed (in the pleated surface world) that when $\partial_{\mathcal{E}} C$ is incompressible, there exists a compact set $K \subset Z$ such that each T_i can be homotoped in Z to a simplicial hyperbolic surface which non trivially hits K. When T_i is incompressible in Z, but compressible in N, Canary—Minsky [CaM] proved the corresponding statement. Generalizing Thurston [T], Souto [So] showed that there are only finitely many homotopy classes (homotopies within Z) of genus g simplicial hyperbolic surfaces which non trivially intersect K. Therefore, by passing to subsequence we can assume that each T_i is homotopic to T_1 via a homotopy supported within Z.

Using the fact [G1] that $x = x_s$, it follows that there exists a sequence of embedded genus-g surfaces $A_1, A_2, \cdots$ such that for each i, A_i lies in a small neighborhood of T_i and $[A_i] = [T_i] = [\partial_{\mathcal{E}} C] \in H_2(Z)$. By passing to subsequence we can assume that the A_i's are pairwise disjoint and each A_i separates $\mathcal{E}$ from T_1. Let $A_{[p,q]}$ denote the compact region between A_p and A_q. To establish tameness it suffices to show that each $A_{[p,p+1]}$ is a product. Let j be sufficiently large so that T_j separates $A_{[p,p+1]}$ from $\mathcal{E}$. Let T be a surface of genus g and let $F : T \times I \to Z$ be a homotopy from T_j to T_1. By homotoping F rel ∂F, we can assume that $F^{-1}(A_p \cup A_{p+1})$ is an incompressible surface in $T \times [0, 1]$, which by construction is disjoint from $\partial T \times [0, 1]$. After a further homotopy, we can assume that $F^{-1}(A_p \cup A_{p+1}) = T \times B$, where B is a finite set of points. After another homotopy we can assume that for each $b \in B$, $F|T \times b$ is a homeomorphism. Therefore there exists $b, b' \in B$ such that $F|T \times [b, b']$ maps onto $A_{[p,p+1]}$ and the restriction of F to $\partial T \times [b, b']$ is a homeomorphism. Note that since $T \times [b, b']$ is homotopic into either boundary component, F is π_1–injective. It follows that this map is actually a homotopy equivalence. By [Wa] such a map is homotopic rel. boundary to a homeomorphism. $\square$

Remarks 6.4. The point of this proof is that in the presence of an escaping sequence of CAT(-1) surfaces, *hyperbolic surface interpolation* and the finiteness of confined simplicial hyperbolic surfaces is all the hyperbolic geometry needed to establish tameness. Note that the finiteness lemma in turn is powered by the Bounded Diameter Lemma.

Proof of Theorem 0.4 It suffices to consider the case that N is orientable, since it readily follows using [Tu], that N is tame if and only if its orientable cover is tame. If $\mathcal{E}$ is geometrically finite, then as in [EM] it follows that $\mathcal{E}$ is tame. Now assume that $\mathcal{E}$ is geometrically infinite. Theorem 0.9 provides us with a collection $\{T_i\}$ which satisfies Tameness Criteria (1)-(4). Now apply Theorem 6.3. $\square$

Proof of Theorem 0.2. It suffices to prove Theorem 0.2 for the geometrically infinite ends of orientable manifolds. It follows from Theorems 0.9 and 0.4 that $\mathcal{E}$ is topologically of the form $T \times [0, \infty)$, where T is a surface of genus g. Theorem 0.9 provides for us a sequence $\{T_i\}$ of surfaces satisfying Tameness Criteria (1)-(4). Since for i sufficiently large $T_i \subset T \times [0, \infty)$ and homologically separates $T \times 0$ from $\mathcal{E}$, it follows that the projection T_i to $T \times 0$ is a degree 1 map of a genus g surface to itself and hence is homotopic to a homeomorphism. $\square$

7. The Parabolic Case

Thanks to the careful expositions in [Bo], [Ca] and [So] it is now routine to obtain general theorems in the presence of parabolics from the corresponding results in the parabolic free case.

We now give the basic definitions and provide statements of our results in the parabolic setting.

The following is well known, e.g. see [Ca] for an expanded version of more or less the following discussion. Let N be a complete hyperbolic 3-manifold, then for sufficiently small ϵ, the ϵ-thin part, $N_{\leq\epsilon}$ of N is a union of solid tori (Margulis tubes), rank-1 cusps and rank-2 cusps. Let $N_{\geq\epsilon}$ denote $N \backslash \text{int}(N_{\leq\epsilon})$. The space $N_0^\epsilon = N_{\geq\epsilon} \cup$ Margulis tubes is called the *neutered space* of N, though we often drop the ϵ. The parabolic locus $\partial N_0^\epsilon = P^\epsilon$ (usually just denoted P) is a finite union of tori $T_1, \cdots, T_m$ and open annuli $A_1, \cdots A_n$. Each annulus A_i is of the form $S^1 \times \mathbb{R}$ such that for $t \in \mathbb{R}$, each $S^1 \times t$ bounds a standard 2-dimensional cusp in $N_{\leq\epsilon}$. By [Mc], N has a compact core $C \subset N_0$ which is also a core of N_0 and the restriction to each component P' of P is a core of P'. Such a core for N_0 is called a *relative core*. In particular if P' is an annulus, then we can assume that $C \cap P' = S^1 \times [t, s]$. If $N = \mathbb{H}^3/\Gamma$, then an end of N_0 is *geometrically finite* if it has a neighborhood disjoint from $C(\Gamma)/\Gamma$, the convex core of N. Such an end has an exponential expanding geometry similar to that of a geometrically finite end of a parabolic free manifold. The end $\mathcal{E}$ of N_0 is *topologically tame* if it is a relative product, i.e. there is a compact surface S and an embedding $S \times [0, \infty) \to N_0$ which parametrizes $\mathcal{E}$. If U is a neighborhood of $\mathcal{E}$, then by passing to a smaller neighborhood we can assume that

$U \cap A_i$ is either $\emptyset$ or of the form $S^1 \times (t, \infty)$ or $S^1 \times ((-\infty, s) \cup (t, \infty))$. Adding the corresponding 2-dimensional cusps to $S^1 \times$pts. we obtain $U_{\mathcal{E}}$ the *parabolic extension* of U. So if $\mathcal{E}$ is topologically tame, $U_{\mathcal{E}}$ is topologically $S^P \times [0, \infty)$, where S^P is topologically int(S) and geometrically S with cusps added.

Following [Bo] and [Ca] we say that the end $\mathcal{E}$ of N_0 is *simply degenerate* if it is topologically tame, has a neighborhood U with a sequence $f_i : S^P \to U_{\mathcal{E}}$ such that f_i induces a CAT(-1) structure on S^P, the f_i's eventually miss given compact sets and each f_i is properly homotopic in $U_{\mathcal{E}}$ to a homeomorphism of S^P onto $S^P \times 0$. We say that $\mathcal{E}$ is *geometrically tame* if it is simply degenerate or geometrically finite. The manifold N is *geometrically tame* if each end of N_0 is geometrically tame.

Francis Bonahon showed that if ϵ is sufficiently small, then an end $\mathcal{E}$ of N_0^ϵ is geometrically infinite if and only if there exists a sequence $\Delta = \{\delta_i\}$ of closed geodesics lying in N_0^ϵ and exiting $\mathcal{E}$.

We can now state the general version of the results stated in the introduction.

Theorem 7.1. *Let N be a complete hyperbolic 3-manifold with finitely generated fundamental group with neutered space N_0. The end $\mathcal{E}$ of N_0 is simply degenerate if there exists a sequence of closed geodesics exiting the end.*

Theorem 7.2. *A complete hyperbolic 3-manifold with finitely generated fundamental group is geometrically tame.*

Theorem 7.3. *If N is a complete hyperbolic 3-manifold with finitely generated fundamental group, then each end of N_0 is topologically tame. In particular, each end of N is topologically tame.*

Theorem 7.4. *If $N = \mathbb{H}^3/\Gamma$ is a complete hyperbolic 3-manifold with finitely generated fundamental group, then the Lebesgue measure of the limit set of Γ is either of full or zero measure. If $L_\Gamma = S^2_\infty$, then Γ acts ergodically on S^2_∞.*

Theorem 7.5 (Classification Theorem)**.** *If N is a complete hyperbolic 3-manifold with finitely generated fundamental group, then N is determined up to isometry by its topological type, its parabolic structure, the conformal boundary of N_0's geometrically finite ends and the ending laminations of N_0's geometrically infinite ends.*

The following result was conjectured by Bers, Sullivan and Thurston. Theorem 7.3 is one of many results, many of them recent, needed to build a proof. See [BCM] for a more detailed discussion.

Theorem 7.6 (Density Theorem)**.** *If $N = \mathbb{H}^3/\Gamma$ is a complete finitely generated 3-manifold with finitely generated fundamental group, then Γ is the algebraic limit of geometrically finite Kleinian groups.*

Theorem 7.7. *Let N be a complete hyperbolic 3-manifold with finitely generated fundamental group and with associated neutered space N_0. Let $\mathcal{E}$ be an end of N_0 with relative compact core C. Let S be a compact surface with the topological type of $\delta_{\mathcal{E}} C$, the component of the frontier of C which faces $\mathcal{E}$. Let $U_{\mathcal{E}}$ denote a parabolic extension of a neighborhood of $\mathcal{E}$. If there exists a sequence of closed geodesics exiting $\mathcal{E}$, then there exists a sequence $\{S_i\}$ of proper CAT(-1) surfaces in $U_{\mathcal{E}}$ homeomorphic to int(S) which eventually miss every compact set and such that each $S_i \cap N_0$ homologically separates C from $\mathcal{E}$.*

Proof of Theorem 7.7 Given the manifold N with neutering N_0 and end $\mathcal{E}$ of N_0 we explain how to find a *relative end-manifold* M containing $\mathcal{E}$.

Definition 7.8. If A is a cod-0 submanifold of a manifold with boundary, then the *frontier* δA of A is the closure of $\partial A - \partial M$. If $(R, \partial R) \subset (N_0, P)$ is a mapped surface (resp. R is a properly mapped surface in N whose ends exit the cusps), then a *P-essential annulus* for R is annulus (resp. half open annulus) A with one component mapped to an essential curve of R which cannot be homotoped in R into ∂R (resp. an end of R) and another component (resp. the end of A) mapped into P (resp. properly mapped into a cusp). Let C_0 be a 3-manifold relative core of N_0. Using [Mc] we can assume that C_0 is of the form $P_0 \times I \cup H_0 \cup$ 1-handles where $P_0 = C_0 \cap P$ is a core of P and H_0 has incompressible frontier and is attached to P_0 along annuli and tori. Furthermore δH_0 has no P-essential annuli. Define $\delta_{\mathcal{E}} C_0$ to be the component of δC_0 which faces $\mathcal{E}$ and $\delta_{\mathcal{E}} H$ to be the components of δH_0 which face $\mathcal{E}$. Define M to be the closure of the component of N split along $\delta_{\mathcal{E}} H_0$ which contains $\mathcal{E}$. Define $\partial_p(M) = P \cap M$ and $\partial_h(M) = \delta_{\mathcal{E}} H_0$. We call M a *relative end-manifold*. By slightly thickening $\partial_h(M)$ and retaining the 1-handles of $C_0 \cap M$ we obtain a core C of M.

By passing to the $\pi_1(M)$ cover of N we reduce to the case that $\mathrm{in} : M \to N_0$ is a homotopy equivalence.

By passing to a subsequence we can assume that $\Delta = \{\delta_i\}$ is a collection of geodesics escaping $\mathcal{E}$ and is weakly 1000-separating. As in Lemma 5.5 [Ca] we slightly perturb the hyperbolic metric in the 1-neighborhood of Δ to a metric μ such that for each i, δ_i is ϵ-homotopic to a simple geodesic γ_i and μ has pinched negative curvature in $(-1.01, -.99)$ and is 1.01-bilipshitz equivalent to the hyperbolic metric. Let Γ be the resulting collection of simple closed curves.

Lemma 7.9. *Let M be a relative end-manifold. Given a sequence Γ of homotopically essential simple closed curves we can pass to an infinite subsequence also called Γ which is the disjoint union $\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \cdots$ where γ_1 has finitely many components and the other γ_i's have one component. If Γ_i denotes $\cup_{j=1}^{i} \gamma_i$, then there exists a manifold $\mathcal{W}$ open in M, exhausted by a sequence of compact manifolds $\{W_i\}$ with the following properties,*

(1) *$\mathcal{W}$ is π_1 and H_1 injective (in $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ coeficients) in M and hence N.*
(2) *For all $i, \partial_h(W_1) = \partial_h(W_i)$ and is a union of components of $\partial_h(M)$. At most one component of $\partial_p(W_1)$ can lie in a component of $\partial_p(M)$. For all $i, \partial_p(W_i)$ is a union of essential annuli and tori, each of which contains a component of $\partial_p(W_1)$. The frontier $\delta(W_i)$ is connected, separates Γ_i from $\mathcal{E}$ and is 2-incompressible rel Γ_i.*
(3) *There exists a compact submanifold core $F \subset W_1$ of $\mathcal{W}$ such that each γ_i can be homotoped into F via a homotopy supported in W_i. F is of the form $W_1 \cap \partial M \times I$ with 1-handles attached to the 1-side. Finally $|\chi(\delta F)| \leq |\chi(\delta C)|$.*

Proof. This lemma is just the relative form of that part of Theorem 5.25 which was used to prove Theorem 0.9. Let J be a connected compact set and $V_1 \subset V_2 \subset V_3 \cdots$ an exhaustion of M such that $\partial_h(M) \subset \partial V_1$ and the $\partial_p(V_i)$ are the tori of M and essential annuli which meet each annular component of $\partial_p(M)$ in exactly one component. Define the *relative end reduction $\mathcal{W}_J$* of J to be the manifold exhausted by submanifolds $\{W_i\}$ where V_i passes to W_i via the operations of compression, 2-handle addition, deletion and isotopy, where the compressions and 2-handle additions are done only to $\delta(V_i)$ and its successors. The same arguments as before show that $\mathcal{W}_J$ is both π_1 and H_1-injective and as before we can define the relative notion of end non separable and therefore end-reductions to finite and locally finite infinite collections of homotopy essential pairwise disjoint simple closed curves. Similarly, relative versions of Remark 5.24 and formula $(*)$ from the proof of Theorem 0.9 complete the proof. $\square$

Let G_i denote $\text{in}_*(\pi_1(F)) \subset \pi_1(W_i)$. Fix a basepoint $f \in F$. Let X_i denote the covering space of W_i (based at f) with group G_i. The homotopy of Γ_i into F supported in W_i lifts to X_i, hence provides us with a canonical $\hat{\Gamma}_i$ of closed lifts of Γ_i in 1-1 correspondence with Γ_i. Since W_i is an atoroidal Haken manifold with nonzero Euler characteristic, it follows by Thurston that $\text{int}(X_i)$ is topologically tame (see Proposition 3.2 [Ca]). By [Tu2] a compactification $\bar{X}_i$ of $\text{int}(X_i)$ extends $\text{int}(X_i) \cup \partial_h \hat{F} \cup \partial_p \hat{F}$, where $\hat{F}$ is the lift of F to X_i. Since $\hat{F}$ is a core of $\bar{X}_i$ it follows that $\bar{X}_i$ is aunion of a closed (possibly disconnected or empty) orientable surface $\times$ I with 1-handles attached to the surface $\times$ 1 side. Let $\bar{S}_i$ denote the unique boundary component of $\bar{X}_i$ which is not a closed component of $\hat{F}$. Push $\bar{S}_i \backslash \text{int}(\hat{F} \cap \partial \bar{X}_i)$ slightly to obtain a properly embedded surface $\hat{S}_i \subset X_i$ with $\partial \hat{S}_i = \partial \delta \hat{F}$ via a homotopy disjoint from $\hat{\Gamma}_i$. Note that $\hat{S}_i$ is of the same topological type as δF. Let $\hat{Z}_i''$ be the compact region with frontier $\hat{S}_i$. Let $\chi := |\chi(\hat{S}_i)| = |\chi(\delta F)|$. Define $\partial_p X_i$ and $\partial_h X_i$ the respective preimages of $\partial_p(W_i)$ and $\partial_h(W_i)$.

If possible compress $\hat{Z}_i''$ along $\delta \hat{Z}_i'' = \hat{S}_i$ via a compression that hits at most one component of $\hat{\Gamma}_i$. Continue in this manner to obtain the region $\hat{Z}_i'$ whose frontier is 2-incompressible rel $\hat{B}^i := \hat{\Gamma}_i \backslash \{\gamma_{i_1}, \cdots, \gamma_{i_m}\}$ where $m, \chi(\delta \hat{Z}_i') \leq \chi$. Since $X_i \backslash \hat{\Gamma}_i$ is irreducible, we can assume that no component of $\partial \hat{Z}_i'$ is a 2-sphere.

Before we shrinkwrap δW_i and $\delta Z_i'$ we need to *annulate* them, i.e. compress them along essential annuli into P and the corresponding $\hat{P}$. Geometrically we are eliminating accidental parabolics so that we can invoke the parabolic version of Lemma 2.1.

Lemma 7.10. *If $(E, \partial E) \subset (N_0, P)$ is a compact Γ-minimal surface (possibly non embedded), then E cannot be homotoped rel ∂E into P.*

Suppose $f : R \to N$ is a properly mapped Γ-minimal surface such that for each $\epsilon > 0$, $f^{-1}(N_0^\epsilon)$ is compact and R has no P-essential annuli disjoint from Γ. If f is transverse to N_0^ϵ, then each component of $R \backslash int(f^{-1}(N_0^\epsilon))$ is either a compact disc or a half open annulus.

Proof. If such a homotopy exists, then the lift $\tilde{E}$ of E to $\mathbb{H}^3$ has the property that there exists a closed horoball H with $\partial \tilde{E} \subset \text{int}(H)$ and $E \cap \partial H \neq \emptyset$. This violates the maximum principle.

Therefore if σ is a component of $f^{-1}(P)$ which bounds a disc D in R, then $f(D) \cap N_0 \subset P$. If σ is essential in R, then σ can be homotoped into an end of R, since there are no P-essential annuli for R disjoint from Γ. Again the maximum principle implies that the entire annular region bounded by σ is mapped into a component of $N \backslash \text{int}(N_0)$. $\square$

Let $L_1, \cdots, L_k$ be a maximal collection of pairwise disjoint, embedded, essential annuli disjoint from Γ_i such that for each j, ∂L_j has one component on δW_i and one component on P. Furthermore assume that $\text{int}(L_i) \cap \delta W_i = \emptyset$. Now annulate δW_i along each L_i to obtain the surface δW_i^L. So if L_i lies to the outside of W_i, then the effect on W_i is to add $N(L_i)$. If $L_i \subset W_i$ and $L_i \times I$ is a product neighborhood, then this annulation deletes $L_i \times \text{int}(I)$ from W_i. There are no P-essential annuli for δW_i^L disjoint from Γ_i and ∂W_i^L is 2-incompressible rel Γ_i. We can assume that components $\partial\delta W_i^L$ are of the form $S^1 \times x \subset A_i \subset P$. The modification $W_i \to W_i^L$ induces a modification of X_i as follows. If L_j annulates W_i to the outside, then enlarge X_i in the natural way. This will enlarge the parabolic boundary $\partial_p(X_i)$. If W_i gets annulated to the inside, then do not change X_i. By abuse of notation, we relabel the space obtained from X_i as X_i. Let $X_i^L \subset X_i$ denote the preimage of the annulated W_i. It differs from X_i by preimages of inner annulations. In like manner, annulate $\delta\hat{Z}_i' \subset X_i^L$ along a maximal collection of pairwise disjoint annuli which are disjoint from $\hat{B}_i$. Let $\hat{Z}_i$ denote the result of annulating $\hat{Z}_i'$. Note that $\hat{Z}_i$ can be constructed so that $\hat{Z}_i \subset X_i^L$ and there are no $\partial_p X_i^L$-essential annuli for $\delta\hat{Z}_i$ disjoint from $\hat{B}_i$.

Here is the parabolic version of Lemma 2.1. The reader may want to refresh their memory by first rereading Lemma 2.1:

Lemma 7.11 (Parabolic construction lemma)**.** *Let $\mathcal{E}$ be an end of the complete open orientable Riemannian 3–manifold N with finitely generated fundamental group, and neutering N_0 with parabolic locus P. Let $W \subset N$ be a submanifold such that $\partial W \cap int(N)$ separates W from $\mathcal{E}$, and whose ends are standardly embedded cusps in the cusps of N. Let $\Delta_1 \subset N_0 \backslash \partial W$ be a finite collection of simple closed geodesics with $\Delta = W \cap \Delta_1$ a non–empty proper subset of Δ_1. Suppose futher that ∂W is 2–incompressible rel. Δ and has no P–essential annuli disjoint from Δ_1.*

Let the Riemannian metric μ on N agree with the hyperbolic metric outside tubular neighborhoods $N_\epsilon(\Delta_1)$ and inside tubular neighborhoods $N_{\epsilon/2}(\Delta_1)$, having Δ_1 as core geodesics, and such that μ is a metric with sectional curvature pinched between -1.01 and 0.99.

Let G be a finitely generated subgroup of $\pi_1(W)$, and let X be the covering space of W corresponding to G. Let Σ be the preimage of Δ in X with $\hat{\Delta} \subset \Sigma$ a subset which maps homeomorphically onto Δ under the covering projection, and let $B \subset \hat{\Delta}$ be a nonempty union of geodesics. Suppose there exists a properly embedded surface $S \subset X\backslash B$ of finite topological type, whose ends are standard cusps in the cusps of X such that S is 2–incompressible rel. B in X and has no P-essential annuli disjoint from B, and which separates every component of B from ∂X.

Then ∂W can be properly homotoped to a Δ_1–minimal surface which, by abuse of notation, we call $\partial W'$, and the map of S into N given by the covering projection is properly homotopic to a map whose image T' is Δ_1–minimal and whose ends exit the cusps of N.

Also, $\partial W'$ (resp. T') can be perturbed by an arbitrarily small perturbation to be an embedded (resp. smoothly immersed) surface ∂W_t (resp. T_t) bounding W_t with the following properties:

1. *There exists a proper isotopy from ∂W to ∂W_t which never crosses Δ_1, and which induces a proper isotopy from W to W_t, and a corresponding deformation of pinched negatively curved manifolds X to X_t which fixes Σ pointwise.*
2. *There exists a proper isotopy from S to $S_t \subset X_t$ which never crosses B, such that T_t is the projection of S_t to N.*
3. *Each of the limit surfaces $F \in \{\partial W', T'\}$* relatively exits the manifold as its restriction exits the neutered part. *That is, if $\mathcal{C}$ is a rank 1 cusp foliated by totally geodesic 2 dimensional cusps $C \times \mathbb{R}$ perpendicular to the boundary annulus $S^2 \times \mathbb{R}$, then if the intersection of F with $\partial\mathcal{C}$ is contained in the region $S^1 \times [t, \infty)$,*

the intersection of F with $\mathcal{C}$ is contained in the region $C \times [t, \infty)$, and similarly if the intersection is contained in $S^1 \times (-\infty, t]$.

Proof. The essential differences between the statements of Lemma 2.1 and Lemma 7.11 are firstly that the metric in the parabolic case is pinched, so that the geodesics can be chosen to be simple; secondly that the surfaces in question are all properly embedded, and the isotopies and homotopies are all proper; and thirdly that the limit surfaces relatively exit the manifold as their restriction exits the neutered part.

These issues are all minor, and do not introduce any real complications in the proof. The only question whose answer might not be immediatly apparent is how to perturb the hyperbolic metric to the g_t metrics near cusps; it turns out that this is straightforward to do, and technically easier that deformations along geodesics, since the perturbed metrics actually have curvature bounded above by 0.

We will find an exhaustion of N by increasingly larger neutered spaces N_0^t, each endowed with a metric g_t, which is obtained from the μ-metric by deforming it along the geodesics Δ_1 and along ∂N_0^t. Our ∂W_t will restrict to g_t-area minimizing representatives of the isotopy class of $\partial W \cap N_0^t$. The convergence and regularity of the limit surface $\partial W'$ near the geodesics will proceed exactly as in §1 and §2. The convegence and regularity in the cusps will follow from §1 using the absence of P-essential annuli disjoint from Δ_1.

To describe the deformed geometry along the cusps, we first recall the usual hyperbolic geometry of the (rank 1) cusps. We parameterize a rank 1 cusp $\mathcal{C}$ as $S^1 \times [1, \infty) \times \mathbb{R}$, where the initial $S^1 \times [1, \infty)$ factor is a 2–dimensional cusp C. With the hyperbolic metric, the three co–ordinate vector fields are orthogonal; we denote these by $\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}$ and $\frac{\partial}{\partial y}$ respectively, so that $\theta \in S^1$, $z \in [1, \infty)$ and $y \in \mathbb{R}$. An orthonormal basis in the hyperbolic metric is $z\frac{\partial}{\partial \theta}, z\frac{\partial}{\partial z}, z\frac{\partial}{\partial y}$. Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotone increasing function with $h(z) = z$ for $z < 1$, and $h(z) = 2$ for $z \geq 3$. Then let

$$h_t(z) = \frac{1}{1-t} h((1-t)z)$$

and define g_t on $\mathcal{C}$ to be the metric with orthonormal basis $h_t(z)\frac{\partial}{\partial \theta}, h_t(z)\frac{\partial}{\partial z}, h_t(z)\frac{\partial}{\partial y}$. Notice that the group of Euclidean symmetries of the boundary $\partial\mathcal{C}$ extends to an isometry of $\mathcal{C}$ for the g_t metric, for all t. In particular, the surface

$$H_s = S^1 \times [1, \infty) \times s$$

is totally geodesic for the g_t metric, and therefore acts as a barrier surface *for all t*.

Moreover, as $t \to 1$, the g_t metrics converge to the hyperbolic metrics on compact subsets, and in fact for every compact $K \subset \mathcal{C}$, there is an $s > 0$ such that the g_t and the hyperbolic metrics agree for $t \leq s$. Finally, for each $t > 0$, the subset $S^1 \times [3/(1-t), \infty) \times \mathbb{R} \subset \mathcal{C}$ is isometric to a Euclidean product, for the g_t metric, and therefore the surface

$$F_t = S^1 \times \frac{3}{1-t} \times \mathbb{R}$$

is totally geodesic for the g_t metric, and also acts as a barrier surface.

Finally, notice that the g_t metrics lift to a family of *isometric* metrics on $\mathbb{H}^3$, and by the symmetries above, therefore have uniformly pinched sectional curvatures, and are uniformly bilipschitz to the hyperbolic metric in the region bounded away from the cusps by F_t.

Let N_0^t be the neutered space whose boundary consists of the surfaces of type F_t constructed above. Endow N_0^t with the g_t metric. Now apply [MSY], as in Lemma 2.1, to the surface $\partial W \cap N_0^t$ to obtain the surface ∂W_t^1 which is g_t-least area among all surfaces properly isotopic to $\partial W \cap N_0^t$. By extending ∂W_t^1 *vertically* we obtain the surface $\partial W_t \subset N$ which is properly isotopic to ∂W. As in Lemma 2.1 these surfaces weakly converge geometrically to a surface $\partial W'$. We will show that there is aproper isotopy of ∂W to $\partial W'$.

Let N_0^ϵ denote a fixed neutered space transverse to $\partial W'$ and countably many ∂W_t's which converge to $\partial W'$. Define ∂W_t^ϵ to be $\partial W_t \cap N_0^\epsilon$ together with the disc components of $\partial W_t \backslash N_0^\epsilon$. Since $\mathrm{area}_{g_t}(\partial W_t^\epsilon)$ is uniformly bounded, and the hyperbolic area form is dominated on all 2–planes by g_t, the hyperbolic area of ∂W_t^ϵ is uniformly bounded. We show that a disc D of $\partial W_t \backslash N_0^\epsilon$ cannot stray too far into the cusp and hence for all t, $\partial W_t^\epsilon \subset N_0^\eta$ for some sufficiently small η. Indeed, the lift $\tilde{D}$ to the universal cover $\tilde{N}$ of N is an embedded disc of uniformly bounded area. If t is very close to 1, then $d_\rho(\partial N_0^\epsilon, \partial N_0^t) = d_t > 0$. If $x \in \tilde{D}$ and $d_\rho(x, \partial N_0^\epsilon \cup N_0^t) = d_t/2$, then $\mathrm{area}_\rho(\tilde{D} \cap (N_0^\epsilon - N_0^t)) > \pi d_t^2/4$. Therefore, for t sufficiently large, $\tilde{D}$ and hence D has uniformly bounded ρ-diamter.

Therefore the ∂W_t^ϵ's converge weakly to the limit surface $\partial W'_\epsilon \subset \partial W'$. For t sufficiently large ∂W_t^ϵ and $\partial W'_\epsilon$ are of the same topological type and very close geometrically. By Lemma 7.10, $\partial W'_\epsilon \cap N_0^\epsilon$ has no components which can be homotoped rel boundary into ∂N_0^ϵ, hence ∂W_t^ϵ shares the similar property. Since ∂W has no P-essential annuli disjoint from Δ_1 it follows that each component of $\partial W_t^\epsilon \backslash \operatorname{int} N_0$ can be properly homotoped in ∂W_t^ϵ into an end of that surface. Therefore, if some non disc component of $\partial W_t - \operatorname{int} N_0^\epsilon$ was not a half open annulus, then one can find a component E of $\partial W_t \cap N_0^\epsilon$ which can be homotoped rel ∂E into ∂N_0^ϵ, which is a contradiction. Note that $\partial W'_\epsilon$ is of the same topological type as $\partial W \cap N_0$ and and that the $\partial W''_\epsilon$'s form an exhaustion of $\partial W'$. By arguing as in the proof of Lemma 2.1, there exists a homotopy $F : \partial W \times I \to N$ with the property that $F(\partial W \times 0) = \partial W$, for infinitely many $t < 1$, $F(\partial W \times t) = \partial W_t$ and $F(\partial W \times 1) = \partial W'$.

If $\partial W'$ intersects $\partial \mathcal{C}$ in the subset $S^1 \times [s, \infty)$ for some s, then for t sufficiently large ∂W_t must intersect $\partial \mathcal{C}$ in the subset $S^1 \times [s - \epsilon, \infty)$. Since projection to the barrier surface $H_{s-\epsilon}$ along horoannuli and horotori is area reducing, this implies that $\partial W_t \cap \mathcal{C}$ is contained in $S^1 \times [1, \infty) \times [s - \epsilon, \infty)$, which in turn implies that $W' \cap \mathcal{C} \subset S^1 \times [1, \infty) \times [s, \infty)$ As in Lemma 2.1, the surfaces ∂W_t converge on compact subsets to $\partial W'$. The main results of §1 imply that $\partial W'$ is Δ_1–minimal.

A similar argument proves similar facts about S_t, T_t and T'. □

Now fix i. Let $\gamma \in \Gamma - W_i^L$. Let W'_i be the parabolic extension of the union of W_i^L and the components of $N_0 \backslash \mathrm{int}(M)$ which non trivially intersect ∂W_i^L. Let Y_i denote the covering space of W_i^L which naturally extends X_i^L. View $X_i^L, \hat{\Gamma}_i$, $\hat{Z}_i$ etc. as sitting naturally in Y_i. Let $\gamma \in \Gamma \backslash W_i^L$. Apply Lemma 7.11 using the following dictionary between our setting and the setting of Lemma 7.11: $\delta \hat{Z}_i$ corresponds to the surface S, W'_i corresponds to W, Y_i corresponds to X, $\gamma \cup \Gamma_i$ corresponds to Γ_1, Γ_i corresponds to Γ, and $\hat{B}^i$ corresponds to B. We conclude that if S^i denotes the projection of $\delta \hat{Z}_i$ into N, then S^i is homotopic to a CAT(-1) surface T^i with the following properties. W_i' is isotopic to a manifold W_i^{new}, via an isotopy fixing Γ_i pointwise which induces a deformation of Riemannian spaces Y_i to Y_i^{new} via a deformation which fixes $\hat{\Gamma}_i$ pointwise. This deformation transforms $\delta\hat{Z}_i$ to a new surface also called $\delta\hat{Z}_i \subset Y_i^{\text{new}}$ via a deformation disjoint from $\hat{B}^i$. The surface T^i is homotopic to a surface P^i so that $P^i \subset W_i^{\text{new}}$ and P^i lifts to an embedded surface $\hat{P}^i \subset Y_i^{\text{new}}$ which is isotopic to $\delta\hat{Z}_i$ via an isotopy disjoint from $\hat{B}^i$. Given $\epsilon > 0$, the P^i can be chosen so that the homotopy restricted to $P^i \cap N_0^\epsilon$ lies in an ϵ-neighborhood of $P^i \cap N_0^\epsilon$ and sends $P^i \cap N_0^\epsilon$ to $\partial W' \cap N_0^\epsilon$. By abuse of notation we will view $\hat{P}^i$ as bounding the region $\hat{Z}_i$ and we will drop the superscripts new, etc.

Let $\{\alpha_i\}$ be a locally finite collection of embedded proper rays in N_0 to $\mathcal{E}$ emanating from $\{\gamma_i\}$.

Let $\pi : Y_i \to N$ be the composition of the covering map to W_i' and inclusion. Let $B^i = \pi(\hat{B}^i)$. If $b \in B^i$ and is disjoint from $N(P^i, 1)$, then some component of T^i homologically separates b from $\mathcal{E}$. Indeed if α_b is the ray from b to $\mathcal{E}$, then $\pi^{-1}(\alpha_b) \cap \hat{Z}_i$ is a finite union of compact segments. If both endpoints lie in ∂Y_i, then it contributes nothing to the algebraic intersection number $\langle \alpha_b, P^i \rangle$. Otherwise it has one endpoint in $\pi^{-1}(\alpha_b)$ and one in ∂Y_i and hence contributes +1. Therefore

$$\langle \alpha_b, T^i \rangle = \langle \alpha_b, P^i \rangle > 0.$$

We have shown that some subsequence of components of $\{T^i\}$ exits $\mathcal{E}$.

By reducing ϵ, if necessary, we can assume that ∂N_0^ϵ is transverse to all the T^i's. By Lemma 7.10, for each i, each component of $T^i \cap (N \setminus \text{int}(N_0))$ is either a disc or a half open annulus. Therefore, the restriction of each component of T^i to N_0 is a connected surface.

We next show that if some component P of T^i has the property that $P \cap N_0$ homologically separates C from $\mathcal{E}$, then $|\chi(P)| = \chi$ and represents the class $[\partial_{\mathcal{E}} C] \in H_2(N_0, P)$. Suppose that $[P \cap N_0] = n[\partial_{\mathcal{E}} C] \in H_2(N_0, P)$. By a homotopy supported in a small neighborhood of the cusps we can push the disc components of $P \cap N \setminus (\text{int}(N_0))$ into N_0 and get $\chi \geq |\chi(P)| = \chi(P \cap N_0)$. On the other hand it follows from [T2] and [G1], that $|\chi(P)| \geq n\chi$. The only possibility is that $n = 1$ and P is homeomorphic to T^i. In particular no compressions or annulations occurred to $\hat{S}^i$.

We claim that the sequence $\{T^i \cap N_0\}$ exits N_0. Otherwise, there exists a $m \leq \chi$, a subsequence $T^{i_1}, T^{i_2}, \cdots$ and a compact connected submanifold $K_1 \subset N_0$ such that $C \subset K_1$ and for each j, m components of T^{i_j} non trivially intersect K_1 and if R^{i_j} are the components of T^{i_j} which miss K_1, then $R^{i_j} \cap N_0$ is an exiting sequence. Furthermore, m is the largest value with this property. Since each component T of T^{i_j} has $T \cap N_0$ connected, it follows from the bounded diameter lemma that there exists a compact set K_2 such that for all j, if T is a component of T^{i_j} with $T \cap K_1 \neq \emptyset$, then $T \cap N_0 \subset K_2$. Let N be so large that $\gamma_N \cap \alpha_N \cap N_2(K_2) = \emptyset$ and $\gamma_N \subset B_{i_j}$ for infinitely many values of j. Let β_N be a path from γ_N to K_2. Since R^{i_j} exits N_0 it follows that for j sufficiently large $(\gamma_N \cup \beta_N) \cap N_2(T^{i_j}) = \emptyset$. This implies that some component P of T^{i_j} homologically separates γ_N and hence C from $\mathcal{E}$. Therefore $|\chi(P)| = \chi$. Since $P \cap \alpha_N \neq \emptyset$, this implies that $P \subset R^{i_j}$ and hence $m = 0$, which is a contradiction.

Since the sequence $\{T^i \cap N_0\}$ exits N_0 it follows from the previous paragraphs that for i sufficiently large, T^i is homeomorphic to $\hat{S}^i$, and $T^i \cap N_0$ represents the class $[\partial_{\mathcal{E}} C] \in H_2(N_0, P)$. Since $\{T^i\}$ exits $\mathcal{E}$, if B is a cusp of N parametrized by $S^1 \times [0,\infty) \times \mathbb{R}$, then by Proposition 7.11, given $n \in \mathbb{R},\ T^i \cap B \subset S^1 \times [0,\infty) \times (n,\infty)$ □

Remark 7.12. Since for i sufficiently large, T^i is of topological type of $\partial_{\mathcal{E}} C$, it follows *a posteriori* that no compressions or annulations occurred in the passage from $\bar{S}^i$ to $\partial \hat{Z}_i$. This mirrors the similar phenomena seen in the proofs of Canary's theorem and Theorem 0.9.

Proof of Theorem 7.3. Tameness of the ends of N_0 follows as in the proof of Theorem 0.4. In particular if the end $\mathcal{E}$ of N_0 is not geometrically finite, then by applying the proof of Theorem 2 [So] to $\{T^i\}$ (with the disc components of $\{T^i\}\cap$ (cusps) pushed into N_0) it follows that $\mathcal{E}$ is tame. Alternatively, as in the proof that Criteria (1)-(4) implies tameness, we can use the hyperbolic surface interpolation technique and basic 3-manifold topology to prove that $\mathcal{E}$ is tame. Finally tameness of N_0 implies tameness of N. □

Proof of Theorem 7.1. It suffices to prove Theorem 7.1 for orientable manifolds which have the homotopy type of a relative end-manifold. It follows from Theorems 7.7 and 7.3 that a parabolic extension $U_{\mathcal{E}}$ of a neighborhood U of $\mathcal{E}$ is topologically of the form $\mathrm{int}(T) \times [0,\infty)$, where T is a surface homeomorphic to $\partial_{\mathcal{E}} C$ and C is a core of N_0. By Proposition 7.11, if $(T^i \cap \partial N_0) \subset \mathrm{int}(T) \times [t,\infty)$, then $T^i \setminus \mathrm{int}\, N_0 \subset \mathrm{int}(T) \times [t,\infty)$. Therefore $\{T^i\}$ exits compact sets in $\mathrm{int}(T) \times [0,\infty)$. Since for i sufficiently large, T_i is properly immersed in $\mathrm{int}(T) \times [0,\infty)$ and homologically separates $\mathrm{int}(T) \times 0$ from $\mathcal{E}$, it follows that the projection T_i to $\mathrm{int}(T) \times 0$ is a proper degree 1 map of a surface of finite type to itself and hence is properly homotopic to a homeomorphism. □

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