C^m Extension by Linear Operators

by

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§0. Introduction and Statement of Results

Let $E \subset \mathbb{R}^n$, and $m \geq 1$. We write $C^m(E)$ for the Banach space of all real-valued functions φ on E such that $\varphi = F$ on E for some $F \in C^m(\mathbb{R}^n)$. The natural norm on $C^m(E)$ is given by

$$\| \varphi \|_{C^m(E)} = \inf\{\| F \|_{C^m(\mathbb{R}^n)}: F \in C^m(\mathbb{R}^n) \text{ and } F = \varphi \text{ on } E\}.$$

Here, as usual, $C^m(\mathbb{R}^n)$ is the space of real-valued functions on $\mathbb{R}^n$ with continuous and bounded derivatives through order m; and

$$\| F \|_{C^m(\mathbb{R}^n)} = \max_{|\beta| \leq m} \sup_{x \in \mathbb{R}^n} |\partial^\beta F(x)|.$$

The first main result of this paper is as follows.

Theorem 1: *For $E \subset \mathbb{R}^n$ and $m \geq 1$, there exists a linear map*

$T : C^m(E) \to C^m(\mathbb{R}^n)$, *such that*

(A) $T\varphi = \varphi$ *on E, for each $\varphi \in C^m(E)$; and*

(B) *The norm of T is bounded by a constant depending only on m and n.*

This result was announced in [15].

To prove Theorem 1, it is enough to treat the case of compact E. In fact, given an arbitrary $E \subset \mathbb{R}^n$, we may first pass to the closure of E without difficulty, and then reduce matters to the compact case via a partition of unity.

Theorem 1 is a special case of a theorem involving ideals of m-jets. To state that result, we fix $m, n \geq 1$.

For $x \in \mathbb{R}^n$, we write $\mathcal{R}_x$ for the ring of m-jets (at x) of smooth, real-valued functions on $\mathbb{R}^n$. For $F \in C^m(\mathbb{R}^n)$, we write $J_x(F)$ for the m-jet of F at x. Our generalization of Theorem 1 is as follows.

Theorem 2: *Let $E \subset \mathbb{R}^n$ be compact. For each $x \in E$, let $I(x)$ be an ideal in $\mathcal{R}_x$. Set*

$\mathcal{J} = \{F \in C^m(\mathbb{R}^n) : J_x(F) \in I(x)$ *for all* $x \in E\}$.

Thus, $\mathcal{J}$ is an ideal in $C^m(\mathbb{R}^n)$, and $C^m(\mathbb{R}^n)/\mathcal{J}$ is a Banach space.

Let $\pi : C^m(\mathbb{R}^n) \to C^m(\mathbb{R}^n)/\mathcal{J}$ be the natural projection.

Then there exists a linear map

$T : C^m(\mathbb{R}^n)/\mathcal{J} \to C^m(\mathbb{R}^n)$,

such that

(A) $\pi T[\varphi] = [\varphi]$ *for all* $[\varphi] \in C^m(\mathbb{R}^n)/\mathcal{J}$*; and*

(B) *The norm of T is less than a constant depending only on m and n.*

Specializing to the case $I(x) = \{J_x(F) : F = 0 \text{ at } x\}$, we recover Theorem 1.

The study of C^m extension by linear operators goes back to Whitney [23,24,25]; and Theorems 1 and 2 are closely connected to the following classical question.

Whitney's Extension Problem:

Given $E \subset \mathbb{R}^n$, $f : E \to \mathbb{R}$, and $m \geq 1$, how can we tell whether $f \in C^m(E)$?

The relevant literature on this problem and its relation to Theorem 1 includes Whitney [23,24,25], Glaeser [16], Brudnyi and Shvartsman [6, ..., 9 and 18,19,20], Bierstone-Milman-Pawłucki [1,2], and my own papers [10,...,15]. (See, e.g., the historical discussions in [1,7,12]. See also Zobin [27] for a related problem.) In particular, Whitney proved Theorem 1 for $C^m(\mathbb{R}^1)$, and Theorem 2 for $I(x) \equiv \{0\}$; and Glaeser proved Theorem 1 for $C^1(\mathbb{R}^n)$. Brudnyi and Shvartsman proved the analogue of Theorem 1 for $C^{1,\omega}(\mathbb{R}^n)$, the space of functions whose gradients have modulus of continuity ω. On the other hand, they exhibited a counterexample to the analogue of Theorem 1 for the space of functions with uniformly continuous gradients on $\mathbb{R}^2$. In [3,8], they explicitly conjectured Theorem 1 and its analogue for $C^{m,\omega}(\mathbb{R}^n)$. As far as I know, no one has previously conjectured Theorem 2.

We turn our attention to the proof of Theorem 2.

Theorem 2 reduces easily to the case in which the family of ideals $(I(x))_{x \in E}$ is "Glaeser stable", in the following sense.

Let $E \subset \mathbb{R}^n$ be compact. Suppose that, for each $x \in E$, we are given an ideal $I(x)$ in $\mathcal{R}_x$ and an m-jet $f(x) \in \mathcal{R}_x$. Then the family of cosets $(f(x) + I(x))_{x \in E}$ will be called "Glaeser stable" if either of the following two equivalent conditions holds:

(GS1) Given $x_0 \in E$ and $P_0 \in f(x_0) + I(x_0)$, there exists $F \in C^m(\mathbb{R}^n)$, with $J_{x_0}(F) = P_0$, and $J_x(F) \in f(x) + I(x)$ for all $x \in E$.

(GS2) Given $x_0 \in E$ and $P_0 \in f(x_0) + I(x_0)$, there exist a neighborhood U of x_0 in $\mathbb{R}^n$, and a function $F \in C^m(U)$, such that $J_{x_0}(F) = P_0$, and $J_x(F) \in f(x) + I(x)$ for all $x \in E \cap U$.

To see the equivalence of (GS1) and (GS2), we use a partition of unity, and exploit the compactness of E and the fact that each $I(x)$ is an ideal. (See Section 1.) Conditions (GS1) and (GS2) are also equivalent to the assertion that $(f(x)+I(x))_{x \in E}$ is its own "Glaeser refinement" in the sense of [12], by virtue of the Corollary to Theorem 2 in [12]. We emphasize that compactness of E is part of the definition of Glaeser stability.

To reduce our present Theorem 2 to the case of Glaeser stable families of ideals, we set

$\tilde{I}(x) = \{J_x(F) : F \in \mathcal{J}\}$ for each $x \in E$.

One checks easily that $\tilde{I}(x)$ is an ideal in $\mathcal{R}_x$, that $(\tilde{I}(x))_{x \in E}$ is Glaeser stable, and that

$\mathcal{J} = \{F \in C^m(\mathbb{R}^n) : J_x(F) \in \tilde{I}(x)$ for each $x \in E\}$.

Thus, Theorem 2 for the general family of ideals $(I(x))_{x \in E}$ is equivalent to Theorem 2 for the Glaeser stable family $(\tilde{I}(x))_{x \in E}$. From now on, we restrict attention to the Glaeser stable case.

To explain our proof of Theorem 2, in the Glaeser stable case, we start with the following result, which follows immediately from Theorem 3 in [12].

Theorem 3: *There exist constants $\bar{k}$ and C_1, depending only on m and n, for which the following holds.*

Let $A > 0$. Suppose that, for each point x in a compact set $E \subset \mathbb{R}^n$, we are given an m-jet $f(x) \in \mathcal{R}_x$ and an ideal $I(x)$ in $\mathcal{R}_x$.

Assume that

(I) $(f(x) + I(x))_{x \in E}$ *is Glaeser stable, and*

(II) *Given* $x_1, \ldots, x_{\bar{k}} \in E$, *there exists* $\tilde{F} \in C^m(\mathbb{R}^n)$, *with* $\| \tilde{F} \|_{C^m(\mathbb{R}^n)} \leq A$, *and* $J_{x_i}(\tilde{F}) \in f(x_i) + I(x_i)$ *for* $i = 1, \ldots, \bar{k}$.

Then there exists $F \in C^m(\mathbb{R}^n)$, *with*

$\| F \|_{C^m(\mathbb{R}^n)} \leq C_1 A$, *and* $J_x(F) \in f(x) + I(x)$ *for all* $x \in E$.

In principle, this result lets us calculate the order of magnitude of the infimum of the C^m-norms of the functions F satisfying $J_x(F) \in f(x) + I(x)$ for all $x \in E$.

We will prove a variant of Theorem 3, in which the m-jets $f(x)(x \in E)$ and the function F depend linearly on a parameter ξ belonging to a vector space Ξ. That variant (Theorem 4 below) is easily seen to imply Theorem 2, as we spell out in Section 1. (The spirit of the reduction of Theorem 2 to Theorem 4 is as follows. Suppose we want to prove that a given map $y = \Phi(x)$ is linear. To do so, we may assume that x depends linearly on a parameter $\xi \in \Xi$, and then prove that $y = \Phi(x)$ also depends linearly on ξ.)

The main content of this paper is the proof of Theorem 4.

To state Theorem 4, we first introduce a few definitions.

Let $E \subset \mathbb{R}^n$ be compact. If $I(x)$ is an ideal in $\mathcal{R}_x$ for each $x \in E$, then we will call $(I(x))_{x \in E}$ a "family of ideals". Similarly, if, for each $x \in E$, $I(x)$ is an ideal in $\mathcal{R}_x$ and $f(x) \in \mathcal{R}_x$, then we will call $(f(x) + I(x))_{x \in E}$ a "family of cosets".

More generally, let Ξ be a vector space, and let $E \subset \mathbb{R}^n$ be compact. Suppose that for each $x \in E$ we are given an ideal $I(x)$ in $\mathcal{R}_x$, and a linear map $\xi \mapsto f_\xi(x)$, from Ξ into $\mathcal{R}_x$. We will call $(f_\xi(x) + I(x))_{x \in E, \xi \in \Xi}$ a "family of cosets depending linearly on $\xi \in \Xi$".

We will say that $(f_\xi(x) + I(x))_{x \in E,\, \xi \in \Xi}$ is "Glaeser stable" if, for each fixed $\xi \in \Xi$, the family of cosets $(f_\xi(x) + I(x))_{x \in E}$ is Glaeser stable.

We can now state our analogue of Theorem 3 with parameters.

Theorem 4: *Let* Ξ *be a vector space, with seminorm* $| \cdot |$.

Let $(f_\xi(x)+I(x))_{x\in E,\xi\in\Xi}$ be a Glaeser stable family of cosets depending linearly on $\xi \in \Xi$.

Assume that for each $\xi \in \Xi$ with $|\xi| \leq 1$, there exists $F \in C^m(\mathbb{R}^n)$, with $\| F \|_{C^m(\mathbb{R}^n)} \leq 1$, and $J_x(F) \in f_\xi(x) + I(x)$ for all $x \in E$.

Then there exists a linear map $\xi \mapsto F_\xi$, from Ξ into $C^m(\mathbb{R}^n)$, such that

(A) $J_x(F_\xi) \in f_\xi(x) + I(x)$ *for all* $x \in E$, $\xi \in \Xi$*; and*

(B) $\| F_\xi \|_{C^m(\mathbb{R}^n)} \leq C|\xi|$ *for all* $\xi \in \Xi$*, with* C *depending only on* m *and* n*.*

It is an elementary exercise to show that Theorem 4 implies Theorem 2 in the case of Glaeser stable $(I(x))_{x\in E}$.

Since we have just seen that this case of Theorem 2 implies the general case, it follows that Theorems 1 and 2 are reduced to Theorem 4.

The rest of this paper gives the proof of Theorem 4.

In this introduction, we explain some of the main ideas in that proof. It is natural to try to adapt the proof of Theorem 3 from [12]. There, we partition E into finitely many "strata", including a "lowest stratum" E_1.

Theorem 3 is proven in [12] by induction on the number of strata, with the main work devoted to a study of the lowest stratum. Unfortunately, the analysis on the lowest stratum in [12] is fundamentally non-linear; hence it cannot be used for Theorem 4. (It is based on an operation analogous to passing from a continuous function F to its modulus of continuity ω_F.)

To prove Theorem 4, we partition E into finitely many "slices", including a "first slice" E_0; and we proceed by induction on the number of slices. We analyze the first slice E_0 in a way that maintains linear dependence on the parameter $\xi \in \Xi$. This is the essentially new part of our proof. Once we have understood the first slice, we can proceed as in [12].

Let us explain the notion of a "slice." To define this notion, we introduce the ring $\mathcal{R}_x^k$ of k-jets of smooth (real-valued) functions at x. For $0 \leq k \leq m$, let $\pi_x^k : \mathcal{R}_x = \mathcal{R}_x^m \to \mathcal{R}_x^k$ be the natural projection. To each $x \in E$ we associate the $(m+1)$-tuple of integers $\text{type}(x) = (\dim[\pi_x^0 I(x)], \dim[\pi_x^1 I(x)], \ldots, \dim[\pi_x^m I(x)])$.

For each fixed $(m+1)$-tuple of integers $(d_0, \ldots, d_m)$, the set

$E(d_0, d_1, \ldots, d_m) = \{x \in E : \text{ type}(x) = (d_0, \ldots, d_m)\}$

will be called a "slice". Thus, E is partitioned into slices.

The "number of slices" in E means simply the number of distinct $(d_0, \ldots, d_m)$ for which $E(d_0, \ldots, d_m)$ is non-empty.

Note that

$0 \le d_0 \le d_1 \le \cdots \le d_m \le D$

for a non-empty slice, where

$D = \dim \mathcal{R}_x$ (any x).

Hence, the number of slices is bounded by a constant depending only on m and n.

Next, we define the "first slice".

To do so, we order $(m+1)$-tuples lexicographically as follows:

$(d_0, \ldots, d_m) < (D_0, \ldots, D_m)$ means that $d_\ell < D_\ell$ for the largest ℓ with $d_\ell \neq D_\ell$.

If E is non-empty, then the $(m+1)$-tuples $\{\text{type}(x) : x \in E\}$ have a minimal element $(d_0^*, d_1^*, \ldots, d_m^*)$, with respect to the above order. We call $E(d_0^*, d_1^*, \ldots, d_m^*)$ the "first slice", and denote it by E_0. It is easy to see that E_0 is compact. (See Section 1.)

We partition $\mathbb{R}^n \setminus E_0$ into "Whitney cubes" $\{Q_\nu\}$, with the following geometrical properties:

For each ν, let δ_ν be the diameter of Q_ν, and let Q_ν^* be the (closed) cube obtained by dilating Q_ν by a factor of 3 about its center.

Then

(a) $\delta_\nu \le 1$ for each ν,

(b) $Q_\nu^* \subset \mathbb{R}^n \smallsetminus E_0$ for each ν, and

(c) If $\delta_\nu < 1$, then distance $(Q_\nu^*, E_0) \le C\delta_\nu$, with C depending only on the dimension n.

In particular, (b) shows that $E \cap Q_\nu^*$ has fewer slices than E. This will play a crucial rôle in our proof of Theorem 4.

Corresponding to the Whitney cubes $\{Q_\nu\}$, there is a "Whitney partition of unity" $\{\theta_\nu\}$, with

- $\sum_\nu \theta_\nu \;=\; 1$ on $\mathbb{R}^n \smallsetminus E_0$,
- $\mathsf{supp}\,\theta_\nu \;\subset\; Q_\nu^*$ for each ν, and
- $|\partial^\beta \theta_\nu| \;\le\; C\,\delta_\nu^{-|\beta|}$ on $\mathbb{R}^n$ for $|\beta| \le m+1$ and for all ν.

Here, C depends only on m and n.

See, e.g., [17,21,23] for the construction of such Q_ν, θ_ν.

Now we can start to explain our proof of Theorem 4. We give a self-contained explanation, without assuming familiarity with [12].

We use induction on the number of slices in E. If the number of slices is zero, then E is empty, and the conclusion of Theorem 4 holds trivially, with $F_\xi = 0$. For the induction step, fix $\Lambda \ge 1$, and assume that Theorem 4 holds whenever the number of slices is less than Λ. Fix Ξ, $|\cdot|$, $(f_\xi(x) + I(x))_{x\in E,\xi\in\Xi}$ as in the hypotheses of Theorem 4, and assume that the number of slices in E is equal to Λ. Under these assumptions, we will prove that there exists a linear map $\xi \mapsto F_\xi$ from Ξ into $C^m(\mathbb{R}^n)$, satisfying conclusions (A) and (B) of Theorem 4. This will complete our induction, and establish Theorem 4.

To achieve (A) and (B), we begin by working on the first slice E_0.

We construct a linear map $\xi \mapsto F_\xi^0$ from Ξ into $C^m(\mathbb{R}^n)$, satisfying

(A′) $J_x(F_\xi^0) \in f_\xi(x) + I(x)$ for all $x \in E_0$, $\xi \in \Xi$; and

(B′) $\|\, F_\xi^0 \,\|_{C^m(\mathbb{R}^n)} \le\; C|\xi|$ for all $\xi \in \Xi$, with C depending only on m and n.

Comparing (A′) with (A), we see that $J_x(F_\xi^0)$ does what we want only for $x \in E_0$.

We will correct F_ξ^0 away from E_0. To do so, we work separately on each Whitney cube $Q_\nu^* \subset \mathbb{R}^n \setminus E_0$. For each fixed ν, we can apply our induction hypothesis (a rescaled version of Theorem 4 for fewer than Λ slices) to the family of cosets

$(f_\xi(x) - J_x(F_\xi^0) + I(x))_{x\in E\cap Q_\nu^*,\, \xi\in\Xi}$, depending linearly on $\xi \in \Xi$.

The crucial point is that our induction hypothesis applies, since as we observed before, $E \cap Q_\nu^*$ has fewer slices than E. From the induction hypothesis, we obtain, for each ν, a linear map $\xi \mapsto F_{\xi,\nu}$ from Ξ into $C^m(\mathbb{R}^n)$, with the following properties:

(A)$_\nu$: $J_x(F_{\xi,\nu}) \in J_x(\theta_\nu) \odot [f_\xi(x) - J_x(F_\xi^0)] + I(x)$ for all $x \in E \cap Q_\nu^*$, $\xi \in \Xi$; and

(B)$_\nu$: $|\partial^\beta F_{\xi,\nu}(x)| \le C\,|\xi|\,\delta_\nu^{m-|\beta|}$ for $x \in \mathbb{R}^n$, $\xi \in \Xi$, $|\beta| \le m$, with C depending only on m and n.

Here $\{\theta_\nu\}$ is our Whitney partition of unity, and $\odot$ denotes multiplication in $\mathcal{R}_x$.

In view of (A)$_\nu$, the function $F_{\xi,\nu}$ corrects F_ξ^0 on $E \cap Q_\nu^*$.

Now, we combine our F_ξ^0 and $F_{\xi,\nu}$ into

$F_\xi = F_\xi^0 + \sum_\nu \theta_\nu^+ F_{\xi,\nu}$, where θ_ν^+ is a smooth cutoff function supported in Q_ν^*. Using (A′), (B′), (A)$_\nu$, (B)$_\nu$ and Glaeser stability, we will show that $F_\xi \in C^m(\mathbb{R}^n)$, and that the linear map $\xi \mapsto F_\xi$ satisfies conditions (A) and (B) in the statement of Theorem 4. This will complete our induction on the number of slices, and establish Theorem 4.

As in [12], the above plan cannot work, unless we can construct the linear map $\xi \mapsto F_\xi^0$ to satisfy something stronger than (A′). More precisely, for a convex set $\Gamma_\xi(x, \bar{k}, C)$ to be defined below, we need to make sure that $\xi \mapsto F_\xi^0$ satisfies

(A″): $J_x(F_\xi^0) \in \Gamma_\xi(x, \bar{k}, C)$ for all $x \in E_0$, $\xi \in \Xi$ with $|\xi| \le 1$.

Here, $\Gamma_\xi(x, \bar{k}, C) \subseteq f_\xi(x) + I(x)$, so (A″) is stronger than (A′).

To define $\Gamma_\xi(x, \bar{k}, C)$ and understand why we need (A″), we introduce some notation and conventions.

Unless we say otherwise, C always denotes a constant depending only on m and n. The value of C may change from one occurrence to the next. For $x', x'' \in \mathbb{R}^n$, we adopt the convention that $|x' - x''|^{m-|\beta|} = 0$ in the degenerate case $x' = x''$, $|\beta| = m$.

We identify the m-jet $J_x(F)$ with the Taylor polynomial $y \mapsto \sum_{|\alpha| \leq m} \frac{1}{\alpha!}(\partial^\alpha F(x)) \cdot (y-x)^\alpha$.

Thus, as a vector space $\mathcal{R}_x$ is identified with the vector space $\mathcal{P}$ of all m^{th} degree (real) polynomials on $\mathbb{R}^n$.

Now suppose $H = (f(x) + I(x))_{x \in E}$ is a family of cosets, and let $x_0 \in E$, $k \geq 1$, $A > 0$ be given. Then we define $\Gamma_H(x_0, k, A)$ as the set of all $P_0 \in f(x_0) + I(x_0)$ with the following property:

Given $x_1, \ldots, x_k \in E$, there exist $P_1 \in f(x_1) + I(x_1), \ldots, P_k \in f(x_k) + I(x_k)$, such that

$|\partial^\beta P_i(x_i)| \leq A$ for $|\beta| \leq m$, $0 \leq i \leq k$; and

$|\partial^\beta (P_i - P_j)(x_j)| \leq A\, |x_i - x_j|^{m-|\beta|}$ for $|\beta| \leq m$, $0 \leq i, j \leq k$.

Here, we regard $P_0, \ldots, P_k$ as m^{th} degree polynomials.

Note that $\Gamma_H(x_0, k, A)$ is a compact, convex subset of $f(x_0) + I(x_0)$.

The point of this definition is that, if we are given $F \in C^m(\mathbb{R}^n)$, with

$\| F \|_{C^m(\mathbb{R}^n)} \leq A$, and $J_x(F) \in f(x) + I(x)$ for each $x \in E$,

then, trivially, $J_{x_0}(F) \in \Gamma_H(x_0, k, CA)$ for any $k \geq 1$.

(To see this, just take $P_i = J_{x_i}(F)$ in the definition of $\Gamma_H(x_0, k, CA)$. The desired estimates on $P_i - P_j$ follow from Taylor's theorem.)

More generally, suppose $(f_\xi(x) + I(x))_{x \in E, \xi \in \Xi}$ is a family of cosets depending linearly on $\xi \in \Xi$. For each $\xi \in \Xi$, we set $H_\xi = (f_\xi(x) + I(x))_{x \in E}$, and we define

$\Gamma_\xi(x_0, k, A) = \Gamma_{H_\xi}(x_0, k, A)$ for $x_0 \in E$, $k \geq 1$, $A > 0$.

Thus, if $\xi \mapsto F_\xi$ is a linear map as in the conclusion of Theorem 4, then we must have

$J_x(F_\xi) \in \Gamma_\xi(x, k, C)$ for all $x \in E$, $\xi \in \Xi$ with $|\xi| \leq 1$.

Recall that our plan for the proof of Theorem 4 was to set

$$F_\xi \;=\; F_\xi^0 \;+\; \sum_\nu \theta_\nu^+ \, F_{\xi,\nu}, \text{ with } \mathsf{supp}\ \theta_\nu^+ \;\subset\; Q_\nu^* \;\subset\; \mathbb{R}^n \smallsetminus E_0.$$

Hence, for $x \in E_0$, we expect that $J_x(F_\xi) \;=\; J_x(F_\xi^0)$.

Therefore, unless $\xi \mapsto F_\xi^0$ has been carefully prepared to satisfy (A″), we will never be able to prove Theorem 4 by defining F_ξ as above.

Conversely, if F_ξ^0 satisfies (A″), then we will gain the quantitative control needed to establish estimates $(\mathrm{B})_\nu$ above.

Thus, (A″) necessarily plays a crucial rôle in our proof of Theorem 4.

We discuss very briefly how to construct $\xi \mapsto F_\xi^0$ satisfying (A″).

Let η be a small enough positive number determined by $(I(x))_{x\in E}$. We pick out a large, finite subset $E_{00} \subset E_0$, such that every point of E_0 lies within distance η of some point of E_{00}. We then construct a linear map $\xi \mapsto F_\xi^{00}$ from Ξ into $C^m(\mathbb{R}^n)$, with norm of most C, satisfying the following condition.

(A‴) $\quad J_x(F_\xi^{00}) \in \Gamma_\xi(x, \bar{k}, C)$ for all $x \in E_{00}$, $\xi \in \Xi$ with $|\xi| \le 1$.

Thus, $J_x(F_\xi^{00})$ does what we want only for $x \in E_{00}$. For $x \in E_0 \smallsetminus E_{00}$, we don't even have $J_x(F_\xi^{00}) \in f_\xi(x) + I(x)$.

On the other hand, for $|\xi| \le 1$, $x \in E_0 \smallsetminus E_{00}$, we hope that $J_x(F_\xi^{00})$ lies very close to $f_\xi(x) + I(x)$, since $J_y(F_\xi^{00}) \in \Gamma_\xi(y, \bar{k}, C) \subseteq f_\xi(y) + I(y)$ for a point $y \in E_{00}$ within distance η of x. We confirm this intuition by constructing a linear map $\xi \mapsto \tilde{F}_\xi$ from Ξ into $C^m(\mathbb{R}^n)$, with the following two properties:

- $\tilde{F}_\xi$ is "small" for $|\xi| \;\le\; 1$.
- $J_x(F_\xi^{00} + \tilde{F}_\xi) \;\in\; f_\xi(x) \;+\; I(x)$ for $x \in E_0$, $\xi \in \Xi$ with $|\xi| \le 1$.

The "corrected" operator $\xi \mapsto F_\xi^0 = F_\xi^{00} + \tilde{F}_\xi$ will then satisfy (A″).

To construct F_ξ^{00}, we combine our previous results from [12,15].

The construction of $\tilde{F}_\xi$ requires new ideas and serious work. (See Sections 6 ... 11 below.)

This concludes our summary of the proof of Theorem 4.

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§1. Elementary Verifications

In this section, we prove some of the elementary assertions made in the introduction. We retain the notation of the introduction.

First of all, we check that the two conditions (GS1) and (GS2) are equivalent. Obviously, (GS1) implies (GS2). Suppose $(f(x)+I(x))_{x\in E}$ satisfies (GS2). We recall that E is compact, and that each $I(x)$ is an ideal in $\mathcal{R}_x$. Suppose $x_0 \in E$ and $P_0 \in f(x_0) + I(x_0)$. For each $y \in E$, (GS2) produces an open neighborhood U_y of y in $\mathbb{R}^n$, and a C^m function F_y on U_y, such that

$$J_x(F_y) \in f(x) + I(x) \text{ for all } x \in U_y \cap E\,,$$

and

$$J_{x_0}(F_y) = P_0 \text{ if } y = x_0\,.$$

If $y \neq x_0$, then by shrinking U_y, we may suppose x_0 does not belong to the closure of U_y.

By compactness of E, finitely many U_y's cover E. Say, $E \subset U_{y_0} \cup \cdots \cup U_{y_N}$. Since $x_0 \in E$, one of the y_j must be x_0. Say, $y_0 = x_0$, and $y_\nu \neq x_0$ for $\nu \neq 0$. We introduce a partition of unity $\{\theta_\nu\}$, such that

- Each $\theta_\nu \in C_0^m(U_{y_\nu})$,

  and

- $\sum_{\nu=0}^{N} \theta_\nu = 1$ in a neighborhood of E.

Since x_0 cannot belong to $\mathsf{supp}\ \theta_\nu$ for $\nu \neq 0$, we have

$J_{x_0}(\theta_0) = 1,\ J_{x_0}(\theta_\nu) = 0$ for $\nu \neq 0$.

Now set $F = \sum_{\nu=0}^{N} \theta_\nu F_{y_\nu} \in C^m(\mathbb{R}^n)$.

For $x \in E$, and for any ν with $\mathsf{supp}\ \theta_\nu \ni x$, we have $J_x(F_{y_\nu}) - f(x) \in I(x)$, hence

$J_x(\theta_\nu F_{y_\nu}) - J_x(\theta_\nu) \odot f(x) \in I(x)$, since $I(x)$ is an ideal.

Here, $\odot$ denotes multiplication in $\mathcal{R}_x$. Summing over ν, we obtain $J_x(F) - f(x) \in I(x)$.

Also, since $J_{x_0}(F_{y_0}) = P_0$ and $J_{x_0}(\theta_\nu) = \delta_{0\nu}$ (Kronecker δ), we have

$J_{x_0}(F) = P_0$.

This proves (GS1).

Next, we check that Theorem 4 implies Theorem 2 in the case of Glaeser stable $(I(x))_{x\in E}$. Let E, $I(x)$, $\mathcal{J}$, π be as in the hypotheses of Theorem 2, with $(I(x))_{x\in E}$ Glaeser stable.

We take Ξ to be the space $C^m(E, I)$, which consists of all families of m-jets $\xi = (f(x))_{x\in E}$, with $f(x) \in \mathcal{R}_x$ for $x \in E$, such that $(f(x) + I(x))_{x\in E}$ is Glaeser stable. (We use Glaeser stability of $(I(x))_{x\in E}$ to check that Ξ is a vector space.)

As a seminorm on Ξ, we take $|\xi| = 2 \parallel (f(x))_{x\in E} \parallel_{C^m(E,I)}$, where

$\parallel (f(x))_{x\in E} \parallel_{C^m(E,I)} = \mathsf{inf}\{\parallel F \parallel_{C^m(\mathbb{R}^n)} :\ F \in C^m(\mathbb{R}^n)$ and $J_x(F) \in f(x) + I(x)$ for $x \in E\}$.

Here, the inf is finite, since $(f(x) + I(x))_{x\in E}$ is Glaeser stable.

Next, we define a linear map $\xi \mapsto f_\xi(x)$ from Ξ into $\mathcal{R}_x$, for each $x \in E$.

For $\xi = (f(x))_{x\in E}$, we simply define $f_\xi(x) = f(x)$.

One checks easily that the above Ξ, $|\cdot|$, $(f_\xi(x) + I(x))_{x\in E, \xi\in\Xi}$ satisfy the hypotheses of Theorem 4. Hence, Theorem 4 gives a linear map $\mathcal{E} : C^m(E, I) \to C^m(\mathbb{R}^n)$, with norm bounded by a constant depending only on m and n, and satisfying

$$J_x(\mathcal{E}\xi) \in f(x) + I(x) \text{ for all } x \in E, \text{ whenever } \xi = (f(x))_{x\in E} \in C^m(E, I)\,.$$

Next, we define a linear map

$\tau : C^m(\mathbb{R}^n)/\mathcal{J} \to C^m(E, I)$.

To define τ, we fix for each x a subspace $V(x) \subseteq \mathcal{R}_x$ complementary to $I(x)$, and we write $\pi_x : \mathcal{R}_x \to V(x)$ for the projection onto $V(x)$ arising from $\mathcal{R}_x = V(x) \oplus I(x)$.

For $\varphi \in C^m(\mathbb{R}^n)$, we define

$\hat{\tau}\varphi = ((\hat{\tau}\varphi)(x))_{x\in E} = (\pi_x J_x(\varphi))_{x\in E}$.

Since $(\hat{\tau}\varphi)(x) - J_x(\varphi) \in I(x)$ for $x \in E$, it follows that

$$((\hat{\tau}\varphi)(x) + I(x))_{x\in E} = (J_x(\varphi) + I(x))_{x\in E}.$$

Since $(I(x))_{x\in E}$ is Glaeser stable and $\varphi \in C^m(\mathbb{R}^n)$, it follows in turn that $((\hat{\tau}\varphi)(x)+I(x))_{x\in E}$ is Glaeser stable.

Thus, $\hat{\tau}\varphi \in C^m(E, I)$. Moreover, since $\varphi \in C^m(\mathbb{R}^n)$ and $J_x(\varphi) \in (\hat{\tau}\varphi)(x) + I(x)$ for all $x \in E$, the definition of the $C^m(E, I)$-seminorm shows that $\| \hat{\tau}\varphi \|_{C^m(E,I)} \leq \| \varphi \|_{C^m(\mathbb{R}^n)}$.

Thus, $\hat{\tau} : C^m(\mathbb{R}^n) \to C^m(E, I)$ is a linear map of norm ≤ 1.

Next, note that $J_x(\varphi) \in I(x)$ implies $(\hat{\tau}\varphi)(x) = 0$ by definition of $\hat{\tau}$ and π_x. Hence, $\varphi \in \mathcal{J}$ implies $\hat{\tau}\varphi = 0$, and therefore $\hat{\tau}$ collapses to a linear map

$\tau : C^m(\mathbb{R}^n)/\mathcal{J} \to C^m(E, I)$.

We now define $T = \mathcal{E}\tau$. Thus, $T : C^m(\mathbb{R}^n)/\mathcal{J} \to C^m(\mathbb{R}^n)$ is a linear map with norm bounded by a constant depending only on m and n. For $\varphi \in C^m(\mathbb{R}^n)$ and $[\varphi] \in C^m(\mathbb{R}^n)/\mathcal{J}$ the equivalence class of φ, we have (for $x \in E$):

$$J_x(\mathcal{E}\tau[\varphi]) = J_x(\mathcal{E}\hat{\tau}\varphi) \in (\hat{\tau}\varphi)(x) + I(x) \quad \text{(by the defining property of } \mathcal{E})$$

$$= J_x(\varphi) + I(x) \quad \text{(by definition of } \hat{\tau}).$$

Thus,

$$J_x(\mathcal{E}\tau[\varphi] - \varphi) \in I(x) \text{ for all } x \in E, \text{ i.e., } \mathcal{E}\tau[\varphi] - \varphi \in \mathcal{J}.$$

Therefore, $\pi T[\varphi] = \pi\mathcal{E}\tau[\varphi] = [\varphi]$ for $[\varphi] \in C^m(\mathbb{R}^n)/\mathcal{J}$.

Thus, $T : C^m(\mathbb{R}^n)/\mathcal{J} \to C^m(\mathbb{R}^n)$ has all the properties asserted in Theorem 2. We have succeeded in reducing Theorem 2 (for $(I(x))_{x\in E}$ Glaeser stable) to Theorem 4.

We close this section by checking that the first slice E_0 is compact.

For $x \in E$, we have $\text{type}(x) = (d_0(x), \ldots, d_m(x))$, with $d_k(x) = \dim \pi_x^k I(x)$.

Fix $x_0 \in E$, $k \in \{0, 1, \ldots, m\}$. Since $\pi_{x_0}^k I(x_0)$ has dimension $d_k(x_0)$, we may pick $P_\mu \in I(x_0)$ $(1 \le \mu \le d_k(x_0))$ such that the images $\pi_{x_0}^k P_\mu$ $(1 \le \mu \le d_k(x_0))$ are linearly independent. Since $(I(x))_{x\in E}$ is Glaeser stable, there exist C^m functions F_μ on $\mathbb{R}^n$ such that

$J_x(F_\mu) \in I(x)$ for all $x \in E$, and $J_{x_0}(F_\mu) = P_\mu$.

The k-jets $\pi_x^k J_x(F_\mu)$ $(1 \le \mu \le d_k(x_0))$ are linearly independent at $x = x_0$, hence also at all x close enough to x_0. Consequently, $d_k(x) = \dim \pi_x^k\ I(x) \ge d_k(x_0)$ for all $x \in E$ near enough to x_0.

Thus, we have proven the following:

Given $x_0 \in E$ there exists a neighborhood U of x_0 in E, such that $d_k(x) \ge d_k(x_0)$ for all $x \in U$, $k \in \{0, 1, \ldots, m\}$.

In particular, $\text{type}(x) \ge \text{type}(x_0)$ for all $x \in U$, where the inequality sign refers to our lexicographic order on $(m+1)$-tuples.

It follows at once that the set E_0 of all $x \in E$ of the minimal type is a closed subset of the compact set E.

Thus, E_0 is compact.

§2. Review of Previous Results

In this section, we collect from previous literature some ideas and results that will play a rôle in our proof of Theorem 4. We retain the notation of Section 0.

We start with the classical Whitney Extension Theorem.

Let $E \subset \mathbb{R}^n$. Then we write $C^m_{\mathsf{jet}}(E)$ for the space of all families of m^{th} degree polynomials $(P^x)_{x\in E}$, satisfying the following conditions;

(a) Given $\epsilon > 0$ there exists $\delta > 0$ such that, for any $x, y \in E$ with $|x - y| < \delta$, we have $|\partial^\beta (P^x - P^y)(y)| \le \epsilon |x - y|^{m-|\beta|}$ for $|\beta| \le m$.

(b) There exists a finite constant $M > 0$ such that
$|\partial^\beta P^x(x)| \le M$ for $|\beta| \le m$, $x \in E$; and
$|\partial^\beta (P^x - P^y)(y)| \le M|x - y|^{m-|\beta|}$ for $|\beta| \le m$, $x, y \in E$.

(Here and throughout this paper $\partial^\beta P^x(x)$ always denotes the value at $y = x$ of $\left(\frac{\partial}{\partial y}\right)^\beta P^x(y)$, never $\partial^\beta \phi(x)$ with $\phi(x) = P^x(x)$.)

The norm $\| \, (P^x)_{x \in E} \, \|_{C^m_{\mathsf{jet}}(E)}$ is defined to be the infimum of all possible M in (b). Note that condition (a) holds vacuously when E is finite.

In terms of these definitions, the classical Whitney Extension Theorem may be stated as follows.

Theorem 2.1: *Given a compact set $E \subset \mathbb{R}^n$, there exists a linear map*

$\mathcal{E} : C^m_{\mathsf{jet}}(E) \to C^m(\mathbb{R}^n)$, such that

(A) *The norm of $\mathcal{E}$ is bounded by a constant C depending only on m and n; and*

(B) $J_{x_0}(\mathcal{E}[(P^x)_{x \in E}]) = P^{x_0}$ *for any $x_0 \in E$ and $(P^x)_{x \in E} \in C^m_{\mathsf{jet}}(E)$.*

(See, e.g., [17,21,23] for a proof of Theorem 2.1.)

Next, we recall some definitions and results from [12].

We introduce a convex set $\sigma(x_0, k)$ that will play a key rôle.

Let $(I(x))_{x \in E}$ be a family of ideals, and let $x_0 \in E$, $k \ge 1$ be given.

Then we define $\sigma(x_0, k)$ as the set of all $P_0 \in I(x_0)$ with the following property:

Given $x_1, \ldots, x_k \in E$, there exist $P_1 \in I(x_1), \ldots, P_k \in I(x_k)$, such that

$|\partial^\beta P_i(x_i)| \le 1$ for $|\beta| \le m$, $0 \le i \le k$; and

$|\partial^\beta (P_i - P_j)(x_j)| \le |x_i - x_j|^{m-|\beta|}$ for $|\beta| \le m$, $0 \le i, j \le k$.

One checks easily that $\sigma(x_0, k)$ is a compact, convex, symmetric subset of $I(x_0)$. (By "symmetric", we mean that $P \in \sigma(x_0, k)$ implies $-P \in \sigma(x_0, k)$.) The basic convex set $\Gamma_\xi(x_0, k, A)$ defined in the Introduction is essentially a translate of $\sigma(x_0, k)$, as the following proposition shows.

Proposition 2.1: *Let $H = (f(x) + I(x))_{x \in E}$ be a family of cosets, and suppose*

$P \in \Gamma_H(x_0, k, A)$. *Then, for any $A' > 0$, we have*

$P + A'\sigma(x_0, k) \subseteq \Gamma_H(x_0, k, A + A') \subseteq P + (2A + A')\sigma(x_0, k)$.

The above proposition follows trivially from the definitions.

A basic property of $\sigma(x_0, k)$ is "Whitney convexity", which we now define.

Let σ be a closed, convex, symmetric subset of $\mathcal{R}_{x_0}$, and let A be a positive constant. Then we say that σ is "Whitney convex with Whitney constant A" if the following condition is satisfied:

Let $P \in \sigma$, $Q \in \mathcal{P}$, $\delta \in (0, 1]$ be given. Suppose P and Q satisfy $|\partial^\beta P(x_0)| \leq \delta^{m-|\beta|}$ and $|\partial^\beta Q(x_0)| \leq \delta^{-|\beta|}$, for $|\beta| \leq m$. Then $P \odot Q \in A\sigma$, where $\odot$ denotes multiplication in $\mathcal{R}_{x_0}$.

Let $k^\#$ be a large enough constant, depending only on m and n, to be picked later. Then we have the following results.

Lemma 2.1: *Let $(I(x))_{x \in E}$ be a Glaeser stable family of ideals.*

Then, for $x_0 \in E$ and $1 \leq k \leq k^\#$, the set $\sigma(x_0, k)$ is Whitney convex, with a Whitney constant depending only on m and n.

Lemma 2.2: *Let $(I(x))_{x \in E}$ be a Glaeser stable family of ideals, and suppose $x_0 \in E$ and $1 \leq k \leq k^\#$.*

Then there exists $\delta > 0$ such that any polynomial P, belonging to $I(x_0)$ and satisfying

$|\partial^\beta P(x_0)| \leq \delta$ *for* $|\beta| \leq m$,

also belongs to $\sigma(x_0, k)$.

To prove Lemmas 2.1 and 2.2, we set $f(x) = 0$ for all $x \in E$, and then note that

$(f(x) + I(x))_{x\in E}$ satisfies hypotheses (I) and (II) of Theorem 3 in [12].

(In fact, (I) is immediate from the Glaeser stability of $(I(x))_{x\in E}$; and (II) holds trivially, since we may just set all the P_i in (II) equal to zero.)

Since also $k^{\#}$ is a large enough constant, depending only on m and n, to be picked later, we find ourselves in the setting of Section 5 of [12]. Our present Lemmas 2.1 and 2.2 are simply Lemmas 5.3 and 5.5, respectively, from [12].

We recall from [12] the notion of the "lowest stratum" E_1.

Let $(I(x))_{x\in E}$ be a family of ideals. We set

$\hat{k}_1 = \min\{\dim I(x) : x \in E\}$, and

$\hat{k}_2 = \max\{\dim(I(x) \cap \ker \pi_x^{m-1}) : x \in E\, , \dim I(x) = \hat{k}_1\}$.

The "lowest stratum" E_1 is then defined as

$E_1 = \{x \in E : \dim I(x) = \hat{k}_1 \text{ and } \dim(I(x) \cap \ker \pi_x^{m-1}) = \hat{k}_2\}$.

We compare the lowest stratum E_1 with the first slice E_0.

Since $\dim(I(x) \cap \ker \pi_x^{m-1}) + \dim(\pi_x^{m-1} I(x)) = \dim I(x)$, the set E_1 may be equivalently defined as follows:

A given $x \in E$ belongs to E_1 if and only if

(a) $\dim(I(x))$ is as small as possible; and

(b) $\dim(\pi_x^{m-1} I(x))$ is as small as possible, subject to (a).

On the other hand, recalling our lexicographic order on $(m+1)$-tuples, we see that E_0 may be equivalently defined as follows:

A given $x \in E$ belongs to E_0 if and only if

(a) $\dim(I(x))$ is as small as possible;

(b) $\dim(\pi_x^{m-1} I(x))$ is as small as possible, subject to (a);

(c) $\dim(\pi_x^{m-2} I(x))$ is as small as possible, subject to (a) and (b); and so forth.

Thus, we have proven the following elementary result.

Proposition 2.2: *Let $(I(x))_{x \in E}$ be a family of ideals. Let E_0 be the first slice, and let E_1 be the lowest stratum. Then $E_0 \subseteq E_1$.*

Our next result is again essentially taken from Section 5 in [12].

Recall that $D = \dim \mathcal{P}$.

Lemma 2.3: *Suppose $1 + (D+1) \cdot k_3 \leq k_2$, $1 + (D+1) \cdot k_2 \leq k_1$, $k_1 \leq k^{\#}$.*

Let $(I(x))_{x \in E}$ be a Glaeser stable family of ideals, and let E_1 be the lowest stratum.

Then there exists $\eta > 0$ with the following property:

Suppose $x \in E_1$ and $P \in I(x)$, with $|\partial^\beta P(x)| \leq \eta^{m-|\beta|}$ for $|\beta| \leq m$.

Then $P \in C\sigma(x, k_3)$, with C depending only on m and n.

To prove Lemma 2.3, we again set $f(x) = 0$ for all $x \in E$, and note that we are in the setting of Section 5 of [12], as in our discussion of Lemmas 2.1 and 2.2. Since $f(x) = 0$ for all $x \in E$, one checks trivially from the definitions that (in the notation of [12]) we have $\Gamma_f(x, k, A) = A\sigma(x, k)$. Consequently, Lemma 2.3 is simply the special case $f \equiv 0$, $A_1 = A_2 = 1$, $x' = x'' = x$, $Q' = 0$, $Q'' = P$, of Lemma 5.10 in [12].

Thus, Lemma 2.3 holds.

Again, from Section 5 in [12], we have the following result.

Lemma 2.4: *Let $H = (f(x) + I(x))_{x \in E}$ be a family of cosets.*

Suppose $1 + (D+1) \cdot k_2 \leq k_1$, and $A > 0$.

Let $x', x'' \in E$, and let $P' \in \Gamma_H(x', k_1, A)$.

Then there exists $P'' \in \Gamma_H(x'', k_2, A)$, with

$|\partial^\beta (P'' - P')(x')| \leq A|x' - x''|^{m-|\beta|}$ *for* $|\beta| \leq m$.

The proof of Lemma 5.6 in [12] applies here, and establishes our present Lemma 2.4.

Advancing to Section 6 in [12], we have the following.

Lemma 2.5: *Suppose* $k \geq 1$, $1 + (D+1) \cdot k \leq k^{\#}$.

Let $(I(x))_{x \in E}$ *be a Glaeser stable family of ideals, and let* E_1 *be the lowest stratum.*

Then, given $\epsilon > 0$ *there exists* $\delta > 0$ *such that the following holds*:

Given $x_0 \in E_1$, $P_0 \in I(x_0)$, *and* $x_1, \ldots, x_k \in E \cap B(x_0, \delta)$, *there exist* $P_1 \in I(x_1), \ldots, P_k \in I(x_k)$, *with*

$$|\partial^\alpha (P_i - P_j)(x_j)| \leq \epsilon |x_i - x_j|^{m - |\alpha|} \cdot (1 + \max_{|\beta| \leq m} |\partial^\beta P_0(x_0)|)$$

for $|\alpha| \leq m$, $0 \leq i, j \leq k$.

To prove Lemma 2.5, we again set $f(x) = 0$ for all $x \in E$, and note once more that $(f(x) + I(x))_{x \in E}$ satisfies the hypotheses of Theorem 3 in [12]. Since also $k^{\#}$ is a large enough constant, depending only on m and n, to be picked later, we find ourselves in the setting of Section 6 of [12].

Our present Lemma 2.5 is simply Lemma 6.3 in [12], for the special case $f(x) = 0$ (all $x \in E$).

Next, we recall Lemma 3.3 from [15].

We write $\#(S)$ for the cardinality of a finite set S.

Lemma 2.6: *Suppose* $k^{\#} \geq (D+1)^{10} \cdot k_1$, $k_1 \geq 1$, $A > 0$, $\delta > 0$.

Let Ξ *be a vector space, with seminorm* $|\cdot|$.

Let $E \subseteq \mathbb{R}^n$, *and let* $x_0 \in E$.

For each $x \in E$, *suppose we are given a vector space* $I(x) \subseteq \mathcal{R}_x$, *and a linear map* $\xi \mapsto f_\xi(x)$ *from* Ξ *into* $\mathcal{R}_x$.

Assume that the following conditions are satisfied.

(a) *Given* $\xi \in \Xi$ *and* $S \subseteq E$, *with* $|\xi| \leq 1$ *and* $\#(S) \leq k^{\#}$, *there exists* $F^S_\xi \in C^m(\mathbb{R}^n)$, *with*

$\| F_\xi^S \|_{C^m(\mathbb{R}^n)} \leq A$, *and* $J_x(F_\xi^S) \in f_\xi(x) + I(x)$ *for each* $x \in S$.

(b) *Suppose* $P_0 \in I(x_0)$, *with* $|\partial^\beta P_0(x_0)| \leq \delta$ *for* $|\beta| \leq m$.

Then, given $x_1, \dots, x_{k^\#} \in E$, *there exist* $P_1 \in I(x_1), \dots, P_{k^\#} \in I(x_{k^\#})$,

with

$|\partial^\beta P_i(x_i)| \leq 1$ *for* $|\beta| \leq m$, $0 \leq i \leq k^\#$; *and*

$|\partial^\beta (P_i - P_j)(x_j)| \leq |x_i - x_j|^{m-|\beta|}$ *for* $|\beta| \leq m$, $0 \leq i, j \leq k^\#$.

Then there exists a linear map $\xi \mapsto \tilde{f}_\xi(x_0)$, *from* Ξ *into* $\mathcal{R}_{x_0}$, *with the following property:*

(c) *Given* $\xi \in \Xi$ *with* $|\xi| \leq 1$, *and given* $x_1, \dots, x_{k_1} \in E$, *there exist polynomials* $P_0, P_1, \dots,$ $P_{k_1} \in \mathcal{P}$, *with*

$P_0 = \tilde{f}_\xi(x_0)$;

$P_i \in f_\xi(x_i) + I(x_i)$ *for* $0 \leq i \leq k_1$;

$|\partial^\beta P_i(x_i)| \leq CA$ *for* $|\beta| \leq m$, $0 \leq i \leq k_1$; *and*

$|\partial^\beta (P_i - P_j)(x_j)| \leq CA| x_i - x_j|^{m-|\beta|}$ *for* $|\beta| \leq m$, $0 \leq i, j \leq k_1$.

Here, C *depends only on* m *and* n.

The version of Lemma 2.6 stated here differs slightly from Lemma 3.3 in [15], since there the constant $k^\#$ is arbitrary, and the constant C is determined by m, n and $k^\#$.

Here, we have taken $k^\#$ to be a (large enough) constant determined by m and n. Consequently, the constant C in our present Lemma 2.6 depends only on m and n, as stated there.

For a family of cosets depending linearly on $\xi \in \Xi$, conclusion (c) of Lemma 2.6 says that we can find $\tilde{f}_\xi(x_0) \in \Gamma_\xi\ (x_0, k_1, CA)$ depending linearly on ξ.

To state the next result, we recall another definition from [15].

Let $E \subset \mathbb{R}^n$ be non-empty. For each $x \in E$, suppose we are given a convex, symmetric subset $\sigma(x) \subseteq \mathcal{R}_x$. Let $f = (f(x))_{x \in E}$ be a family of m-jets, with $f(x) \in \mathcal{R}_x$ for each $x \in E$.

Then we say that f belongs to $C^m(E,\sigma(\cdot))$ if there exist a function $F \in C^m(\mathbb{R}^n)$ and a finite constant $M > 0$, such that

(1) $\quad \| F \|_{C^m(\mathbb{R}^n)} \leq M$, and $J_x(F) \in f(x) + M\sigma(x)$ for all $x \in E$.

The seminorm $\| f \|_{C^m(E,\sigma(\cdot))}$ is defined as the infimum of all possible M in (1).

We now recall Theorem 5 from [15].

Theorem 2.2: *Let $E_{00} \subset \mathbb{R}^n$ be a finite set. For each $x \in E_{00}$, let $\sigma(x) \subseteq \mathcal{R}_x$ be Whitney convex, with Whitney constant A.*

Then there exists a linear map

$T : C^m(E_{00},\sigma(\cdot)) \to C^m(\mathbb{R}^n)$,

with the following properties.

(A) *The norm of T is bounded by a constant determined by m, n and A.*

(B) *Given $f = (f(x))_{x\in E} \in C^m(E_{00},\sigma(\cdot))$ with $\| f \|_{C^m(E_{00},\sigma(\cdot))} \leq 1$,*

we have

$J_x(Tf) \in f(x) + A'\sigma(x)$ *for all $x \in E_{00}$, with A' determined by m, n and A.*

We close this section by pointing out that several of the above results could have been given in a more general or natural form than the versions stated here. We were motivated by the desire to quote from [12,15] rather than prove slight variants of known results.

§3. Consequences of Previous Results

In this section, we prove some simple consequences of the results of Section 2, as well as a Corollary of Theorem 3 (which, we recall, was proven in [12]).

Lemma 3.1: *There exist C, $\bar{k}$, depending only on m and n, for which the following holds.*

Let $(f(x) + I(x))_{x\in E}$ be a Glaeser stable family of cosets.

Suppose we are given $A > 0$, $x_0 \in E$, and $P_0 \in f(x_0) + I(x_0)$.

Assume that, given $x_1, \ldots, x_{\bar{k}} \in E$, there exist $P_1 \in f(x_1) + I(x_1), \ldots, P_{\bar{k}} \in f(x_{\bar{k}}) + I(x_{\bar{k}})$, with

$|\partial^\beta P_i(x_i)| \le A$ *for* $|\beta| \le m\,,\ 0 \le i \le \bar{k}\,;$ *and*

$|\partial^\beta (P_i - P_j)(x_j)| \le A|x_i - x_j|^{m-|\beta|}$ *for* $|\beta| \le m\,,\ 0 \le i, j \le \bar{k}\,.$

Then there exists $F \in C^m(\mathbb{R}^n)$, with

$\| F \|_{C^m(\mathbb{R}^n)} \le CA$, *$J_x(F) \in f(x) + I(x)$ for all $x \in E$, and $J_{x_0}(F) = P_0$.*

Proof: Define $\hat{f}(x_0) = P_0$, $\hat{I}(x_0) = \{0\}$; and, for $x \in E \setminus \{x_0\}$, define $\hat{f}(x) = f(x)$, $\hat{I}(x) = I(x)$. Using the definition (GS2), we see that $(\hat{f}(x) + \hat{I}(x))_{x \in E}$ is a Glaeser stable family of cosets. Applying Theorem 3 to $(\hat{f}(x) + \hat{I}(x))_{x \in E}$, we obtain the conclusion of Lemma 3.1. (To check hypothesis (II) of Theorem 3, we apply Theorem 2.1 to the set $\{x_0, \ldots, x_{\bar{k}}\}$.)

The proof of the lemma is complete. ■

As in the previous section, we take $k^\#$ to be a large enough constant, determined by m and n, to be picked later.

Lemma 3.2: *Suppose $1 + (D+1) \cdot k_3 \le k_2$, $1 + (D+1) \cdot k_2 \le k_1$, $k_1 \le k^\#$; and $A_1, A_2 > 0$.*

Let $(I(x))_{x \in E}$ be a Glaeser stable family of ideals, and let E_1 be the lowest stratum.

Then there exists $\eta > 0$, for which the following holds:

For each $x \in E$, suppose we are given an m-jet $f(x) \in \mathcal{R}_x$.

Set $H = (f(x) + I(x))_{x \in E}$.

Suppose we are given $x', x'' \in E_1$, $P' \in \Gamma_H(x', k_1, A_1)$, and $P'' \in f(x'') + I(x'')$.

If $|x' - x''| \le \eta$ and $|\partial^\beta(P' - P'')(x')| \le A_2 \eta^{m-|\beta|}$ for $|\beta| \le m$, then

$P'' \in \Gamma_H(x'', k_3, A')$, with A' depending only on A_1, A_2, m, n.

Proof: In this proof, we write A_3, A_4, etc. for constants depending only on A_1, A_2, m, n.

Let η be as in Lemma 2.3, and let H, x', x'', P', P'' be as in the hypotheses of Lemma 3.2. In particular, we have $P' \in \Gamma_H(x', k_1, A_1)$. Lemma 2.4 gives us a polynomial $\tilde{P} \in \Gamma_H(x'', k_3, A_1) \subseteq f(x'') + I(x'')$, with

$|\partial^\beta(P' - \tilde{P})(x')| \leq A_1|x' - x''|^{m-|\beta|} \leq A_1\eta^{m-|\beta|}$ for $|\beta| \leq m$.

Since also $P'' \in f(x'') + I(x'')$ and $|\partial^\beta(P' - P'')(x')| \leq A_2\eta^{m-|\beta|}$ for $|\beta| \leq m$, it follows that

(1) $P'' - \tilde{P} \in I(x'')$,
and
$|\partial^\beta(P'' - \tilde{P})(x')| \leq (A_1 + A_2) \cdot \eta^{m-|\beta|}$ for $|\beta| \leq m$.

This last estimate implies

(2) $|\partial^\beta(P'' - \tilde{P})(x'')| \leq A_3\eta^{m-|\beta|}$ for $|\beta| \leq m$,
since $|x' - x''| \leq \eta$, and $P'', \tilde{P}$ are m^{th} degree polynomials.

Since $x'' \in E_1$, we learn from (1) and (2) that Lemma 2.3 applies to $(P'' - \tilde{P})/A_3$. Consequently, we have

$P'' - \tilde{P} \in A_4\sigma(x'', k_3)$.

Since also $\tilde{P} \in \Gamma_H(x'', k_3, A_1)$, it now follows from Proposition 2.1 that

$P'' \in \Gamma_H(x'', k_3, A_5)$,

which is the conclusion of Lemma 3.2.

The proof of the Lemma is complete. ■

Note that Lemma 3.2 here sharpens Lemma 5.10 in [12], since our η is independent of f.

Lemma 3.3: *Suppose $k^\# \geq (D+1)^{10} \cdot k_1$, $k_1 \geq 1$, and $A > 0$.*

Let Ξ be a vector space with a seminorm $|\cdot|$, and let $(f_\xi(x) + I(x))_{x \in E, \xi \in \Xi}$ be a Glaeser stable family of cosets, depending linearly on $\xi \in \Xi$.

Assume that, for any $\xi \in \Xi$ with $|\xi| \leq 1$, there exists $F \in C^m(\mathbb{R}^n)$, with

(∗) $\| F \|_{C^m(\mathbb{R}^n)} \leq A$, *and $J_x(F) \in f_\xi(x) + I(x)$ for all $x \in E$.*

Then, given $x_0 \in E$, there exists a linear map $\xi \mapsto \tilde{f}_\xi(x_0)$, from Ξ into $\mathcal{R}_{x_0}$, such that

$\tilde{f}_\xi(x_0) \in \Gamma_\xi(x_0, k_1, CA)$ *for all* $\xi \in \Xi$ *with* $|\xi| \leq 1$.

Here, C depends only on m and n.

<u>Proof</u>: By definition, $(f_\xi(x) + I(x))_{x \in E}$ is Glaeser stable for each $\xi \in \Xi$.

Setting $\xi = 0$, we learn that $(I(x))_{x \in E}$ is Glaeser stable, hence Lemma 2.2 applies. Thus, there exists $\delta > 0$ such that

(**) any $P \in I(x_0)$ satisfying $|\partial^\beta P(x_0)| \leq \delta$ for $|\beta| \leq m$ belongs to $\sigma(x_0, k^\#)$.

We now invoke Lemma 2.6. Hypotheses (a) and (b) of that Lemma follow at once from (*) and (**), and from the definition of $\sigma(x_0, k^\#)$.

Hence, there exists a linear map $\xi \mapsto \tilde{f}_\xi(x_0)$ from Ξ into $\mathcal{R}_{x_0}$, satisfying condition (c) in the statement of Lemma 2.6.

Comparing condition (c) with the definition of $\Gamma_\xi(x_0, k_1, CA)$, we see that

$\tilde{f}_\xi(x_0) \in \Gamma_\xi(x_0, k_1, CA)$ for $|\xi| \leq 1$, with C depending only on m and n.

The proof of Lemma 3.3 is complete. ∎

The next result involves the space $C^m(E, \sigma(\cdot))$ from Section 2. (See Theorem 2.2 and the paragraph before it.)

<u>Lemma 3.4</u>: *Suppose $k^\# \geq (D+1)^{10} \cdot k_1, k_1 \geq 1$ and $A > 0$.*

Let Ξ be a vector space with a seminorm $|\cdot|$.

Let $(f_\xi(x) + I(x))_{x \in E, \xi \in \Xi}$ be a Glaeser stable family of cosets depending linearly on $\xi \in \Xi$.

Assume that, given $\xi \in \Xi$ with $|\xi| \leq 1$, there exists $F \in C^m(\mathbb{R}^n)$, with

$\| F \|_{C^m(\mathbb{R}^n)} \leq A$, *and* $J_x(F) \in f_\xi(x) + I(x)$ *for all* $x \in E$.

For each $x_0 \in E$, let $\xi \mapsto \tilde{f}_\xi(x_0)$ be a linear map from Ξ into $\mathcal{R}_{x_0}$, as in the conclusion of Lemma 3.3.

Set $\sigma(x) = \sigma(x, k_1)$ *for all* $x \in E$,

and set $\tilde{f}_\xi = (\tilde{f}_\xi(x_0))_{x_0 \in E}$ *for each* $\xi \in \Xi$.

Then, for each $\xi \in \Xi$, *we have* $\tilde{f}_\xi \in C^m(E, \sigma(\cdot))$.

Moreover, if $|\xi| \leq 1$, *then* $\| \tilde{f} \|_{C^m(E,\sigma(\cdot))} \leq CA$,

with C *depending only on* m *and* n.

Proof: Since $\xi \mapsto \tilde{f}_\xi$ is linear, we may restrict attention to the case $|\xi| \leq 1$. Fix $\xi \in \Xi$ with $|\xi| \leq 1$, and fix $F \in C^m(\mathbb{R}^n)$, with

$\| F \|_{C^m(\mathbb{R}^n)} \leq A$, and $J_x(F) \in f_\xi(x) + I(x)$ for all $x \in E$.

We then have

$(*)$ $\quad J_{x_0}(F) \in \Gamma_\xi(x_0, k, CA)$ for any $x_0 \in E$, $k \geq 1$.

To see this, suppose we are given $x_1, \ldots, x_k \in E$.

Setting $P_i = J_{x_i}(F)$ for $i = 0, 1, \ldots, k$, we have:

$P_0 \ = \ J_{x_0}(F)$;

$P_i \in f_\xi(x_i) \ + \ I(x_i)$ for $0 \leq i \leq k$;

$|\partial^\beta P_i(x_i)| \leq CA$ for $|\beta| \leq m$, $0 \leq i \leq k$; and

$|\partial^\beta (P_i - P_j)(x_j)| \ \leq \ CA|x_i - x_j|^{m-|\beta|}$ for $|\beta| \leq m$, $0 \leq i, j \leq k$.

Hence, $(*)$ holds, by definition of $\Gamma_\xi(x_0, k, CA)$.

For $x_0 \in E$, we have

$J_{x_0}(F), \tilde{f}_\xi(x_0) \in \Gamma_\xi(x_0, k_1, CA)$,

since $(*)$ holds and $\tilde{f}_\xi(x_0)$ is as in the conclusion of Lemma 3.3. Consequently,

$J_{x_0}(F) - \tilde{f}_\xi(x_0) \in CA\sigma(x_0, k_1) \ = \ CA\sigma(x_0)$ for $x_0 \in E$, by Proposition 2.1.

Thus, $F \in C^m(\mathbb{R}^n)$, with

$\| F \|_{C^m(\mathbb{R}^n)} \leq CA$, and $J_x(F) \in \tilde{f}_\xi(x) + CA\sigma(x)$ for all $x \in E$.

By definition of $C^m(E, \sigma(\cdot))$, this means that

$\tilde{f}_\xi \in C^m(E, \sigma(\cdot))$, and that $\| \tilde{f}_\xi \|_{C^m(E,\sigma(\cdot))} \leq CA$.

The proof of Lemma 3.4 is complete. ■

Lemma 3.5: *Suppose $k^\# \geq (D+1)^{10} \cdot k_1$, $k_1 \geq 1$, $A > 0$.*

Let Ξ be a vector space with a seminorm $|\cdot|$, and let $(f_\xi(x) + I(x))_{x \in E, \xi \in \Xi}$ be a Glaeser stable family of cosets depending linearly on $\xi \in \Xi$.

Assume that, given any $\xi \in \Xi$ with $|\xi| \leq 1$, there exists $F \in C^m(\mathbb{R}^n)$, with

$\| F \|_{C^m(\mathbb{R}^n)} \leq A$, and $J_x(F) \in f_\xi(x) + I(x)$ for all $x \in E$.

Let $E_{00} \subseteq E$ be a finite set.

Then there exists a linear map $\xi \mapsto F_\xi^{00}$, from Ξ into $C^m(\mathbb{R}^n)$, with norm at most CA, such that, for $|\xi| \leq 1$, we have

$J_x(F_\xi^{00}) \in \Gamma_\xi(x, k_1, CA) \subseteq f_\xi(x) + I(x)$ for all $x \in E_{00}$.

Here, C depends only on m and n.

Proof: We recall that C denotes a constant determined by m and n.

For each $x \in E_{00}$, set $\sigma(x) = \sigma(x, k_1)$. By Lemma 2.1, each $\sigma(x)$ is Whitney convex, with Whitney constant C.

Hence, Theorem 2.2 provides a linear map

$T : C^m(E_{00}, \sigma(\cdot)) \to C^m(\mathbb{R}^n)$,

with norm at most C, satisfying the following property:

(∗) Suppose $f = (f(x))_{x \in E_{00}} \in C^m(E_{00}, \sigma(\cdot))$, with $\| f \|_{C^m(E_{00},\sigma(\cdot))} \leq 1$.
Then $J_x(Tf) \in f(x) + C\sigma(x, k_1)$ for all $x \in E_{00}$.

Next, note that our present hypotheses include those of Lemma 3.3.

Hence, Lemma 3.3 lets us pick out, for each $x \in E_{00}$, a linear map $\xi \mapsto \tilde{f}_\xi(x)$, from Ξ into $\mathcal{R}_x$, such that

$$(**) \quad \tilde{f}_\xi(x) \in \Gamma_\xi(x, k_1, CA) \text{ for all } x \in E_{00}, \xi \in \Xi \text{ with } |\xi| \le 1.$$

For $\xi \in \Xi$, we set $\tilde{f}^{00}_\xi = (\tilde{f}_\xi(x))_{x \in E_{00}}$. Immediately from Lemma 3.4, we learn that $\xi \mapsto \tilde{f}^{00}_\xi$ is a linear map from Ξ into $C^m(E_{00}, \sigma(\cdot))$, with norm at most CA.

For $\xi \in \Xi$, we now define $F^{00}_\xi = T\tilde{f}^{00}_\xi$. Thus, $\xi \mapsto F^{00}_\xi$ is a linear map from Ξ into $C^m(\mathbb{R}^n)$, of norm at most CA. Moreover, suppose $|\xi| \le 1$. Then we have $\| \tilde{f}^{00}_\xi \|_{C^m(E_{00}, \sigma(\cdot))} \le CA$. Applying $(*)$ to $f = \tilde{f}^{00}_\xi/(CA)$, we learn that

$J_x(F^{00}_\xi) \in \tilde{f}_\xi(x) + CA\sigma(x, k_1)$ for all $x \in E_{00}$.

Together with (**) and Proposition 2.1, this shows that

$J_x(F^{00}_\xi) \in \Gamma_\xi(x, k_1, CA)$ for all $x \in E_{00}$.

Thus, the map $\xi \mapsto F^{00}_\xi$ has all the properties asserted in the statement of Lemma 3.5.

The proof of the lemma is complete. ∎

Lemma 3.6: *Suppose $k \ge 1$, and $1 + (D+1) \cdot k \le k^\#$.*

Let $(f(x) + I(x))_{x \in E}$ be a Glaeser stable family of cosets, and let E_1 be the lowest stratum for $(I(x))_{x \in E}$.

Then, given $\epsilon > 0$, there exists $\delta > 0$ such that the following holds:

Given $x_0 \in E_1$, $P_0 \in f(x_0) + I(x_0)$, and $x_1, \ldots, x_k \in E \cap B(x_0, \delta)$, there exist $P_1 \in f(x_1) + I(x_1), \ldots, P_k \in f(x_k) + I(x_k)$ such that

$|\partial^\alpha(P_i - P_j)(x_j)| \le \epsilon |x_i - x_j|^{m-|\alpha|} \cdot (1 + \max_{|\beta| \le m} |\partial^\beta P_0(x_0)|)$ *for* $|\alpha| \le m$, $0 \le i, j \le k$.

Proof: Since $(f(x) + I(x))_{x \in E}$ is Glaeser stable, it follows easily that $(I(x))_{x \in E}$ is Glaeser stable. Moreover, by definition (GS1) of Glaeser stability, there exists $F \in C^m(\mathbb{R}^n)$, with

$$(*0) \quad J_x(F) \in f(x) + I(x) \text{ for all } x \in E.$$

We fix an F as above, and let $\epsilon > 0$ be given. Set $\epsilon' = \frac{\epsilon}{2+\|F\|_{C^m(\mathbb{R}^n)}}$.

Since $F \in C^m(\mathbb{R}^n)$ and E is compact, there exists $\delta_1 > 0$ with the following property:

(∗1) Given $x_0 \in E$ and $x_1, \ldots, x_k \in E \cap B(x_0, \delta_1)$, we have

$|\partial^\alpha(J_{x_i}(F) - J_{x_j}(F))(x_j)| \le \epsilon' \, |x_i - x_j|^{m-|\alpha|}$ for $|\alpha| \le m$, $0 \le i, j \le k$.

We apply Lemma 2.5, with ϵ' in place of ϵ. Thus, we obtain $\delta_2 > 0$, for which the following holds.

(∗2) Given $x_0 \in E_1, \hat{P}_0 \in I(x_0)$, and $x_1, \ldots, x_k \in E \cap B(x_0, \delta_2)$, there exist

$\hat{P}_1 \in I(x_1), \ldots, \hat{P}_k \in I(x_k)$, with

$|\partial^\alpha(\hat{P}_i - \hat{P}_j)(x_j)| \le \epsilon' \, |x_i - x_j|^{m-|\alpha|} \cdot (1 + \max_{|\beta| \le m} |\partial^\beta \hat{P}_0(x_0)|)$ for $|\alpha| \le m$, $0 \le i, j \le k$.

We set $\delta = \min(\delta_1, \delta_2)$.

Now suppose we are given $x_0 \in E_1, P_0 \in f(x_0) + I(x_0)$, and $x_1, \ldots, x_k \in E \cap B(x_0, \delta)$.

Then

(∗3) $\hat{P}_0 \; = \; P_0 - J_{x_0}(F)$

belongs to $I(x_0)$, thanks to (∗0).

We apply (∗2), to obtain $\hat{P}_1 \in I(x_1), \ldots, \hat{P}_k \in I(x_k)$ as indicated there.

Setting

(∗4) $P_i \; = \; \hat{P}_i \; + \; J_{x_i}(F)$ for $i = 1, \ldots, k$,

we have $P_i \in f(x_i) \; + \; I(x_i)$ for $i = 1, \ldots, k$,

thanks to (∗0).

Note that (∗4) holds also for $i = 0$.

From (∗1),…,(∗4), we learn that

$$|\partial^\alpha(P_i - P_j)(x_j)| \; \le \; \epsilon' |x_i - x_j|^{m-|\alpha|} \cdot (2 + \max_{|\beta| \le m} |\, \partial^\beta \hat{P}_0(x_0)|)$$

$$\le \epsilon' |x_i - x_j|^{m-|\alpha|} (2 + \; \| \; F \; \|_{C^m(\mathbb{R}^n)} \; + \; \max_{|\beta| \le m} |\partial^\beta P_0(x_0)|)$$

for $|\alpha| \leq m$, $0 \leq i, j \leq m$.

Since we have taken $\epsilon' = \epsilon/(2+ \parallel F \parallel_{C^m(\mathbb{R}^n)})$, it follows that

$$|\partial^\alpha(P_i - P_j)(x_j)| \ \leq \ \epsilon|x_i - x_j|^{m-|\alpha|} \cdot (1 \ + \ \max_{|\beta| \leq m} |\partial^\beta P_0(x_0)|)$$

for $|\alpha| \leq m$, $0 \leq i, j \leq m$.

Thus, the polynomials $P_1, \dots, P_k$ have all the properties asserted in the statement of Lemma 3.6.

The proof of the lemma is complete. ■

§4. Picking the Constants

Let $\bar{k}$ be as in Lemma 3.1. Thus, $\bar{k}$ depends only on m, n.

We recall that D is the dimension of the vector space of all m^{th} degree polynomials on $\mathbb{R}^n$.

We set

$k_3 \ = \ \bar{k}$,

$k_2 \ = \ 1 \ + \ (D \ + \ 1) \ \cdot \ k_3$,

$k_1 \ = \ 1 \ + \ (D \ + 1) \ \cdot \ k_2$,

and we pick

$k^\# \ \geq (D+1)^{10} \ \cdot \ k_1$.

§5. The First Main Lemma

In this section, we complete the analysis of F_ξ^{00} as described in the Introduction. Our result is as follows. Recall that $\mathcal{P}$ is the vector space of m^{th} degree polynomials on $\mathbb{R}^n$.

First Main Lemma: *Let Ξ be a vector space with a seminorm $|\cdot|$, let $(f_\xi(x) + I(x))_{x \in E, \xi \in \Xi}$ be a Glaeser stable family of cosets depending linearly on $\xi \in \Xi$, and let E_0 be the first slice for $(I(x))_{x \in E}$.*

Assume that, given $\xi \in \Xi$ with $|\xi| \leq 1$, there exists $F \in C^m(\mathbb{R}^n)$, with $\| F \|_{C^m(\mathbb{R}^n)} \leq 1$, and $J_x(F) \in f_\xi(x) + I(x)$ for all $x \in E$.

Then, given $A > 0$, there exists $\eta_0 > 0$ for which the following holds:

Suppose $E_{00} \subseteq E_0$ is finite, and suppose that no point of E_0 lies farther than distance η_0 from E_{00}.

Then there exists a linear map $\xi \mapsto F_\xi^{00}$, from Ξ into $C^m(\mathbb{R}^n)$, such that, for any $\xi \in \Xi$ with $|\xi| \leq 1$, we have:

(I) $\| F_\xi^{00} \|_{C^m(\mathbb{R}^n)} \leq C$, *with C depending only on m, n.*

(II) $J_x(F_\xi^{00}) \in f_\xi(x) + I(x)$ *for all $x \in E_{00}$.*

(III) *Let $x \in E_0$, $Q \in \mathcal{P}$ be given, with $|\partial^\beta Q(x)| \leq A\eta_0^{m-|\beta|}$ for $|\beta| \leq m$.*
If $J_x(F_\xi^{00}) + Q \in f_\xi(x) + I(x)$, then

$$J_x(F_\xi^{00}) + Q \in \Gamma_\xi(x, \bar{k}, A'),$$

where $\bar{k}$ is as in Lemma 3.1, and A' is a constant depending only on A, m, n.

Proof: We take $k^\#, k_1, k_2, k_3$ as in Section 4.

Let $\Xi, |\cdot|$, $(f_\xi(x) + I(x))_{x \in E, \xi \in \Xi}$ be as in the hypotheses of the First Main Lemma, and let $A > 0$ be given.

We know that $(I(x))_{x \in E}$ is Glaeser stable, since $(f_\xi(x) + I(x))_{x \in E, \xi \in \Xi}$ is Glaeser stable. Also, from Section 4, we have $1 + (D+1) \cdot k_3 \leq k_2$, $1 + (D+1) \cdot k_2 \leq k_1$, and $k_1 \leq k^\#$. Hence, we may apply Lemma 3.2, for any constants $A_1, A_2 > 0$. We will take $A_1 = \hat{C}$ and $A_2 = C^* + C^* A$, where $\hat{C}$ and C^* are constants, depending only on m and n, to be picked below.

Applying Lemma 3.2 with the above A_1, A_2; and recalling Proposition 2.2, we obtain $\eta_0 > 0$, for which the following holds.

(1) Suppose $\xi \in \Xi$, $x_0 \in E_0$, $x \in E_0$, $P_0 \in \Gamma_\xi(x_0, k_1, \hat{C})$, $P \in f_\xi(x) + I(x)$, $|x - x_0| \leq \eta_0$, and $|\partial^\beta (P - P_0)(x_0)| \leq (C^* + C^* A)\ \eta_0^{m-|\beta|}$ for $|\beta| \leq m$.

Then $P \in \Gamma_\xi(x, k_3, A')$, with A' depending only on m, n, A.

Now suppose $E_{00} \subseteq E_0$ is a finite set, and suppose that no point of E_0 lies farther than distance η_0 from E_{00}.

The hypotheses of Lemma 3.5 (with $A = 1$ there) are satisfied by Ξ, $|\cdot|$, $(f_\xi(x) + I(x))_{x\in E, \xi\in\Xi}$, and E_{00}. (In particular, we have $k^\# \geq (D+1)^{10} \cdot k_1$, as we recall from Section 4.)

Let $\xi \mapsto F_\xi^{00}$ be the linear map, from Ξ into $C^m(\mathbb{R}^n)$, given by Lemma 3.5. Thus, for $\xi \in \Xi$ with $|\xi| \leq 1$, we have

(2) $\| F_\xi^{00} \|_{C^m(\mathbb{R}^n)} \leq C_1,$

and

(3) $J_{x_0}(F_\xi^{00}) \in \Gamma_\xi(x_0, k_1, C_2) \subseteq f_\xi(x_0) + I(x_0)$ for all $x_0 \in E_{00}$.

We now take $\hat{C}$ to be the constant C_2 in (3). As promised, $\hat{C}$ depends only on m and n.

From (2) and (3), we see that the linear map $\xi \mapsto F_\xi^{00}$ satisfies (I) and (II) in the statement of the First Main Lemma. We check that it also satisfies (III).

Thus, let $\xi \in \Xi$ with $|\xi| \leq 1$, and let $x \in E_0$, $Q \in \mathcal{P}$ be given, with

(4) $|\partial^\beta Q(x)| \leq A\eta_0^{m-|\beta|}$ for $|\beta| \leq m,$

and

(5) $J_x(F_\xi^{00}) + Q \in f_\xi(x) + I(x).$

We must show that $J_x(F_\xi^{00}) + Q \in \Gamma_\xi(x, \bar{k}, A')$, where $\bar{k}$ is as in Lemma 3.1, and A' is a constant depending only on m, n, A.

By our assumption on E_{00}, there exists $x_0 \in E_{00}$, with $|x - x_0| \leq \eta_0$. From (2), we then have

$$|\partial^\beta(J_x(F_\xi^{00}) - J_{x_0}(F_\xi^{00}))(x)| \leq C|x-x_0|^{m-|\beta|} \leq C\eta_0^{m-|\beta|} \text{ for } |\beta| \leq m\,.$$

Together with (4), this yields

$$|\partial^\beta\{[J_x(F_\xi^{00}) + Q] - J_{x_0}(F_\xi^{00})\}(x)| \leq (C+A) \cdot \eta_0^{m-|\beta|} \text{ for } |\beta| \leq m\,.$$

Since $|x - x_0| \leq \eta_0$ and the expression in curly brackets is an m^{th} degree polynomial , it follows that

(6) $|\partial^\beta\{[J_x(F_\xi^{00})+Q]-J_{x_0}(F_\xi^{00})\}(x_0)| \le (C'+C'A)\cdot\eta_0^{m-|\beta|}$ for $|\beta|\le m$.

We now take C^* to be the constant C' in (6). As promised, C^* depends only on m and n.

We set $P=J_x(F_\xi^{00})+Q$, and $P_0=J_{x_0}(F_\xi^{00})$.

We make the following observations:

- $\xi\in\Xi$ and $x_0,x\in E_0$ (since $E_{00}\subseteq E_0$).
- $P_0\in\Gamma_\xi(x_0,k_1,\hat{C})$ (by (3) and our choice of $\hat{C}$).
- $P\in f_\xi(x)+I(x)$ (by (5)).
- $|x-x_0|\le\eta_0$ (by the defining properties of x_0).
- $|\partial^\beta(P-P_0)(x_0)|\le(C^*+C^*A)\cdot\eta_0^{m-|\beta|}$ for $|\beta|\le m$ (by (6) and our choice of C^*).

Consequently, (1) applies, and it tells us that $P\in\Gamma_\xi(x,k_3,A')$, with A' determined by A,m,n. Recalling that $P=J_x(F_\xi^{00})+Q$, and that $k_3=\bar{k}$ (as in Lemma 3.1; see Section 4), we conclude that $J_x(F_\xi^{00})+Q\in\Gamma_\xi(x,\bar{k},A')$, with A' determined by A,m,n.

This completes the proof of (III), hence also that of the First Main Lemma. ■

§6. Dominant Monomials

In the next several sections, we will construct the linear map $\xi\mapsto\tilde{F}^\xi$ described in the Introduction. We begin with an elementary rescaling lemma that will be used in Section 8 below.

Lemma 6.1: *Let $P_1,\dots,P_L\in\mathcal{P}$ be given, non-zero polynomials.*

Let $0<a<1$ be given. Then there exists a linear map $T:\mathbb{R}^n\to\mathbb{R}^n$, of the form $T:(x_1,\dots,x_n)\mapsto(\lambda_1x_1,\dots,\lambda_nx_n)$, with the following properties:

(1) *$\kappa\le\lambda_i\le1$ for $i=1,\dots,n$; where κ is a positive constant depending only on a,L,m,n.*

(2) *For each* $\ell (1 \leq \ell \leq L)$, *there exists a multi-index* $\beta(\ell)$, *with* $|\beta(\ell)| \leq m$, *such that*

$$|\partial^\beta (P_\ell \circ T)(0)| \leq a |\partial^{\beta(\ell)} (P_\ell \circ T)(0)| \ for \ |\beta| \leq m, \ \beta \neq \beta(\ell)\,.$$

<u>Proof</u>: Let A be a large, positive constant, to be picked later.

For $1 \leq i \leq n$, let $\lambda_i = \exp(-s_i)$ with $0 \leq s_i \leq A$.

Thus,

(3) $\exp(-A) \leq \lambda_i \leq 1$ for $i = 1, \ldots, n$.

Note that (2) holds unless there exist

$$\ell (1 \leq \ell \leq L)\,, \beta' = (\beta'_1, \ldots, \beta'_n)\,, \ \beta'' = (\beta''_1, \ldots, \beta''_n)\,, \text{ with}$$

(4) $|\beta'|$, $|\beta''| \leq m$, $\beta' \neq \beta''$, $\partial^{\beta'} P_\ell(0) \neq 0$, $\partial^{\beta''} P_\ell(0) \neq 0$,
for which $(s_1, \ldots, s_n)$ satisfies

(5) $$\left| \sum_{i=1}^{n} (\beta'_i - \beta''_i) \cdot s_i - \log \left| \frac{\partial^{\beta'} P_\ell(0)}{\partial^{\beta''} P_\ell(0)} \right| \right| \leq |\log a|\,.$$

For fixed ℓ, β', β'' satisfying (4), the volume of the set of all $(s_1, \ldots, s_n) \in [0, A]^n$ for which (5) holds is at most $2|\log a| \cdot A^{n-1}$. To see this, fix i_0 with $\beta'_{i_0} \neq \beta''_{i_0}$, and then fix all the s_i except for s_{i_0}. The set of all $s_{i_0} \in [0, A]$ for which (5) holds forms an interval of length $\leq \frac{2|\log a|}{|\beta'_{i_0} - \beta''_{i_0}|} \leq 2|\log a|$.

Integrating over all $(s_1, \ldots, s_{i_0 - 1}, \ s_{i_0 + 1}, \ldots, s_n) \in [0, A]^{n-1}$, we see that the set where (5) holds has volume at most $2|\log a| \cdot A^{n-1}$, as claimed.

Note also that the number of distinct (ℓ, β', β'') satisfying (4) is bounded by a constant depending only on m, n, L. Consequently, the set

$\Omega = \{(s_1, \ldots, s_n) \in [0, A]^n$ satisfying (5) for some (ℓ, β', β'') satisfying (4) $\}$ has volume at most $C|\log a| \cdot A^{n-1}$, with C depending only on m, n, L.

Hence, if we take A to be a large enough constant depending only on m, n, L, a, then we will have $\mathsf{vol}\ \Omega < \frac{1}{2}\ \mathsf{vol}\ ([0, A]^n)$, and thus $[0, A]^n \setminus \Omega$ will be non-empty.

Taking $(s_1, \ldots, s_n) \in [0, A]^n \smallsetminus \Omega$, we conclude that (5) never holds for any (ℓ, β', β'') satisfying (4), and therefore (2) holds for $\lambda_i = \exp(-s_i)$. Also, (3) shows that (1) holds, since A depends only on m, n, L, a.

The proof of Lemma 6.1 is complete. ■

§7. Definitions and Notation

We write $\mathcal{M}$ for the set of all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ of order $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq m$.

A subset $\mathcal{A} \subseteq \mathcal{M}$ will be called "monotonic" if, for any $\alpha \in \mathcal{A}$, and any multi-index γ with $|\gamma| \leq m - |\alpha|$, we have $\alpha + \gamma \in \mathcal{A}$.

(We warn the reader that this differs from the standard use of the word "monotonic" in the literature on resolution of singularities. We thank the referee of [11] for bringing this to our attention.)

If α, β are multi-indices, then $\delta_{\beta\alpha}$ denotes the Kronecker delta, equal to 1 if $\alpha = \beta$, and equal to zero otherwise.

Now suppose we are given a point $x_0 \in \mathbb{R}^n$, and an ideal I in $\mathcal{R}_{x_0}$. Then we make the following definitions.

- A subset $\mathcal{A} \subseteq \mathcal{M}$ is called "adapted to I" if $\mathcal{A}$ is monotonic, and, for each r $(0 \leq r \leq m)$, we have

$$\dim(\pi^r_{x_0} I) \;=\; \#\{\alpha \in \mathcal{A} : |\alpha| \leq r\}\,.$$

- If $\mathcal{A} \subseteq \mathcal{M}$, and if $(P_\alpha)_{\alpha \in \mathcal{A}}$ forms a basis for I and satisfies $\partial^\beta P_\alpha(x_0) = \delta_{\beta\alpha}$ for all $\beta, \alpha \in \mathcal{A}$, then we will say that $(P_\alpha)_{\alpha \in \mathcal{A}}$ is an "$\mathcal{A}$-basis" for I.

- If $\mathcal{A} \subseteq \mathcal{M}$, then we say that "$I$ admits an $\mathcal{A}$-basis" if there exists an $\mathcal{A}$-basis for I.

- Let $\eta, A > 0$, suppose $\mathcal{A} \subseteq \mathcal{M}$, and let $(P_\alpha)_{\alpha \in \mathcal{A}}$ be a family of polynomials, indexed by $\mathcal{A}$. Then we say that $(P_\alpha)_{\alpha \in \mathcal{A}}$ is "(η, A)-controlled" if we have

(a) $|\partial^\beta P_\alpha(x_0)| \le A\eta^{|\alpha|-|\beta|}$ for $\alpha \in \mathcal{A}$, $\beta \in \mathcal{M}$; and

(b) $\partial^\beta P_\alpha(x_0) = 0$ for $|\beta| < |\alpha|$, $\alpha \in \mathcal{A}$.

- Let $\eta, A > 0$, and suppose $\mathcal{A} \subseteq \mathcal{M}$. Then we say that I "admits an (η, A)-controlled $\mathcal{A}$-basis" if there exists an $\mathcal{A}$-basis $(P_\alpha)_{\alpha\in\mathcal{A}}$ for I, such that $(P_\alpha)_{\alpha\in\mathcal{A}}$ is (η, A)-controlled.

Note that, whenever $(P_\alpha)_{\alpha\in\mathcal{A}}$ is (η, A)-controlled, it is also (η', A')-controlled for $0 < \eta' \le \eta$, $A' \ge A$.

§8. An $\mathcal{A}$-Basis at a Point

Let $x_0 \in \mathbb{R}^n$, and let I be an ideal in $\mathcal{R}_{x_0}$. In this section, we show that I admits an (η, A)-controlled $\mathcal{A}$-basis, for suitable $\eta, A, \mathcal{A}$. We begin with the elementary properties of an $\mathcal{A}$-basis.

Proposition 8.1: *There exists at most one $\mathcal{A}$-basis for I.*

Proof: Suppose $(P_\alpha)_{\alpha\in\mathcal{A}}$, $(\tilde{P}_\alpha)_{\alpha\in\mathcal{A}}$ are two $\mathcal{A}$-bases for I.

Then we have

$\tilde{P}_{\alpha'} = \sum_{\alpha\in\mathcal{A}} M_{\alpha'\alpha} P_\alpha$ (all $\alpha' \in \mathcal{A}$), for some matrix $(M_{\alpha'\alpha})$.

Hence, for any $\beta, \alpha' \in \mathcal{A}$, we have

$$\delta_{\beta\alpha'} \;=\; \partial^\beta \tilde{P}_{\alpha'}(x_0) \;=\; \sum_{\alpha\in\mathcal{A}} M_{\alpha'\alpha}\, \partial^\beta P_\alpha(x_0) \;=\; \sum_{\alpha\in\mathcal{A}} M_{\alpha'\alpha}\, \delta_{\beta\alpha} \;=\; M_{\alpha'\beta}\,,$$

and therefore $\tilde{P}_{\alpha'} = P_{\alpha'}$ for all $\alpha' \in \mathcal{A}$. ■

In view of the above proposition, we may speak of "the $\mathcal{A}$-basis for I" whenever I admits an $\mathcal{A}$-basis.

Proposition 8.2: *Suppose $\mathcal{A} \subseteq \mathcal{M}$ is adapted to I, and suppose I admits an $\mathcal{A}$-basis. Then the $\mathcal{A}$-basis $(P_\alpha)_{\alpha\in\mathcal{A}}$ for I satisfies*

$\partial^\beta P_\alpha(x_0) = 0$ *for* $|\beta| < |\alpha|$, $\alpha \in \mathcal{A}$.

Proof: Fix $\bar{\beta}, \bar{\alpha}$, with $|\bar{\beta}| < |\bar{\alpha}|$ and $\bar{\alpha} \in \mathcal{A}$. Set $r = \bar{\beta}$; thus $r < |\bar{\alpha}|$.

Also, set $\mathcal{B} = \{\alpha \in \mathcal{A} : |\alpha| \leq r\}$. For $\beta \in \mathcal{B}$, we have $\delta_{\beta\bar{\alpha}} = 0$.

We know that the $\pi^r_{x_0} P_\alpha$ $(\alpha \in \mathcal{B})$ belong to $\pi^r_{x_0} I$. We know also that, for $\beta, \alpha \in \mathcal{B}$, we have $\partial^\beta [\pi^r_{x_0} P_\alpha](x_0) = \partial^\beta P_\alpha(x_0) = \delta_{\beta\alpha}$. Hence, the $\pi^r_{x_0} P_\alpha(\alpha \in \mathcal{B})$ are linearly independent in $\pi^r_{x_0} I$. On the other hand, since $\mathcal{A}$ is adapted to I, the dimension of $\pi^r_{x_0} I$ is equal to the number of elements of $\mathcal{B}$. Hence, the $\pi^r_{x_0} P_\alpha(\alpha \in \mathcal{B})$ form a basis for $\pi^r_{x_0} I$. In particular, for some coefficients $A_\alpha(\alpha \in \mathcal{B})$, we have

$\pi^r_{x_0} P_{\bar{\alpha}} = \sum_{\alpha \in \mathcal{B}} A_\alpha \pi^r_{x_0} P_\alpha$. Consequently, for any $\beta \in \mathcal{B}$, we have

$$\begin{aligned} 0 = \delta_{\beta\bar{\alpha}} &= \partial^\beta P_{\bar{\alpha}}(x_0) = \partial^\beta [\pi^r_{x_0} P_{\bar{\alpha}}](x_0) = \sum_{\alpha \in \mathcal{B}} A_\alpha \, \partial^\beta [\pi^r_{x_0} P_\alpha](x_0) \\ &= \sum_{\alpha \in \mathcal{B}} A_\alpha \, \partial^\beta P_\alpha(x_0) = \sum_{\alpha \in \mathcal{B}} A_\alpha \delta_{\beta\alpha} = A_\beta \,. \end{aligned}$$

Thus, the coefficients A_β all vanish, and therefore $\pi^r_{x_0} P_{\bar{\alpha}} = 0$.

Since $|\bar{\beta}| = r$, it follows that $\partial^{\bar{\beta}} P_{\bar{\alpha}}(x_0) = 0$.

The proof of Proposition 8.2 is complete. ■

We begin the work of constructing an (η, A)-controlled $\mathcal{A}$ basis.

Recall that c, C, C', etc. denote constants depending only on m and n. We call such constants "controlled".

Lemma 8.1: *There exist a monotonic set $\mathcal{A} \subseteq \mathcal{M}$, and a basis $(P_\alpha)_{\alpha \in \mathcal{A}}$ for I, with the following properties.*

(1) $\partial^\beta P_\alpha(x_0) = 0$ *for* $|\beta| < |\alpha|, \alpha \in \mathcal{A}$.

(2) $|\partial^\beta P_\alpha(x_0)| \leq C$ *for* $|\beta| = |\alpha|, \alpha \in \mathcal{A}$.

(3) $\partial^\beta P_\beta(x_0) = 1$ *for* $\beta \in \mathcal{A}$.

(4) *For each $r (0 \leq r \leq m)$, we can order the set $\mathcal{A}(r) = \{\alpha \in \mathcal{A} : |\alpha| = r\}$ so that the matrix $(\partial^\beta P_\alpha(x_0))_{\beta, \alpha \in \mathcal{A}(r)}$ is triangular.*

(If $\mathcal{A}(r)$ is empty, then (4) *holds vacuously.)*

Proof: Without loss of generality, we may suppose $x_0 = 0$. For $0 \le r \le m$, set

$\mathcal{M}_r = \{\alpha \in \mathcal{M} : |\alpha| = r\}$.

For each $r (0 \le r \le m)$ and $\mathcal{B} \subseteq \mathcal{M}_r$, we say that $\mathcal{B} \in \Omega(r)$ if and only if there exists $P \in I$ such that:

(5) $\partial^\beta P(0) = 0$ for $|\beta| < r$;

(6) $\partial^\beta P(0) = 0$ for all $\beta \in \mathcal{B}$; and

(7) $\partial^\beta P(0) \neq 0$ for some $\beta \in \mathcal{M}_r$.

For each $r (0 \le r \le m)$, and for each $\mathcal{B} \in \Omega(r)$, fix a polynomial $P_{r,\mathcal{B}} \in I$ satisfying (5), (6), (7); and let $\hat{P}_{r,\mathcal{B}}$ be the part of $P_{r,\mathcal{B}}$ that is homogeneous of degree r.

(That is, if $P_{r,\mathcal{B}}(x) = \displaystyle\sum_{\alpha \in \mathcal{M}} A_\alpha x^\alpha$, then $\hat{P}_{r,\mathcal{B}}(x) = \displaystyle\sum_{\alpha \in \mathcal{M}_r} A_\alpha x^\alpha$.)

Since $P_{r,\mathcal{B}}$ satisfies (7), the polynomials $\hat{P}_{r,\mathcal{B}}$ $(0 \le r \le m,\ \mathcal{B} \in \Omega(r))$ are all nonzero.

Let $a \in (0,1)$ be a small constant, to be picked later.

We write $c(a), C(a), C'(a)$, etc. to denote constants determined by a, m, n.

We apply Lemma 6.1 to the polynomials $\hat{P}_{r,\mathcal{B}}$ $(0 \le r \le m, \mathcal{B} \in \Omega(r))$.

Thus, for some linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ of the form

(8) $T : (x_1, \ldots, x_n) \mapsto (\lambda_1 x_1, \ldots, \lambda_n x_n)$,

the following hold.

(9) $c(a) \le \lambda_i \le 1$ for $i = 1, \ldots, n$.

(10) For $0 \le r \le m$ and $\mathcal{B} \in \Omega(r)$, there exists a multi-index $\beta(r, \mathcal{B})$ such that $|\partial^\beta (\hat{P}_{r,\mathcal{B}} \circ T)(0)| \le a |\partial^{\beta(r,\mathcal{B})} (\hat{P}_{r,\mathcal{B}} \circ T)(0)|$ whenever $\beta \neq \beta(r, \mathcal{B})$.

Fix $\beta(r,\mathcal{B})$ as in (10). Since $\hat{P}_{r,\mathcal{B}}$ is the part of $P_{r,\mathcal{B}}$ that is homogeneous of degree r, it follows from (10) that

(11) $|\beta(r,\mathcal{B})| = r$ for $0 \le r \le m$, $\mathcal{B} \in \Omega(r)$;

and

(12) $|\partial^\beta (P_{r,\mathcal{B}} \circ T)(0)| \le a |\partial^{\beta(r,\mathcal{B})} (P_{r,\mathcal{B}} \circ T)(0)|$ for all $\beta \in \mathcal{M}_r \setminus \{\beta(r,\mathcal{B})\}$, $0 \le r \le m$, $\mathcal{B} \in \Omega(r)$.

Also, by definition of $T, \Omega(r)$, $P_{r,\mathcal{B}}$, we have

(13) $\partial^\beta (P_{r,\mathcal{B}} \circ T)(0) = 0$ for $|\beta| < r$, $0 \le r \le m$, $\mathcal{B} \in \Omega(r)$;

(14) $\partial^\beta (P_{r,\mathcal{B}} \circ T)(0) = 0$ for $\beta \in \mathcal{B}$, $\mathcal{B} \in \Omega(r)$, $0 \le r \le m$;

and

(15) $\partial^{\beta(r,\mathcal{B})} (P_{r,\mathcal{B}} \circ T)(0) \neq 0$ for $\mathcal{B} \in \Omega(r)$, $0 \le r \le m$.

For each $r (0 \le r \le m)$, we define a (possibly empty) finite sequence of multi-indices $\gamma_1^r, \gamma_2^r, \dots, \gamma_{L(r)}^r \in \mathcal{M}_r$, and a (possibly empty) finite sequence of polynomials, $Q_1^r, \dots, Q_{L(r)}^r$, by the following induction.

Fix $r (0 \le r \le m)$. For a given $\ell \ge 1$, suppose we have already defined the $\gamma_{\ell'}^r$ and $Q_{\ell'}^r$ for all ℓ' with $1 \le \ell' < \ell$. (For $\ell = 1$, this holds vacuously.) Set

(16) $\mathcal{B}_\ell^r = \{\gamma_1^r, \dots, \gamma_{\ell-1}^r\}$. (Thus, $\mathcal{B}_\ell^r$ is empty if $\ell = 1$.)

If $\mathcal{B}_\ell^r \notin \Omega(r)$, then we set $L(r) = \ell - 1$, and we stop defining additional γ_i^r and Q_i^r.

If instead $\mathcal{B}_\ell^r \in \Omega(r)$, then we set

(17) $Q_\ell^r = P_{r,\mathcal{B}_\ell^r} \circ T$,

and

(18) $\gamma_\ell^r = \beta(r, \mathcal{B}_\ell^r)$.

This completes our induction on ℓ, and produces possibly empty, possibly infinite sequences $\gamma_1^r, \gamma_2^r, \dots$ and $Q_1^r, Q_2^r, \dots$ of multi-indices and polynomials, respectively. We will see that these sequences terminate. Note that $|\gamma_\ell^r| = r$, by (11) and (18). Set

$I \circ T = \{P \circ T : P \in I\}$.

Then, since all $P_{r,\mathcal{B}}$ belong to I and satisfy (12), ..., (15), and since γ_ℓ^r, Q_ℓ^r are defined by (16), (17), (18), we have the following results.

(19) $Q_\ell^r \in I \circ T$ for $0 \le r \le m$, $1 \le \ell \le L(r)$.

(20) $\partial^\beta Q_\ell^r(0) = 0$ for $|\beta| < r$, $0 \le r \le m$, $1 \le \ell \le L(r)$.

(21) $\partial^\beta Q_\ell^r(0) = 0$ for $\beta = \gamma_{\ell'}^r$, $1 \le \ell' < \ell \le L(r)$, $0 \le r \le m$.

(22) $\partial^\beta Q_\ell^r(0) \ne 0$ for $\beta = \gamma_\ell^r$, $1 \le \ell \le L(r)$, $0 \le r \le m$.

(23) $|\partial^\beta Q_\ell^r(0)| \le a|\partial^{\gamma_\ell^r} Q_\ell^r(0)|$ for $\beta \in \mathcal{M}_r \smallsetminus \{\gamma_\ell^r\}$, $0 \le r \le m$, $1 \le \ell \le L(r)$.

Here, we define $L(r) = 0$ if our sequences $\gamma_1^r, \gamma_2^r, \dots$ and $Q_1^r, Q_2^r, \dots$ are empty; and we define $L(r) = \infty$ if those sequences never terminate.

Comparing (21) with (22), we see that, for fixed r, the γ_ℓ^r are all distinct. Since also $|\gamma_\ell^r| = r$ for each ℓ, the sequence $\gamma_1^r, \gamma_2^r, \dots$ must terminate. Thus,

(24) $L(r) < \infty$ for $0 \le r \le m$,

as promised. This tells us that

$$\mathcal{B}_{L(r)+1}^r = \{\gamma_1^r, \dots, \gamma_{L(r)}^r\} \notin \Omega(r)\,.$$

By definition of $\Omega(r)$, this in turn tells us the following.

(25) Let $0 \le r \le m$ and $P \in I$ be given. If $\partial^\beta P(0) = 0$ for $|\beta| < r$, and for $\beta = \gamma_1^r, \gamma_2^r, \dots, \gamma_{L(r)}^r$, then $\partial^\beta P(0) = 0$ for $|\beta| \le r$.

Since $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map given by a diagonal matrix, (25) is equivalent to the following result.

(26) Let $0 \le r \le m$ and $P \in I \circ T$ be given. Suppose $\partial^\beta P(0) = 0$ for $|\beta| < r$, and for $\beta = \gamma_\ell^r$ ($\ell = 1, \ldots, L(r)$). Then $\partial^\beta P(0) = 0$ for $|\beta| \le r$.

Next, suppose we are given $r (0 \le r \le m)$ and $P \in I \circ T$, with

(27) $\partial^\beta P(0) = 0$ for $|\beta| < r$.

Then, since the matrix $(\partial^{\gamma_{\ell'}^r} Q_\ell^r(0))_{1 \le \ell, \ell' \le L(r)}$

is invertible (thanks to (21), (22)), there exist coefficients $A_\ell (1 \le \ell \le L(r))$ such that

(28) $\tilde{P} = P - \sum_{1 \le \ell \le L(r)} A_\ell Q_\ell^r$

satisfies

(29) $\partial^{\gamma_\ell^r} \tilde{P}(0) = 0$ for $1 \le \ell \le L(r)$.

From (19), (20), (27), (28), we have also

(30) $\partial^\beta \tilde{P}(0) = 0$ for $|\beta| < r$,

and

(31) $\tilde{P} \in I \circ T$.

From (26) and (29), (30), (31), we find that $\partial^\beta \tilde{P}(0) = 0$ for $|\beta| \le r$.

Thus, we have established the following.

(32) Let $P \in (I \circ T) \cap \ker \pi_0^{r-1}$. (For $r = 0$, this means simply that $P \in I \circ T$.)

Then there exist coefficients $A_\ell (1 \le \ell \le L(r))$, such that $P - \sum_{1 \le \ell \le L(r)} A_\ell \, Q_\ell^r \in (I \circ T) \cap \ker \pi_0^r$.

Here, π_0^r denotes $\pi_{x_0}^r$ with $x_0 = 0$. Since $\pi_{x_0}^m$ is the identity map on $\mathcal{R}_{x_0}$, an obvious induction on r using (32) shows that

(33) $I \circ T$ is contained in the linear span of the $Q_\ell^r (1 \le \ell \le L(r), 0 \le r \le m)$.

Now we define

(34) $\mathcal{A} = \{\gamma_\ell^r : 0 \le r \le m, 1 \le \ell \le L(r)\}$,

and for $\alpha \in \mathcal{A}$ we define P_α, by setting

(35) $P_{\gamma_\ell^r} = \frac{Q_\ell^r \circ T^{-1}}{\partial^{\gamma_\ell^r}(Q_\ell^r \circ T^{-1})(0)}$ for $0 \le r \le m$, $1 \le \ell \le L(r)$.

Note that the denominator in (35) is non-zero, thanks to (22) and the diagonal form of the linear map T.

Note also that the set $\mathcal{A}(r) = \{\alpha \in \mathcal{A} : |\alpha| = r\}$ from (4) is given by

(36) $\mathcal{A}(r) = \{\gamma_\ell^r : 1 \le \ell \le L(r)\}$ for $0 \le r \le m$,

since $|\gamma_\ell^r| = r$ for $0 \le r \le m$, $1 \le \ell \le L(r)$.

We prepare to show that $\mathcal{A}$ is monotonic, provided we take the constant a to be small enough. To see this, we introduce the vector space of polynomials

(37) $V_r = \pi_0^r[\ker \pi_0^{r-1} \cap (I \circ T)]$ for $0 \le r \le m$.

(If $r = 0$, this means simply $V_0 = \pi_0^0[I \circ T]$.)

We set

(38) $\tilde{Q}_\ell^r = \frac{\pi_0^r Q_\ell^r}{\partial^{\gamma_\ell^r} Q_\ell^r(0)}$ for $0 \le r \le m$, $1 \le \ell \le L(r)$.

The denominator in (38) is non-zero, by (22).

In view of (19), (20), (37), we have

(39) $\tilde{Q}_\ell^r \in V_r$ for $0 \le r \le m$, $1 \le \ell \le L$.

From (22), (23), (38) (and the fact that $|\gamma_\ell^r| = r$), we have

(40) $|\partial^\beta \tilde{Q}^r_\ell(0)| \leq a$ for $0 \leq r \leq m$, $1 \leq \ell \leq L$, $\beta \neq \gamma^r_\ell$.

(Note that $\partial^\beta \tilde{Q}^r_\ell(0) = 0$ for $|\beta| \neq r$, by (37) and (39).)

Also, since $|\gamma^r_\ell| = r$, we have $\partial^{\gamma^r_\ell}[\pi^r_0 Q^r_\ell](0) = \partial^{\gamma^r_\ell} Q^r_\ell(0)$, and therefore (38) yields

(41) $\partial^{\gamma^r_\ell} \tilde{Q}^r_\ell(0) = 1$ for $0 \leq r \leq m$, $1 \leq \ell \leq L$.

Next, we check that

(42) $V_r = \mathsf{span}\{\tilde{Q}^r_\ell : 1 \leq \ell \leq L(r)\}$ for $0 \leq r \leq m$.

In fact, an obvious induction using (32) shows that any polynomial $P \in \ker \pi^{r-1}_0 \cap (I \circ T)$ may be written as a linear combination of the $Q^{r'}_\ell$ for $1 \leq \ell \leq L(r')$, $r' \geq r$. We have also $\pi^r_0 Q^{r'}_\ell = 0$ for $r' > r$, by (20); and $\pi^r_0 Q^r_\ell$ is a constant multiple of $\tilde{Q}^r_\ell$, by (38). Consequently, $\pi^r_0 P \in \mathsf{span}\{\tilde{Q}^r_\ell : 1 \leq \ell \leq L(r)\}$ for every $P \in \ker \pi^{r-1}_0 \cap (I \circ T)$. Together with (37) and (39), this completes the proof of (42).

Now, from (36), (40), (41) , (42), we see that

(43) $\max\limits_{\beta \notin \mathcal{A}(r)} |\partial^\beta P(0)| \leq Ca \cdot \max\limits_{\beta \in \mathcal{A}(r)} |\partial^\beta P(0)|$ for all $P \in V_r$, $0 \leq r \leq m$, provided

(44) $0 < a < c$

for a small enough controlled constant c.

We recall here that c, C, C', etc. denote "controlled constants", i.e., constants depending only on m and n.

We are now ready to show that

(45) $\mathcal{A}$ is monotonic.

To see this, suppose $0 \leq r < s \leq m$, and $1 \leq \ell \leq L(r)$; and let γ be a multi-index with

(48) $|\gamma| = s - r$.

We must show that

(49) $\gamma_\ell^r + \gamma \in \mathcal{A}(s)$.

This will establish (45), in view of (34), (36).

Let $P(x) = x^\gamma \odot Q_\ell^r(x)$, the symbol $\odot$ denoting multiplication in $\mathcal{R}_0 (= \mathcal{R}_{x_0}$ with $x_0 = 0)$.

Since $I \circ T$ is an ideal , (19) shows that $P \in I \circ T$.

(This is the only place in the proof of Lemma 8.1 where we use the hypothesis that I is an ideal.)

Also, (20) and (48) show that $P \in \ker \pi_0^{s-1}$. Hence, by (37), we have

(50) $\pi_0^s P \in V_s$.

From (22), (23), (48), and the definition of P, we see that

$$|\partial^\beta P(0)| \leq Ca|\partial^{\gamma_\ell^r + \gamma} P(0)| \neq 0 \text{ for } \beta \in \mathcal{M}_s \setminus \{\gamma_\ell^r + \gamma\}\,,$$

and therefore

(51) $|\partial^\beta (\pi_0^s P)(0)| \leq Ca|\partial^{\gamma_\ell^r + \gamma} (\pi_0^s P)(0)| \neq 0$ for $\beta \in \mathcal{M}_s \setminus \{\gamma_\ell^r + \gamma\}$.

Also, from (50) and the definition of V_s, we see that $\pi_0^s P$ is homogeneous of degree s.

Consequently, (51) implies

(52) $|\partial^\beta (\pi_0^s P)(0)| \leq Ca|\partial^{\gamma_\ell^r + \gamma} (\pi_0^s P)(0)| \neq 0$ for $\beta \in \mathcal{M} \setminus \{\gamma_\ell^r + \gamma\}$.

In particular, $\pi_0^s P \neq 0$.

Now suppose we take our constant a to be a small enough controlled constant. Then (44) holds, and therefore (43) and (50) show that

(53) $\max_{\beta \notin \mathcal{A}(s)} |\partial^\beta(\pi_0^s P)(0)| \leq \max_{\beta \in \mathcal{A}(s)} |\partial^\beta(\pi_0^s P)(0)|$,

while (52) shows that

(54) $\max_{\beta \in \mathcal{M} \setminus \{\gamma_\ell^r + \gamma\}} |\partial^\beta(\pi_0^s P)(0)| < |\partial^{\gamma_\ell^r + \gamma}(\pi_0^s P)(0)|$.

If $\gamma_\ell^r + \gamma \notin \mathcal{A}(s)$, then (53) and (54) would show that

$$|\partial^{\gamma_\ell^r+\gamma}(\pi_0^s P)(0)| \leq \max_{\beta \in \mathcal{A}(s)} |\partial^\beta(\pi_0^s P)(0)| \leq \max_{\beta \in \mathcal{M} \setminus \{\gamma_\ell^r + \gamma\}} |\partial^\beta(\pi_0^s P)(0)| < |\partial^{\gamma_\ell^r+\gamma}(\pi_0^s P)(0)|\,,$$

which is absurd.

This completes the proof of (49), hence also that of (45).

From now on, we fix a to be a controlled constant, picked small enough to make the above arguments work. In particular, since a is a controlled constant, (9) yields

(55) $c \leq \lambda_i \leq 1$, for $i = 1, \ldots, n$.

From (8) and (55), we obtain

$$c|\partial^\beta Q_\ell^r(0)| \;\leq\; |\partial^\beta(Q_\ell^r \circ T^{-1})(0)| \;\leq\; C|\partial^\beta Q_\ell^r(0)| \text{ for } \beta \in \mathcal{M},\ 0 \leq r \leq m,\ 1 \leq \ell \leq L(r)\,.$$

Together with (22), (23), (35), this shows that

$$|\partial^\beta P_{\gamma_\ell^r}(0)| \leq C \text{ for } |\beta| = r\,, 0 \leq r \leq m\,, 1 \leq \ell \leq L(r)\,.$$

Since $|\gamma_\ell^r| = r$, it therefore follows from (34) that

(56) $|\partial^\beta P_\alpha(0)| \leq C$ for $|\beta| = |\alpha|$, $\alpha \in \mathcal{A}$.

It is now easy to check the conclusions of Lemma 8.1 for $\mathcal{A}$, $(P_\alpha)_{\alpha \in \mathcal{A}}$ as in (34) , (35).

In fact, we have already checked that $\mathcal{A}$ is monotonic (see (45)). Conclusion (1) follows easily from (8), (20), (34), (35) and the fact that $|\gamma_\ell^r| = r$.

Conclusion(2) is simply our result (56).

Conclusion(3) is immediate from (34), (35).

Conclusion(4) follows easily from (8), (21), (35), and (36).

Thus, it remains only to check that the $P_\alpha(\alpha \in \mathcal{A})$ form a basis for I. From (19), (33), we see that $I \circ T = \mathsf{span}\,\{Q_\ell^r : 0 \le r \le m,\ 1 \le \ell \le L(r)\}$. Hence (34), (35) show that $I = \mathsf{span}\,\{P_\alpha : \alpha \in \mathcal{A}\}$. Moreover, by (1), (3), (4) (which we already know), we may order $\mathcal{A}$ in such a way that the matrix $(\partial^\beta P_\alpha(0))_{\beta,\alpha\in\mathcal{A}}$ is triangular, with 1's on the main diagonal. Hence the $P_\alpha(\alpha \in \mathcal{A})$ are linearly independent.

Since we have now shown that the $(P_\alpha)_{\alpha\in\mathcal{A}}$ form a basis for I, the proof of Lemma 8.1 is complete. ■

Proposition 8.3: *Let $\mathcal{A}$, $(P_\alpha)_{\alpha\in\mathcal{A}}$ be as in Lemma 8.1.*

Then $\mathcal{A}$ is adapted to I.

Proof: Already from Lemma 8.1, we know that $\mathcal{A}$ is monotonic.

We must show that $\dim(\pi_{x_0}^r I) = \#\{\alpha \in \mathcal{A} : |\alpha| \le r\}$ for $0 \le r \le m$.

Fix r, and note that

$\pi_{x_0}^r P_\alpha = 0$ for $\alpha \in \mathcal{A}$, $|\alpha| > r$, by conclusion (1) of Lemma 8.1.

On the other hand, conclusions (1), (3), (4) of Lemma 8.1 show that we may order $\mathcal{B} = \{\alpha \in \mathcal{A} : |\alpha| \le r\}$ in such a way that the matrix

$$(\partial^\beta[\pi_{x_0}^r P_\alpha](x_0))_{\beta,\alpha\in\mathcal{B}} \;=\; (\partial^\beta P_\alpha(x_0))_{\beta,\alpha\in\mathcal{B}}$$

is triangular, with 1's on the main diagonal. Consequently, the polynomials $\pi_{x_0}^r P_\alpha(\alpha \in \mathcal{B})$ are linearly independent.

Recalling from Lemma 8.1 that the $P_\alpha(\alpha \in \mathcal{A})$ form a basis for I, we conclude that

$$\dim(\pi_{x_0}^r I) \;=\; \dim\,\mathsf{span}\,\{\pi_{x_0}^r P_\alpha : \alpha \in \mathcal{A}\} = \dim\,\mathsf{span}\,\{\pi_{x_0}^r P_\alpha : \alpha \in \mathcal{A}, |\alpha| \le r\}$$

$$= \#\{\alpha \in \mathcal{A} : |\alpha| \le r\}\,.$$

The proof of Proposition 8.3 is complete. ■

The main result of this section is as follows.

Lemma 8.2: *There exist a controlled constant C, a positive number η, and a subset $\mathcal{A} \subseteq \mathcal{M}$, such that $\mathcal{A}$ is adapted to I and I admits an (η, C)-controlled $\mathcal{A}$-basis.*

Proof: Let $\mathcal{A}$, $(P_\alpha)_{\alpha\in\mathcal{A}}$ be as in Lemma 8.1. By Proposition 8.3, $\mathcal{A}$ is adapted to I. Moreover, by conclusions (1) and (2) of Lemma 8.1, there exists a positive real number η, such that

(57) $\eta^{|\beta|-|\alpha|}\,|\partial^\beta P_\alpha(x_0)| \,\le\, C$ for $\alpha \in \mathcal{A}$, $|\beta| \le m$.

We fix $\eta > 0$ as in (57).

Next, as we noted before, conclusions (1), (3), (4) of Lemma 8.1 show that we can order $\mathcal{A}$ in such a way that the matrix $(\partial^\beta P_\alpha(x_0))_{\beta,\alpha\in\mathcal{A}}$ is triangular, with 1's on the main diagonal.

Hence, the same is true of the matrix

$$\tilde{M} \,=\, (\eta^{|\beta|-|\alpha|}\,\partial^\beta P_\alpha(x_0))_{\beta,\alpha\in\mathcal{A}}\,.$$

Moreover (57) shows that the entries of $\tilde{M}$ are bounded in absolute value by a controlled constant.

It follows that $\tilde{M}$ is invertible, and that its inverse matrix

$$M \,=\, (M_{\alpha'\alpha})_{\alpha',\alpha\in\mathcal{A}}$$

satisfies

(58) $|M_{\alpha'\alpha}| \,\le\, C$ for $\alpha', \alpha \in \mathcal{A}$.

By definition, we have

(59) $\delta_{\beta\alpha} \,=\, \sum_{\alpha'\in\mathcal{A}} \eta^{|\beta|-|\alpha'|}\,\partial^\beta P_{\alpha'}(x_0)\,\cdot\, M_{\alpha'\alpha}$ for $\beta, \alpha \in \mathcal{A}$.

We now set

(60) $\tilde{P}_\alpha \; = \; \eta^{|\alpha|} \sum_{\alpha' \in \mathcal{A}} \eta^{-|\alpha'|} \, P_{\alpha'} M_{\alpha'\alpha}$ for $\alpha \in \mathcal{A}$.

Since $(M_{\alpha'\alpha})_{\alpha',\alpha\in\mathcal{A}}$ is invertible, so is $(\eta^{|\alpha|-|\alpha'|} \, M_{\alpha'\alpha})_{\alpha',\alpha\in\mathcal{A}}$.

Since the $P_{\alpha'}(\alpha' \in \mathcal{A})$ form a basis for I (by Lemma 8.1), it therefore follows from (60) that the $\tilde{P}_\alpha(\alpha \in \mathcal{A})$ also form a basis for I. Moreover, (59) and (60) show that, for $\beta, \alpha \in \mathcal{A}$, we have

$$\partial^\beta \tilde{P}_\alpha(x_0) \; = \; \eta^{|\alpha|-|\beta|} \cdot \sum_{\alpha' \in \mathcal{A}} \eta^{|\beta|-|\alpha'|} \, \partial^\beta P_{\alpha'}(x_0) \, \cdot \, M_{\alpha'\alpha} \; = \; \eta^{|\alpha|-|\beta|} \cdot \delta_{\beta\alpha} \; = \; \delta_{\beta\alpha} \, .$$

Thus, $(\tilde{P}_\alpha)_{\alpha\in\mathcal{A}}$ is an $\mathcal{A}$-basis for I.

We show that the $\mathcal{A}$-basis $(\tilde{P}_\alpha)_{\alpha\in\mathcal{A}}$ is (η, C)-controlled.

In fact, since $(\tilde{P}_\alpha)_{\alpha\in\mathcal{A}}$ is an $\mathcal{A}$-basis for I, and since $\mathcal{A}$ is adapted to I, Proposition 8.2 shows that

(61) $\partial^\beta \tilde{P}_\alpha(x_0) = 0$ for $|\beta| < |\alpha|$, $\alpha \in \mathcal{A}$.

Moreover, (57), (58), (60) show that, for $\alpha \in \mathcal{A}$, $|\beta| \leq m$, we have

$$\eta^{|\beta|-|\alpha|} | \, \partial^\beta \tilde{P}_\alpha(x_0)| \; \leq \; \sum_{\alpha' \in \mathcal{A}} \eta^{|\beta|-|\alpha'|} | \, \partial^\beta P_{\alpha'}(x_0)| \, \cdot \, |M_{\alpha'\alpha}| \; \leq \; C \, .$$

Thus,

(62) $|\partial^\beta \tilde{P}_\alpha(x_0)| \; \leq \; C\eta^{|\alpha|-|\beta|}$ for $\alpha \in \mathcal{A}$, $|\beta| \leq m$.

Our results (61), (62) tell us that $(\tilde{P}_\alpha)_{\alpha\in\mathcal{A}}$ is (η, C)-controlled.

Thus, we know that $\mathcal{A}$ is adapted to I, that $(\tilde{P})_{\alpha\in\mathcal{A}}$ is an $\mathcal{A}$-basis, and that $(\tilde{P}_\alpha)_{\alpha\in\mathcal{A}}$ is (η, C)-controlled.

The proof of Lemma 8.2 is complete. ∎

§9. An $\mathcal{A}$-basis on a Small Ball

In this section, we suppose we are given a Glaeser stable family of ideals, $(I(x))_{x\in E}$. We write E_0 for the first slice of E.

We say that an open ball $B(y_0, \eta) \subset \mathbb{R}^n$ with radius $\eta < 1$ is an "excellent ball" if there exists $\mathcal{A} \subseteq \mathcal{M}$ for which the following hold:

(I) For each $x \in E_0 \cap B(y_0, \eta)$, $\mathcal{A}$ is adapted to $I(x)$, and $I(x)$ admits an (η, C_1)-controlled $\mathcal{A}$-basis.

(II) Let $x, x' \in E_0 \cap B(y_0, \eta)$, and let $(P_\alpha)_{\alpha\in\mathcal{A}}$ be the $\mathcal{A}$-basis for $I(x)$. Then, for each $\alpha \in \mathcal{A}$, there exists $P'_\alpha \in I(x')$, with

$$|\partial^\beta(P'_\alpha - P_\alpha)(x)| \le |x - x'|^{m-|\beta|} \text{ for } |\beta| \le m\,.$$

(III) Given $\epsilon > 0$, there exists $\delta > 0$ for which the following holds:

Let $x, x' \in E_0 \cap B(y_0, \eta)$, with $|x - x'| \le \delta$. Let $(P_\alpha)_{\alpha\in\mathcal{A}}$ be the $\mathcal{A}$-basis for $I(x)$. Then, for each $\alpha \in \mathcal{A}$, there exists $P'_\alpha \in I(x')$, with

$$|\partial^\beta(P'_\alpha - P_\alpha)(x)| \le \epsilon|x - x'|^{m-|\beta|} \text{ for } |\beta| \le m\,.$$

Here, C_1 is a large enough controlled constant, to be picked later.

We recall that, in view of (I) and Proposition 8.1, there exists a unique $\mathcal{A}$-basis for $I(x)$, at each $x \in E_0 \cap B(y_0, \eta)$.

Note that any open ball of radius < 1 that does not meet E_0 satisfies (I), (II), (III) vacuously, and is therefore excellent.

The goal of this section is to prove that every sufficiently small open ball is excellent.

Lemma 9.1: *Let $x_0 \in E_0$ be given. Then there exists $\rho > 0$ such that any open ball contained in $B(x_0, \rho)$ is excellent.*

Proof: We recall from the start of Section 2 a small remark about notation. In our proof of Lemma 9.1, we will introduce polynomials P^x_α, $P^{x,x'}_\alpha$ depending on parameters $x, x' \in \mathbb{R}^n$.

When we write $\partial^\beta P^x_\alpha(x)$ or $\partial^\beta P^{x,x'}_\alpha(x)$, we mean $\left(\frac{\partial}{\partial y}\right)^\alpha P^x_\alpha(y)$ or $\left(\frac{\partial}{\partial y}\right)^\alpha P^{x,x'}(y)$ evaluated at $y = x$, rather than the derivative of order α of $x \mapsto P^x_\alpha(x)$ or $x \mapsto P^{x,x'}_\alpha(x)$.

Let us apply Lemma 8.2 to the ideal $I(x_0)$ and the point $x_0 \in \mathbb{R}^n$. We obtain a set $\mathcal{A} \subseteq \mathcal{M}$ adapted to $I(x_0)$, a positive number η_0, and an (η_0, C_0)-controlled $\mathcal{A}$-basis $(P^0_\alpha)_{\alpha\in\mathcal{A}}$ for $I(x_0)$.

By definition of a slice, we know that the function $x \mapsto \dim \pi^r_x I(x)$ is constant on E_0, for each $r(0 \le r \le m)$. Therefore, since $\mathcal{A}$ is adapted to $I(x_0)$ with $x_0 \in E_0$, it follows that

(1) $\mathcal{A}$ is adapted to $I(x)$ for all $x \in E_0$.

Also, since $(P^0_\alpha)_{\alpha\in\mathcal{A}}$ is an (η_0, C_0)-controlled $\mathcal{A}$-basis for $I(x_0)$, we have

(2) $P^0_\alpha \in I(x_0)$ for $\alpha \in \mathcal{A}$;

(3) $\partial^\beta P^0_\alpha(x_0) = \delta_{\beta\alpha}$ for $\beta, \alpha \in \mathcal{A}$; and

(4) $|\partial^\beta P^0_\alpha(x_0)| \le C_0 \eta_0^{|\alpha|-|\beta|}$ for $\alpha \in \mathcal{A}, |\beta| \le m$.

Next, since $(I(x))_{x\in E}$ is Glaeser stable, (2) shows that there exist $F_\alpha \in C^m(\mathbb{R}^n)$ $(\alpha \in \mathcal{A})$, with

(5) $J_x(F_\alpha) \in I(x)$ for $\alpha \in \mathcal{A}$, $x \in E$; and

(6) $J_{x_0}(F_\alpha) = P^0_\alpha$ for $\alpha \in \mathcal{A}$.

We fix F_α as above. From (3), (4), (6), we have

(7) $|\partial^\beta F_\alpha(x_0)| \le C_0 \eta_0^{|\alpha|-|\beta|}$ for $\alpha \in \mathcal{A}$, $|\beta| \le m$; and

(8) $\partial^\beta F_\alpha(x_0) = \delta_{\beta\alpha}$ for $\beta, \alpha \in \mathcal{A}$.

The matrix-valued function $x \mapsto (\partial^\beta F_\alpha(x))_{\beta,\alpha\in\mathcal{A}}$ is continuous on $\mathbb{R}^n$, and equal to the identity matrix at $x = x_0$ (see (8)).

Hence, for $\rho_1 > 0$ small enough, $(\partial^\beta F_\alpha(x))_{\beta,\alpha\in\mathcal{A}}$ is invertible for $x \in B(x_0, \rho_1)$, and its inverse matrix $(M_{\alpha'\alpha}(x))_{\alpha',\alpha\in\mathcal{A}}$ satisfies

(9) $x \mapsto (M_{\alpha'\alpha}(x))_{\alpha',\alpha \in \mathcal{A}}$ is continuous on $B(x_0, \rho_1)$ and equal to the identity matrix at $x = x_0$.

By definition of $(M_{\alpha'\alpha})$, we have

(10) $$\sum_{\alpha' \in \mathcal{A}} \partial^\beta F_{\alpha'}(x) \cdot M_{\alpha'\alpha}(x) = \delta_{\beta\alpha} \text{ for } \beta, \alpha \in \mathcal{A} \text{ and } x \in B(x_0, \rho_1).$$

Now define

(11) $$P_\alpha^x = \sum_{\alpha' \in \mathcal{A}} J_x(F_{\alpha'}) \cdot M_{\alpha'\alpha}(x) \text{ for } \alpha \in \mathcal{A},\ x \in E_0 \cap B(x_0, \rho_1); \text{ and}$$

(12) $$P_\alpha^{x,x'} = \sum_{\alpha' \in \mathcal{A}} J_{x'}(F_{\alpha'}) \cdot M_{\alpha'\alpha}(x) \text{ for } \alpha \in \mathcal{A},\ x, x' \in E_0 \cap B(x_0, \rho_1).$$

From (5), (11), (12), we have

(13) $P_\alpha^x \in I(x)$ for $\alpha \in \mathcal{A}$, $x \in E_0 \cap B(x_0, \rho_1)$; and

(14) $P_\alpha^{x,x'} \in I(x')$ for $\alpha \in \mathcal{A}$, $x, x' \in E_0 \cap B(x_0, \rho_1)$.

Also, from (10), (11), we have

(15) $\partial^\beta P_\alpha^x(x) = \delta_{\beta\alpha}$ for $\beta, \alpha \in \mathcal{A}$, $x \in E_0 \cap B(x_0, \rho_1)$.

In particular, the $P_\alpha^x (\alpha \in \mathcal{A})$ are linearly independent for $x \in E_0 \cap B(x_0, \rho_1)$. Moreover, the number of $P_\alpha^x (\alpha \in \mathcal{A})$ is equal to $\dim I(x)$ for $x \in E_0 \cap B(x_0, \rho_1)$, thanks to (1). Together with (13), these remarks imply

(16) $(P_\alpha^x)_{\alpha \in \mathcal{A}}$ is the $\mathcal{A}$-basis for $I(x)$, for each $x \in E_0 \cap B(x_0, \rho_1)$.

Also, since $F_\alpha \in C^m(\mathbb{R}^n)$, we learn from (7), (9), (11) that:

For $\alpha \in \mathcal{A}$, $|\beta| \leq m$, the function $x \mapsto \partial^\beta P_\alpha^x(x)$ is continuous on $E_0 \cap B(x_0, \rho_1)$, and has absolute value at most $C_0 \eta^{|\alpha|-|\beta|}$ at $x = x_0$. Consequently, for a positive number $\rho_2 < \rho_1$, we have

(17) $|\partial^\beta P^x_\alpha(x)| \leq C'\eta_0^{|\alpha|-|\beta|}$ for $|\beta| \leq m$, $\alpha \in \mathcal{A}$, $x \in E_0 \cap B(x_0, \rho_2)$.

Also, from (1), (16), and Proposition 8.2, we have

(18) $\partial^\beta P^x_\alpha(x) = 0$ for $|\beta| < |\alpha|$, $\alpha \in \mathcal{A}$, $x \in E_0 \cap B(x, \rho_2)$.

We now take the constant C_1 in the definition of an "excellent ball" to be equal to C' from (17). Thus, (16), (17), (18) show that

(19) $I(x)$ admits an (η_0, C_1)-controlled $\mathcal{A}$-basis, for each $x \in E_0 \cap B(x_0, \rho_2)$.

Next, let $\epsilon > 0$ be given. Since each F_α belongs to $C^m(\mathbb{R}^n)$ and E_0 is compact, there exists $\delta > 0$ such that, for $x, x' \in E_0$ with $|x - x'| \leq \delta$, we have

(20) $|\partial^\beta(J_x(F_\alpha) - J_{x'}(F_\alpha))(x)| \leq \frac{1}{2}\epsilon|x - x'|^{m-|\beta|}$ for $|\beta| \leq m$, $\alpha \in \mathcal{A}$.

From (9), (11), (12) we obtain a positive number $\rho_3 < \rho_2$, independent of ϵ, such that, for $x, x' \in E_0 \cap B(x_0, \rho_3)$, (20) implies

$$|\partial^\beta(P^x_\alpha - P^{x,x'}_\alpha)(x)| \leq \epsilon|x - x'|^{m-|\beta|} \text{ for } |\beta| \leq m,\ \alpha \in \mathcal{A}\,.$$

Consequently, we obtain:

(21) Given $\epsilon > 0$, there exists $\delta > 0$ such that, for any $x, x' \in E_0 \cap B(x_0, \rho_3)$ with $|x - x'| \leq \delta$, we have

$$|\partial^\beta(P^x_\alpha - P^{x,x'}_\alpha)(x)| \leq \epsilon|x - x'|^{m-|\beta|} \text{ for } |\beta| \leq m,\ \alpha \in \mathcal{A}\,.$$

In particular, (21) gives us a positive number δ_1, such that

(22) For any $x, x' \in E_0 \cap B(x_0, \rho_3)$ with $|x - x'| \leq \delta_1$, we have

$$|\partial^\beta(P^x_\alpha - P^{x,x'}_\alpha)(x)| \leq |x - x'|^{m-|\beta|} \text{ for } |\beta| \leq m,\ \alpha \in \mathcal{A}.$$

If we take ρ to be a positive number less than the minimum of $\rho_3, \frac{1}{2}\,\delta_1,\ 1, \eta_0$, then (22) yields

(23) For any $x, x' \in E_0 \cap B(x_0, \rho)$, and for $|\beta| \le m$, $\alpha \in \mathcal{A}$, we have

$$|\partial^\beta (P_\alpha^x - P_\alpha^{x,x'})(x)| \le |x - x'|^{m-|\beta|} .$$

Now let $B(y_0, \eta)$ be any open ball contained in $B(x_0, \rho)$. We will show that $B(y_0, \eta)$ is an excellent ball, thus proving Lemma 9.1.

In fact, we have $\eta \le \rho < 1$ by our choice of ρ. We must show that (I), (II), (III) hold for $B(y_0, \eta)$.

To check (I), we first note that $\mathcal{A}$ is adapted to $I(x)$ for all $x \in E_0 \cap B(y_0, \eta)$ (see (1)). Moreover, since $\eta \le \rho \le \eta_0$ and

(24) $B(y_0, \eta) \subset B(x_0, \rho) \subset B(x_0, \rho_3) \subset B(x_0, \rho_2) \subset B(x_0, \rho_1)$

we know from (16), (17), (18) that

$$|\partial^\beta P_\alpha^x(x)| \ \le \ C_1 \eta^{|\alpha| - |\beta|} \text{ for } \alpha \in \mathcal{A}\,, |\alpha| \le |\beta| \le m,\ x \in E_0 \cap B(y_0, \eta)$$

and

$$\partial^\beta P_\alpha^x(x) = 0 \text{ for } |\beta| < |\alpha|,\ \alpha \in \mathcal{A},\ x \in E_0 \cap B(y_0, \eta)\,,$$

where

$(P_\alpha^x)_{\alpha \in \mathcal{A}}$ is the $\mathcal{A}$-basis for $I(x)$, $x \in E_0 \cap B(y_0, \eta)$.

Thus, $I(x)$ admits an (η, C_1)-controlled $\mathcal{A}$-basis for each $x \in E_0 \cap B(y_0, \eta)$.

This completes the proof of (I) for the ball $B(y_0, \eta)$.

To check (II), we just recall (16) , (14), (23), and the inclusions (24).

Thus, for any $x, x' \in E_0 \cap B(y_0, \eta)$ and any $\alpha \in \mathcal{A}$, we obtain

$$|\partial^\beta (P_\alpha^x - P_\alpha^{x,x'})(x)| \le |x - x'|^{m-|\beta|} \text{ for } |\beta| \le m\,,$$

where $P_\alpha^{x,x'} \in I(x')$ and $(P_\beta^x)_{\beta \in \mathcal{A}}$ is the $\mathcal{A}$-basis for $I(x)$.

This completes the proof of (II) for the ball $B(y_0, \eta)$.

Finally, to check (III), we just recall (16), (14), (21), and the inclusions (24).

Thus, given $\epsilon > 0$ there exists $\delta > 0$ such that, for $x, x' \in E_0 \cap B(y_0, \eta)$ with $|x - x'| \leq \delta$, we have

$$|\partial^\beta (P_\alpha^x - P_\alpha^{x,x'})(x)| \leq \epsilon |x - x'|^{m-|\beta|} \text{ for } |\beta| \leq m,\ \alpha \in \mathcal{A},$$

where $P_\alpha^{x,x'} \in I(x')$ for $\alpha \in \mathcal{A}$, and $(P_\alpha^x)_{\alpha \in \mathcal{A}}$ is the $\mathcal{A}$-basis for $I(x)$.

This completes the proof of (III) for the ball $B(y_0, \eta)$.

We have now shown that any open ball $B(y_0, \eta) \subset B(x_0, \rho)$ has radius less than 1 and satisfies (I), (II), (III).

Thus, any open ball contained in $B(x_0, \rho)$ is excellent.

The proof of Lemma 9.1 is complete. ■

Lemma 9.2: *Let $(I(x))_{x \in E}$ be a Glaeser stable family of ideals.*

Then there exists $\bar{\eta} > 0$ such that any open ball of radius less than $\bar{\eta}$ is excellent.

Proof: Let $B(x_0, \rho)$ be an open ball in $\mathbb{R}^n$. We will call $B(x_0, \rho)$ a "useful" ball if every open ball $B(y_0, \eta) \subset B(x_0, 10\rho)$ with radius $\eta < \rho$ is excellent. By Lemma 9.1, every point of E_0 is the center of a useful ball. Since E_0 if compact, it follows that E_0 is covered by finitely many useful balls $B(x_1, \rho_1), \ldots, B(x_N, \rho_N)$. We take $\bar{\eta} = \min(1, \rho_1, \ldots, \rho_N) > 0$; we show that every open ball $B(y_0, \eta)$ of radius $\eta < \bar{\eta}$ is excellent. In fact, if $B(y_0, \eta)$ is disjoint from E_0, then (as we observed earlier) $B(y_0, \eta)$ is an excellent ball, for trivial reasons. If $B(y_0, \eta)$ is not disjoint from E_0, then let $\tilde{x} \in B(y_0, \eta) \cap E_0$. We have $\tilde{x} \in B(x_\nu, \rho_\nu)$ for some $\nu (1 \leq \nu \leq N)$. For that ν, we have also $\tilde{x} \in B(y_0, \eta)$ and $\eta < \bar{\eta} \leq \rho_\nu$. Consequently, $B(y_0, \eta) \subset B(x_\nu, 10\rho_\nu)$. Since $B(x_\nu, \rho_\nu)$ is useful, it follows that $B(y_0, \eta)$ is excellent.

The proof of Lemma 9.2 is complete. ■

§10. Analysis on an Excellent Ball

In this section, we suppose we are given the following data:

- A vector space Ξ with a seminorm $|\cdot|$.
- A Glaeser stable family of cosets $(g_\xi(x) + I(x))_{x \in E, \xi \in \Xi}$ depending linearly on $\xi \in \Xi$.

- An open ball $B(y_0, \eta) \subset \mathbb{R}^n$.
- A positive constant A.

Note that the family of ideals $(I(x))_{x \in E}$ is Glaeser stable.

Let E_0 be the first slice for the family of ideals $(I(x))_{x \in E}$.

We make the following

Assumptions

(1) $y_0 \in E_0$.

(2) $B(y_0, \eta)$ is an excellent ball for the family of ideals $(I(x))_{x \in E}$. (See Section 9.)

(3) We have $g_\xi(y_0) \in I(y_0)$ for all $\xi \in \Xi$.

(4) For any $\xi \in \Xi$ with $|\xi| \leq 1$, there exists $G \in C^m(\mathbb{R}^n)$, with

$$\| G \|_{C^m(\mathbb{R}^n)} \leq A\,, \text{ and } J_x(G) \in g_\xi(x) + I(x) \text{ for all } x \in E\,.$$

Let $\mathcal{A} \subseteq \mathcal{M}$ be as in the definition of an "excellent ball" in Section 9.

For $x \in E_0 \cap B(y_0, \eta)$, let $(P_\alpha^x)_{\alpha \in \mathcal{A}}$ be the $\mathcal{A}$-basis for $I(x)$.

Then, for $x \in E_0 \cap B(y_0, \eta)$, we define a linear map $\mathsf{proj}_x : \mathcal{P} \to \mathcal{P}$,

by setting

(5) $$\mathsf{proj}_x\, P = P - \sum_{\alpha \in \mathcal{A}} [\partial^\alpha P(x)] \cdot P_\alpha^x.$$

Recall that $\mathcal{P}$ is the vector space of m^{th} degree polynomials on $\mathbb{R}^n$.

We note a few elementary properties of proj_x.

Proposition 10.1: *Let $x \in E_0 \cap B(y_0, \eta)$, and let $P \in \mathcal{P}$. Then:*

(6) $P - \mathsf{proj}_x\, P \in I(x)$;

(7) *If* $P \in I(x)$, *then* $\mathsf{proj}_x\, P = 0$; *and*

(8) $\partial^\beta(\mathsf{proj}_x P)(x) = 0$ *for* $\beta \in \mathcal{A}$.

Here, $\partial^\beta(\mathsf{proj}_x P)(x)$ denotes $\left(\frac{\partial}{\partial y}\right)^\beta (\mathsf{proj}_x P)(y)$ evaluated at $y = x$.

Proof: We have (6), simply because each P^x_α belongs to $I(x)$.

To prove (8), we note that, for $\beta \in \mathcal{A}$, we have

$$\partial^\beta(\mathsf{proj}_x P)(x) = \partial^\beta P(x) - \sum_{\alpha \in \mathcal{A}} [\partial^\alpha P(x)] \cdot \partial^\beta P^x_\alpha(x) = 0\,,$$

since $\partial^\beta P^x_\alpha(x) = \delta_{\beta\alpha}$ for $\beta, \alpha \in \mathcal{A}$ (because $(P^x_\alpha)_{\alpha\in\mathcal{A}}$ is the $\mathcal{A}$-basis for $I(x)$).

To prove (7), suppose $P \in I(x)$. Then (6) gives $\mathsf{proj}_x\, P \in I(x)$. Since $(P^x_\alpha)_{\alpha\in\mathcal{A}}$ is a basis for $I(x)$, we therefore have

(9) $\mathsf{proj}_x P = \sum_{\alpha\in\mathcal{A}} A_\alpha P^x_\alpha$ for some coefficients A_α.

Since $\partial^\beta P^x_\alpha(x) = \delta_{\beta\alpha}$ for $\beta, \alpha \in \mathcal{A}$, we learn from (8) and (9) that

$$0 = \partial^\beta(\mathsf{proj}_x P)(x) = \sum_{\alpha \in A} A_\alpha \partial^\beta P^x_\alpha(x) = A_\beta \text{ for any } \beta \in \mathcal{A}\,.$$

Therefore, (9) gives $\mathsf{proj}_x P = 0$, completing the proof of (7). ■

Lemma 10.1: *Let* $\xi \in \Xi$, *with* $|\xi| \le 1$. *Then, for* $x, x' \in E_0 \cap B(y_0, \eta)$, *we have*

$$|\partial^\beta(\mathsf{proj}_x g_\xi(x) - \mathsf{proj}_{x'} g_\xi(x'))(x)| \le CA|x - x'|^{m-|\beta|} \text{ } for\ |\beta| \le m\,.$$

Proof: Recall that C denotes a constant depending only on m and n.

Let $(P'_\alpha)_{\alpha\in\mathcal{A}}$ satisfy the following conditions.

(10) $P'_\alpha \in I(x')$ for $\alpha \in \mathcal{A}$;

(11) $|\partial^\beta(P^x_\alpha - P'_\alpha)(x)| \le |x - x'|^{m-|\beta|}$ for $|\beta| \le m$, $\alpha \in \mathcal{A}$.

Such P'_α exist, by part (II) of the definition of an excellent ball.

Also, we fix $G \in C^m(\mathbb{R}^n)$, with

(12) $\| G \|_{C^m(\mathbb{R}^n)} \leq A$

and

(13) $J_x(G) \in g_\xi(x) + I(x)$ for all $x \in E$.

We set $P = J_x(G)$ and $P' = J_{x'}(G)$. From (12) we have

(14) $|\partial^\beta P(x)| \leq CA$ for $|\beta| \leq m$, and

(15) $|\partial^\beta (P - P')(x)| \leq CA|x - x'|^{m-|\beta|}$ for $|\beta| \leq m$.

Note that

(16) $\mathsf{proj}_x g_\xi(x) = \mathsf{proj}_x J_x(G) = \mathsf{proj}_x P$, and

(17) $\mathsf{proj}_{x'} g_\xi(x') = \mathsf{proj}_{x'} J_{x'}(G) = \mathsf{proj}_{x'} P'$,

thanks to (13) and (7).

We define

(18) $\tilde{P} = P - \sum_{\alpha \in \mathcal{A}} [\partial^\alpha P(x)] \cdot P^x_\alpha$

and

(19) $\tilde{P}' = P' - \sum_{\alpha \in \mathcal{A}} [\partial^\alpha P(x)] \cdot P'_\alpha$.

Then $\tilde{P} = \mathsf{proj}_x P$ by definition, and $\mathsf{proj}_{x'} P' = \mathsf{proj}_{x'} \tilde{P}'$ by (7) and (10).

Hence, (16), (17) yield

(20) $\mathsf{proj}_x g_\xi(x) = \tilde{P}$, and

(21) $\mathsf{proj}_{x'} g_\xi(x') = \mathsf{proj}_{x'} \tilde{P}'$.

For $|\beta| \le m$, we have

$$(22)\quad |\partial^\beta(\tilde{P} - \tilde{P}')(x)| \le |\partial^\beta(P - P')(x)| + \sum_{\alpha \in \mathcal{A}} |\partial^\alpha P(x)| \cdot |\partial^\beta(P^x_\alpha - P'_\alpha)(x)|$$

$$\le CA|x - x'|^{m-|\beta|} + \sum_{\alpha \in \mathcal{A}} [CA] \cdot |x - x'|^{m-|\beta|}$$

$$\le C'A|x - x'|^{m-|\beta|}, \text{ by (18), (19), (15), (14), (11)}.$$

In particular, for $\beta \in \mathcal{A}$, we have

$\partial^\beta \tilde{P}(x) = \partial^\beta(\mathsf{proj}_x P)(x) = 0$ by (8). Hence (22) yields

$|\partial^\beta \tilde{P}'(x)| \le C'A|x - x'|^{m-|\beta|}$ for $\beta \in \mathcal{A}$.

Since $\mathcal{A}$ is monotonic (by definition of an "excellent ball"), it follows that

$$(23)\quad |\partial^{\gamma+\beta} \tilde{P}'(x)| \le C'A|x - x'|^{m-|\beta|-|\gamma|} \text{ for } \beta \in \mathcal{A},\ |\gamma| \le m - |\beta|.$$

Since $\partial^\beta \tilde{P}'$ is a polynomial of degree at most $m - |\beta|$, (23) implies

$$(24)\quad |\partial^\beta \tilde{P}'(x')| \le CA|x - x'|^{m-|\beta|} \text{ for } \beta \in \mathcal{A}.$$

Now, for $|\beta| \le m$, we have

$$(25)\quad |\partial^\beta[\tilde{P}' - \mathsf{proj}_{x'} \tilde{P}'](x')| = |\sum_{\alpha \in \mathcal{A}} [\partial^\alpha \tilde{P}'(x')] \cdot \partial^\beta P^{x'}_\alpha(x')|$$

(by definition of $\mathsf{proj}_{x'}$)

$$\le \sum_{\substack{\alpha \in \mathcal{A} \\ |\alpha| \le |\beta|}} |\partial^\alpha \tilde{P}'(x')| \cdot |\partial^\beta P^{x'}_\alpha(x')|$$

(since $\partial^\beta P^{x'}_\alpha(x') = 0$ for $\alpha \in \mathcal{A}$, $|\beta| < |\alpha|$; see Proposition 8.2)

$$\leq \sum_{\substack{\alpha \in \mathcal{A} \\ |\alpha| \leq |\beta|}} [CA|x-x'|^{m-|\alpha|}] \cdot [C\eta^{|\alpha|-|\beta|}]$$

(by (24) and the fact that $(P_\alpha^{x'})_{\alpha \in \mathcal{A}}$ is (η, C_1)-controlled; see the definition of an "excellent ball", and also the definition of "(η, C_1)-controlled")

$$\leq \sum_{\substack{\alpha \in \mathcal{A} \\ |\alpha| \leq |\beta|}} [CA|x-x'|^{m-|\alpha|}] \cdot [C|x-x'|^{|\alpha|-|\beta|}]$$

(since $|x-x'| \leq 2\eta$ because $x, x' \in B(y_0, \eta)$)

$$\leq CA\,|x-x'|^{m-|\beta|}.$$

Since $\tilde{P}' - \mathsf{proj}_{x'}\tilde{P}'$ is an m^{th} degree polynomial on $\mathbb{R}^n$, (25) implies

(26) $|\partial^\beta[\tilde{P}' - \mathsf{proj}_{x'}\tilde{P}']\ (x)| \leq CA|x-x'|^{m-|\beta|}$ for $|\beta| \leq m$.

From (22) and (26), we have

$$|\partial^\beta(\tilde{P} - \mathsf{proj}_{x'}\tilde{P}')(x)| \leq\ CA|x-x'|^{m-|\beta|} \text{ for } |\beta| \leq m\,.$$

In view of (20), (21), this means that

$$|\partial^\beta(\mathsf{proj}_x g_\xi(x) - \mathsf{proj}_{x'} g_\xi(x'))(x)|\ \leq\ CA|x-x'|^{m-|\beta|} \text{ for } |\beta| \leq m\,,$$

which is the conclusion of Lemma 10.1.

The proof of the Lemma is complete. ■

Similarly, we have the following result.

Lemma 10.2: *Let $\xi \in \Xi$ and $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that, for any $x, x' \in E_0 \cap B(y_0, \eta)$ with $|x-x'| \leq \delta$, we have*

$$|\partial^\beta(\mathsf{proj}_x g_\xi(x) - \mathsf{proj}_{x'} g_\xi(x'))(x)| \leq \epsilon|x-x'|^{m-|\beta|} \; for \; |\beta| \leq m\,.$$

Proof: Since $\xi \mapsto g_\xi(x)$ is linear for each $x \in E$, we may assume without loss of generality that $|\xi| \leq 1$.

Let $\epsilon' > 0$ be a small, positive number, to be picked later.

By part (III) of the definition of an excellent ball, there exists $\delta_1 > 0$, for which the following holds.

(27) Given $x, x' \in E_0 \cap B(y_0, \eta)$ with $|x - x'| \leq \delta_1$, there exists a family of polynomials $(P'_\alpha)_{\alpha \in \mathcal{A}}$, such that

(a) $P'_\alpha \in I(x')$ for $\alpha \in \mathcal{A}$;

and

(b) $|\partial^\beta (P^x_\alpha - P'_\alpha)(x)| \leq \epsilon' |x - x'|^{m - |\beta|}$ for $|\beta| \leq m$, $\alpha \in \mathcal{A}$.

Also, we fix $G \in C^m(\mathbb{R}^n)$, with

(28) $\| G \|_{C^m(\mathbb{R}^n)} \leq A$, and

(29) $J_x(G) \in g_\xi(x) + I(x)$ for all $x \in E$.

Since $G \in C^m(\mathbb{R}^n)$ and E_0 is compact, there exists $\delta_2 > 0$ for which the following holds.

(30) Given $x, x' \in E_0$ with $|x - x'| \leq \delta_2$, we have

$$|\partial^\beta (J_x(G) - J_{x'}(G))(x)| \leq \epsilon' |x - x'|^{m - |\beta|} \text{ for } |\beta| \leq m\,.$$

Now suppose $x, x' \in E_0 \cap B(y_0, \eta)$ with $|x - x'| \leq \min(\delta_1, \delta_2)$.

Then (27) and (30) apply. We fix $(P'_\alpha)_{\alpha \in \mathcal{A}}$ as in (27); and we set

$P = J_x(G), P' = J_{x'}(G)$,

$\tilde{P} = P - \sum_{\alpha \in \mathcal{A}} [\partial^\alpha P(x)] \cdot P^x_\alpha$,

$\tilde{P}' = P' - \sum_{\alpha \in \mathcal{A}} [\partial^\alpha P(x)] \cdot P'_\alpha$,

as in the proof of Lemma 10.1. As in that proof, we have

(31) $\mathsf{proj}_x g_\xi(x) = \tilde{P}$ and $\mathsf{proj}_{x'} g_\xi(x') = \mathsf{proj}_{x'} \tilde{P}'$.

By (28) and (30), we have

(32) $|\partial^\beta P(x)| \leq CA$ for $|\beta| \leq m$, and

(33) $|\partial^\beta (P - P')(x)| \leq \epsilon' |x - x'|^{m-|\beta|}$ for $|\beta| \leq m$.

For $|\beta| \leq m$, we have

$$(34)\quad |\partial^\beta(\tilde{P} - \tilde{P}')(x)| \leq |\partial^\beta(P - P')(x)| + \sum_{\alpha\in\mathcal{A}} |\partial^\alpha P(x)| \cdot |\partial^\beta(P_\alpha^x - P'_\alpha)(x)|$$
$$\leq \epsilon'|x - x'|^{m-|\beta|} + \sum_{\alpha\in\mathcal{A}} [CA] \cdot [\epsilon'|x - x'|^{m-|\beta|}] \text{ (by (27)(b), (32), (33))}$$
$$\leq [C + CA] \cdot \epsilon'|x - x'|^{m-|\beta|}.$$

From (34) it follows, as in the proof of Lemma 10.1, that

(35) $|\partial^\beta \tilde{P}'(x')| \leq [C + CA] \cdot \epsilon'|x - x'|^{m-|\beta|}$ for $\beta \in \mathcal{A}$.

Proceeding as in our derivation of (25), we obtain for $|\beta| \leq m$ the estimates

$$(36)\quad |\partial^\beta(\tilde{P}' - \mathsf{proj}_{x'}\tilde{P}')(x')| = \left| \sum_{\alpha\in\mathcal{A}} [\partial^\alpha \tilde{P}'(x')] \cdot \partial^\beta P_\alpha^{x'}(x') \right|$$
$$\leq \sum_{\substack{\alpha\in\mathcal{A} \\ |\alpha|\leq|\beta|}} |\partial^\alpha \tilde{P}'(x')| \cdot |\partial^\beta P_\alpha^{x'}(x')|$$
$$\leq \sum_{\substack{\alpha\in\mathcal{A} \\ |\alpha|\leq|\beta|}} [(CA + C) \cdot \epsilon' \cdot |x - x'|^{m-|\alpha|}] \cdot [C\eta^{|\alpha|-|\beta|}]$$
$$\leq \sum_{\substack{\alpha\in\mathcal{A} \\ |\alpha|\leq|\beta|}} [(CA + C) \cdot \epsilon' \cdot |x - x'|^{m-|\alpha|}] \cdot [C|x - x'|^{|\alpha|-|\beta|}]$$
$$\leq (CA + C) \cdot \epsilon' \cdot |x - x'|^{m-|\beta|},$$

thanks to (35). Since $\tilde{P}' - \mathsf{proj}_{x'}\tilde{P}'$ is an m^{th} degree polynomial, (36) implies

(37) $|\partial^\beta(\tilde{P}' - \mathsf{proj}_{x'}\tilde{P}')(x)| \leq (CA + C) \cdot \epsilon' \cdot |x - x'|^{m-|\beta|}$ for $|\beta| \leq m$.

From (31), (34), (37), we see that

(38) $|\partial^\beta[\mathsf{proj}_x g_\xi(x) - \mathsf{proj}_{x'} g_\xi(x')](x)| \le [CA + C]\,\epsilon'|x - x'|^{m-|\beta|}$ for $|\beta| \le m$.

Taking $\epsilon' = \epsilon/[CA + C]$ with C as in (38), we obtain the conclusion of Lemma 10.2.

The proof of the Lemma is complete. ■

We prepare to apply the classical Whitney extension theorem, i.e., Theorem 2.1.

Recall that this result produces a linear extension operator $\mathcal{E} : C^m_{\mathsf{jet}}(E) \to C^m(\mathbb{R}^n)$, for any compact $E \subset \mathbb{R}^n$.

The main result of this section is as follows.

Lemma 10.3: *There exists a linear map $\xi \mapsto G_\xi$, from Ξ into $C^m(\mathbb{R}^n)$, with norm at most CA, such that the following properties hold.*

(a) $J_x(G_\xi) \in g_\xi(x) + I(x)$ *for all* $x \in E_0 \cap B(y_0, \frac{1}{2}\eta)\,, \xi \in \Xi\,.$

(b) $J_{y_0}(G_\xi) = 0$ *for all* $\xi \in \Xi$.

Proof: We start with a corollary of Lemma 10.1, namely

(39) $|\partial^\beta[\mathsf{proj}_x g_\xi(x)](x)| \le CA$ for $|\beta| \le m$, $x \in E_0 \cap B(y_0, \eta)$, $|\xi| \le 1$.

To prove (39), we note that assumptions (1), (3), and property (7) show that

(40) $\mathsf{proj}_{y_0} g_\xi(y_0) = 0$ for all $\xi \in \Xi$.

Hence, putting $x' = y_0$ in Lemma 10.1, we learn that

(41) $|\partial^\beta[\mathsf{proj}_x g_\xi(x)]| \le CA|x - y_0|^{m-|\beta|}$ for $|\beta| \le m$, $x \in E_0 \cap B(y_0, \eta)$.

For $x \in E_0 \cap B(y_0, \eta)$, we have $|x - y_0|^{m-|\beta|} \le \eta^{m-|\beta|} \le 1$, since an excellent ball $B(y_0, \eta)$ has radius $\eta \le 1$ by definition. Hence, (41) implies (39).

Let $\bar{B}(y_0, \frac{1}{2}\eta)$ denote the closed ball about y_0 with radius $\frac{1}{2}\eta$.

In view of (39) and Lemmas 10.1 and 10.2, the linear map $\xi \mapsto (\mathsf{proj}_x g_\xi(x))_{x \in E_0 \cap \bar{B}(y_0, \frac{1}{2}\eta)}$ carries Ξ into $C^m_{\mathsf{jet}}\,(E_0 \cap \bar{B}(y_0, \frac{1}{2}\eta))$, and has norm at most CA.

Now let $\mathcal{E} : C^m_{\mathsf{jet}}(E_0 \cap \bar{B}(y_0, \frac{1}{2}\eta)) \to C^m(\mathbb{R}^n)$ be as in Theorem 2.1.

Thus, $\mathcal{E}$ has norm at most C, and $J_x(\mathcal{E}\vec{f}\,) = f(x)$ for $\vec{f} = (f(x))_{x \in E} \in C^m_{\mathsf{jet}}(E_0 \cap \bar{B}(y_0, \frac{1}{2}\eta))$ and $x \in E_0 \cap \bar{B}(y_0, \frac{1}{2}\eta)$. We define

$$G_\xi = \mathcal{E}([\mathsf{proj}_x g_\xi(x)]_{x \in E_0 \cap \bar{B}(y_0, \frac{1}{2}\eta)}) \ \textit{for} \ \xi \in \Xi\,.$$

Thus, $\xi \mapsto G_\xi$ is a linear map from Ξ into $C^m(\mathbb{R}^n)$, with norm at most CA. Moreover, for $\xi \in \Xi$, $x \in E_0 \cap \bar{B}(y_0, \frac{1}{2}\eta)$, we have

(42) $J_x(G_\xi) = J_x(\mathcal{E}([\mathsf{proj}_x g_\xi(x)]_{x \in E_0 \cap \bar{B}(y_0, \frac{1}{2}\eta)})) = \mathsf{proj}_x g_\xi(x)$, by the defining property of $\mathcal{E}$.

From (6) and (42), we obtain

$J_x(G_\xi) \in g_\xi(x) \,+\, I(x)$ for $x \in E_0 \cap \bar{B}(y_0, \frac{1}{2}\eta)\,,\ \xi \in \Xi\,;$

which is conclusion (a) of Lemma 10.3.

Also, from (40) and (42), we obtain

$$J_{y_0}(G_\xi) = 0 \text{ for all } \xi \in \Xi\,,$$

which is conclusion (b).

The proof of Lemma 10.3 is complete. ■

§11. The Second Main Lemma

In this section, we pass from F^{00}_ξ to the "corrected" linear map $\xi \mapsto F^0_\xi = F^{00}_\xi + \tilde{F}_\xi$, as described in the Introduction.

Second Main Lemma: *Let Ξ be a vector space with a seminorm $|\cdot|$, and suppose $(f_\xi(x) + I(x))_{x \in E, \xi \in \Xi}$ is a Glaeser stable family of cosets, depending linearly on $\xi \in \Xi$. Let E_0 be the first slice.*

Assume that, given $\xi \in \Xi$ with $|\xi| \le 1$, there exists $F \in C^m(\mathbb{R}^n)$, with $\| F \|_{C^m(\mathbb{R}^n)} \le 1$, and $J_x(F) \in f_\xi(x) + I(x)$ for all $x \in E$.

Then there exists a linear map $\xi \mapsto F^0_\xi$, from Ξ into $C^m(\mathbb{R}^n)$, with norm at most C, and satisfying the following properties.

(A) *If $\xi \in \Xi$ with $|\xi| \le 1$, then $J_x(F^0_\xi) \in \Gamma_\xi(x, \bar{k}, C)$ for all $x \in E_0$, with $\bar{k}$ as in Lemma 3.1.*

(B) *For any $\xi \in \Xi$, we have $J_x(F^0_\xi) \in f_\xi(x) + I(x)$ for all $x \in E_0$.*

Here, C depends only on m and n.

Proof: Since $(f_\xi(x) + I(x))_{x \in E, \xi \in \Xi}$ is Glaeser stable, it follows that

(1) $(I(x))_{x \in E}$ is Glaeser stable.

Let $A > 0$ be a large number, and let $\eta > 0$ be a small number, to be picked later. We introduce a partition of unity

(2) $\sum_\nu \theta_\nu = 1$ on $\mathbb{R}^n$,

with

(3) $\mathsf{supp}\, \theta_\nu \subset B(x_\nu, \frac{1}{2}\eta)$

and

(4) $|\partial^\beta \theta_\nu| \le C \eta^{-|\beta|}$ on $\mathbb{R}^n$, for $|\beta| \le m+1$.

Here, the points x_ν in (3) may be taken to satisfy

(5) Any given ball of radius η intersects at most C of the balls $B(x_\nu, \eta)$.

Let Ω be the set of ν for which $B(x_\nu, \frac{1}{2}\eta) \cap E_0$ is non-empty.

Note that Ω is finite, thanks to (5) and the compactness of E_0.

For $x \in E_0$, we have $x \in \mathsf{supp}\, \theta_\nu$ only for $\nu \in \Omega$. Hence, (2) implies

(6) $\sum_{\nu \in \Omega} J_x(\theta_\nu) = 1$ for all $x \in E_0$.

For each $\nu \in \Omega$, we pick $y_\nu \in E_0 \cap B(x_\nu, \frac{1}{2}\eta)$. From (3), we have

(7) $\mathsf{supp}\, \theta_\nu \subset B(y_\nu, \eta)$ for $\nu \in \Omega$.

Let

(8) $E_{00} = \{y_\nu : \nu \in \Omega\}$.

Thus, E_{00} is a finite subset of E_0; and (6), (7) show that

(9) No point of E_0 lies farther than distance η from E_{00}.

In view of (9) and the hypotheses of the Second Main Lemma, we are in position to apply the First Main Lemma. We write $\eta_0(A)$ for the small constant called η_0 in the statement of the First Main Lemma. Recall that $\eta_0(A)$ is determined by A and by the family of cosets $(f_\xi(x) + I(x))_{x \in E, \xi \in \Xi}$.

From the First Main Lemma, we learn the following.

Suppose η satisfies

(10) $\eta < \eta_0(A)$.

Then there exists a linear map $\xi \mapsto F_\xi^{00}$, from Ξ into $C^m(\mathbb{R}^n)$, with norm at most C, such that the following hold.

(11) $J_x(F_\xi^{00}) \in f_\xi(x) + I(x)$ for all $x \in E_{00}, \xi \in \Xi$.

(12) Let $\xi \in \Xi$ with $|\xi| \leq 1$. Let $x \in E_0$ and $Q \in \mathcal{P}$. Suppose $J_x(F_\xi^{00}) + Q \in f_\xi(x) + I(x)$, and suppose also that $|\partial^\beta Q(x)| \leq A\eta^{m-|\beta|}$ for $|\beta| \leq m$.

Then $J_x(F_\xi^{00}) + Q \in \Gamma_\xi(x, \bar{k}, A')$, where $\bar{k}$ is as in Lemma 3.1, and A' is determined by A, m, n.

We fix $\xi \mapsto F_\xi^{00}$ as above.

Next, we apply Lemma 9.2. (The hypothesis of Lemma 9.2 holds here, thanks to (1).) Thus, we obtain $\bar{\eta} > 0$, determined by $(I(x))_{x\in E}$, such that

(13) $B(y_\nu, 2\eta)$ is an excellent ball, for each $\nu \in \Omega$, provided η satisfies

(14) $\eta < \bar{\eta}$.

We define a new family of cosets $(g_\xi(x) + I(x))_{x\in E, \xi\in\Xi}$ depending linearly on $\xi \in \Xi$, by taking

(15) $g_\xi(x) = f_\xi(x) - J_x(F_\xi^{00})$ for $\xi \in \Xi$, $x \in E$.

Since $F_\xi^{00} \in C^m(\mathbb{R}^n)$ and $(f_\xi(x) + I(x))_{x\in E,\xi\in\Xi}$ is Glaeser stable, it follows that

(16) $(g_\xi(x) + I(x))_{x\in E,\xi\in\Xi}$ is Glaeser stable.

Also, from (11) and (8), we see that

(17) $J_{y_\nu}(g_\xi) \in I(y_\nu)$ for $\nu \in \Omega$, $\xi \in \Xi$.

Moreover, suppose $\xi \in \Xi$ with $|\xi| \leq 1$. Then we have $\| F_\xi^{00} \|_{C^m(\mathbb{R}^n)} \leq C$, since $\xi \mapsto F_\xi^{00}$ has norm at most C.

Also, by hypothesis of the Second Main Lemma, there exists $F \in C^m(\mathbb{R}^n)$, with

$$\| F \|_{C^m(\mathbb{R}^n)} \leq 1, \text{ and } J_x(F) \in f_\xi(x) + I(x) \text{ for all } x \in E\,.$$

Setting $G = F - F_\xi^{00}$, we therefore have $G \in C^m(\mathbb{R}^n)$, $\| G \|_{C^m(\mathbb{R}^n)} \leq C$, $J_x(G) \in g_\xi(x) + I(x)$ for all $x \in E$.

We have proven the following:

(18) Given $\xi \in \Xi$ with $|\xi| \leq 1$, there exists $G \in C^m(\mathbb{R}^n)$, with $\| G \|_{C^m(\mathbb{R}^n)} \leq C$, and $J_x(G) \in g_\xi(x) + I(x)$ for all $x \in E$.

Thanks to (13), (17), (18) and the defining properties of y_ν, we see that the standing assumptions (10.1),..., (10.4) of Section 10 hold here, with our present y_ν in place of y_0 in Section 10, with 2η in place of η, and with a controlled constant C in place of A in Section 10. Hence, we may apply Lemma 10.3.

For each $\nu \in \Omega$, Lemma 10.3 gives us a linear map $\xi \mapsto G_{\nu,\xi}$, from Ξ into $C^m(\mathbb{R}^n)$, with norm at most C, such that the following properties hold.

(19) $J_x(G_{\nu,\xi}) \in g_\xi(x) + I(x)$ for all $x \in E_0 \cap B(y_\nu, \eta)$, $\xi \in \Xi$.

(20) $J_{y_\nu}(G_{\nu,\xi}) = 0$ for all $\xi \in \Xi$.

In particular, for $\xi \in \Xi$ with $|\xi| \leq 1$, we have

$\| G_{\nu,\xi} \|_{C^m(\mathbb{R}^n)} \leq C$ and $J_{y_\nu}(G_{\nu,\xi}) = 0$. Consequently, we have

(21) $|\partial^\beta G_{\nu,\xi}(x)| \leq C\eta^{m-|\beta|}$ for $|\beta| \leq m$, $x \in B(y_\nu, \eta)$, $|\xi| \leq 1$.

Our results (19), (20), (21) hold for all $\nu \in \Omega$.

We now define

(22) $$\tilde{F}_\xi = \sum_{\nu \in \Omega} \theta_\nu \cdot G_{\nu,\xi} \text{ for } \xi \in \Xi.$$

Thus, $\xi \mapsto \tilde{F}_\xi$ is a linear map from Ξ into $C^m(\mathbb{R}^n)$.

Suppose $\xi \in \Xi$, with $|\xi| \leq 1$. From (4), (7), (21), we see that

$|\partial^\beta \{\theta_\nu \cdot G_{\nu,\xi}\}| \leq C\eta^{m-|\beta|}$ on $\mathbb{R}^n$, for $|\beta| \leq m$, $\nu \in \Omega$.

Also, from (3) and (5), we see that any given ball of radius η intersects at most C of the supports of the functions $\{\theta_\nu \cdot G_{\nu,\xi}\}$, $\nu \in \Omega$. Consequently, we have

(23) $$|\partial^\beta \tilde{F}_\xi| = |\sum_{\nu \in \Omega} \partial^\beta \{\theta_\nu \cdot G_{\nu,\xi}\}| \leq C\eta^{m-|\beta|} \text{ on } \mathbb{R}^n, \text{ for } |\beta| \leq m, |\xi| \leq 1.$$

It follows that the linear map $\xi \mapsto \tilde{F}_\xi$ from Ξ to $C^m(\mathbb{R}^n)$ has norm at most C, provided we take η to satisfy

(24) $\eta \leq 1$.

Suppose once more that $\xi \in \Xi$ with $|\xi| \leq 1$, and let $x \in E_0$. From (19) and (7), we learn that

$$J_x(G_{\nu,\xi}) \in g_\xi(x) + I(x) = f_\xi(x) - J_x(F^{00}_\xi) + I(x)\,,$$

whenever $\mathsf{supp}\ \theta_\nu \ni x$ and $\nu \in \Omega$. Consequently,

(25) $J_x(\theta_\nu \cdot G_{\nu,\xi}) \in J_x(\theta_\nu) \odot [f_\xi(x) - J_x(F^{00}_\xi)] + I(x)$ for all $\nu \in \Omega$, where $\odot$ denotes multiplication in $\mathcal{R}_x$, and we have used the fact that $I(x)$ is an ideal.

Summing (25) over all $\nu \in \Omega$, and recalling (6), we find that

$$J_x(\tilde{F}_\xi) = \sum_{\nu \in \Omega} J_x(\theta_\nu \cdot G_{\nu,\xi}) \in f_\xi(x) - J_x(F^{00}_\xi) + I(x)\,.$$

Thus, we have shown that

(26) $J_x(F^{00}_\xi) + J_x(\tilde{F}_\xi) \in f_\xi(x) + I(x)$ for $x \in E_0$, $|\xi| \leq 1$.

Also, from (23), we have

(27) $|\partial^\beta[J_x(\tilde{F}_\xi)](x)| \leq C_1 \eta^{m-|\beta|}$ for $|\beta| \leq m$, $|\xi| \leq 1$.

If we take

(28) $A > C_1$,

with C_1 as in (27), then from (26), (27) and (12), we learn that

(29) $J_x(F^{00}_\xi) + J_x(\tilde{F}_\xi) \in \Gamma_\xi(x, \bar{k}, A')$ for $x \in E_0$, $|\xi| \leq 1$.

Here, $\bar{k}$ is as in Lemma 3.1, and A' is determined by A, m, n.

We now pick the constants A and η. First, we take A to be a controlled constant, large enough to satisfy (28).

We then pick $\eta > 0$ small enough to satisfy the smallness assumptions (10), (14) and (24). With A, η picked in this manner, the above arguments go through, and the constant A' in (29) is controlled (i.e., it depends only on m and n).

Thus, from (29), we have

(30) $J_x(F_\xi^{00} + \tilde{F}_\xi) \in \Gamma_\xi(x, \bar{k}, C)$ for $x \in E_0$, $|\xi| \le 1$,

with $\bar{k}$ as in Lemma 3.1.

Finally, as promised in the Introduction, we define

$$F_\xi^0 \;=\; F_\xi^{00} \;+\; \tilde{F}_\xi \text{ for } \xi \in \Xi\,.$$

Since $\xi \mapsto F_\xi^{00}$ and $\xi \mapsto \tilde{F}_\xi$ are linear maps from Ξ into $C^m(\mathbb{R}^n)$ with norm at most C, the same is true for $\xi \mapsto F_\xi^0$.

Moreover, conclusion (A) of the Second Main Lemma is precisely our result (30). Since $\Gamma_\xi(x, \bar{k}, C) \subseteq f_\xi(x) + I(x)$, it follows that

(31) $J_x(F_\xi^0) \in f_\xi(x) + I(x)$ for $x \in E_0$, $|\xi| \le 1$.

Since the maps $\xi \mapsto F_\xi^0$ and $\xi \mapsto f_\xi(x)$ $(x \in E)$ are both linear, we may drop the assumption $|\xi| \le 1$ from (31).

This proves conclusion (B) of the Second Main Lemma.

The proof of the Second Main Lemma is complete. ■

§12. The Error Outside the First Slice

In this section, we study $f_\xi(x) - J_x(F_\xi^0)$ for x outside the first slice E_0, where f_ξ and F_ξ^0 are as in the Second Main Lemma.

Lemma 12.1: *Let* $\Xi, |\cdot|$, $(f_\xi(x) + I(x))_{x\in E,\xi\in\Xi}$, E_0, *and* $\xi \mapsto F_\xi^0$ *be as in the* Second Main Lemma.

Then, given $\xi \in \Xi$ *with* $|\xi| \leq 1$, *and given* $x_0 \in E_0$, *there exists* $G \in C^m(\mathbb{R}^n)$, *with*

$$\| G \|_{C^m(\mathbb{R}^n)} \leq C,\ J_x(G) \in f_\xi(x) - J_x(F_\xi^0) + I(x) \textit{ for all } x \in E, \textit{ and } J_{x_0}(G) = 0\,.$$

Proof: Set $P_0 = J_{x_0}(F_\xi^0)$. By the Second Main Lemma, we have $P_0 \in \Gamma_\xi(x_0, \bar{k}, C)$ with $\bar{k}$ as in Lemma 3.1. That is, given $x_1, \dots, x_{\bar{k}} \in E$, there exist

$P_1 \in f_\xi(x_1) + I(x_1), \dots, P_{\bar{k}} \in f_\xi(x_{\bar{k}}) + I(x_{\bar{k}})$, with

$|\partial^\beta P_i(x_i)| \leq C$ for $|\beta| \leq m$, $0 \leq i \leq \bar{k}$; and

$|\partial^\beta (P_i - P_j)(x_j)| \leq C|x_i - x_j|^{m-|\beta|}$ for $|\beta| \leq m$, $0 \leq i, j \leq \bar{k}$.

Hence, Lemma 3.1 shows that there exists $F \in C^m(\mathbb{R}^n)$, with

(1) $\| F \|_{C^m(\mathbb{R}^n)} \leq C$, $J_x(F) \in f_\xi(x) + I(x)$ for $x \in E$, and $J_{x_0}(F) = P_0$.

Setting $G = F - F_\xi^0 \in C^m(\mathbb{R}^n)$, and recalling that

(2) $\| F_\xi^0 \|_{C^m(\mathbb{R}^n)} \leq C$

by the Second Main Lemma, we conclude from (1) and (2) that G satisfies the conditions asserted in Lemma 12.1.

The proof of the lemma is complete. ■

Lemma 12.2: *Let* $\Xi, |\cdot|$, $(f_\xi(x) + I(x))_{x\in E,\xi\in\Xi}$, E_0 *and* $\xi \mapsto F_\xi^0$ *be as in the* Second Main Lemma.

Then, given $\xi \in \Xi$ *and* $\epsilon > 0$, *there exists* $\delta > 0$ *for which the following holds*:

Given $x_0 \in E_0$, *there exists* $G \in C^m(\mathbb{R}^n)$, *with*

$$\| G \|_{C^m(\mathbb{R}^n)} < \epsilon, J_x(G) \in f_\xi(x) - J_x(F_\xi^0) + I(x) \textit{ for all } x \in E \cap B(x_0, \delta)\,, \textit{ and } J_{x_0}(G) = 0\,.$$

Proof: Fix $\xi \in \Xi$. Then $(f_\xi(x) + I(x))_{x \in E}$ is Glaeser stable, and $F_\xi^0 \in C^m(\mathbb{R}^n)$. Hence, $(f_\xi(x) - J_x(F_\xi^0) + I(x))_{x \in E}$ is Glaeser stable.

We take $k = \bar{k}$ from Lemma 3.1, and we take $k^\#$ as in Section 4. Thus, $k \geq 1$ and $1 + (D+1) \cdot k \leq k^\#$, as in the hypotheses of Lemma 3.6. We apply Lemma 3.6 to the family of cosets $H = (f_\xi(x) - J_x(F_\xi^0) + I(x))_{x \in E}$, for $x_0 \in E_0$ and $P_0 = 0$.

We recall from Proposition 2.2 that the first slice E_0 is contained in the lowest stratum E_1. From the Second Main Lemma, we recall also that $0 \in f_\xi(x_0) - J_{x_0}(F_\xi^0) + I(x_0)$. Consequently, given $\epsilon > 0$, Lemma 3.6 applied to H provides a positive number δ, for which the following holds:

(3) Given $x_0 \in E_0$ and $x_1, \ldots, x_{\bar{k}} \in E \cap B(x_0, \delta)$, there exist

$$P_1 \in f_\xi(x_1) - J_{x_1}(F_\xi^0) + I(x_1), \ldots, P_{\bar{k}} \in f_\xi(x_{\bar{k}}) - J_{x_{\bar{k}}}(F_\xi^0) + I(x_{\bar{k}}),$$

with

(4) $|\partial^\beta (P_i - P_j)(x_j)| \leq \epsilon |x_i - x_j|^{m-|\beta|}$ for $|\beta| \leq m$, $0 \leq i, j \leq \bar{k}$; where $P_0 = 0$.

By taking δ smaller in (3), we may assume that $B(x_0, \delta)$ is a closed ball, and that $\delta < 1$.

Taking $i = 0$ in (4), we learn that

(5) $|\partial^\beta P_j(x_j)| \leq \epsilon |x_0 - x_j|^{m-|\beta|} \leq \epsilon \delta^{m-|\beta|} \leq \epsilon$ for $|\beta| \leq m$, $0 \leq j \leq m$.

In view of (3), (4) and (5), Lemma 3.1 applies to the Glaeser stable family of cosets $(f_\xi(x) - J_x(F_\xi^0) + I(x))_{x \in E \cap B(x_0, \delta)}$, with $A = \epsilon$ and $P_0 = 0$. Therefore, there exists $G \in C^m(\mathbb{R}^n)$, with

(6) $\| G \|_{C^m(\mathbb{R}^n)} \leq C\epsilon$, $J_x(G) \in f_\xi(x) - J_x(F_\xi^0) + I(x)$ for $x \in E \cap B(x_0, \delta)$, and $J_{x_0}(G) = 0$.

We can achieve (6) for any $x_0 \in E_0$. Lemma 12.2 follows trivially. ∎

§13. The Rescaled Induction Hypothesis

For $\delta > 0$ and $F \in C^m(\mathbb{R}^n)$, we introduce the norm

$$\| F \|_{C^m_\delta(\mathbb{R}^n)} = \max_{|\beta| \le m} \sup_{x \in \mathbb{R}^n} |\partial^\beta F(x)| \cdot \delta^{|\beta| - m} .$$

We write $C^m_\delta(\mathbb{R}^n)$ for the vector space $C^m(\mathbb{R}^n)$ equipped with the norm $\| \cdot \|_{C^m_\delta(\mathbb{R}^n)}$.

Next, suppose we are given a Glaeser stable family of ideals $\mathcal{J} = (I(x))_{x \in E}$, and a positive number δ. Let $f = (f(x))_{x \in E}$ be a family of m-jets, with $f(x) \in \mathcal{R}_x$ for all $x \in E$. We say that $f \in C^m_\delta(E, \mathcal{J})$ if there exists $F \in C^m_\delta(\mathbb{R}^n)$ with $J_x(F) \in f(x) + I(x)$ for all $x \in E$; and we write $\| f \|_{C^m_\delta(E,\mathcal{J})}$ for all infimum of $\| F \|_{C^m_\delta(\mathbb{R}^n)}$ over all such F.

Thus, $C^m_\delta(E, \mathcal{J})$ is a vector space equipped with a seminorm.

Note that $(f(x) + I(x))_{x \in E}$ is Glaeser stable for $f = (f(x))_{x \in E} \in C^m_\delta(E, \mathcal{J})$.

The purpose of this section is to establish the following simple result.

Lemma 13.1 (Rescaled Induction Hypothesis): *Fix $\wedge \ge 1$, and assume that Theorem 4 holds whenever the number of slices is less than $\wedge$.*

Let $\delta > 0$, and let $\mathcal{J} = (I(x))_{x \in E}$ be a Glaeser stable family of ideals, with fewer than $\wedge$ slices.

Then there exists a bounded linear map $T : C^m_\delta(E, \mathcal{J}) \to C^m_\delta(\mathbb{R}^n)$, with the following properties.

(A) *The norm of T is less than a constant C depending only on m and n.*

(B) *Let $f = (f(x))_{x \in E}$ belong to $C^m_\delta(E, \mathcal{J})$. Then*
$J_x(Tf) \in f(x) + I(x)$ for all $x \in E$.

Proof: By an obvious rescaling, we may assume that $\delta = 1$.

We now follow the reduction of Theorem 2 to Theorem 4 in Section 1.

We take $\Xi = C^m_1(E, \mathcal{J})$, with the seminorm $|\xi| = 2 \| \xi \|_{C^m_1(E,\mathcal{J})}$. For $x \in E$, there is a natural tautological map $\xi \mapsto f_\xi(x)$ from Ξ to $\mathcal{R}_x$, defined by $f_\xi(x) = g(x)$ for $\xi = (g(x))_{x \in E} \in \Xi$.

Thus, Ξ is a vector space with a seminorm $|\cdot|$, and $(f_\xi(x) + I(x))_{x\in E, \xi\in\Xi}$ is a Glaeser stable family of cosets, depending linearly on $\xi \in \Xi$.

Moreover, given $\xi \in \Xi$ with $|\xi| \leq 1$, there exists $F \in C^m(\mathbb{R}^n)$, with $\| F \|_{C^m(\mathbb{R}^n)} \leq 1$, and $J_x(F) \in f_\xi(x) + I(x)$ for all $x \in E$.

Thus, $\Xi, |\cdot|, (f_\xi(x) + I(x))_{x\in E,\xi\in\Xi}$ satisfy the hypotheses of Theorem 4. Also, by hypothesis, the number of slices is less than $\wedge$, and Theorem 4 holds whenever the number of slices is less than $\wedge$. Consequently, we obtain a linear map $T : \xi \mapsto F_\xi$, from Ξ into $C^m(\mathbb{R}^n)$, with norm at most C, such that

$$J_x(F_\xi) \in f_\xi(x) + I(x) \text{ for all } x \in E,\, \xi \in \Xi\,.$$

Recalling the definitions of $\Xi, |\cdot|, f_\xi(x)$, we conclude that the linear map T behaves as asserted in the statement of Lemma 13.1.

The proof of the lemma is complete. ■

§14. Whitney Cubes

Let E_0 be a compact subset of $\mathbb{R}^n$. We define a partition of $\mathbb{R}^n \smallsetminus E_0$ into "Whitney cubes" Q_ν, and we introduce cutoff functions θ_ν, θ_ν^+ adapted to the Q_ν.

We begin with some notation. Let Q be a cube in $\mathbb{R}^n$. To "bisect" Q means to partition it into 2^n congruent subcubes in the obvious way. Also, we write Q^* for the closed cube having the same center as Q, but with three times the diameter of Q. Similarly, we write Q^+ for the cube having the same center as Q, but with $(1 + c_1)$ times the diameter of Q. Here, c_1 is a small enough constant depending only on the dimension n.

To construct the Whitney cubes, we first partition $\mathbb{R}^n$ into a grid of cubes $Q_i^0, i = 1, 2, \ldots,$ with diameter 1. We than successively "bisect" each Q_i^0 in Calderón-Zygmund fashion, stopping at a cube Q whenever we have

$$\mathsf{dist}(Q^*, E_0) > \mathsf{diam}(Q^*)\,.$$

Let $\{Q_\nu\}$ be the collection of all the cubes obtained in this manner from all the Q_i^0; and let δ_ν be the diameter of Q_ν.

Then the Whitney cubes Q_ν have the following geometrical properties. (See, e.g., the proof of the classical Whitney extension theorem in [17,21,23].)

(1) The Q_ν form a partition of $\mathbb{R}^n \smallsetminus E_0$.

(2) Each Q_ν^* is a closed cube disjoint from E_0.

(3) $\delta_\nu \leq 1$.

(4) If $\delta_\nu < 1$, then there exists $x_\nu \in E_0$, with distance $(x_\nu, Q_\nu) \leq C\delta_\nu$, hence $Q_\nu^* \subset B(x_\nu, C'\delta_\nu)$.

(5) If Q_μ^+ and Q_ν^+ intersect, then $c < \delta_\mu/\delta_\nu < C$.

(6) For each ν, there are at most C distinct μ for which Q_μ^+ meets Q_ν^+.

(7) Each point of $\mathbb{R}^n \smallsetminus E_0$ has a neighborhood that meets at most C of the Q_ν^+.

(8) Given $x \in E_0$ and $\delta > 0$, there exists a neighborhood of x that intersects none of the Q_ν^+ with $\delta_\nu \geq \delta$.

(9) Given $x \in \mathbb{R}^n \smallsetminus E_0$, there exist a neighborhood U of x and a positive number $\delta(x)$, such that $\delta_\nu > \delta(x)$ for any ν such that Q_ν^+ intersects U.

Next, we introduce a "Whitney partition of unity". We can find functions $\theta_\nu, \theta_\nu^+ \in C^m(\mathbb{R}^n)$, with the following properties.

(10) $\sum\limits_\nu \theta_\nu = 1$ on $\mathbb{R}^n \smallsetminus E_0$.

(11) $\theta_\nu^+ = 1$ on $\mathsf{supp}\,(\theta_\nu)$, and $\mathsf{supp}\,\theta_\nu^+ \subset Q_\nu^+$.

(12) $|\partial^\beta \theta_\nu(x)|$, $|\partial^\beta \theta_\nu^+(x)| \leq C\delta_\nu^{-|\beta|}$ for $|\beta| \leq m$, $x \in \mathbb{R}^n$, all ν.

Again, see the proof of Whitney's classical theorem in [17,21,23].

We will use the above cubes and cutoff functions in the next section, taking E_0 to be the first slice.

§15. Proof of the Main Result

In this section, we give the proof of Theorem 4. As explained in the Introduction, we use induction on the number of slices. If the number of slices is zero, then Theorem 4 holds trivially (as also noted in the Introduction).

Fix $\wedge \geq 1$, and assume that

(1) Theorem 4 holds for the case of fewer than $\wedge$ slices.

Let Ξ, $|\cdot|$, $(f_\xi(x)+I(x))_{x\in E,\xi\in\Xi}$ be as in the hypotheses of Theorem 4, and assume that

(2) The number of slices for $(I(x))_{x\in E}$ is equal to $\wedge$.

Under these assumptions, we will prove the conclusion of Theorem 4.

This will complete our induction and establish Theorem 4.

Let E_0 be the first slice. We recall that E_0 is compact.

We use the Whitney cubes $\{Q_\nu\}$ and cutoff functions θ_ν, θ_ν^+ from the preceding section. These satisfy (14.1) , . . . , (14.12), with $\delta_\nu =$ diameter (Q_ν).

We apply the Second Main Lemma, and Lemmas 12.1 and 12.2.

Thus, we obtain a linear map $\xi \mapsto F_\xi^0$, from Ξ into $C^m(\mathbb{R}^n)$, with the following properties.

(3) If $\xi \in \Xi$ with $|\xi| \leq 1$, then $\| F_\xi^0 \|_{C^m(\mathbb{R}^n)} \leq C$.

(4) $J_x(F_\xi^0) \in f_\xi(x) + I(x)$ for all $x \in E_0$, $\xi \in \Xi$.

(5) Suppose $\xi \in \Xi$ with $|\xi| \leq 1$, and suppose $y_0 \in E_0$. Then there exists $G \in C^m(\mathbb{R}^n)$, with $\| G \|_{C^m(\mathbb{R}^n)} \leq C$, $J_x(G) \in f_\xi(x) - J_x(F_\xi^0) + I(x)$ for $x \in E$, $J_{y_0}(G) = 0$.

(6) Given $\xi \in \Xi$ and $\epsilon > 0$, there exists $\delta > 0$ with the following property:

Suppose $y_0 \in E_0$. Then there exists $G \in C^m(\mathbb{R}^n)$, with

$$\| G \|_{C^m(\mathbb{R}^n)} < \epsilon\, , J_x(G) \in f_\xi(x) - J_x(F_\xi^0) + I(x) \text{ for } x \in E \cap B(y_0, \delta)\, , \text{ and } J_{y_0}(G) = 0\, .$$

For each ν, we define

(7) $g_{\nu,\xi}(x) = J_x(\theta_\nu) \odot [f_\xi(x) - J_x(F^0_\xi)]$ for $x \in E \cap Q^*_\nu$, $\xi \in \Xi$.

Here, $\odot$ denotes multiplication in $\mathcal{R}_x$. Note that

(8) $\xi \mapsto g_{\nu,\xi}(x)$ is a linear map from Ξ into $\mathcal{R}_x$, for each $x \in E \cap Q^*_\nu$.

Note also that

(9) $\mathcal{J}_\nu \;=\; (I(x))_{x \in E \cap Q^*_\nu}$

is Glaeser stable, with fewer than $\wedge$ slices, thanks to (2) and (14.2).

Hence, Lemma 13.1 and (1) yield a linear map

(10) $T_\nu : C^m_{\delta_\nu}(E \cap Q^*_\nu, \mathcal{J}_\nu) \to C^m_{\delta_\nu}(\mathbb{R}^n)$, for each ν, with the following properties.

(11) The norm of T_ν is at most C.

(12) Let $g = (g(x))_{x \in E \cap Q^*_\nu} \;\in\; C^m_{\delta_\nu}(E \cap Q^*_\nu, \mathcal{J}_\nu)$. Then

$$J_x(T_\nu g) \;\in\; g(x) \;+\; I(x) \text{ for all } x \in E \cap Q^*_\nu\,.$$

We remark that the functions called $F_{\xi,\nu}$ in the Introduction are given here by

$$F_{\xi,\nu} \;=\; T_\nu(g_{\nu,\xi}) \text{ with } g_{\nu,\xi} \text{ as in (7)}\,.$$

The next two lemmas estimate the $C^m_{\delta_\nu}(E \cap Q^*_\nu,\, \mathcal{J}_\nu)$-seminorms of the $g_{\nu,\xi}$.

Lemma 15.1: *Let $\xi \in \Xi$, with $|\xi| \le 1$. Then $g_{\nu,\xi} \in C^m_{\delta_\nu}(E \cap Q^*_\nu, \mathcal{J}_\nu)$ and* $\| \, g_{\nu,\xi} \, \|_{C^m_{\delta_\nu}(E \cap Q^*_\nu, \mathcal{J}_\nu)} \le \; C$ for each ν.

Proof: We look separately at the cases $\delta_\nu < 1$ and $\delta_\nu = 1$. (See (14.3).)

Suppose first that $\delta_\nu < 1$. We let x_ν and C' be as in (14.4), and then apply (5), with $y_0 = x_\nu$. Let G be as in (5). Since $\| \, G \, \|_{C^m(\mathbb{R}^n)} \le C$ and $J_{x_\nu}(G) = 0$, we have $|\partial^\beta G(x)| \le C \delta_\nu^{m-|\beta|}$ for $|\beta| \le m$, $x \in B(x_\nu, C'\delta_\nu)$, and therefore for $|\beta| \le m$, $x \in Q^*_\nu$. Together with (14.11) and (14.12), this shows that

(13) $\theta_\nu G \in C^m(\mathbb{R}^n)$, with $|\partial^\beta(\theta_\nu G)(x)| \le C \delta_\nu^{m-|\beta|}$ for $|\beta| \le m$, $x \in \mathbb{R}^n$.

Also, for $x \in E \cap Q_\nu^*$, we have (with $\odot$ denoting multiplication in $\mathcal{R}_x$):

(14) $J_x(\theta_\nu G) \in J_x(\theta_\nu) \odot [f_\xi(x) - J_x(F_\xi^0) + I(x)]$ (by (5))

$\subseteq J_x(\theta_\nu) \odot [f_\xi(x) - J_x(F_\xi^0)] + I(x)$ (since $I(x)$ is an ideal)

$= g_{\nu,\xi}(x) + I(x)$ (by (7)).

The conclusion of Lemma 15.1 is immediate from (13), (14), and the definition of the $C^m_{\delta_\nu}(E \cap Q_\nu^*, \mathcal{J}_\nu)$-seminorm.

This proves Lemma 15.1 in the case $\delta_\nu < 1$.

On the other hand, suppose that $\delta_\nu = 1$. Since Ξ, $|\cdot|$, $(f_\xi(x) + I(x))_{x\in E, \xi \in \Xi}$ satisfy the hypotheses of Theorem 4, there exists $F \in C^m(\mathbb{R}^n)$, with

(15) $\| F \|_{C^m(\mathbb{R}^n)} \leq 1$, and $J_x(F) \in f_\xi(x) + I(x)$ for all $x \in E$.

From (15), (3), and (14.12) with $\delta_\nu = 1$, we learn that

(16) $G = \theta_\nu \cdot (F - F_\xi^0) \in C^m(\mathbb{R}^n)$, with $\| G \|_{C^m(\mathbb{R}^n)} \leq C$.

Moreover, for $x \in E \cap Q_\nu^*$, we have (with $\odot$ denoting multiplication in $\mathcal{R}_x$):

(17) $J_x(G) \in J_x(\theta_\nu) \odot [f_\xi(x) - J_x(F_\xi^0) + I(x)]$ (by (15))

$\subseteq J_x(\theta_\nu) \odot [f_\xi(x) - J_x(F_\xi^0)] + I(x)$ (since $I(x)$ is an ideal)

$= g_{\nu,\xi}(x) + I(x)$ (by (7)).

Comparing (16) and (17) with the definition of the $C^m_\delta(E, \mathcal{J})$-seminorm (with $\delta = 1$), we conclude that

$g_{\nu,\xi} \in C^m_{\delta_\nu}(E \cap Q_\nu^*, \mathcal{J}_\nu)$, with $\| g_{\nu,\xi} \|_{C^m_{\delta_\nu}(E\cap Q_\nu^*, \mathcal{J}_\nu)} \leq C$, in the case $\delta_\nu = 1$.

The proof of Lemma 15.1 is complete ■

Lemma 15.2: *Given $\xi \in \Xi$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$\| g_{\nu,\xi} \|_{C^m_{\delta_\nu}(E\cap Q_\nu^*, \mathcal{J}_\nu)} < \epsilon$ *for* $\delta_\nu < \delta$.

Proof: Let $\delta > 0$ be as in (6), and suppose $\delta_\nu < 1$.

Let x_ν and C' be as in (14.4). If $C'\delta_\nu < \delta$, then from (6) with $y_0 = x_\nu$, we obtain $G \in C^m(\mathbb{R}^n)$, with

(18) $\| G \|_{C^m(\mathbb{R}^n)} < \epsilon$ and $J_{x_\nu}(G) = 0$,

and

(19) $J_x(G) \in f_\xi(x) - J_x(F_\xi^0) + I(x)$ for $x \in E \cap B(x_\nu, C'\delta_\nu)$, hence for $x \in E \cap Q_\nu^*$.

From (18), we obtain

$|\partial^\beta G(x)| \leq C\epsilon\delta_\nu^{m-|\beta|}$ for $x \in B(x_\nu, C'\delta_\nu)$, $|\beta| \leq m$; hence for $x \in Q_\nu^*$, $|\beta| \leq m$.

Together with (14.11) and (14.12), this shows that

(20) $|\partial^\beta[\theta_\nu G]| \leq C\epsilon\delta_\nu^{m-|\beta|}$ on $\mathbb{R}^n$, for $|\beta| \leq m$.

Also, for $x \in E \cap Q_\nu^*$, we have (with $\odot$ denoting multiplication in $\mathcal{R}_x$):

(21)
$$\begin{aligned} J_x(\theta_\nu G) &\in J_x(\theta_\nu) \odot [f_\xi(x) - J_x(F_\xi^0) + I(x)] \text{ (by (19))} \\ &\subseteq J_x(\theta_\nu) \odot [f_\xi(x) - J_x(F_\xi^0)] + I(x) \text{ (since } I(x) \text{ is an ideal)} \\ &= g_{\nu,\xi}(x) + I(x) \text{ (see (7))}. \end{aligned}$$

Comparing (20), (21) with the definition of the $C^m_{\delta_\nu}(E \cap Q_\nu^*, \mathcal{J}_\nu)$-seminorm, we see that

(22) $\| g_{\nu,\xi} \|_{C^m_{\delta_\nu}(E\cap Q_\nu^*, \mathcal{J}_\nu)} \leq C\epsilon$.

We have proven (22) under the assumptions $\delta_\nu < 1$, $C'\delta_\nu < \delta$.

This trivially implies Lemma 15.2. ■

Now, for $\delta > 0$, we define

(23) $$F_\xi^{[\delta]} = F_\xi^0 + \sum_{\delta_\nu > \delta} \theta_\nu^+ \cdot T_\nu(g_{\nu,\xi}) \text{ for } \xi \in \Xi.$$

Note that

(24) For fixed $\delta > 0$, each $x \in \mathbb{R}^n$ has a neighborhood on which the sum in (23) includes at most C non-zero terms.

(This follows from (14.7), (14.8), (14.11).)

Also, we recall that F_ξ^0, θ_ν^+, and $T_\nu(g_{\nu,\xi})$ are C^m functions on $\mathbb{R}^n$.

It follows that $F_\xi^{[\delta]}$ is well-defined by (23), belongs to $C^m_{\ell oc}(\mathbb{R}^n)$, and satisfies the estimates:

(25) $\| F_\xi^{[\delta]} \|_{C^m(\mathbb{R}^n)} \leq \| F_\xi^0 \|_{C^m(\mathbb{R}^n)} + C \sup_\nu \| \theta_\nu^+ \cdot T_\nu(g_{\nu,\xi}) \|_{C^m(\mathbb{R}^n)}$ for $\xi \in \Xi$,

and

(26) $\| F_\xi^{[\bar{\delta}_1]} - F_\xi^{[\bar{\delta}_2]} \|_{C^m(\mathbb{R}^n)} \leq C \sup_{\bar{\delta}_1 \leq \delta_\nu \leq \bar{\delta}_2} \| \theta_\nu^+ \cdot T_\nu(g_{\nu,\xi}) \|_{C^m(\mathbb{R}^n)}$ for $0 < \bar{\delta}_1 < \bar{\delta}_2$ and $\xi \in \Xi$.

In particular, if the right-hand side of (25) is finite, then $F_\xi^{[\delta]}$ belongs to $C^m(\mathbb{R}^n)$.

Since $\xi \mapsto F_\xi^0$ is linear, and since each T_ν is linear, (23) gives

(27) $\xi \mapsto F_\xi^{[\delta]}$ is a linear map from Ξ into $C^m_{\ell oc}(\mathbb{R}^n)$, for each $\delta > 0$.

Next, we examine $J_x(F_\xi^{[\delta]})$ for $x \in \mathbb{R}^n$.

From (23) and (14.8), we obtain

(28) $J_x(F_\xi^{[\delta]}) = J_x(F_\xi^0)$ for $x \in E_0$, $\xi \in \Xi$, $\delta > 0$.

On the other hand, suppose $x \in \mathbb{R}^n \setminus E_0$. We define

(29) $\Omega(x) = \{\nu : x \in \mathrm{supp}\theta_\nu^+\}$ for $x \in \mathbb{R}^n \setminus E_0$.

From (14.7) and (14.11), we see that

(30) $\Omega(x)$ contains at most C elements.

From (14.10), (14.11), and (29), we have

$$(31) \quad \sum_{\nu \in \Omega(x)} J_x(\theta_\nu) \;=\; 1 \text{ for } x \in \mathbb{R}^n \setminus E_0.$$

Also, from (14.9), (14.11), and (23), (29), we see that

$$(32) \quad J_x(F_\xi^{[\delta]}) \;=\; J_x(F_\xi^0) \;+ \sum_{\nu \in \Omega(x)} J_x(\theta_\nu^+ \;\cdot\; T_\nu\, g_{\nu,\xi}) \text{ for } x \in \mathbb{R}^n \setminus E_0,\, 0 < \delta < \delta(x),\, \xi \in \Xi.$$

Here, $\delta(x)$ is the small positive number from (14.9).

We estimate the right-hand sides of (25) and (26). To do this, note that

$\| F \|_{C^m(\mathbb{R}^n)} \leq \| F \|_{C^m_\delta(\mathbb{R}^n)}$ for $\delta \leq 1$, and that $\delta_\nu \leq 1$. (See (14.3).)

Now, suppose $|\xi| \leq 1$. From (3), (10), (11), and Lemma 15.1, we learn that

$\| F_\xi^0 \|_{C^m(\mathbb{R}^n)} \leq C$ and

$$\| \theta_\nu^+ \;\cdot\; T_\nu(g_{\nu,\xi}) \|_{C^m(\mathbb{R}^n)} \leq \| \theta_\nu^+ \;\cdot\; T_\nu(g_{\nu,\xi}) \|_{C^m_{\delta_\nu}(\mathbb{R}^n)}$$

$$\leq\; C \,\| T_\nu(g_{\nu,\xi}) \|_{C^m_{\delta_\nu}(\mathbb{R}^n)} \text{ (see (14.12))} \leq$$

$$C \,\| g_{\nu,\xi} \|_{C^m_{\delta_\nu}(E \cap Q_\nu^*, \mathcal{J}_\nu)} \leq\; C.$$

Therefore, (25) shows that $F_\xi^{[\delta]} \in C^m(\mathbb{R}^n)$, with $\| F_\xi^{[\delta]} \|_{C^m(\mathbb{R}^n)} \leq\; C$.

Thus, we may sharpen (27) as follows.

(33) $\xi \mapsto F_\xi^{[\delta]}$ is a bounded linear map from Ξ into $C^m(\mathbb{R}^n)$, with norm at most C, for each $\delta > 0$.

Turning to the right-hand side of (26), we apply (14.12), (10), (11), and Lemma 15.2. Thus, let $\xi \in \Xi$ and $\epsilon > 0$ be given. If $0 < \bar{\delta}_1 < \bar{\delta}_2$ and $\bar{\delta}_2$ is small enough, then for $\bar{\delta}_1 \leq \delta_\nu \leq \bar{\delta}_2$ we have

$$\| \theta_\nu^+ \;\cdot\; T_\nu(g_{\nu,\xi}) \|_{C^m(\mathbb{R}^n)} \leq \| \theta_\nu^+ \;\cdot\; T_\nu(g_{\nu,\xi}) \|_{C^m_{\delta_\nu}(\mathbb{R}^n)}$$

$$\leq C \parallel T_\nu(g_{\nu,\xi}) \parallel_{C^m_{\delta_\nu}(\mathbb{R}^n)} \leq C \parallel g_{\nu,\xi} \parallel_{C^m_{\delta_\nu}(E\cap Q^*_\nu,\mathcal{J}_\nu)} < \epsilon\,.$$

Consequently, for fixed $\xi \in \Xi$ and $\epsilon > 0$, the right-hand side of (26) will be less than ϵ if $\bar{\delta}_2$ is small enough.

Hence, (26) shows that, for each fixed $\xi \in \Xi$, the function $\delta \mapsto F^{[\delta]}_\xi$, from $(0,1]$ into $C^m(\mathbb{R}^n)$, is Cauchy as $\delta \to 0+$. Consequently, there exists $F_\xi \in C^m(\mathbb{R}^n)$, such that

(34) $\lim\limits_{\delta\to 0+} F^{[\delta]}_\xi = F_\xi$ in $C^m(\mathbb{R}^n)$, for each $\xi \in \Xi$.

From (33) and (34), we see that

(35) $\xi \mapsto F_\xi$ is a bounded linear map from Ξ into $C^m(\mathbb{R}^n)$, with norm at most C.

We examine the jet $J_x(F_\xi)$ for $x \in E$.

From (4), (28), and (34), we obtain

(36) $J_x(F_\xi) \in f_\xi(x) + I(x)$ for $x \in E_0$, $\xi \in \Xi$.

On the other hand, suppose $x \in E \setminus E_0$, $\xi \in \Xi$. Then (7) and (12) yield

(37) $J_x(T_\nu g_{\nu,\xi}) \in g_{\nu,\xi}(x) + I(x) = J_x(\theta_\nu) \odot [f_\xi(x) - J_x(F^0_\xi)] + I(x)$ for all ν with $Q^+_\nu \ni x$, with $\odot$ denoting multiplication in $\mathcal{R}_x$.

We have $Q^+_\nu \ni x$ for all $\nu \in \Omega(x)$. (See (29) and (14.11).) Hence, for $\nu \in \Omega(x)$, (37) holds, and consequently

(38) $$\begin{aligned} J_x(\theta^+_\nu \cdot T_\nu g_{\nu,\xi}) &\in J_x(\theta^+_\nu) \odot [J_x(\theta_\nu) \odot (f_\xi(x) - J_x(F^0_\xi)) + I(x)] \\ &\subseteq J_x(\theta^+_\nu) \odot J_x(\theta_\nu) \odot [f_\xi(x) - J_x(F^0_\xi)] + I(x) \text{ (since } I(x) \text{ is an ideal)} \\ &= J_x(\theta_\nu) \odot [f_\xi(x) - J_x(F^0_\xi)] + I(x). \text{ (See (14.11).)} \end{aligned}$$

Summing (38) over all $\nu \in \Omega(x)$, and applying (31), we learn that

$$\sum_{\nu \in \Omega(x)} J_x(\theta_\nu^+ \cdot T_\nu g_{\nu,\xi}) \in f_\xi(x) - J_x(F_\xi^0) + I(x).$$

Hence, from (32) and (34), we obtain

$J_x(F_\xi) \in f_\xi(x) + I(x)$ for $x \in E \setminus E_0$, $\xi \in \Xi$.

Together with (36), this yields

(39) $J_x(F_\xi) \in f_\xi(x) + I(x)$ for $x \in E$, $\xi \in \Xi$.

Our results (35) and (39) are the conclusions of Theorem 4.

This completes our induction on $\wedge$, and thus proves Theorem 4.

Since we have already shown that Theorem 4 implies Theorem 2, which in turn implies Theorem 1, we have proven those results as well. ■

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