A SHARP FORM OF WHITNEY'S EXTENSION THEOREM

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0. Introduction

In this paper, we solve the following extension problem.

Problem 1. Suppose we are given a function $f : E \to \mathbb{R}$, where E is a given subset of $\mathbb{R}^n$. How can we decide whether f extends to a $C^{m-1,1}$ function F on $\mathbb{R}^n$?

Here, $m \geq 1$ is given. As usual, $C^{m-1,1}$ denotes the space of functions whose $(m-1)^{rst}$ derivatives are Lipschitz 1. We make no assumption on the set E or the function f.

This problem, with C^m in place of $C^{m-1,1}$, goes back to Whitney [15,16,17]. To answer it, we prove the following sharp form of the Whitney extension theorem.

Theorem A. *Given $m, n \geq 1$, there exists k, depending only on m and n, for which the following holds.*

Let $f : E \to \mathbb{R}$ be given, with E an arbitrary subset of $\mathbb{R}^n$.

Suppose that, for any k distinct points $x_1, \dots, x_k \in E$, there exist $(m-1)^{rst}$ degree polynomials $P_1, \dots, P_k$ on $\mathbb{R}^n$, satisfying

(a) *$P_i(x_i) = f(x_i)$ for $i = 1, \dots, k$;*
(b) *$|\partial^\beta P_i(x_i)| \leq M$ for $i = 1, \dots, k$ and $|\beta| \leq m-1$; and*
(c) *$|\partial^\beta (P_i - P_j)(x_i)| \leq M|x_i - x_j|^{m-|\beta|}$ for $i, j = 1, \dots, k$ and $|\beta| \leq m-1$; with M independent of $x_1, \dots, x_k$.*

Then f extends to a $C^{m-1,1}$ function on R^n

The converse of Theorem A is obvious, and the order of magnitude of the best possible M in (a), (b), (c) may be computed from $f(x_1), \dots, f(x_k)$ by elementary linear algebra, as we spell out in sections 1 and 2 below. Thus, Theorem A provides a solution to Problem 1. The point is that, in Theorem A, we need only extend the function value $f(x_i)$ to a jet P_i at a fixed, finite number of points $x_1, \dots, x_k$. To apply the standard Whitney extension theorem (see [9,13]) to Problem 1, we would first need to extend $f(x)$ to a jet P_x at every point $x \in E$. Note that each P_i in (a), (b), (c) is allowed to depend on $x_1, \dots, x_k$, rather than on x_i alone.

To prove Theorem A, it is natural to look for functions F of bounded $C^{m-1,1}$–norm on $\mathbb{R}^n$, that agree with f on arbitrarily large finite subsets $E_1 \subset E$. Thus, we arrive at a "finite extension problem".

Problem 2. Given a function $f : E \to \mathbb{R}$, defined on a finite subset $E \subset \mathbb{R}^n$, compute the order of magnitude of the infimum of the C^m norms of all the smooth functions $F : \mathbb{R}^n \to \mathbb{R}$ that agree with f on E.

To "compute the order of magnitude" here means to give computable upper and lower bounds M_{lower}, M_{upper}, with $M_{\text{upper}} \leq A\, M_{\text{lower}}$, for a constant A depending only on m and n. (In particular, A must be independent of the number and position of the points of E.) Here, we have passed from $C^{m-1,1}$ to C^m. For finite sets E, Problem 2 is completely equivalent to its analogue for $C^{m-1,1}$. (See section 18 below for the easy argument.)

Problem 2 calls to mind an experimentalist trying to determine an unknown function $F : \mathbb{R}^n \to \mathbb{R}$ by making finitely many measurements, i.e., determining $F(x)$ for x in a large finite set E. Of course, the experimentalist can never decide whether $F \in C^m$ by making finitely many measurements, but he or she can ask whether the data force the C^m norm of F to be large (or perhaps increasingly large as more data are collected). Real measurements of $f(x)$ will be subject to experimental error $\sigma(x) > 0$. Thus, we are led to a more general version of Problem 2, a "finite extension problem with error bars".

Problem 3. Let $E \subset \mathbb{R}^n$ be a finite set, and let $f : E \to \mathbb{R}$ and $\sigma : E \to [0, \infty)$ be given. How can we tell whether there exists a function $F : \mathbb{R}^n \to \mathbb{R}$, with $|F(x) - f(x)| \lesssim \sigma(x)$ for all $x \in E$, and $\|F\|_{C^m(\mathbb{R}^n)} \lesssim 1$?

Here, $P \lesssim Q$ means that $P \leq A \cdot Q$ for a constant A depending only on m and n. (In particular, A must be independent of the set E.)

This problem is solved by the following analogue of Theorem A for finite sets E.

Theorem B. *Given $m, n \geq 1$, there exists $k^\#$, depending only on m and n, for which the following holds.*

Let $f : E \to \mathbb{R}$ and $\sigma : E \to [0, \infty)$ be functions defined on a finite set $E \subset \mathbb{R}^n$. Let M be a given, positive number. Suppose that, for any k distinct points $x_1, \ldots, x_k \in E$, with $k \leq k^\#$, there exist $(m-1)^{rst}$ degree polynomials $P_1, \ldots, P_k$ on $\mathbb{R}^n$, satisfying

(a) $|P_i(x_i) - f(x_i)| \leq \sigma(x_i)$ *for* $i = 1, \ldots, k$;
(b) $|\partial^\beta P_i(x_i)| \leq M$ *for* $i = 1, \ldots, k$ *and* $|\beta| \leq m-1$; *and*
(c) $|\partial^\beta (P_i - P_j)(x_i)| \leq M \cdot |x_i - x_j|^{m-|\beta|}$ *for* $i, j = 1, \ldots, k$ *and* $|\beta| \leq m-1$.

Then there exists $F \in C^m(\mathbb{R}^n)$, with $\|F\|_{C^m(\mathbb{R}^n)} \leq A \cdot M$, and $|F(x) - f(x)| \leq A \cdot \sigma(x)$ for all $x \in E$.

Here, the constant A depends only on m and n.

Again, the point of Theorem B is that we need look only at a fixed number $k^\#$ of points of E, even though E may contain arbitrarily many points. Theorem B solves Problem 3; by specialization to $\sigma \equiv 0$, it also solves Problem 2. Once we know Theorem B, a compactness argument using Ascoli's theorem allows us to deduce Theorem A, in a more general form involving error bars. In turn, Theorem B may be reduced to the following result, by applying the standard Whitney extension theorem.

Theorem C. *Given $m, n \geq 1$, there exist $k^\#$ and A, depending only on m and n, for which the following holds. Let $f : E \to \mathbb{R}$ and $\sigma : E \to [0, \infty)$ be functions on a finite set $E \subset \mathbb{R}^n$. Suppose that, for every subset $S \subset E$ with at most $k^\#$ elements, there exists a function $F^S \in C^m(\mathbb{R}^n)$, with $\|F^S\|_{C^m(\mathbb{R}^n)} \leq 1$, and $|F^S(x) - f(x)| \leq \sigma(x)$ for all $x \in S$.*

Then there exists a function $F \in C^m(\mathbb{R}^n)$, with $\|F\|_{C^m(\mathbb{R}^n)} \leq A$, and $|F(x) - f(x)| \leq A \cdot \sigma(x)$ for all $x \in E$.

Thus, Theorem C is the heart of the matter. In a moment, we sketch some of the ideas in the proof of Theorem C.

First, however, we make a few remarks on the analogue of Problem 1 with C^m in place of $C^{m-1,1}$. This is the most classical form of Whitney's extension problem. Whitney himself solved the one–dimensional case in terms of finite differences (see [16]). A geometrical solution for the case of $C^1(\mathbb{R}^n)$ was given by Glaeser [8], who introduced the notion of an "iterated paratangent bundle". The correct notion of an iterated paratangent bundle relevant to $C^m(\mathbb{R}^n)$ was introduced by Bierstone–Milman–Pawl ucki. (See [1], which proves an extension theorem for subanalytic sets.) It would be very interesting to generalize the extension theorem of [1] from subanalytic to arbitrary subsets of $\mathbb{R}^n$. I hope that the ideas in this paper will be helpful in carrying this out. I have been greatly helped by discussions with Bierstone and Milman.

Y. Brudnyi and P. Shvartsman conjectured a result analogous to our Theorem C, but without the function σ, and with $C^{m-1,1}$ replaced by more general function spaces. They conjectured also that the extension F may be taken to depend linearly on f. For function spaces between C^0 and $C^{1,1}$, they succeeded in proving their conjectures by the elegant method of "Lipschitz selection," obtaining in particular an optimal $k^\#$. Their results solve our Problem 1 in the simplest nontrivial case, $m = 2$. We refer the reader to [2,3,4,5,6,10,11,12] for the above, and for additional results and conjectures. A forthcoming paper [7] will settle some of the issues raised by Brudnyi and Shvartsman, to whom I am grateful for bringing these matters to my attention.

Next, we explain some ideas from the proof of Theorem C, sacrificing accuracy for ease of understanding.

One ingredient in our proof is the following standard result on convex sets.

Helly's Theorem (see, e.g., [14]). *Let $\mathcal{J}$ be a family of compact, convex subsets of $\mathbb{R}^d$, any (d+1) of which have non–empty intersection. Then the whole family $\mathcal{J}$ has non–empty intersection.*

The following observation is typical of our repeated applications of Helly's theorem in the proof of Theorem C. Let $\mathcal{P}$ denote the vector space of $(m-1)^{rst}$ degree polynomials on $\mathbb{R}^n$, and let D be its dimension. For $F \in C^m(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$, let $J_y(F)$ denote the $(m-1)$ jet of F at y. Let E, f, σ be as in the hypotheses of Theorem C. Fix $y \in \mathbb{R}^n$. Then there exists a polynomial $P_y \in \mathcal{P}$, with the following property:

(1) Given $S \subset E$ with at most $k^{\#}/(D+1)$ elements, there exists $F^S \in C^m(\mathbb{R}^n)$, with $\|F^S\|_{C^m(\mathbb{R}^n)} \leq 1$, $|F^S(x) - f(x)| \leq \sigma(x)$ on S, and $J_y(F^S) = P_y$.

Thus, we can pin down the $(m-1)$ jet of F^S at a single point y, at the cost of passing from $k^{\#}$ to $k^{\#}/(D+1)$. We may regard P_y as a plausible guess for the $(m-1)$ jet at y of the function F in the conclusion of Theorem C. Let us call P_y a "putative Taylor polynomial".

To prove (1), let $\mathcal{S}$ denote the family of subsets $S \subset E$ with at most $k^{\#}/(D+1)$ elements. To each $S \subset E$ (not necessarily in $\mathcal{S}$), we associate a subset $\mathcal{K}(S) \subset \mathcal{P}$, defined by $\mathcal{K}(S) = \{J_y(F) : \|F\|_{C^m(\mathbb{R}^n)} \leq 1 \ , \ |F(x) - f(x)| \leq \sigma(x) \text{ on } S\}$. Each $\mathcal{K}(S)$ is convex and bounded. In this heuristic introduction, we ignore the question of whether $\mathcal{K}(S)$ is compact. If $S_1, \ldots, S_{D+1} \in \mathcal{S}$ are given, then $S = S_1 \cup \cdots \cup S_{D+1} \subset E$ has at most $k^{\#}$ elements, hence $\mathcal{K}(S)$ is non–empty, thanks to the hypothesis of Theorem C. On the other hand, we have the obvious inclusion $\mathcal{K}(S) \subseteq \mathcal{K}(S_i)$ for each i. Therefore, $\mathcal{K}(S_1) \cap \cdots \cap \mathcal{K}(S_{D+1})$ is non–empty, for any $S_1, \ldots, S_{D+1} \in \mathcal{S}$. Applying Helly's theorem, we obtain a polynomial $P_y \in \mathcal{P}$ belonging to $\mathcal{K}(S)$ for every $S \in \mathcal{S}$. Property (1) is now immediate from the definition of $\mathcal{K}(S)$.

Unfortunately, property (1) needn't uniquely specify the polynomial P_y. Therefore, if we are not careful, we may associate to two nearby points y and y' putative Taylor polynomials P_y and $P_{y'}$ that have nothing to do with each other. If we are hoping that P_y and $P_{y'}$ will be the jets of a single C^m function at the points y and y', then we will be in for a surprise.

To express the ambiguity in choosing a putative Taylor polynomial, we introduce the notion of a polynomial that is "small on E near y". If $y \in \mathbb{R}^n$ and $\hat{P} \in \mathcal{P}$ is a polynomial, then we say that $\hat{P}$ is small on E near y, provided the following holds:

(2) Given $S \subset E$ with at most $k^{\#}/(D+1)$ elements, there exists $\varphi^S \in C^m(\mathbb{R}^n)$, with

$\|\varphi^S\|_{C^m(\mathbb{R}^n)} \le A$, $|\varphi^S(x)| \le A\sigma(x)$ on S, and $J_y(\varphi^S) = \hat{P}$.

Here, A is a suitable constant. The connection of this notion to the ambiguity of the putative Taylor polynomial P_y is immediately clear. If two polynomials $P_y^{(1)}$ and $P_y^{(2)}$ both satisfy (1), then their difference $P_y^{(1)} - P_y^{(2)}$ evidently satisfies (2), with $A = 2$. Conversely, if P_y satisfies (1), and $\hat{P}$ satisfies (2), then one sees easily that $P_y + \hat{P}$ satisfies the following condition, which is essentially as good as (1):

(3) Given $S \subset E$ with at most $k^{\#}/(D+1)$ elements, there exists $\tilde{F}^S \in C^m(\mathbb{R}^n)$, with $\|\tilde{F}^S\|_{C^m(\mathbb{R}^n)} \le A+1$, $|\tilde{F}^S(x) - f(x)| \le (A+1)\cdot\sigma(x)$ on S, and $J_y(\tilde{F}^S) = P_y + \hat{P}$.

Thus, the ambiguity in the putative Taylor polynomial lies precisely in the freedom to add an arbitrary polynomial $\hat{P} \in \mathcal{P}$ that is "small on E near y".

It is therefore essential to keep track of which polynomials $\hat{P}$ are small on E near y. If $\mathcal{A}$ is a set of multi-indices $\beta = (\beta_1, \dots, \beta_n)$ of order $|\beta| = \beta_1 + \cdots + \beta_n \le m-1$, then let us say that E has "type $\mathcal{A}$" at y (with respect to σ) if there exist polynomials $P_\alpha \in \mathcal{P}$, indexed by $\alpha \in \mathcal{A}$, that satisfy the conditions

(4) Each P_α is small on E near y, and

(5) $\partial^\beta P_\alpha(y) = \delta_{\beta\alpha}$ (Kronecker delta) for $\beta, \alpha \in \mathcal{A}$.

Note that if E has type $\mathcal{A}$, then automatically E has type $\mathcal{A}'$ for any subset $\mathcal{A}' \subset \mathcal{A}$.

A crucial idea in our proof is to formulate a "MAIN LEMMA for $\mathcal{A}$", for each set $\mathcal{A}$ of multi–indices of order $\le m-1$. The Main Lemma for $\mathcal{A}$ says roughly that if E has "type $\mathcal{A}$" at y, then a local form of Theorem C holds in a fixed neighborhood of y. Suppose we can prove the Main Lemma for all $\mathcal{A}$. Taking $\mathcal{A}$ to be the empty set, we know that (trivially) E has type $\mathcal{A}$ at every point $y \in \mathbb{R}^n$. Hence, a local form of Theorem C holds in a ball of fixed radius about any point y. A partition of unity allows us to patch together these local results, and deduce Theorem C.

Thus, we have reduced matters to the task of proving the Main Lemma for any set $\mathcal{A}$ of multi–indices of order $\le m-1$. We proceed by induction on $\mathcal{A}$, where the sets $\mathcal{A}$ are given a natural order $<$. In particular, if $\mathcal{A}' \subset \mathcal{A}$, then $\mathcal{A} < \mathcal{A}'$ under our order; thus, the empty set is maximal, and the set $\mathcal{M}$ of all multi–indices of order $\le m-1$ is minimal under $<$. The induction on $\mathcal{A}$ thus starts with $\mathcal{A} = \mathcal{M}$ and ends with $\mathcal{A} =$ empty set.

For $\mathcal{A} = \mathcal{M}$, the Main Lemma is trivial, essentially because the hypothesis that E is of type $\mathcal{M}$ forces $\sigma(x)$ to be so big that we may take $F \equiv 0$ in the conclusion of Theorem C, without noticing the error.

For the induction step, we fix $\mathcal{A} \neq \mathcal{M}$, and assume that the MAIN LEMMA holds for all $\mathcal{A}' < \mathcal{A}$. We have to prove the MAIN LEMMA for $\mathcal{A}$. Thus, suppose E is of type $\mathcal{A}$ at y. We start with a cube Q° of small, fixed sidelength, centered at y. We then make a Calderón–Zygmund decomposition of Q° into subcubes $\{Q_\nu\}$. To construct the Q_ν, we repeatedly "bisect" Q° into ever smaller subcubes, stopping at Q_ν when, after rescaling Q_ν to the unit cube, we find that E has type $\mathcal{A}'$ for some $\mathcal{A}' < \mathcal{A}$. Using the induction hypothesis, we can deal with each Q_ν locally. We can patch together the local solutions using a partition of unity adapted to the Calderón–Zygmund decomposition. This completes the induction step, establishing the Main Lemma for every $\mathcal{A}$, and completing the proof of Theorem C.

We again warn the reader that the above summary is oversimplified. For instance, there are actually two Main Lemmas for each $\mathcal{A}$. The phrases "putative Taylor polynomial", "small on E near y", and "type $\mathcal{A}$" do not appear in the rigorous discussion below; they are meant here to motivate some of the rigorous developments in sections 1 through 19.

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1. Notation

Fix $m, n \geq 1$ throughout this paper.

$C^m(\mathbb{R}^n)$ denotes the space of functions $F : \mathbb{R}^n \to \mathbb{R}$ whose derivatives of order $\leq m$ are continuous and bounded on $\mathbb{R}^n$. For $F \in C^m(\mathbb{R}^n)$, we define $\|F\|_{C^m(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} \max_{|\beta| \leq m} |\partial^\beta F(x)|$, and $\|\partial^m F\|_{C^0(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} \max_{|\beta| = m} |\partial^\beta F(x)|$. For $F \in C^m(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$, we define $J_y(F)$ to be the $(m-1)$ jet of F at y, i.e., the polynomial

$$J_y(F)(x) = \sum_{|\beta| \leq m-1} \frac{1}{\beta!} \big(\partial^\beta F(y)\big) \cdot (x-y)^\beta.$$

$C^{m-1,1}(\mathbb{R}^n)$ denotes the space of all functions $F : \mathbb{R}^n \to \mathbb{R}$, whose derivatives of order $\leq m-1$ are continuous, and for which the norm

$$\|F\|_{C^{m-1,1}(\mathbb{R}^n)} = \max_{|\beta| \leq m-1} \left\{ \sup_{x \in \mathbb{R}^n} |\partial^\beta F(x)| + \sup_{\substack{x,y \in \mathbb{R}^n \\ x \neq y}} \frac{|\partial^\beta F(x) - \partial^\beta F(y)|}{|x-y|} \right\}$$

is finite.

Let $\mathcal{P}$ denote the vector space of polynomials of degree $\leq m-1$ on $\mathbb{R}^n$ (with real coefficients), and let D denote the dimension of $\mathcal{P}$.

Let $\mathcal{M}$ denote the set of all multi–indices $\beta = (\beta_1, \dots, \beta_n)$ with $|\beta| = \beta_1 + \cdots + \beta_n \leq m-1$.

Let $\mathcal{M}^+$ denote the set of multi–indices $\beta = (\beta_1, \dots, \beta_n)$ with $|\beta| \leq m$.

If α and β are multi–indices, then $\delta_{\beta\alpha}$ denotes the Kronecker delta, equal to 1 if $\beta = \alpha$ and 0 otherwise.

We will be dealing with functions of x parametrized by y ($x, y \in \mathbb{R}^n$). We will often denote these by $\varphi^y(x)$, or by $P^y(x)$ in case $x \mapsto P^y(x)$ is a polynomial for fixed y. When we write $\partial^\beta P^y(y)$, we always mean the value of $\left(\frac{\partial}{\partial x}\right)^\beta P^y(x)$ at $x = y$; we never use $\partial^\beta P^y(y)$ to denote the derivative of order β of the function $y \mapsto P^y(y)$.

We write $B(x, r)$ to denote the ball with center x and radius r in $\mathbb{R}^n$. If Q is a cube in $\mathbb{R}^n$, then δ_Q denotes the diameter of Q; and $Q^\star$ denotes the cube whose center is that of Q, and whose diameter is 3 times that of Q.

If Q is a cube in $\mathbb{R}^n$, then to "bisect" Q is to partition it into 2^n congruent subcubes in the obvious way. Later on, we will fix a cube $Q^\circ \subset \mathbb{R}^n$, and define the class of "dyadic" cubes to consist of Q°, together with all the cubes arising from Q° by repeated bisection. Each dyadic cube Q other than Q° arises from bisecting a dyadic cube $Q^+ \subseteq Q^\circ$, with $\delta_{Q^+} = 2\delta_Q$. We call Q^+ the dyadic "parent" of Q. Note that $Q^+ \subset Q^\star$.

For any finite set X, write $\#(X)$ to denote the number of elements of X. If X is infinite, then we define $\#(X) = \infty$.

This paper is divided into sections. The label (p.q) refers to formula (q) in section p. Within section p, we abbreviate (p. q) to (q).

Let $\vec{x} = (x_1, \dots, x_k)$ be a finite sequence consisting of k distinct points of $\mathbb{R}^n$. On the vector space $\mathcal{P} \oplus \cdots \oplus \mathcal{P}$ (k copies), we define quadratic forms $\mathcal{Q}_\circ(\cdot; \vec{x})$, $\mathcal{Q}_1(\cdot; \vec{x})$, $\mathcal{Q}(\cdot; \vec{x})$ as follows. Given $\vec{P} = (P_\mu)_{1 \leq \mu \leq k} \in \mathcal{P} \oplus \cdots \oplus \mathcal{P}$, we define

$$\begin{aligned}
\mathcal{Q}_\circ(\vec{P}; \vec{x}) &= \sum_{1 \leq \mu \leq k} \sum_{|\beta| \leq m-1} (\partial^\beta (P_\mu)(x_\mu))^2 \\
\mathcal{Q}_1(\vec{P}; \vec{x}) &= \sum_{\substack{1 \leq \mu, \nu \leq k \\ (\mu \neq \nu)}} \sum_{|\beta| \leq m-1} (\partial^\beta (P_\mu - P_\nu)(x_\nu))^2 \cdot |x_\mu - x_\nu|^{-2(m - |\beta|)} \\
\mathcal{Q}(\vec{P}; \vec{x}) &= \mathcal{Q}_\circ(\vec{P}; \vec{x}) + \mathcal{Q}_1(\vec{P}; \vec{x}).
\end{aligned}$$

If $f : E \to \mathbb{R}$ with $x_1, \dots, x_k \in E$, then we define $\|f\|^2_{C^m(\vec{x})}$ to be the minimum of $\mathcal{Q}(\vec{P}; \vec{x})$ over all $\vec{P} = (P_\mu)_{1 \le \mu \le k} \in \mathcal{P} \oplus \cdots \oplus \mathcal{P}$ subject to the constraints $P_\mu(x_\mu) = f(x_\mu)$ for all $\mu = 1, \dots, k$.

Note that elementary linear algebra gives

$$\|f\|^2_{C^m(\vec{x})} = \sum_{\mu,\nu=1}^{k} a_{\mu\nu}(\vec{x}) f(x_\mu) f(x_\nu)$$

for a positive–definite matrix $(a_{\mu\,\nu}(\vec{x}))$ whose entries are rational functions of $x_1, \dots, x_k$.

2. Statement of Results

Theorem 1. *Given $m, n \ge 1$, there exist constants $k^\#, A$, depending only on m and n, for which the following holds.*

Let $E \subset \mathbb{R}^n$ be a finite set, and let $f : E \to \mathbb{R}$ and $\sigma : E \to [0, \infty)$ be functions on E.

Assume that, for every subset $S \subset E$ with $\#(S) \le k^\#$, there exists a function $F^S \in C^m(\mathbb{R}^n)$, with $\|F^S\|_{C^m(\mathbb{R}^n)} \le 1$, and $|F^S(x) - f(x)| \le \sigma(x)$ for all $x \in S$.

Then there exists a function $F \in C^m(\mathbb{R}^n)$, with $\|F\|_{C^m(\mathbb{R}^n)} \le A$, and $|F(x) - f(x)| \le A\sigma(x)$ for all $x \in E$.

Theorem 2. *Given $m, n \ge 1$, there exist constants $k^\#, A$, depending only on m and n, for which the following holds.*

Let $E \subset \mathbb{R}^n$ be an arbitrary subset, and let $f : E \to \mathbb{R}$ and $\sigma : E \to [0, \infty)$ be functions on E.

Assume that, for every subset $S \subset E$ with $\#(S) \le k^\#$, there exists a function $F^S \in C^{m-1,1}(\mathbb{R}^n)$, with $\|F^S\|_{C^{m-1,1}(\mathbb{R}^n)} \le 1$, and $|F^S(x) - f(x)| \le \sigma(x)$ for all $x \in S$.

Then there exists a function $F \in C^{m-1,1}(\mathbb{R}^n)$, with $\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \le A$, and $|F(x) - f(x)| \le A\sigma(x)$ for all $x \in E$.

Theorem 3. *Given $m, n \ge 1$, there exists $k^\#$, depending only on m and n, for which the following holds.*

Let $E \subset \mathbb{R}^n$ be an arbitrary subset, and let $f : E \to \mathbb{R}$ be a function on E. Then f

extends to a $C^{m-1,1}$ function on $\mathbb{R}^n$, if and only if

$$\sup_{\vec{x}} \|f\|_{C^m(\vec{x})} < \infty,$$

where $\vec{x}$ varies over all sequences $(x_1, \dots, x_k)$ consisting of at most $k^\#$ distinct elements of E.

3. Order Relations Involving Multi–indices

We introduce an order relation on multi-indices. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ be distinct multi–indices. Since α and β are distinct, we cannot have $\alpha_1 + \cdots + \alpha_k = \beta_1 + \cdots + \beta_k$ for all $k = 1, \dots, n$. Let $\bar{k}$ be the largest k for which $\alpha_1 + \cdots + \alpha_k \neq \beta_1 + \cdots + \beta_k$. Then we say that $\alpha < \beta$ if and only if $\alpha_1 + \cdots + \alpha_{\bar{k}} < \beta_1 + \cdots + \beta_{\bar{k}}$. One checks easily that this defines an order relation. We use this order relation on multi–indices throughout this paper.

Next, we introduce an order relation on subsets of $\mathcal{M}$, the set of multi–indices of order at most $m-1$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are distinct subsets of $\mathcal{M}$. Then the symmetric difference $\mathcal{A} \vartriangle \mathcal{B} = (\mathcal{A} \setminus \mathcal{B}) \cup (\mathcal{B} \setminus \mathcal{A})$ is non–empty. Let α be the least element of $\mathcal{A} \vartriangle \mathcal{B}$ (under the above ordering on multi–indices). We say that $\mathcal{A} < \mathcal{B}$ if α belongs to $\mathcal{A}$. Again, one checks easily that this defines an order relation; and we use this order relation on sets of multi–indices throughout this paper.

We need a few simple results on the above order relations.

Lemma 3.1. *If α and β are multi–indices with $|\alpha| < |\beta|$, then $\alpha < \beta$.*

Lemma 3.2. *If $\mathcal{A}, \bar{\mathcal{A}} \subset \mathcal{M}$, and if $\mathcal{A} \subseteq \bar{\mathcal{A}}$, then $\bar{\mathcal{A}} \leq \mathcal{A}$.*

Lemma 3.3. *Let $\mathcal{A} \subset \mathcal{M}$, and let $\phi : \mathcal{A} \to \mathcal{M}$. Suppose that*

(1) *$\phi(\alpha) \leq \alpha$ for all $\alpha \in \mathcal{A}$.*
(2) *For each $\alpha \in \mathcal{A}$, either $\phi(\alpha) = \alpha$ or $\phi(\alpha) \notin \mathcal{A}$.*

Then $\phi(\mathcal{A}) \leq \mathcal{A}$, with equality if and only if ϕ is the identity map.

Lemmas 3.1 and 3.2 are immediate from the definitions. We give the proof of Lemma 3.3. First we show that $\phi(\mathcal{A}) \leq \mathcal{A}$. We use induction on $\#(\mathcal{A})$, the number of elements of $\mathcal{A}$. For $\#(\mathcal{A}) = 0$, the lemma holds trivially, since $\mathcal{A} = \phi(\mathcal{A}) =$ empty set. For the induction step, fix $k \geq 1$, assume that (1) and (2) imply $\phi(\mathcal{A}) \leq \mathcal{A}$ whenever $\#(\mathcal{A}) = k-1$, and fix $\mathcal{A} \subset \mathcal{M}$ with $\#(\mathcal{A}) = k$. Let α be the least element of $\mathcal{A}$, and let β be the least element of $\phi(\mathcal{A})$. From (1) we see that $\beta \le \alpha$. If $\beta < \alpha$, then β is the least element of $\phi(\mathcal{A}) \vartriangle \mathcal{A}$, hence $\phi(\mathcal{A}) < \mathcal{A}$ by definition. If instead $\beta = \alpha$, then we apply our induction hypothesis to $\mathcal{A} \smallsetminus \{\alpha\}$. Note that $\#(\mathcal{A} \smallsetminus \{\alpha\}) = k-1$, and that

$$(\mathcal{A} \smallsetminus \{\alpha\}) \vartriangle \phi(\mathcal{A} \smallsetminus \{\alpha\}) = \mathcal{A} \vartriangle \phi(\mathcal{A}). \tag{3}$$

Inductive hypothesis gives $\phi(\mathcal{A} \smallsetminus \{\alpha\}) \le \mathcal{A} \smallsetminus \{\alpha\}$, and therefore $\phi(\mathcal{A}) \le \mathcal{A}$, thanks to (3). This completes the induction step. Hence, (1) and (2) imply $\phi(\mathcal{A}) \le \mathcal{A}$. Also, (2) shows at once that $\phi(\mathcal{A}) \ne \mathcal{A}$ whenever ϕ is not the identity map. The proof of Lemma 3.3 is complete. ■

Note that in view of Lemma 3.2, the empty set is maximal, and the set $\mathcal{M}$ is minimal, under the order $<$.

4. Statement of Two Main Lemmas

Fix $\mathcal{A} \subseteq \mathcal{M}$. We state two results involving $\mathcal{A}$.

Weak Main Lemma for $\mathcal{A}$. *Given $m, n \ge 1$, there exist constants $k^{\#}, a_0$, depending only on m and n, for which the following holds.*

Suppose we are given a finite set $E \subset \mathbb{R}^n$ and functions $f : E \to \mathbb{R}$ and $\sigma : E \to (0, \infty)$. Suppose we are also given a point $y^0 \in \mathbb{R}^n$ and a family of polynomials $P_\alpha \in \mathcal{P}$, indexed by $\alpha \in \mathcal{A}$. Assume that the following conditions are satisfied:

(WL1). $\partial^\beta P_\alpha(y^0) = \delta_{\beta\,\alpha}$ *for all* $\beta, \alpha \in \mathcal{A}$.

(WL2). $|\partial^\beta P_\alpha(y^0) - \delta_{\beta\alpha}| \le a_0$ *for all* $\alpha \in \mathcal{A}$, $\beta \in \mathcal{M}$.

(WL3). *Given $S \subset E$ with $\#(S) \le k^{\#}$, and given $\alpha \in \mathcal{A}$, there exists $\varphi^S_\alpha \in C^m(\mathbb{R}^n)$, with*

(a) $\|\partial^m \varphi^S_\alpha\|_{C^0(\mathbb{R}^n)} \le a_0$.
(b) $|\varphi^S_\alpha(x)| \le C\sigma(x)$ *for all* $x \in S$.
(c) $J_{y^0}(\varphi^S_\alpha) = P_\alpha$.

(WL4). *Given $S \subset E$ with $\#(S) \le k^{\#}$, there exists $F^S \in C^m(\mathbb{R}^n)$, with*

(a) $\|F^S\|_{C^m(\mathbb{R}^n)} \le C$.
(b) $|F^S(x) - f(x)| \le C\sigma(x)$ *for all* $x \in S$.

Then there exists $F \in C^m(\mathbb{R}^n)$, with

(WL5). $\|F\|_{C^m(\mathbb{R}^n)} \le C'$ *and*

(WL6). $|F(x) - f(x)| \le C'\sigma(x)$ *for all* $x \in E \cap B(y^0, c')$.

Here, C' *and* c' *in (WL5, 6) depend only on* C, m, n *in (WL1,* $\dots, 4$*).*

Strong Main Lemma for $\mathcal{A}$. *Given* $m, n \ge 1$*, there exists* $k^\#$*, depending only on* m *and* n*, for which the following holds.*

Suppose we are given a finite set $E \subset \mathbb{R}^n$*, and functions* $f : E \to \mathbb{R}$ *and* $\sigma : E \to (0, \infty)$*. Suppose we are also given a point* $y^0 \in \mathbb{R}^n$*, and a family of polynomials* $P_\alpha \in \mathcal{P}$*, indexed by* $\alpha \in \mathcal{A}$*. Assume that the following conditions are satisfied:*

(SL1). $\partial^\beta P_\alpha(y^0) = \delta_{\beta\,\alpha}$ *for all* $\alpha, \beta \in \mathcal{A}$.

(SL2). $|\partial^\beta P_\alpha(y^0)| \le C$ *for all* $\alpha \in \mathcal{A}, \beta \in \mathcal{M}$ *with* $\beta \ge \alpha$.

(SL3). *Given* $S \subset E$ *with* $\#(S) \le k^\#$*, and given* $\alpha \in \mathcal{A}$ *there exists* $\varphi_\alpha^S \in C^m(\mathbb{R}^n)$*, with*

(a) $\|\partial^m \varphi_\alpha^S\|_{C^0(\mathbb{R}^n)} \le C$.
(b) $|\varphi_\alpha^S(x)| \le C\sigma(x)$ *for all* $x \in S$.
(c) $J_{y^0}(\varphi_\alpha^S) = P_\alpha$.

(SL4). *Given* $S \subset E$ *with* $\#(S) \le k^\#$*, there exists* $F^S \in C^m(\mathbb{R}^n)$*, with*

(a) $\|F^S\|_{C^m(\mathbb{R}^n)} \le C$.
(b) $|F^S(x) - f(x)| \le C\sigma(x)$ *for all* $x \in S$.

Then there exists $F \in C^m(\mathbb{R}^n)$, with

(SL5). $\|F\|_{C^m(\mathbb{R}^n)} \le C'$

and

(SL6). $|F(x) - f(x)| \le C'\sigma(x)$ *for all* $x \in E \cap B(y^0, c')$.

Here, C' *and* c' *in (SL5, 6) depend only on* C, m, n *in (SL1,* $\dots, 4$*).*

5. Plan of the Proof

We explain here the plan of our proof of our two MAIN LEMMAS, and we indicate briefly how these lemmas imply Theorems 1, 2, 3. To prove the MAIN LEMMAS for $\mathcal{A}$, we proceed by induction on $\mathcal{A}$, where subsets $\mathcal{A} \subseteq \mathcal{M}$ are ordered by $<$ as described in section 3. More precisely, we will prove the following results.

Lemma 5.1. *The* WEAK MAIN LEMMA *and the* STRONG MAIN LEMMA *both hold for $\mathcal{A} = \mathcal{M}$.* (Recall that $\mathcal{M}$ is minimal under $<$.)

Lemma 5.2. *Fix $\mathcal{A} \subset \mathcal{M}$, with $\mathcal{A} \neq \mathcal{M}$. Assume that the* STRONG MAIN LEMMA *holds for each $\bar{\mathcal{A}} < \mathcal{A}$. Then the* WEAK MAIN LEMMA *holds for $\mathcal{A}$.*

Lemma 5.3. *Fix $\mathcal{A} \subseteq \mathcal{M}$, and assume that the* WEAK MAIN LEMMA *holds for all $\bar{\mathcal{A}} \leq \mathcal{A}$. Then the* STRONG MAIN LEMMA *holds for $\mathcal{A}$.*

Once we have established these three lemmas, the two MAIN LEMMAS must hold for all $\mathcal{A}$, by induction on $\mathcal{A}$.

Next, we explain how to deduce Theorems 1, 2, 3 from the above MAIN LEMMAS. Taking $\mathcal{A}$ to be the empty set in, say, the WEAK MAIN LEMMA, we see that hypotheses (WL 1, 2, 3) hold vacuously; hence we obtain the following result.

Local Theorem 1. *Given $m, n \geq 1$, there exist $k^{\#}$, A, c' depending only on m and n, for which the following holds.*

Let $E \subset \mathbb{R}^n$ be finite, and let $f : E \to \mathbb{R}$ and $\sigma : E \to (0, \infty)$ be functions. Let $y^0 \in \mathbb{R}^n$. Assume that, given $S \subset E$ with $\#(S) \leq k^{\#}$, there exists $F^S \in C^m(\mathbb{R}^n)$, with $\|F^S\|_{C^m(\mathbb{R}^n)} \leq 1$, and $|F^S(x) - f(x)| \leq \sigma(x)$ for all $x \in S$.

Then there exists $F \in C^m(\mathbb{R}^n)$, with $\|F\|_{C^m(\mathbb{R}^n)} \leq A$, and $|F(x) - f(x)| \leq A\sigma(x)$ for all $x \in E \cap B(y^0, c')$.

Once we have the above Local Theorem 1, it is easy to relax the hypothesis $\sigma : E \to (0, \infty)$ to $\sigma : E \to [0, \infty)$ by a limiting argument. We may then deduce a local version of Theorem 2 by a compactness argument, reducing matters to the Local Theorem 1 by Ascoli's theorem. Next, a partition of unity allows us to pass from the local versions of Theorems 1 and 2 to the full results as given in section 2. Finally, Theorem 3 follows from the special case $\sigma \equiv 0$ of Theorem 2, by applying the standard Whitney extension theorem for $C^{m-1,1}$ to each $S \subset E$ with $\#(S) \leq k^{\#}$. The details of how we pass from our MAIN LEMMAS to Theorems 1, 2, 3 are given in Section 18 below.

We end this section with a few remarks on the proofs of Lemmas 5.1, 5.2, 5.3. We will see that Lemma 5.1 is easy, and Lemma 5.3 may be proven without much trouble, by making a rescaling of the form $(x_1, \dots, x_n) \mapsto (\lambda_1 x_1, \dots, \lambda_n x_n)$ on $\mathbb{R}^n$, for properly chosen $\lambda_1, \dots, \lambda_n$. The hard work goes into the proof of Lemma 5.2. A key property of subsets $\mathcal{A} \subseteq \mathcal{M}$, relevant to the proof of Lemma 5.2, is as follows.

We say that $\mathcal{A} \subseteq \mathcal{M}$ is *monotonic* if, for any $\alpha \in \mathcal{A}$, we have $\alpha + \gamma \in \mathcal{A}$ for all multi–indices γ of order $|\gamma| \leq m - 1 - |\alpha|$.

Lemma 5.2 is easy for non–monotonic $\mathcal{A}$. The main work in our proof lies in establishing Lemma 5.2 for monotonic $\mathcal{A}$.

This completes our discussion of the Plan of the Proof.

6. Starting the Main Induction

In this section, we give the proof of Lemma 5.1. We will show here that the STRONG MAIN LEMMA holds for $\mathcal{A} = \mathcal{M}$. The argument for the WEAK MAIN LEMMA is nearly identical.

Suppose E, f, σ, y^0, $P_\alpha(\alpha \in \mathcal{A})$ satisfy hypotheses (SL1, ..., 4) with $\mathcal{A} = \mathcal{M}$. From (SL1) with $\mathcal{A} = \mathcal{M}$, we see that $P_\alpha(x) = \frac{1}{\alpha!}(x - y^0)^\alpha$ for all $\alpha \in \mathcal{A}$. In particular, $P_0(x) = 1$. Hence, (SL3) with $\alpha = 0$, tells us the following:

Given $S \subset E$ with $\#(S) \leq k^\#$, there exists $\varphi^S \in C^m(\mathbb{R}^n)$, with

(a) $\|\partial^m \varphi^S\|_{C^0(\mathbb{R}^n)} \leq C$.
(b) $|\varphi^S(x)| \leq C\sigma(x)$ for all $x \in S$.
(c) $J_{y^0}(\varphi^S) = 1$.

We take $k^\# = 1$, and apply the above result to $S = \{y\}$ for an arbitrary $y \in E$. From (a) and (c) above, we conclude that

(1) $|\varphi^S - 1| \leq \frac{1}{2}$ on $B(y^0, c')$, with c' determined by C, m, n in (a), (b), (c).

In particular, if $y \in E \cap B(y^0, c')$, then (b) and (1) give $\frac{1}{2} \leq |\varphi^S(y)| \leq C\sigma(y)$. Thus,

(2) $\sigma(y) \geq \frac{1}{2C}$ for all $y \in E \cap B(y^0, c')$.

Next, we apply (SL4) with $k^\# = 1$, $S = \{y\}$, $y \in E \cap B(y^0, c')$. We conclude that there exists $F^S \in C^m(\mathbb{R}^n)$, with $\|F^S\|_{C^m(\mathbb{R}^n)} \leq C$ and $|F^S(y) - f(y)| \leq C\sigma(y)$.

In particular, $|F^S(y)| \leq C$ and $|F^S(y) - f(y)| \leq C\sigma(y)$. Hence, $|f(y)| \leq C + C\sigma(y) \leq 2C^2\sigma(y) + C\sigma(y)$, thanks to (2).

Thus, $|f(y)| \leq (2C^2 + C) \cdot \sigma(y)$ for all $y \in E \cap B(y^0, c')$.

Consequently, the conclusions (SL5, 6) hold, with $F \equiv 0$.

The proof of Lemma 5.1 is complete. ■

7. Non–Monotonic Sets

In this section, we will prove Lemma 5.2 in the (easy) case of non–monotonic $\mathcal{A}$.

Lemma 7.1. *Fix a non–monotonic set $\mathcal{A} \subset \mathcal{M}$, and assume that the STRONG MAIN LEMMA holds for all $\bar{\mathcal{A}} < \mathcal{A}$. Then the* WEAK MAIN LEMMA *holds for $\mathcal{A}$.*

Proof. Since $\mathcal{A}$ is not monotonic, there exist multi-indices $\bar{\alpha}, \bar{\gamma}$, with

$$\bar{\alpha} \in \mathcal{A}, \bar{\alpha} + \bar{\gamma} \in \mathcal{M} \setminus \mathcal{A}. \tag{1}$$

We set

$$\bar{\mathcal{A}} = \mathcal{A} \cup \{\bar{\alpha} + \bar{\gamma}\}, \tag{2}$$

and we take $k^{\#}$ as in the STRONG MAIN LEMMA for $\bar{\mathcal{A}}$. By Lemma 3.2 we have $\bar{\mathcal{A}} < \mathcal{A}$; hence, we may assume here that the STRONG MAIN LEMMA holds for $\bar{\mathcal{A}}$.

Let E, f, σ, y^0, $P_\alpha (\alpha \in \mathcal{A})$ be as in the WEAK MAIN LEMMA for $\mathcal{A}$. Thus, (WL1, ... , 4) hold. We must prove that there exists $F \in C^m(\mathbb{R}^n)$ satisfying (WL5, 6).

Define

$$P_{\bar{\alpha}+\bar{\gamma}}(x) = \sum_{|\bar{\beta}| \leq m-1-|\bar{\gamma}|} \left(\frac{1}{\bar{\beta}!} \partial^{\bar{\beta}} P_{\bar{\alpha}}(y^0) \right) \cdot (x - y^0)^{\bar{\beta}+\bar{\gamma}} \cdot \frac{\bar{\alpha}!}{(\bar{\alpha}+\bar{\gamma})!}. \tag{3}$$

Thus, $P_\alpha \in \mathcal{P}$ is defined for all $\alpha \in \bar{\mathcal{A}}$. From (3) we obtain easily, for any $\beta \in \mathcal{M}$, that

$$\partial^\beta P_{\bar{\alpha}+\bar{\gamma}}(y^0) = \left\{ \begin{array}{cl} \dfrac{\beta!}{\bar{\beta}!} \partial^{\bar{\beta}} P_{\bar{\alpha}}(y^0) \cdot \dfrac{\bar{\alpha}!}{(\bar{\alpha}+\bar{\gamma})!} & \text{if } \beta = \bar{\beta} + \bar{\gamma} \text{ for a multi–index } \bar{\beta} \\ 0 & \text{if } \beta \text{ doesn't have the form } \bar{\beta} + \bar{\gamma} \end{array} \right\}.$$

Consequently, (WL2) gives

$$|\partial^\beta P_{\bar{\alpha}+\bar{\gamma}}(y^0) - \delta_{\beta, \bar{\alpha}+\bar{\gamma}}| \leq C' a_0 \qquad \text{for all } \beta \in \mathcal{M}, \tag{4}$$

with C' determined by m and n.

From (4) and another application of (WL2), we see that

$$|\partial^\beta P_\alpha(y^0) - \delta_{\beta\alpha}| \leq C' a_0 \quad \text{for all } \alpha \in \bar{\mathcal{A}},\ \beta \in \mathcal{M}. \tag{5}$$

with C' depending only on m and n.

If a_0 is a small enough constant determined by m and n, then (5) shows that the matrix

$$(\partial^\beta P_\alpha(y^0))_{\alpha,\beta\in\bar{\mathcal{A}}}$$

is invertible, and that the inverse matrix $(M_{\alpha'\alpha})_{\alpha',\,\alpha\in\bar{\mathcal{A}}}$ satisfies

$$|M_{\alpha'\alpha}| \leq C'' \tag{6}$$

with C'' depending only on m and n. We fix a_0 to be a small enough constant, depending only on m and n, guaranteeing (6). By definition of $(M_{\alpha'\alpha})$, we have

$$\delta_{\beta\alpha} = \sum_{\alpha'\in\bar{\mathcal{A}}} \partial^\beta P_{\alpha'}(y^0)\cdot M_{\alpha'\alpha} \qquad \text{for all } \beta,\alpha\in\bar{\mathcal{A}}. \tag{7}$$

We define

$$\bar{P}_\alpha = \sum_{\alpha'\in\bar{\mathcal{A}}} P_{\alpha'}\cdot M_{\alpha'\alpha} \qquad \text{for all } \alpha\in\bar{\mathcal{A}}. \tag{8}$$

From (7), (8), we have

$$\partial^\beta \bar{P}_\alpha(y^0) = \delta_{\beta\alpha} \qquad \text{for all } \alpha,\beta\in\bar{\mathcal{A}}. \tag{9}$$

From (5), (6), (8) we have

$$|\partial^\beta \bar{P}_\alpha(y^0)| \leq C''' \qquad \text{for all } \alpha\in\mathcal{A}, \beta\in\mathcal{M}, \tag{10}$$

with C''' depending only on m and n.

Next, let $S\subset E$ be given, with $\#(S)\leq k^\#$. For $\alpha\in\mathcal{A}$, let φ_α^S be as in (WL3). We define also

$$\varphi_{\bar{\alpha}+\bar{\gamma}}(x) = (x-y^0)^{\bar{\gamma}}\chi(x-y^0)\cdot\varphi_{\bar{\alpha}}^S(x)\cdot\frac{\bar{\alpha}!}{(\bar{\alpha}+\bar{\gamma})!} \quad \text{on } \mathbb{R}^n, \tag{11}$$

where χ satisfies

$$\|\chi\|_{C^m(\mathbb{R}^n)} \leq C_1', \chi = 1 \text{ on } B(0,1),\ \text{supp }\chi\subset B(0,2), \tag{12}$$

with C_1' determined by m and n.

From (WL2, 3(a), 3(c)), we see that $\|\varphi^S_{\bar{\alpha}}\|_{C^m(B(0,2))} \leq C_1''$, with C_1'' determined by m and n. Together with (12), this implies

$$\|\partial^m \varphi_{\bar{\alpha}+\bar{\gamma}}\|_{C^0(\mathbb{R}^n)} \leq C_1''' \text{ with } C_1^{'''} \text{ determined by } m \text{ and } n. \tag{13}$$

Also, for $x \in S$ we have $|\varphi_{\bar{\alpha}+\bar{\gamma}}(x)| \leq C_2'|\varphi^S_{\bar{\alpha}}(x)|$ with C_2' determined by m and n, simply because $|(x-y^0)^{\bar{\gamma}}\chi(x-y^0)| \leq C_2'$ on $\mathbb{R}^n$ (see (11), (12)). Hence, (WL3 (b)) implies

$$|\varphi_{\bar{\alpha}+\bar{\gamma}}(x)| \leq C_2'|\varphi^S_{\bar{\alpha}}(x)| \leq C_3\sigma(x) \quad \text{ for all } x \in S, \tag{14}$$

with C_3 determined by C, m, n in (WL1, ... , 4).

Also, from (11), (12) and (WL3 (c)), we find that $\varphi_{\bar{\alpha}+\bar{\gamma}}(x) - \frac{\bar{\alpha}!}{(\bar{\alpha}+\bar{\gamma})!}\cdot (x-y^0)^{\bar{\gamma}}$ $P_{\bar{\alpha}}(x) = O(|x-y^0|^m)$ as $x \to y^0$. On the other hand, (3) implies $P_{\bar{\alpha}+\bar{\gamma}}(x) - \frac{\bar{\alpha}!}{(\bar{\alpha}+\bar{\gamma})!}\cdot$ $(x-y^0)^{\bar{\gamma}}$ $P_{\bar{\alpha}}(x) = O(|x-y^0|^m)$ as $x \to y^0$. Hence,

$$\varphi^S_{\bar{\alpha}+\bar{\gamma}}(x) - P_{\bar{\alpha}+\bar{\gamma}}(x) = O(|x-y^0|^m) \quad \text{as } x \to y^0. \tag{15}$$

Since $\varphi_{\bar{\alpha}+\bar{\gamma}} \in C^m(\mathbb{R}^n)$ and $P_{\bar{\alpha}+\bar{\gamma}} \in \mathcal{P}$, (15) implies

$$J_{y^0}(\varphi^S_{\bar{\alpha}+\bar{\gamma}}) = P_{\bar{\alpha}+\bar{\gamma}}. \tag{16}$$

From (13), (14), (16) together with (WL3), we have the following result.

(17) Let $S \subset E$ with $\#(S) \leq k^\#,$ and let $\alpha \in \bar{\mathcal{A}}$.
Then there exists $\varphi^S_\alpha \in C^m(\mathbb{R}^n),$ with
(a) $\|\partial^m \varphi^S_\alpha\|_{C^0(\mathbb{R}^n)} \leq C_4,$
(b) $|\varphi^S_\alpha(x)| \leq C_4\sigma(x)$ for all $x \in S,$
(c) $J_{y^0}(\varphi^S_\alpha) = P_\alpha,$

where C_4 is determined by C, m, n in (WL1, ..., 4).

Next, given $S \subset E$ with $\#(S) \leq k^\#$, and given $\alpha \in \bar{\mathcal{A}}$, define

$$\bar{\varphi}^S_\alpha = \sum_{\alpha' \in \bar{\mathcal{A}}} \varphi^S_{\alpha'} M_{\alpha'\alpha}. \tag{18}$$

From (17)(a),(b) and (6), we see that

$$\|\partial^m \bar{\varphi}^S_\alpha\|_{C^0(\mathbb{R}^n)} \leq C_5 \tag{19}$$

and

$$|\bar{\varphi}^S_\alpha(x)| \leq C_5\sigma(x) \quad \text{for all } x \in S, \tag{20}$$

with C_5 determined by C, m, n in (WL1, …, 4).

Also, (8), (18), (17)(c) together yield

$$J_{y^0}(\bar{\varphi}_\alpha^S) = \bar{P}_\alpha. \tag{21}$$

Now we can check that E, f, σ, y^0, $\bar{P}_\alpha (\alpha \in \bar{\mathcal{A}})$ satisfy the hypotheses (SL1, …, 4) of the STRONG MAIN LEMMA for $\bar{\mathcal{A}}$, with a constant determined by C, m, n in (WL1, …, 4). In fact, (SL1) for the $\bar{P}_\alpha$ is just (9); (SL2) for the $\bar{P}_\alpha$ is immediate from (10); (SL3) for the $\bar{P}_\alpha$ is immediate from (19), (20), (21); and (SL4) for the $\bar{P}_\alpha$ is just (WL4). (Note that, to prove (SL2) for the $\bar{P}_\alpha$, we need (10) only for $\beta \geq \alpha$.)

Applying the STRONG MAIN LEMMA for $\bar{\mathcal{A}}$, we conclude that there exists $F \in C^m(\mathbb{R}^n)$, with

$$\|F\|_{C^m(\mathbb{R}^n)} \leq C_6, \quad \text{and} \quad |F(x) - f(x)| \leq C_6 \sigma(x) \quad \text{for all } x \in E \cap B(y^0, c_7), \tag{22}$$

where C_6 and c_7 are determined by C, m, n in (WL1, …, 4) for the P_α $(\alpha \in \mathcal{A})$.

However, (22) is the conclusion of the WEAK MAIN LEMMA for $\mathcal{A}$. Thus, the WEAK MAIN LEMMA holds for $\mathcal{A}$. The proof of Lemma 7.1 is complete. ∎

8. A Consequence of the Main Inductive Assumption

In this section, we establish the following result.

Lemma 8.1. *Fix $\mathcal{A} \subset \mathcal{M}$, and assume that the* STRONG MAIN LEMMA *holds, for all $\bar{\mathcal{A}} < \mathcal{A}$. Then there exists $k^{\#}_{\text{old}}$, depending only on m and n, for which the following holds. Let $A > 0$ be given. Let $Q \subset \mathbb{R}^n$ be a cube, $\hat{E} \subset \mathbb{R}^n$ a finite set, $\hat{f} : \hat{E} \to \mathbb{R}$ and $\sigma : \hat{E} \to (0, \infty)$ functions on $\hat{E}$. Suppose that, for each $y \in Q^{\star\star}$, we are given a set $\bar{\mathcal{A}}^y < \mathcal{A}$ and a family of polynomials $\bar{P}^y_\alpha \in \mathcal{P}$, indexed by $\alpha \in \bar{\mathcal{A}}^y$. Assume that the following conditions are satisfied:*

(G1). $\partial^\beta \bar{P}^y_\alpha(y) = \delta_{\beta\alpha}$ *for all* $\beta, \alpha \in \bar{\mathcal{A}}^y, \quad y \in Q^{\star\star}$.

(G2). $|\partial^\beta \bar{P}^y_\alpha(y)| \leq A\delta_Q^{|\alpha|-|\beta|}$ *for all* $\alpha \in \bar{\mathcal{A}}^y, \beta \geq \alpha, \quad y \in Q^{\star\star}$.

(G3). *Given $S \subset \hat{E}$ with $\#(S) \leq k^{\#}_{\text{old}}$, and given $y \in Q^{\star\star}$ and $\alpha \in \bar{\mathcal{A}}^y$, there exists $\varphi^S_\alpha \in C^m(\mathbb{R}^n)$, with*

(a) $\|\partial^m \varphi^S_\alpha\|_{C^0(\mathbb{R}^n)} \leq A\delta_Q^{|\alpha|-m}$,

(b) $|\varphi_\alpha^S(x)| \leq A\delta_Q^{|\alpha|-m}\sigma(x)$ *for all* $x \in S$,
(c) $J_y(\varphi_\alpha^S) = \bar{P}_\alpha^y$.

(G4). *Given* $S \subset \hat{E}$ *with* $\#(S) \leq k_{\text{old}}^{\#}$, *there exists* $F^S \in C^m(\mathbb{R}^n)$, *with*
(a) $\|\partial^\beta F^S\|_{C^0(\mathbb{R}^n)} \leq A\delta_Q^{m-|\beta|}$ *for all* β *with* $|\beta| \leq m$,
(b) $|F^S(x) - \hat{f}(x)| \leq A\sigma(x)$ *for all* $x \in S$.

Then there exists $F \in C^m(\mathbb{R}^n)$, *with*

(G5). $\|\partial^\beta F\|_{C^0(\mathbb{R}^n)} \leq A'\delta_Q^{m-|\beta|}$ *for all* β *with* $|\beta| \leq m$, *and*

(G6). $|F(x) - \hat{f}(x)| \leq A'\sigma(x)$ *for all* $x \in \hat{E} \cap Q^\star$.

Here, A' *depends only on* A, m, n.

Proof. By a rescaling, we may reduce matters to the case $\delta_Q = 1$.

In fact, we set $\bar{\bar{Q}} = \delta_Q^{-1}Q$, $\bar{\bar{P}}_\alpha(x) = \delta_Q^{-|\alpha|} \cdot \bar{P}_\alpha(\delta_Q x)$, $\bar{\bar{\varphi}}_\alpha^{\bar{\bar{S}}}(x) = \delta_Q^{-|\alpha|} \cdot \varphi_\alpha^S\ (\delta_Q x)$, $\bar{\bar{S}} = \delta_Q^{-1}S$, $\bar{\bar{F}}^{\bar{\bar{S}}}(x) = \delta_Q^{-m} \cdot F^S(\delta_Q x)$, $\bar{\bar{f}}(x) = \delta_Q^{-m} \cdot \hat{f}(\delta_Q x)$, $\bar{\bar{\sigma}}(x) = \delta_Q^{-m} \cdot \sigma(\delta_Q x)$, $\bar{\bar{E}} = \delta_Q^{-1} \cdot \hat{E}$.

If (G1, …, 4) hold for $Q, \hat{E}, \hat{f}, \sigma$, then (G1, …, 4) hold also for $\bar{\bar{Q}}, \bar{\bar{E}}, \bar{\bar{f}}, \bar{\bar{\sigma}}$ with the same constant A. If Lemma 8.1 holds in the case $\delta_Q = 1$, then in particular it holds for $\bar{\bar{Q}}, \bar{\bar{E}}, \bar{\bar{f}}, \bar{\bar{\sigma}}$. Hence, there exists $\bar{\bar{F}} \in C^m(\mathbb{R}^n)$, with $\|\bar{\bar{F}}\|_{C^m(\mathbb{R}^n)} \leq A'$, and $|\bar{\bar{F}}(x) - \bar{\bar{f}}(x)| \leq A'\bar{\bar{\sigma}}(x)$ for all $x \in \bar{\bar{E}} \cap \bar{\bar{Q}}^\star$.

Defining $F(x) = \delta_Q^m \cdot \bar{\bar{F}}(\delta_Q^{-1}x)$, we conclude that F satisfies (G5, 6). Thus, as claimed, it's enough to prove Lemma 8.1 in the case $\delta_Q = 1$.

Let $\delta_Q = 1$, and assume (G1, …, 4). For each $y \in Q^{\star\star}$, the hypotheses (SL1, …, 4) for the STRONG MAIN LEMMA for $\bar{\mathcal{A}}^y$ hold, with $\hat{E}, \hat{f}, \sigma, y, \bar{P}_\alpha^y(\alpha \in \bar{\mathcal{A}}^y), A$ in place of $E, f, \sigma, y^0, \bar{P}_\alpha(\alpha \in \mathcal{A}), C$ in (SL1, … , 4). In fact, (SL1, … , 4) for $\hat{E}, \hat{f}, \sigma, y, \bar{P}_\alpha^y(\alpha \in \bar{\mathcal{A}}^y), A$ are immediate from (G1, …, 4), where we define $k_{\text{old}}^{\#}$ to be the maximum of all the $k^{\#}$ arising in the STRONG MAIN LEMMA for all $\bar{\mathcal{A}}(\bar{\mathcal{A}} < \mathcal{A})$.

Hence, for each $y \in Q^{\star\star}$, the STRONG MAIN LEMMA for $\bar{\mathcal{A}}^y$ produces a function $F^y \in C^m(\mathbb{R}^n)$, with

(1) $\|F^y\|_{C^m(\mathbb{R}^n)} \leq A'$, and $|F^y(x) - \hat{f}(x)| \leq A'\sigma(x)$ for all $x \in E \cap B(y, c')$,

where A' and c' are determined by A, m, n in (G1, … , 4).

To exploit (1), we use a partition of unity

$$
\begin{aligned}
&(2) && 1 = \sum_{\nu=1}^{\nu_{\max}} \theta_\nu(x) \quad \text{on} \quad Q^\star, && \text{where} \\
&(3) && 0 \le \theta_\nu \le 1 \quad \text{on} \quad \mathbb{R}^n; && \\
&(4) && \operatorname{supp}\theta_\nu \subset B(y_\nu, c') && \text{with} \\
&(5) && y_\nu \in Q^{\star\star} && \text{and } c' \text{ as in (1);} \\
&(6) && \|\theta_\nu\|_{C^m(\mathbb{R}^n)} \le C'; && \text{and} \\
&(7) && \nu_{\max} \le C'; && \text{where } C' \text{ is determined by } c', m, n, \text{ hence by } A, m, n.
\end{aligned}
$$

We then define $F = \sum_{\nu=1}^{\nu_{\max}} \theta_\nu \cdot F^{y_\nu}$. From (1), (6), (7) we conclude that

$$(8) \qquad \|F\|_{C^m(\mathbb{R}^n)} \le C'' \qquad \text{with } C'' \text{ determined by } A, m, n.$$

From (1), … , (5), we conclude that every $x \in E \cap Q^\star$ satisfies

$$
\begin{aligned}
|F(x) - \hat{f}(x)| = \left| \sum_{\nu=1}^{\nu_{\max}} \theta_\nu(x) F^{y_\nu}(x) - \sum_{\nu=1}^{\nu_{\max}} \theta_\nu(x)\hat{f}(x) \right| \le \\
\sum_{\nu=1}^{\nu_{\max}} \theta_\nu(x) \cdot |F^{y_\nu}(x) - \hat{f}(x)| \le \\
\sum_{\nu=1}^{\nu_{\max}} \theta_\nu(x) \cdot A'\sigma(x) \\
= A'\sigma(x).
\end{aligned}
$$

Thus

$$(9) \qquad |F(x) - \hat{f}(x)| \le A'\sigma(x) \qquad \text{for all} \quad x \in E \cap Q^\star,$$

with A' determined by A, m, n.

Estimates (8) and (9) are the conclusions of Lemma 8.1, since we are assuming that $\delta_Q = 1$. The proof of the Lemma is complete. ∎

9. Setup for the Main Induction

In this section, we give the setup for the proof of Lemma 5.2 in the monotonic case.

We fix $m, n \geq 1$ and $\mathcal{A} \subset \mathcal{M}$. We let $k^{\#}$ be a large enough integer, determined by m and n, to be picked later. We suppose we are given a finite set $E \subset \mathbb{R}^n$, functions $f : E \to \mathbb{R}$ and $\sigma : E \to (0, \infty)$, a point $y^0 \in \mathbb{R}^n$, a family of polynomials $P_\alpha \in \mathcal{P}$ indexed by $\alpha \in \mathcal{A}$, and a positive number a_1. We fix $\mathcal{A}, k^{\#}, E, f, \sigma, y^0$, $(P_\alpha)_{\alpha \in \mathcal{A}}, a_1$ until the end of section 15. We make the following assumptions.

(SU0). $\mathcal{A}$ is monotonic, and $\mathcal{A} \neq \mathcal{M}$.

(SU1). The STRONG MAIN LEMMA holds for all $\bar{\mathcal{A}} < \mathcal{A}$.

(SU2). $\partial^\beta P_\alpha(y^0) = \delta_{\beta\alpha}$ for all $\beta, \alpha \in \mathcal{A}$.

(SU3). $|\partial^\beta P_\alpha(y^0) - \delta_{\beta\alpha}| \leq a_1$ for all $\alpha \in \mathcal{A}, \beta \in \mathcal{M}$.

(SU4). a_1 is less than a small enough constant determined by m and n.

(SU5). Given $S \subset E$ with $\#(S) \leq k^{\#}$, and given $\alpha \in \mathcal{A}$, there exists $\varphi_\alpha^S \in C^m(\mathbb{R}^n)$, with

(a) $\|\partial^m \varphi_\alpha^S\|_{C^0(\mathbb{R}^n)} \leq a_1$
(b) $|\varphi_\alpha^S(x)| \leq \sigma(x)$ for all $x \in S$.
(c) $J_{y^0}(\varphi_\alpha^S) = P_\alpha$.

(SU6). Given $S \subset E$ with $\#(S) \leq k^{\#}$, there exists $F^S \in C^m(\mathbb{R}^n)$, with

(a) $\|F^S\|_{C^m(\mathbb{R}^n)} \leq 1$,
(b) $|F^S(x) - f(x)| \leq \sigma(x)$ for all $x \in S$.

The main effort of this paper goes into proving the following result.

Lemma 9.1. *Assume* (SU0, ... , 6). *Then there exists $F \in C^m(\mathbb{R}^n)$, with*

(a) $\|F\|_{C^m(\mathbb{R}^n)} \leq A$,
(b) $|F(x) - f(x)| \leq A\sigma(x)$ *for all $x \in E \cap B(y^0, a)$; where A and a are determined by a_1, m, n.*

Once we establish Lemma 9.1, then Lemma 5.2 will follow easily, as we explain in a moment. First, however, we point out a few minor differences between Lemmas 9.1 and 5.2. In the WEAK MAIN LEMMA, the constant a_0 depends only on m and n. Hence, the same is true in Lemma 5.2. On the other hand, in Lemma 9.1, the analogous constant a_1 is said merely to be less than a small enough constant determined by m and n. We do not assume in Lemma 9.1 that a_1 is determined by m and n. Also, if we compare (WL3 (b)) and (WL4) with (SU5 (b)) and (SU6), we see that the constant C

in the statement of the WEAK MAIN LEMMA has in effect been set equal to 1 in the statement of Lemma 9.1.

Now we check that Lemma 5.2 follows from Lemma 9.1. Thus, we fix $\mathcal{A} \subset \mathcal{M}$ ($\mathcal{A} \neq \mathcal{M}$) as in Lemma 5.2, and assume that the STRONG MAIN LEMMA holds for all $\bar{\mathcal{A}} < \mathcal{A}$. We must prove that the WEAK MAIN LEMMA holds for $\mathcal{A}$. This follows at once from Lemma 7.1 if $\mathcal{A}$ is non-monotonic. Hence, we may assume that $\mathcal{A}$ is monotonic. We will show that the WEAK MAIN LEMMA for $\mathcal{A}$ holds in the special case $C = 1$. To see this, we invoke Lemma 9.1, with a_1 taken to be a constant determined by m and n, small enough to satisfy (SU4). We take $a_0 = a_1$, and assume the hypotheses (WL1, ... , 4) of the WEAK MAIN LEMMA, with $C = 1$. Let us check that hypotheses (SU0, ... , 6) are satisfied.

In fact, we are assuming (SU0), (SU1), (SU4). The remaining hypotheses (SU2, 3, 5, 6) are precisely the hypotheses (WL1, ... , 4) of the WEAK MAIN LEMMA for $\mathcal{A}$, with $C = 1$. Thus, (SU0, ... , 6) are satisfied. Applying Lemma 9.1, we obtain a function $F \in C^m(\mathbb{R}^n)$, with

$$(1) \quad \|F\|_{C^m(\mathbb{R}^n)} \leq A, \quad \text{and} \quad |F(x) - f(x)| \leq A\sigma(x) \quad \text{for all} \quad x \in E \cap B(y^0, a),$$

where A and a are determined by a_1, m, n. Since we have picked a_1 to depend only on m and n, it follows that also A and a are determined by m and n. Therefore, (1) is precisely the conclusion (WL5, 6) of the WEAK MAIN LEMMA for $\mathcal{A}$, with $C = 1$.

Thus, we have proven the WEAK MAIN LEMMA for $\mathcal{A}$, in the special case $C = 1$. On the other hand, it is trivial to reduce the WEAK MAIN LEMMA for $\mathcal{A}$ to the special case $C = 1$. In fact, if hypotheses (WL1, ... , 4) are satisfied, with $C \neq 1$, then we just set $\tilde{\sigma}(x) = C\sigma(x)$ and $\tilde{f}(x) = (C+1)^{-1} f(x)$ for all $x \in E$. One checks that (WL1, ... , 4) are satisfied, with $C = 1$, by $E, \tilde{f}, \tilde{\sigma}, y^0, \mathcal{A}, P_\alpha (\alpha \in \mathcal{A})$. Applying the WEAK MAIN LEMMA for $\mathcal{A}$, with $C = 1$, to $E, \tilde{f}, \tilde{\sigma}, y^0, P_\alpha (\alpha \in \mathcal{A})$, we obtain the conclusion of the WEAK MAIN LEMMA for $\mathcal{A}$, for our original $E, f, \sigma, y^0, P_\alpha (\alpha \in \mathcal{A})$.

This proves the WEAK MAIN LEMMA for $\mathcal{A}$ in the general case, and completes the proof of the following result.

Lemma 9.2. *Lemma 9.1 implies Lemma 5.2.*

We begin the work of proving Lemma 9.1. We write c, C, C', etc. to denote constants determined entirely by m and n. We call such constants "controlled". We write a, a', A, A', etc. to denote constants determined by a_1, m, n in (SU0, ... , 6). We call such constants "weakly controlled".

We fix a constant $k^{\#}_{\text{old}}$, depending only on m and n, as in Lemma 8.1.

These conventions will remain in effect through the end of section 15.

10. Applying Helly's Theorem on Convex Sets

In this section, we start the proof of Lemma 9.1, by repeatedly applying the following well–known result (Helly's Theorem; see [14]).

Lemma 10.0. *Let $\mathcal{J}$ be a family of compact convex subsets of $\mathbb{R}^d$. Suppose that any $(d+1)$ of the sets in $\mathcal{J}$ have non–empty intersection. Then the whole family $\mathcal{J}$ has non–empty intersection.*

We assume (SU0, ... , 6) and adopt the conventions of section 9. For $M > 0$, $S \subset E$, $y \in \mathbb{R}^n$, define

(1)
$$\mathcal{K}_f(y; S, M) = \left\{ P \in \mathcal{P} : \begin{array}{l} \text{There exists } F \in C^m(\mathbb{R}^n), \text{ with } \|F\|_{C^m(\mathbb{R}^n)} \le M, \\ |F(x) - f(x)| \le M\sigma(x) \text{ for all } x \in S, \text{ and } J_y(F) = P \end{array} \right\}.$$

For $M > 0, k \ge 1, y \in \mathbb{R}^n$, we then define

(2)
$$\mathcal{K}_f(y; k, M) = \bigcap_{\substack{S \subset E \\ \#(S) \le k}} \mathcal{K}_f(y; S, M).$$

Thus, if $P \in \mathcal{K}_f(y; k, M)$, then for any subset $S \subset E$ with $\#(S) \le k$, there exists $F^S \in C^m(\mathbb{R}^n)$, with $\|F^S\|_{C^m(\mathbb{R}^n)} \le M$, $|F^S(x) - f(x)| \le M\sigma(x)$ for all $x \in S$, and $J_y(F^S) = P$.

Lemma 10.1. *Suppose we are given $k_1^\#$, with $k^\# \ge (D+1)k_1^\#$ and $k_1^\# \ge 1$. Then $\mathcal{K}_f(y; k_1^\#, 2)$ is non–empty, for all $y \in \mathbb{R}^n$.*

Proof. We start with a small remark. Given a point $y \in \mathbb{R}^n$ and a polynomial $\tilde{P} \in \mathcal{P}$, there exists $\tilde{G} \in C^m(\mathbb{R}^n)$, with

(3)
$$\|\tilde{G}\|_{C^m(\mathbb{R}^n)} \le C \cdot \max_{|\beta| \le m-1} |\partial^\beta \tilde{P}(y)| \quad \text{and} \quad J_y(\tilde{G}) = \tilde{P}.$$

This remark shows easily that

(4)
$$\text{Closure } (\mathcal{K}_f(y; S, M)) \subset \mathcal{K}_f(y; S, M') \quad \text{whenever} \quad M' > M.$$

To check (4), fix $y \in \mathbb{R}^n$, $S \subset E$, $M' > M$, and $P \in$ Closure $(\mathcal{K}_f(y; S, M))$. Given $\varepsilon > 0$, there exists $P_\varepsilon \in \mathcal{K}_f(y; S, M)$ with $\max_{|\beta| \le m-1} |\partial^\beta (P - P_\varepsilon)(y)| < \varepsilon$.

Applying (3) to $\tilde{P} = P - P_\varepsilon$, we obtain $G_\varepsilon \in C^m(\mathbb{R}^n)$, with $\|G_\varepsilon\|_{C^m(\mathbb{R}^n)} \le C\varepsilon$, and $J_y(G_\varepsilon) = P - P_\varepsilon$.

Moreover, since $P_\varepsilon \in \mathcal{K}_f(y; S, M)$, there exists $F_\varepsilon \in C^m(\mathbb{R}^n)$, with $\|F_\varepsilon\|_{C^m(\mathbb{R}^n)} \le M$, $|F_\varepsilon(x) - f(x)| \le M\ \sigma(x)$ on S, $J_y(F_\varepsilon) = P_\varepsilon$.

Taking $F = F_\varepsilon + G_\varepsilon$ with ε small enough, we find that

$$\|F\|_{C^m(\mathbb{R}^n)} \le M + C\varepsilon, |F(x) - f(x)| \le M\sigma(x) + C\varepsilon \text{ on } S, J_y(F) = P.$$

Recall that $M' > M$, $S \subset E$, E is finite, and $\sigma(x)$ is strictly positive on E. Hence, for $\varepsilon > 0$ small enough, we have

$$M + C\varepsilon < M', \text{ and } M\sigma(x) + C\varepsilon < M'\sigma(x) \text{ on } S.$$

Using such an ε to define F, we obtain

$$\|F\|_{C^m(\mathbb{R}^n)} \le M', |F(x) - f(x)| \le M'\sigma(x) \quad \text{on} \quad S, \quad \text{and} \quad J_y(F) = P. \tag{5}$$

Since we have found an $F \in C^m(\mathbb{R}^n)$ satisfying (5), we know that P belongs to $\mathcal{K}_f(y; S, M')$. The proof of (4) is complete.

Now let $S_1, \dots, S_{D+1} \subset E$ be given, with $\#(S_i) \le k_1^\#$ for each i. Fix $y \in \mathbb{R}^n$, and set $S = S_1 \cup \dots \cup S_{D+1}$. We have $S \subset E$ and $\#(S) \le (D+1) \cdot k_1^\# \le k^\#$. Hence, by (SU6), there exists $F^S \in C^m(\mathbb{R}^n)$, with

$$\|F^S\|_{C^m(\mathbb{R}^n)} \le 1, \quad \text{and} \quad |F^S(x) - f(x)| \le \sigma(x) \quad \text{on} \quad S.$$

Define $P = J_y(F^S)$. Then, for each $i = 1, \dots, D+1$, we have obviously

$$\|F^S\|_{C^m(\mathbb{R}^n)} \le 1, |F^S(x) - f(x)| \le \sigma(x) \quad \text{on} \quad S_i, \quad \text{and} \quad J_y(F^S) = P.$$

Hence, P belongs to $\mathcal{K}_f(y; S_i, 1)$ for each i. Consequently, the sets $\mathcal{K}_f(y; S_i, 1)$ for $i = 1, \dots, D+1$ have non–empty intersection.

Thus, the sets $\mathcal{K}_f(y; S, 1) \subset \mathcal{P}$ $(S \subset E, \#(S) \le k_1^\#)$ have the property that any $(D+1)$ of them have non–empty intersection. Moreover, each $\mathcal{K}_f(y; S, 1)$ is easily seen to be a convex, bounded subset of the D–dimensional vector space $\mathcal{P}$. Hence, by Lemma 10.0, the closures of the $\mathcal{K}_f(y; S, 1)$ $(S \subset E, \#(S) \le k_1^\#)$ have non–empty intersection. Applying (4), we see that the intersection of $\mathcal{K}_f(y; S, 2)$ over all $S \subset E$ with $\#(S) \le k_1^\#$ is non–empty.

That is, $\mathcal{K}_f(y; k_1^\#, 2)$ is non–empty. The proof of Lemma 10.1 is complete. ∎

In the same spirit, we can prove the following result.

Lemma 10.2. *Suppose $k_1^\# \geq (D+1)k_2^\#$, and suppose $P \in \mathcal{K}_f(y; k_1^\#, C)$ is given. Then, for any $y' \in \mathbb{R}^n$, there exists $P' \in \mathcal{K}_f(y'; k_2^\#, C')$, with*

$$|\partial^\beta(P-P')(y)|, |\partial^\beta(P-P')(y')| \leq C''|y-y'|^{m-|\beta|} \quad \text{for all } \beta \in \mathcal{M}.$$

Proof. The result is trivial for $y'=y$; just take $P'=P$. Suppose $y' \neq y$. Then, for a constant $\Gamma(y,y')$ determined by y, y', m and n, we have the following small remark.

(6) Given $\tilde{P} \in \mathcal{P}$ there exists $\tilde{G} \in C^m(\mathbb{R}^n)$with

$$\|\tilde{G}\|_{C^m(\mathbb{R}^n)} \leq \Gamma(y,y') \cdot \max_{|\beta| \leq m-1} |\partial^\beta \tilde{P}(y')|, J_{y'}(\tilde{G}) = \tilde{P}, J_y(\tilde{G}) = 0.$$

Fix P as in the hypotheses of the Lemma. For each $S \subset E$ and $M > 0$, define

$$\mathcal{K}_{\text{temp}}(S,M) = \left\{ P' \in \mathcal{P} : \begin{array}{l} \text{There exists } F \in C^m(\mathbb{R}^n), \text{ with } \|F\|_{C^m(\mathbb{R}^n)} \leq M, \\ |F(x)-f(x)| \leq M\sigma(x) \text{ on } S, J_y(F) = P, \text{ and} \\ J_{y'}(F) = P' \end{array} \right\}.$$

Using the small remark (6), we can show that

(7) $$\text{Closure}(\mathcal{K}_{\text{temp}}(S,M)) \subset \mathcal{K}_{\text{temp}}(S,M') \text{ for } \quad M' > M.$$

To check (7), let $P' \in$ Closure $(\mathcal{K}_{\text{temp}}\ (S,M))$ be given, and let $\varepsilon > 0$. Then there exists $P'_\varepsilon \in \mathcal{K}_{\text{temp}}(S,M)$, with $\max_{|\beta| \leq m-1} |\partial^\beta(P'-P'_\varepsilon)\ (y')| < \varepsilon$.

Since $P'_\varepsilon \in \mathcal{K}_{\text{temp}}(S,M)$, there exists $F_\varepsilon \in C^m(\mathbb{R}^n)$, with $\|F_\varepsilon\|_{C^m(\mathbb{R}^n)} \leq M$, $|F_\varepsilon(x) - f(x)| \leq M\sigma(x)$ on $S, J_y(F_\varepsilon) = P, J_{y'}(F_\varepsilon) = P'_\varepsilon$.

Also, applying (6) to $P'-P'_\varepsilon$, we obtain a function $G_\varepsilon \in C^m(\mathbb{R}^n)$, with

$$\|G_\varepsilon\|_{C^m(\mathbb{R}^n)} \leq \Gamma(y,y')\varepsilon, \quad J_y(G_\varepsilon) = 0, \quad J_{y'}(G_\varepsilon) = P'-P'_\varepsilon.$$

Putting $F = F_\varepsilon + G_\varepsilon$ with ε small enough we obtain the following:

$$\|F\|_{C^m(\mathbb{R}^n)} \leq M + \Gamma(y,y') \cdot \varepsilon \leq M',$$

$$|F(x)-f(x)| \leq M\sigma(x) + \Gamma(y,y') \cdot \varepsilon \leq M'\sigma(x) \text{ on } S$$

(Recall: $S \subset E$, E is finite, $\sigma(x) > 0$ on E.)

$$J_y(F) = P \text{ and } J_{y'}(F) = P'.$$

Hence, $P' \in \mathcal{K}_{\text{temp}}(S,M')$, completing the proof of (7).

Next, let $S_1, \cdots, S_{D+1} \subset E$ be given, with $\#(S_i) \leq k_2^\#$ for each i. Set $S = S_1 \cup \cdots \cup S_{D+1}$. Thus, $S \subset E$, and $\#(S) \leq (D+1)k_2^\# \leq k_1^\#$. Since $P \in \mathcal{K}_f(y; k_1^\#, C)$ it follows that there exists $F^S \in C^m(\mathbb{R}^n)$, with

$$\|F^S\|_{C^m(\mathbb{R}^n)} \leq C, |F^S(x) - f(x)| \leq C\sigma(x) \text{ on } S, \quad J_y(F^S) = P.$$

Define $P' = J_{y'}(F^S)$. Then obviously, for $i = 1, \ldots, D+1$, we have

$$\|F^S\|_{C^m(\mathbb{R}^n)} \leq C, |F^S(x) - f(x)| \leq C\sigma(x) \text{ on } S_i, J_y(F^S) = P, J_{y'}(F^S) = P'.$$

Hence, $P' \in \mathcal{K}_{\text{temp}}(S_i, C)$ for each $i = 1, \ldots, D+1$.

We have shown that any $D+1$ of the sets $\mathcal{K}_{\text{temp}}(S, C)$ (where $S \subset E$, $\#(S) \leq k_2^\#$) have non–empty intersection. Moreover, one checks easily that each $\mathcal{K}_{\text{temp}}(S, C) \subset \mathcal{P}$ is a bounded, convex subset of a D–dimensional vector space. Applying Lemma 10.0, we see that the closures of the sets $\mathcal{K}_{\text{temp}}(S, C)$ (all $S \subset E$ with $\#(S) \leq k_2^\#$) have non–empty intersection.

Hence, $\displaystyle\bigcap_{\substack{S \subset E \\ \#(S) \leq k_2^\#}} \mathcal{K}_{\text{temp}}(S, C')$ is non–empty, for any $C' > C$.

Let $P' \in \displaystyle\bigcap_{\substack{S \subset E \\ \#(S) \leq k_2^\#}} \mathcal{K}_{\text{temp}}(S, C')$. Then, by definition, given $S \subset E$ with $\#(S) \leq k_2^\#$, there exists $F^S \in C^m(\mathbb{R}^n)$, with

$$\|F^S\|_{C^m(\mathbb{R}^n)} \leq C', |F^S(x) - f(x)| \leq C'\sigma(x) \text{ on } S, \qquad J_{y'}(F^S) = P', \tag{8}$$

and $J_y(F^S) = P$.

In particular, this implies $P' \in \mathcal{K}_f(y'; k_2^\#, C')$. Moreover, if we take S to be the empty set in (8), then we obtain a function $F \in C^m(\mathbb{R}^n)$, with $\|F\|_{C^m(\mathbb{R}^n)} \leq C'$, $J_{y'}(F) = P', J_y(F) = P$.

By Taylor's theorem, the polynomials P, P' satisfy

$$\begin{aligned} |\partial^\beta P'(y') - \partial^\beta P(y')| &= \left| \partial^\beta P'(y') - \sum_{|\gamma| \leq m-1-|\beta|} \frac{1}{\gamma!} \left(\partial^{\gamma+\beta} P(y) \right) \cdot (y'-y)^\gamma \right| \\ &= |\partial^\beta F(y') - \sum_{|\gamma| \leq m-1-|\beta|} \frac{1}{\gamma!} \left(\partial^{\gamma+\beta} F(y) \right) \cdot \\ &\qquad \cdot (y'-y)^\gamma| \leq C''|y-y'|^{m-|\beta|} \end{aligned}$$

for $|\beta| \leq m-1$, and similarly $|\partial^\beta P(y) - \partial^\beta P'(y)| \leq C''|y-y'|^{m-|\beta|}$ for $|\beta| \leq m-1$.

The proof of Lemma 10.2 is complete. ∎

The next Lemma is again proven using the same ideas as above. It associates to each point y near y^0 a family of polynomials P_α^y, analogous to the polynomials P_α associated to y^0.

Lemma 10.3. *Suppose $k^\# \geq (D+1)\cdot k_1^\#$, and let $y \in B(y^0, a_1)$ be given. Then there exist polynomials $P_\alpha^y \in \mathcal{P}$, indexed by $\alpha \in \mathcal{A}$, with the following properties:*

(WL1)^y. *$\partial^\beta P_\alpha^y(y) = \delta_{\beta\alpha}$ for all $\beta, \alpha \in \mathcal{A}$.*

(WL2)^y. *$|\partial^\beta P_\alpha^y(y) - \delta_{\beta\alpha}| \leq Ca_1$ for all $\alpha \in \mathcal{A}$, $\beta \in \mathcal{M}$.*

(WL3)^y. *Given $\alpha \in \mathcal{A}$ and $S \subset E$ with $\#(S) \leq k_1^\#$, there exists $\varphi_\alpha^S \in C^m(\mathbb{R}^n)$, with*
- (a) $\|\partial^m \varphi_\alpha^S\|_{C^0(\mathbb{R}^n)} \leq Ca_1$.
- (b) $|\varphi_\alpha^S(x)| \leq C\sigma(x)$ *for all* $x \in S$.
- (c) $J_y(\varphi_\alpha^S) = P_\alpha^y$.

Proof. For $y = y^0$, the Lemma is trivial; we just set $P_\alpha^y = P_\alpha (\alpha \in \mathcal{A})$ and invoke (SU2, 3, 5). Suppose $y \neq y^0$. For a constant $\Gamma(y, y^0)$ determined by y, y^0, m and n we have the small remark:

(9) $$\begin{aligned}&\text{Given } \tilde{P} \in \mathcal{P}, \text{there exists } \tilde{G} \in C^m(\mathbb{R}^n) \text{ with}\\ &\|\tilde{G}\|_{C^m(\mathbb{R}^n)} \leq \Gamma(y, y^0)\cdot \max_{|\beta|\leq m-1} |\partial^\beta \tilde{P}(y)|, J_y(\tilde{G}) = \tilde{P}, J_{y^0}(\tilde{G}) = 0.\end{aligned}$$

Now, given $\alpha \in \mathcal{A}$, $S \subset E$ and $M > 0$, we define

$$\mathcal{K}_\alpha(S, M) = \left\{ P' \in \mathcal{P} : \begin{array}{l}\text{There exists } \varphi_\alpha^S \in C^m(\mathbb{R}^n) \text{ with } \|\partial^m \varphi_\alpha^S\|_{C^0(\mathbb{R}^n)} \leq Ma_1,\\ |\varphi_\alpha^S(x)| \leq M\sigma(x) \text{ on } S,\ J_{y^0}(\varphi_\alpha^S) = P_\alpha, \text{ and } J_y(\varphi_\alpha^S) = P'\end{array} \right\}$$

By a now–familiar argument using (9), we know that

(10) $$\text{Closure}(\mathcal{K}_\alpha(S, M)) \subset \mathcal{K}_\alpha(S, M') \quad \text{for any} \quad M' > M.$$

Each $\mathcal{K}_\alpha(S, M)$ is a bounded convex subset of the D–dimensional vector space $\mathcal{P}$. Moreover, it follows from (SU5) by a now–familiar argument that $\bigcap_{i=1}^{D+1} \mathcal{K}_\alpha(S_i, 1)$ is non–empty, whenever $S_1, \ldots, S_{D+1} \subset E$ with $\#(S_i) \leq k_1^\#$ for each i. Lemma 10.0 therefore shows that, for each $\alpha \in \mathcal{A}$, the closures of all the $\mathcal{K}_\alpha(S, 1)$ ($S \subset E$ with $\#(S) \leq k_1^\#$) have non–empty intersection.

Therefore by (10), there exist polynomials $\bar{P}^y_\alpha$ ($\alpha \in \mathcal{A}$), belonging to $\mathcal{K}_\alpha(S,2)$ for all $S \subset E$ with $\#(S) \le k_1^\#$. Thus, given $S \subset E$ with $\#(S) \le k_1^\#$, and given $\alpha \in \mathcal{A}$, there exists $\bar{\varphi}^S_\alpha \in C^m(\mathbb{R}^n)$, with

$$\|\partial^m \bar{\varphi}^S_\alpha\|_{C^0(\mathbb{R}^n)} \le 2a_1, \tag{11}$$

$$|\bar{\varphi}^S_\alpha(x)| \le 2\sigma(x) \quad \text{for all} \;\; x \in S, \tag{12}$$

$$J_{y^0}(\bar{\varphi}^S_\alpha) = P_\alpha \quad \text{and } J_y(\bar{\varphi}^S_\alpha) = \bar{P}^y_\alpha. \tag{13}$$

We apply (11), (12), (13) with $S =$ empty set. Thus, there exists $\bar{\varphi}_\alpha$ with

$$\|\partial^m \bar{\varphi}_\alpha\|_{C^0(\mathbb{R}^n)} \le 2a_1, J_{y^0}(\bar{\varphi}_\alpha) = P_\alpha,\; J_y(\bar{\varphi}_\alpha) = \bar{P}^y_\alpha. \tag{14}$$

Since also $y \in B(y^0, a_1)$, (14) and (SU3, 4) imply

$$|\partial^\beta \bar{P}^y_\alpha(y) - \partial^\beta P_\alpha(y^0)| \le Ca_1 \quad \text{for all} \;\; \alpha \in \mathcal{A}, \beta \in \mathcal{M},$$

and therefore

$$|\partial^\beta \bar{P}^y_\alpha(y) - \delta_{\beta\alpha}| \le C'a_1 \quad \text{for all} \;\; \alpha \in \mathcal{A}, \beta \in \mathcal{M}, \tag{15}$$

thanks to (SU3). In particular, the matrix $(\partial^\beta \bar{P}^y_\alpha(y))_{\beta,\alpha \in \mathcal{A}}$ has an inverse $(M_{\alpha'\alpha})_{\alpha',\alpha\in\mathcal{A}}$, with

$$|M_{\alpha'\alpha} - \delta_{\alpha'\alpha}| \le C''a_1 \quad \text{for all} \;\; \alpha', \alpha \in \mathcal{A}. \tag{16}$$

(Here and in the next few paragraphs we use (SU4).) By definition, we have

$$\sum_{\alpha' \in \mathcal{A}} \partial^\beta \bar{P}^y_{\alpha'}(y) \cdot M_{\alpha'\alpha} = \delta_{\beta\alpha} \quad \text{for all } \beta, \alpha \in \mathcal{A}. \tag{17}$$

Now define

$$P^y_\alpha = \sum_{\alpha' \in \mathcal{A}} \bar{P}^y_{\alpha'} M_{\alpha'\alpha} \quad \text{for all } \alpha \in \mathcal{A}. \tag{18}$$

From (15), (16), (17), we see that

$$\partial^\beta P^y_\alpha(y) = \delta_{\beta\alpha} \quad \text{for all } \beta, \alpha \in \mathcal{A}, \tag{19}$$

and that

$$|\partial^\beta P^y_\alpha(y) - \delta_{\beta\alpha}| \le C'''a_1 \quad \text{for all } \alpha \in \mathcal{A}, \beta \in \mathcal{M}. \tag{20}$$

Moreover, let $S \subset E$ be given, with $\#(S) \leq k_1^{\#}$. With $\bar{\varphi}_\alpha^S$ as in (11), ... ,(13), define

$$\varphi_\alpha^S = \sum_{\alpha' \in \mathcal{A}} \bar{\varphi}_{\alpha'}^S M_{\alpha'\alpha} \quad \text{for all } \alpha \in \mathcal{A}. \tag{21}$$

From (11), (16), (21) we obtain

$$\|\partial^m \varphi_\alpha^S\|_{C^0(\mathbb{R}^n)} \leq C a_1 \quad \text{for all } \alpha \in \mathcal{A}. \tag{22}$$

From (12), (16), (21) we have

$$|\varphi_\alpha^S(x)| \leq C\sigma(x) \quad \text{for all } \alpha \in \mathcal{A}, x \in S. \tag{23}$$

From (13), (18), (21), we see that

$$J_y(\varphi_\alpha^S) = P_\alpha^y \quad \text{for all } \alpha \in \mathcal{A}. \tag{24}$$

The conclusions of the Lemma are (19), (20), (22), (23), (24). The proof of Lemma 10.3 is complete. ■

Lemma 10.4. *Suppose $k^{\#} \geq (D+1)k_1^{\#}$ and $k_1^{\#} \geq (D+1)k_2^{\#}$. Let $y \in B(y^0, a_1)$, and let $(P_\alpha^y)_{\alpha \in \mathcal{A}}$ satisfy conditions (WL1)$^y \cdots$ (WL3)^y, as in the conclusion of Lemma 10.3. Let $y' \in \mathbb{R}^n$ be given. Then there exist polynomials $(\tilde{P}_\alpha^{y',y})_{\alpha \in \mathcal{A}}$, with the following property:*

Given $\alpha \in \mathcal{A}$ and $S \subset E$ with $\#(S) \leq k_2^{\#}$, there exists $\varphi_\alpha^S \in C^m(\mathbb{R}^n)$, with

(a) $\|\partial^m \varphi_\alpha^S\|_{C^0(\mathbb{R}^n)} \leq C' a_1$.
(b) $|\varphi_\alpha^S(x)| \leq C'\sigma(x)$ *for all* $x \in S$.
(c) $J_y(\varphi_\alpha^S) = P_\alpha^y$.
(d) $J_{y'}(\varphi_\alpha^S) = \tilde{P}_\alpha^{y',y}$.

Proof. The Lemma is trivial for $y' = y$; we just set $\tilde{P}_\alpha^{y',y} = P_\alpha^y$ and invoke (WL3)^y. Suppose $y' \neq y$. Then, for a constant $\Gamma(y, y')$ determined by $y - y', m$ and n, the following small remark holds.

$$\text{Given } \tilde{P} \in \mathcal{P}, \text{ there exists } \tilde{G} \in C^m(\mathbb{R}^n), \text{ with} \\ \|\tilde{G}\|_{C^m(\mathbb{R}^n)} \leq \Gamma(y, y') \cdot \max_{|\beta| \leq m-1} |\partial^\beta \tilde{P}(y')|, J_y(\tilde{G}) = 0, J_{y'}(\tilde{G}) = \tilde{P}. \tag{25}$$

Now, given $\alpha \in \mathcal{A}, M > 0, S \subset E$, we define

$$\mathcal{K}^{[\alpha]}(S, M) = \left\{ P' \in \mathcal{P} : \begin{array}{l} \text{There exists } \varphi \in C^m(\mathbb{R}^n), \text{ with } \|\partial^m \varphi\|_{C^0(\mathbb{R}^n)} \leq \\ M a_1, |\varphi(x)| \leq M\sigma(x) \text{ on } S, J_y(\varphi) = P_\alpha^y, J_{y'}(\varphi) = P' \end{array} \right\}.$$

As usual, (25) shows that

$$\text{Closure}(\mathcal{K}^{[\alpha]}(S,M)) \subset \mathcal{K}^{[\alpha]}(S,M') \quad \text{for all } M' > M. \tag{26}$$

Each $\mathcal{K}^{[\alpha]}(S,M)$ is easily seen to be a bounded convex subset of the D–dimensional vector space $\mathcal{P}$. Moreover, a familiar argument using $(\text{WL3})^y$ shows that $\bigcap_{i=1}^{D+1} \mathcal{K}^{[\alpha]}(S_i,C)$ is non–empty, for any $S_1,\dots,S_{D+1} \subset E$ with $\#(S_i) \le k_2^{\#}$ for each i.

Consequently, Lemma 10.0 shows that the intersection of all the sets $\text{Closure}(\mathcal{K}^{[\alpha]}(S,C))$ ($S \subset E$ with $\#(S) \le k_2^{\#}$) is non–empty. Applying (26), we find that for each $\alpha \in \mathcal{A}$, there exists $\tilde{P}_\alpha^{y',y} \in \mathcal{P}$, with $\tilde{P}_\alpha^{y',y}$ belonging to $\mathcal{K}^{[\alpha]}(S,C')$ for each $S \subset E$ with $\#(S) \le k_2^{\#}$.

The conclusions of Lemma 10.4 are now immediate from the definition of $\mathcal{K}^{[\alpha]}(S,C')$. ■

Next, for $y \in \mathbb{R}^n, k \ge 1, M > 0$, we define

$$\mathcal{K}_f^{\#}(y;k,M) = \{P \in \mathcal{K}_f(y;k,M) : \partial^\beta P(y) = 0 \quad \text{for all } \beta \in \mathcal{A}\}.$$

Lemma 10.5. *Suppose $k^{\#} \ge (D+1)k_1^{\#}$ and $k_1^{\#} \ge 1$. Then, for a large enough controlled constant C, the set $\mathcal{K}_f^{\#}(y;k_1^{\#},C)$ is non–empty for each $y \in B(y^0,a_1)$.*

Proof. By Lemma 10.1, there exists $P \in \mathcal{K}_f(y;k_1^{\#},2)$. By definition, we have

(27) Given $S \subset E$ with $\#(S) \le k_1^{\#}$, there exists $F^S \in C^m(\mathbb{R}^n)$, with $\|F^S\|_{C^m(\mathbb{R}^n)} \le 2, |F^S(x) - f(x)| \le 2\sigma(x)$ on S, and $J_y(F^S) = P$.

Taking S to be the empty set in (27), we learn that

$$|\partial^\beta P(y)| \le C \quad \text{for all } \beta \in \mathcal{M}. \tag{28}$$

By Lemma 10.3, there exist polynomials $P_\alpha^y (\alpha \in \mathcal{A})$ satisfying $(\text{WL1})^y, \dots, (\text{WL3})^y$. We define

$$\tilde{P} = P - \sum_{\alpha \in \mathcal{A}} (\partial^\alpha P(y)) \cdot P_\alpha^y. \tag{29}$$

From $(\text{WL1})^y$ and (29), we have

$$\partial^\beta \tilde{P}(y) = \partial^\beta P(y) - \sum_{\alpha \in \mathcal{A}} (\partial^\alpha P(y)) \cdot \delta_{\beta\alpha} = 0 \quad \text{for all } \beta \in \mathcal{A}. \tag{30}$$

Let $S \subset E$ with $\#(S) \le k_1^{\#}$, and let φ_α^S, F^S be as in $(\mathrm{WL3})^y$ and (27). Also, fix $\theta \in C^m(\mathbb{R}^n)$, with

$$0 \le \theta \le 1 \text{ on } \mathbb{R}^n,\ \operatorname{supp} \theta \subset B(y,1),\ \theta = 1 \ \text{ on } \ B(y,1/2), \|\theta\|_{C^m(\mathbb{R}^n)} \le C. \tag{31}$$

Then define

$$\tilde{F}^S = F^S - \sum_{\alpha \in \mathcal{A}} (\partial^\alpha P(y)) \varphi_\alpha^S \theta. \tag{32}$$

From $(\mathrm{WL2})^y$, $(\mathrm{WL3(a)})^y$, $(\mathrm{WL3(c)})^y$, we conclude that $|\partial^\beta \varphi_\alpha^S| \le C$ on $B(y,1)$, for $|\beta| \le m$. Hence, (31) gives

$$\|\varphi_\alpha^S \theta\|_{C^m(\mathbb{R}^n)} \le C' \quad \text{for each } \alpha \in \mathcal{A}. \tag{33}$$

Putting (27), (28) and (33) into (32), we find that

$$\|\tilde{F}^S\|_{C^m(\mathbb{R}^n)} \le C''. \tag{34}$$

Also, for $x \in S$, we have $|\tilde{F}^S(x) - f(x)| \le |F^S(x) - f(x)| + \sum_{\alpha \in \mathcal{A}} |\partial^\alpha P(y)| \cdot |\varphi_\alpha^S(x)\theta(x)|$ (by (32)) $\le C''\sigma(x)$, by (27), (28), $(\mathrm{WL3(b)})^y$, and (31). Thus,

$$|\tilde{F}^S(x) - f(x)| \le C''\sigma(x) \quad \text{on } S. \tag{35}$$

From $(\mathrm{WL3(c)})^y$, (27), (29), (31), (32), we find that

$$J_y(\tilde{F}^S) = J_y(F^S) - \sum_{\alpha \in \mathcal{A}} (\partial^\alpha P(y)) J_y(\varphi_\alpha^S \theta) = P - \sum_{\alpha \in \mathcal{A}} (\partial^\alpha P(y)) P_\alpha^y = \tilde{P}. \tag{36}$$

Thus, given $S \subset E$ with $\#(S) \le k_1^{\#}$, there exists $\tilde{F}^S \in C^m(\mathbb{R}^n)$, satisfying (34), (35), (36). In other words,

$$\tilde{P} \in \mathcal{K}_f(y; k_1^{\#}, C'').$$

From (30), we then have $\tilde{P} \in \mathcal{K}_f^{\#}(y; k_1^{\#}, C'')$, completing the proof of lemma 10.5. ■

11. A Calderón–Zygmund Decomposition

In this section, we are again in the setting of section 9, and we assume (SU0, . . . , 6). We fix a cube $Q^\circ \subset \mathbb{R}^n$, with the following properties:

$$Q^\circ \text{ is centered at } y^0. \tag{1}$$

$$(Q^\circ)^{\star\star\star} \subset B(y^0, a_1). \tag{2}$$

$$c a_1 < \delta_{Q^\circ} < a_1. \tag{3}$$

A subcube $Q \subseteq Q^\circ$ is called "dyadic" if either $Q = Q^\circ$ or else Q arises from Q° by successive "bisection". A dyadic cube $Q \underset{\neq}{\subset} Q^\circ$ arises by "bisecting" its dyadic "parent" Q^+, which is again a dyadic cube, with $\delta_{Q^+} = 2\delta_Q$. A cube Q not contained in Q° is not dyadic, according to our definition.

Two distinct dyadic cubes Q, Q' are said to "abut" if their closures have non–empty intersection.

We say that a dyadic cube $Q \subseteq Q^\circ$ is "OK" if it satisfies the following condition.

(OK). For every $y \in Q^{\star\star}$, there exist $\bar{\mathcal{A}}^y < \mathcal{A}$ and polynomials $\bar{P}^y_\alpha \in \mathcal{P}$, indexed by $\alpha \in \bar{\mathcal{A}}^y$, with the following properties:

(OK1). $\partial^\beta \bar{P}^y_\alpha(y) = \delta_{\beta\alpha}$ for all $\beta, \alpha \in \bar{\mathcal{A}}^y$.

(OK2). $\delta_Q^{|\beta|-|\alpha|} |\partial^\beta \ \bar{P}^y_\alpha(y)| \leq (a_1)^{-(m+1)}$ for all $\alpha, \beta \in \mathcal{M}$ with $\alpha \in \bar{\mathcal{A}}^y$ and $\beta \geq \alpha$.

(OK3). Given $\alpha \in \bar{\mathcal{A}}^y$ and $S \subset E$ with $\#(S) \leq k^\#_{\text{old}}$, there exists $\varphi^{S,y}_\alpha \in C^m(\mathbb{R}^n)$, with

(a) $\delta_Q^{m-|\alpha|} \ \|\partial^m \varphi^{S,y}_\alpha\|_{C^0(\mathbb{R}^n)} \leq (a_1)^{-(m+1)}$.
(b) $\delta_Q^{m-|\alpha|} |\varphi^{S,y}_\alpha(x)| \leq (a_1)^{-(m+1)} \cdot \sigma(x)$ for all $x \in S$.
(c) $J_y(\varphi^{S,y}_\alpha) = \bar{P}^y_\alpha$.

Here, $k^\#_{\text{old}}$ is as in Lemma 8.1 and section 9.

We say that a dyadic cube $Q \subseteq Q^\circ$ is a "CZ" or "Calderón–Zygmund" cube, if it is OK, but no dyadic cube properly containing Q is OK.

Given any two dyadic cubes Q, Q', either $Q \cap Q' = \phi$, or $Q \subseteq Q'$, or $Q' \subseteq Q$. Hence, any two distinct CZ cubes are disjoint.

Lemma 11.1. *Any dyadic cube Q with $\delta_Q < \min\limits_{x \in E} \sigma(x)$ is OK.*

Proof. Let $y \in Q^{\star\star}$, with Q a dyadic cube of diameter less than $\min\limits_{x \in E} \sigma(x)$.

We set $\bar{\mathcal{A}}^y = \mathcal{M}$. Note that $\bar{\mathcal{A}}^y < \mathcal{A}$, thanks to (SU0) and Lemma 3.2. For $\alpha \in \bar{\mathcal{A}}^y$, we set $\bar{P}^y_\alpha(x) = \frac{1}{\alpha!}(x-y)^\alpha$ We have $\partial^\beta \bar{P}^y_\alpha(y) = \delta_{\beta\alpha}$ for $\alpha, \beta \in \mathcal{M}$.

Hence, (OK1) holds, and (OK2) follows from (SU4). It remains to check (OK3). We fix a function $\theta \in C^m(\mathbb{R}^n)$, with

$$(4) \qquad 0 \le \theta \le 1 \text{ on } \mathbb{R}^n, \;\; \theta = 1 \text{ on } B(0,1/2), \;\; \text{supp } \theta \subset B(0,1), \;\; \|\theta\|_{C^m(\mathbb{R}^n)} \le C.$$

Given $\alpha \in \bar{\mathcal{A}}^y$ and $S \subset E$ with $\#(S) \le k^{\#}_{\text{old}}$, we define

$$(5) \qquad \varphi^{S,y}_\alpha(x) = \frac{1}{\alpha!}(x-y)^\alpha \theta(x-y).$$

From (4), (5) we have $\|\partial^m \varphi^{S,y}_\alpha\|_{C^0(\mathbb{R}^n)} \le C'$.

Also, we have $\delta_Q \le a_1$ by (3), since $Q \subseteq Q^\circ$. Hence,

$$\delta_Q^{m-|\alpha|} \|\partial^m \varphi^{S,y}_\alpha\|_{C^0(\mathbb{R}^n)} \le C'(a_1)^{m-|\alpha|} < (a_1)^{-(m+1)} \quad \text{by (SU4)}.$$

Thus, (OK3(a)) holds. Also for $x \in S$, we have $\delta_Q^{m-|\alpha|}|\varphi^{S,y}_\alpha(x)| \le C' \; \delta_Q^{m-|\alpha|} \le C'\delta_Q$ (since $\delta_Q \le a_1 \le 1$ and $|\alpha| \le m-1$) $< C'\sigma(x)$ (by hypothesis of Lemma 11.1) $< (a_1)^{-(m+1)}\sigma(x)$ (by (SU4)).

Thus, (OK3(b)) holds

Also, (OK3(c)) holds, as we see at once by comparing the definitions of $\bar{P}^y_\alpha$ and $\varphi^{S,y}_\alpha$, and recalling (4).

Thus, (OK1, ... , 3) are satisfied. The proof of Lemma 11.1 is complete. ∎

Corollary. *The CZ cubes form a partition of Q° into finitely many dyadic cubes.*

Lemma 11.2. *If two CZ cubes Q, Q' abut, then*

$$(6) \qquad \frac{1}{2}\delta_Q \le \delta_{Q'} \le 2\delta_Q.$$

Proof. Assume (6) false. Without loss of generality, we may assume that $\delta_Q \le \delta_{Q'}$. Then we have

$$(7) \qquad \delta_Q \le \frac{1}{4}\delta_{Q'}.$$

Note that $Q \neq Q^\circ$, since Q is assumed to abut another CZ cube Q'. Hence, Q has a dyadic parent Q^+, which also abuts Q', and satisfies

$$\delta_{Q^+} \leq \frac{1}{2}\delta_{Q'}. \tag{8}$$

Consequently, we have

$$(Q^+)^{\star\star} \subset (Q')^{\star\star}. \tag{9}$$

We know that Q' is OK, since it is a CZ cube. We will show that Q^+ is also OK. For any $y \in (Q')^{\star\star}$, let $\bar{\mathcal{A}}^y < \mathcal{A}$ and $\bar{P}^y_\alpha(\alpha \in \bar{\mathcal{A}}^y)$ satisfy (OK1, 2, 3) for Q'. Then, for any $y \in (Q^+)^{\star\star}$, we may use the same $\bar{\mathcal{A}}^y$ and $\bar{P}^y_\alpha(\alpha \in \bar{\mathcal{A}}^y)$ for Q^+, thanks to (9). Conditions (OK1, 2, 3) hold for Q^+, because they hold for Q', and thanks to (8). Here we use (8) to show that $(\delta_{Q^+})^{m-|\alpha|} \leq (\delta_{Q'})^{m-|\alpha|}$ for $\alpha \in \mathcal{M}$, and that

$$(\delta_{Q^+})^{|\beta|-|\alpha|} \leq (\delta_{Q'})^{|\beta|-|\alpha|} \quad \text{for} \quad \beta \geq \alpha. \qquad \text{(See Lemma 3.1)} \tag{10}$$

This proves that Q^+ is OK, as claimed. On the other hand, Q^+ is a dyadic cube that properly contains the CZ cube Q. Hence, Q^+ cannot be OK, by the definition of CZ cubes. This contradiction shows that (6) cannot be false, completing the proof of Lemma 2. ∎

Remark. In proving Lemma 2, we made essential use of the restriction to the case $\beta \geq \alpha$ in (OK2). (See (10)).

12. Controlling Auxiliary Polynomials I

We again place ourselves in the setting of Section 9, and we assume (SU0, . . . , 6). In this section only, we fix an integer $k_1^\#$, a dyadic cube Q, a point $y \in \mathbb{R}^n$, and a family of polynomials $P^y_\alpha \in \mathcal{P}$, indexed by $\alpha \in \mathcal{A}$; and we make the following assumptions.

(CAP1). $k^\# \geq (D+1)k_1^\#$ and $k_1^\# \geq (D+1)\cdot k^\#_{\text{old}}$.

(CAP2). $y \in Q^{\star\star\star}$.

(CAP3). Q is properly contained in Q°.

(CAP4). The $P^y_\alpha(\alpha \in \mathcal{A})$ satisfy conditions $(\text{WL1})^y$, $(\text{WL2})^y$, $(\text{WL3})^y$. (See Lemma 10.3.)

(CAP5). $(a_1)^{-m} \le \max_{\substack{\beta\in\mathcal{M}\\ \alpha\in\mathcal{A}}} \delta_Q^{|\beta|-|\alpha|}\ |\partial^\beta P_\alpha^y(y)| \le 2^m \cdot (a_1)^{-m}$.

Note that $\mathcal{A}$ is non–empty, since the max in (CAP5) cannot be zero. Our goal in this section is to show that the dyadic cube Q^+ is OK.

Let

$$y' \in (Q^+)^{\star\star} \tag{1}$$

be given. Then $y, y' \in Q^{\star\star\star} \subseteq (Q^\circ)^{\star\star\star} \subset B(y^0, a_1)$, by (11.2). Applying Lemma 10.4, with $k_2^\# = k_{\text{old}}^\#$, we obtain a family of polynomials $\tilde{P}_\alpha^{y'} \in \mathcal{P}$, indexed by $\alpha \in \mathcal{A}$, with the following property.

$$\text{(2)}\quad \begin{array}{l} \text{Given } S \subset E \text{ with } \#(S) \le k_{\text{old}}^\#, \text{ and given } \alpha \in \mathcal{A}, \text{ there exists } \varphi_\alpha^S \in C^m(\mathbb{R}^n), \\ \text{with} \\ \quad \text{(a)}\quad \|\partial^m \varphi_\alpha^S\|_{C^0(\mathbb{R}^n)} \le Ca_1, \\ \quad \text{(b)}\quad |\varphi_\alpha^S(x)| \le C\sigma(x) \text{ for all } x \in S, \\ \quad \text{(c)}\quad J_y(\varphi_\alpha^S) = P_\alpha^y, \\ \quad \text{(d)}\quad J_{y'}(\varphi_\alpha^S) = \tilde{P}_\alpha^{y'}. \end{array}$$

We fix polynomials $\tilde{P}_\alpha^{y'}$ satisfying (2). The basic properties of $\tilde{P}_\alpha^{y'}$ are as follows.

Lemma 12.1. *We have*

$$c \cdot (a_1)^{-m} \le \max_{\substack{\beta\in\mathcal{M}\\ \alpha\in\mathcal{A}}} \delta_Q^{|\beta|-|\alpha|} |\partial^\beta \tilde{P}_\alpha^{y'}(y')| \le C \cdot (a_1)^{-m}; \tag{3}$$

$$\delta_Q^{|\beta|-|\alpha|} |\partial^\beta \tilde{P}_\alpha^{y'}(y')| \le C \cdot a_1 \quad \textit{for } \alpha \in \mathcal{A},\ \beta > \alpha,\ \beta \in \mathcal{M}; \tag{4}$$

$$|\partial^\alpha \tilde{P}_\alpha^{y'}(y') - 1| \le C \cdot a_1 \quad \textit{for } \alpha \in \mathcal{A}; \tag{5}$$

$$\delta_Q^{|\beta|-|\alpha|} |\partial^\beta \tilde{P}_\alpha^{y'}(y')| \le C \qquad \textit{for } \alpha, \beta \in \mathcal{A}. \tag{6}$$

Proof. We apply (2), with S = empty set. Thus, for each $\alpha \in \mathcal{A}$ we obtain $\varphi_\alpha \in C^m(\mathbb{R}^n)$, with $\|\partial^m \varphi_\alpha\|_{C^0(\mathbb{R}^n)} \le Ca_1$, $J_y(\varphi_\alpha) = P_\alpha^y$, $J_{y'}\ (\varphi_\alpha) = \tilde{P}_\alpha^{y'}$. Taylor's theorem gives

$$\begin{aligned} &\left| \partial^\beta \tilde{P}_\alpha^{y'}(y') \ - \sum_{|\gamma|\le m-1-|\beta|} \frac{1}{\gamma!}\big(\partial^{\gamma+\beta} P_\alpha^y(y)\big) \cdot (y'-y)^\gamma \right| \\ &= \left| \partial^\beta \varphi_\alpha(y') \ - \sum_{|\gamma|\le m-1-|\beta|} \frac{1}{\gamma!}(\partial^{\gamma+\beta}\varphi_\alpha(y)) \cdot (y'-y)^\gamma \right| \\ &\le Ca_1 \cdot |y-y'|^{m-|\beta|} \qquad\qquad \textit{for } \beta \in \mathcal{M}. \end{aligned} \tag{7}$$

Similarly,

$$\left|\partial^\beta P_\alpha^y(y) - \sum_{|\gamma|\le m-1-|\beta|} \frac{1}{\gamma!}\left(\partial^{\gamma+\beta}\tilde{P}_\alpha^{y'}(y')\right)\cdot(y-y')^\gamma\right| \tag{8}$$
$$= \left|\partial^\beta \varphi_\alpha(y) - \sum_{|\gamma|\le m-1-|\beta|} \frac{1}{\gamma!}\left(\partial^{\gamma+\beta}\varphi_\alpha(y')\right)\cdot(y-y')^\gamma\right|$$
$$\le Ca_1\cdot|y-y'|^{m-|\beta|} \qquad\qquad for\ \beta\in\mathcal{M}.$$

In view of (CAP2) and (1), we have

$$|y-y'| \le C\delta_Q \le C\delta_{Q^\circ} \le Ca_1 < 1. \tag{9}$$

(Here we have used also that $Q\subseteq Q^\circ$ since Q is dyadic, as well as (11.3) and (SU4).) From (CAP5) we have

(10)

$|\partial^{\gamma+\beta}P_\alpha^y(y)| \le 2^m\cdot(a_1)^{-m}\cdot\delta_Q^{|\alpha|-|\beta|-|\gamma|}$, for all $\alpha\in\mathcal{A},\ \beta\in\mathcal{M},\ |\gamma|\le m-1-|\beta|$.

Putting (9), (10) into (7), we find that

$$|\partial^\beta\tilde{P}_\alpha^{y'}(y')| \le C\cdot(a_1)^{-m}\cdot\delta_Q^{|\alpha|-|\beta|} \quad \text{for all } \alpha\in\mathcal{A},\ \beta\in\mathcal{M}. \tag{11}$$

On the other hand, if we put

$$\Omega = \max_{\substack{\beta\in\mathcal{M}\\ \alpha\in\mathcal{A}}} \delta_Q^{|\beta|-|\alpha|}|\partial^\beta\tilde{P}_\alpha^{y'}(y')|, \tag{12}$$

then we have

$$|\partial^{\gamma+\beta}\tilde{P}_\alpha^{y'}(y')| \le \Omega\delta_Q^{|\alpha|-|\beta|-|\gamma|} \quad \text{for} \quad \alpha\in\mathcal{A},\ \beta\in\mathcal{M},\ |\gamma|\le m-1-|\beta|. \tag{13}$$

Putting (9) and (13) into (8), we find that

$$|\partial^\beta P_\alpha^y(y)| \le C\Omega\delta_Q^{|\alpha|-|\beta|} + Ca_1\delta_Q^{m-|\beta|} \le (C\Omega+1)\delta_Q^{|\alpha|-|\beta|} \text{ for all } \alpha\in\mathcal{A},\ \beta\in\mathcal{M}. \tag{14}$$

Comparing (14) with (CAP5), we see that $C\Omega+1\ge(a_1)^{-m}$, hence $\Omega > c(a_1)^{-m}$. Together with (11) and (12) this proves conclusion (3).

Next, suppose $\alpha\in\mathcal{A}$ and $\beta>\alpha$ ($\beta\in\mathcal{M}$). From (WL2)^y and Lemma 3.1, we see that $|\partial^{\gamma+\beta}P_\alpha^y(y)| \le Ca_1$ for $|\gamma|\le m-1-|\beta|$. Putting this and (9) into (7), we obtain the estimate

$$|\partial^\beta\tilde{P}_\alpha^{y'}(y')| \le Ca_1 \text{ for } \alpha\in\mathcal{A},\ \beta\in\mathcal{M},\ \beta>\alpha. \tag{15}$$

For $\beta > \alpha$, we have also $\delta_Q^{|\beta|-|\alpha|} \leq 1$; hence, (15) implies conclusion (4).

Next, suppose $\alpha \in \mathcal{A}$ and $\beta = \alpha$. Then we have $\gamma + \beta > \alpha$ for $\gamma \neq 0$, and hence $|\partial^{\gamma+\beta} P_\alpha^y(y)| \leq Ca_1$ by $(\text{WL2})^y$. On the other hand, $\partial^\beta P_\alpha^y(y) = 1$ in this case, by $(\text{WL1})^y$. These remarks and (9) may be substituted into (7), to show that $|\partial^\alpha \tilde{P}_\alpha^{y'}(y') - 1| \leq Ca_1$ for all $\alpha \in \mathcal{A}$, which is conclusion (5).

Next suppose $\alpha, \beta \in \mathcal{A}$. By (SU0), we have $\beta + \gamma \in \mathcal{A}$ for $|\gamma| \leq m - 1 - |\beta|$. Hence, $(\text{WL1})^y$ implies $\partial^{\gamma+\beta} P_\alpha^y(y) = \delta_{\beta+\gamma,\alpha}$. In particular

$$|\partial^{\gamma+\beta} P_\alpha^y(y)| \leq \delta_Q^{|\alpha|-|\beta|-|\gamma|} \quad \text{for} \quad |\gamma| \leq m - 1 - |\beta|.$$

Putting this and (9) into (7), we find that $|\partial^\beta \tilde{P}_\alpha^{y'}(y')| \leq C\delta_Q^{|\alpha|-|\beta|}$ for $\alpha, \beta \in \mathcal{A}$, which is conclusion (6).

The proof of the Lemma is complete. ■

Define a matrix $\tilde{M} = (\tilde{M}_{\beta\alpha})_{\beta,\alpha\in\mathcal{A}}$ by setting

$$\tilde{M}_{\beta\alpha} = \delta_Q^{|\beta|-|\alpha|} \partial^\beta \tilde{P}_\alpha^{y'}(y') \quad \text{for} \quad \beta, \alpha \in \mathcal{A}. \tag{16}$$

From (4), (5), (6), we see that

$$\begin{aligned} |\tilde{M}_{\beta\alpha}| &\leq Ca_1 && \text{for } \beta > \alpha, \\ |\tilde{M}_{\beta\alpha} - 1| &\leq Ca_1 && \text{for } \beta = \alpha, \text{ and} \\ |\tilde{M}_{\beta\alpha}| &\leq C && \text{for all } \beta, \alpha. \end{aligned} \tag{16a}$$

That is, $\tilde{M}$ lies within distance Ca_1 of a triangular matrix with bounded entries and 1's on the main diagonal. It follows that the inverse matrix $M = (M_{\alpha'\alpha})_{\alpha',\alpha\in\mathcal{A}}$ has the same property, i.e.,

$$|M_{\alpha'\alpha}| \leq Ca_1 \quad \text{for} \quad \alpha' > \alpha \quad (\alpha, \alpha' \in \mathcal{A}), \tag{17}$$

$$|M_{\alpha\alpha} - 1| \leq Ca_1 \quad \text{for} \quad \alpha \in \mathcal{A}, \tag{18}$$

$$|M_{\alpha'\alpha}| \leq C \quad \text{for all} \quad \alpha', \alpha \in \mathcal{A}. \tag{19}$$

By definition, we have

$$\sum_{\alpha'\in\mathcal{A}} \tilde{M}_{\beta\alpha'} M_{\alpha'\alpha} = \delta_{\beta\alpha} \quad \text{for all} \quad \beta, \alpha \in \mathcal{A}. \tag{20}$$

That is,

$$\delta_{\beta\alpha} = \sum_{\alpha'\in\mathcal{A}} \delta_Q^{|\beta|-|\alpha'|} \partial^\beta \tilde{P}_{\alpha'}^{y'}(y') \cdot M_{\alpha'\alpha} \quad \text{for all} \quad \beta, \alpha \in \mathcal{A}. \tag{21}$$

We define new polynomials

$$\check{P}^{y'}_\alpha = \delta_Q^{|\alpha|} \sum_{\alpha' \in \mathcal{A}} \delta_Q^{-|\alpha'|} \tilde{P}^{y'}_{\alpha'} M_{\alpha'\alpha} \quad \text{for all} \quad \alpha \in \mathcal{A}. \tag{22}$$

The basic properties of the $\check{P}^{y'}_\alpha$ are as follows.

Lemma 12.2. *We have*

$$\partial^\beta \check{P}^{y'}_\alpha (y') = \delta_{\beta\alpha} \quad \textit{for all} \quad \beta, \alpha \in \mathcal{A}. \tag{23}$$

$$c \cdot (a_1)^{-m} < \max_{\substack{\beta \in \mathcal{M} \\ \alpha \in \mathcal{A}}} |\partial^\beta \check{P}^{y'}_\alpha (y')| \cdot \delta_Q^{|\beta|-|\alpha|} < C \cdot (a_1)^{-m}. \tag{24}$$

$$\delta_Q^{|\beta|-|\alpha|} |\partial^\beta \check{P}^{y'}_\alpha (y')| \leq C \cdot (a_1)^{-(m-1)} \quad \textit{for all} \quad \alpha \in \mathcal{A},\ \beta \in \mathcal{M} \quad \textit{with} \quad \beta > \alpha. \tag{25}$$

(26) *Given $\alpha \in \mathcal{A}$ and $S \subset E$ with $\#(S) \leq k^\#_{\text{old}}$, there exists $\check{\varphi}^S_\alpha \in C^m(\mathbb{R}^n)$, with*

(a) $\delta_Q^{m-|\alpha|} \|\partial^m \check{\varphi}^S_\alpha\|_{C^0(\mathbb{R}^n)} \leq C a_1$,

(b) $\delta_Q^{m-|\alpha|} |\check{\varphi}^S_\alpha(x)| \leq C\sigma(x)$ *for all* $x \in S$,

(c) $J_{y'}(\check{\varphi}^S_\alpha) = \check{P}^{y'}_\alpha$.

Proof. Conclusion (23) is immediate from (21) and (22). From (22) we have

$$\left[\delta_Q^{|\beta|-|\alpha|} \partial^\beta \check{P}^{y'}_\alpha (y')\right] = \sum_{\alpha' \in \mathcal{A}} \left[\delta_Q^{|\beta|-|\alpha'|} \partial^\beta \tilde{P}^{y'}_{\alpha'} (y')\right] \cdot M_{\alpha'\alpha} \quad \text{for} \quad \alpha \in \mathcal{A},\ \beta \in \mathcal{M}. \tag{27}$$

Since M and $\tilde{M}$ are inverse matrices, (27) implies

$$\left[\delta_Q^{|\beta|-|\alpha|} \partial^\beta \tilde{P}^{y'}_\alpha (y')\right] = \sum_{\alpha' \in \mathcal{A}} \left[\delta_Q^{|\beta|-|\alpha'|} \partial^\beta \check{P}^{y'}_{\alpha'} (y')\right] \cdot \tilde{M}_{\alpha'\alpha}. \tag{28}$$

From (16a), (19), (27), (28), we see that

$$c \max_{\substack{\beta \in \mathcal{M} \\ \alpha \in \mathcal{A}}} \delta_Q^{|\beta|-|\alpha|} |\partial^\beta \tilde{P}^{y'}_\alpha (y')| < \max_{\substack{\beta \in \mathcal{M} \\ \alpha \in \mathcal{A}}} \delta_Q^{|\beta|-|\alpha|} |\partial^\beta \check{P}^{y'}_\alpha (y')| < C \max_{\substack{\beta \in \mathcal{M} \\ \alpha \in \mathcal{A}}} \delta_Q^{|\beta|-|\alpha|} |\partial^\beta \tilde{P}^{y'}_\alpha (y')|.$$

Together with (3), this proves conclusion (24).

Next, suppose $\beta \in \mathcal{M}$, $\alpha \in \mathcal{A}$ are given, with $\beta > \alpha$. Then, for each $\alpha' \in \mathcal{A}$, we have either $\beta > \alpha'$ or $\alpha' > \alpha$. If $\beta > \alpha'$, then (4) and (19) yield

$$\left|\left[\delta_Q^{|\beta|-|\alpha'|}\partial^\beta \tilde{P}_{\alpha'}^{y'}(y')\right] \cdot M_{\alpha'\alpha}\right| \leq Ca_1 \leq C(a_1)^{-(m-1)} \quad \text{by (SU4).}$$

If instead $\alpha' > \alpha$, then (3) and (17) yield

$$\left|\left[\delta_Q^{|\beta|-|\alpha'|}\partial^\beta \tilde{P}_{\alpha'}^{y'}(y')\right] \cdot M_{\alpha'\alpha}\right| \leq C \cdot (a_1)^{-m} \cdot Ca_1 = C'(a_1)^{-(m-1)}.$$

Consequently, (27) implies conclusion (25).

Finally, let $S \subset E$, with $\#(S) \leq k_{\text{old}}^{\#}$, and let $\varphi_\alpha^S(\alpha \in \mathcal{A})$ be as in (2). Define

$$\check{\varphi}_\alpha^S = \delta_Q^{|\alpha|} \sum_{\alpha' \in \mathcal{A}} \delta_Q^{-|\alpha'|} \varphi_{\alpha'}^S M_{\alpha'\alpha}. \tag{29}$$

From (2)(a) and (19), we see that, for all $\alpha \in \mathcal{A}$, we have

$$\|\partial^m \check{\varphi}_\alpha^S\|_{C^0(\mathbb{R}^n)} \leq \delta_Q^{|\alpha|} \sum_{\alpha' \in \mathcal{A}} \delta_Q^{-|\alpha'|} \cdot Ca_1 \leq Ca_1 \cdot \delta_Q^{|\alpha|-m} \qquad \text{(see (9))},$$

which proves conclusion (26)(a).

Also, from (2)(b), (19), (29), we have for all $\alpha \in \mathcal{A}$, $x \in S$, that

$$|\check{\varphi}_\alpha^S(x)| \leq \delta_Q^{|\alpha|} \sum_{\alpha' \in \mathcal{A}} \delta_Q^{-|\alpha'|} \cdot C\sigma(x) \leq C'\delta_Q^{|\alpha|-m}\sigma(x),$$

which proves conclusion (26)(b).

For each $\alpha \in \mathcal{A}$, we recall (2)(d), (22), and (29). These imply conclusion (26)(c).

The proof of the Lemma 12.2 is complete. ■

Next, we pick $\bar{\beta} \in \mathcal{M}$ and $\bar{\alpha} \in \mathcal{A}$ to maximize $\delta_Q^{|\bar{\beta}|-|\bar{\alpha}|}|\partial^{\bar{\beta}}\ \check{P}_{\bar{\alpha}}^{y'}(y')|$. By definition of $\bar{\beta}, \bar{\alpha}$, and by (24), we have

$$c \cdot (a_1)^{-m} < \delta_Q^{|\bar{\beta}|-|\bar{\alpha}|}|\partial^{\bar{\beta}}\check{P}_{\bar{\alpha}}^{y'}(y')| < C \cdot (a_1)^{-m}; \tag{30}$$

$$\delta_Q^{|\beta|-|\alpha|}|\partial^{\beta}\check{P}_{\alpha}^{y'}(y')| \leq \delta_Q^{|\bar{\beta}|-|\bar{\alpha}|}|\partial^{\bar{\beta}}\check{P}_{\bar{\alpha}}^{y'}(y')| \qquad \text{for all} \qquad \alpha \in \mathcal{A},\ \beta \in \mathcal{M}; \tag{31}$$

and of course

$$\bar{\alpha} \in \mathcal{A},\ \bar{\beta} \in \mathcal{M}. \tag{32}$$

If $\bar\beta \in \mathcal{A}$, then $\delta_Q^{|\bar\beta|-|\bar\alpha|}\ |\partial^{\bar\beta}\check{P}_{\bar\alpha}^{y'}(y')| = \delta_{\bar\beta\bar\alpha} \le 1$ (see (23)), which contradicts (30), thanks to (SU4). Hence,

$$\bar\beta \notin \mathcal{A}. \tag{33}$$

In particular, $\bar\beta \neq \bar\alpha$.

Moreover, if $\bar\beta > \bar\alpha$, then (25) contradicts (30), again thanks to (SU4). Hence,

$$\bar\beta < \bar\alpha. \tag{34}$$

Now define

$$\bar{\mathcal{A}}^{y'} = (\mathcal{A} \smallsetminus \{\bar\alpha\}) \cup \{\bar\beta\}, \tag{35}$$

$$\overset{+}{P}{}_{\alpha}^{y'} = \check{P}_{\alpha}^{y'} \qquad \text{for all } \alpha \in \mathcal{A} \smallsetminus \{\bar\alpha\}, \tag{36}$$

$$\overset{+}{P}{}_{\bar\beta}^{y'} = \check{P}_{\bar\alpha}^{y'} \big/ \big(\partial^{\bar\beta}\check{P}_{\bar\alpha}^{y'}(y')\big). \qquad \text{(The denominator is non–zero, by (30)).} \tag{37}$$

Thus, $\overset{+}{P}{}_{\alpha}^{y'}$ is defined for all $\alpha \in \bar{\mathcal{A}}^{y'}$.

In view of (33), (34), (35), the least element of the symmetric difference $\mathcal{A}\triangle\bar{\mathcal{A}}^{y'}$ is $\bar\beta$. Hence, by definition of our ordering $<$ on sets of multi-indices, we have

$$\bar{\mathcal{A}}^{y'} < \mathcal{A}. \tag{38}$$

The basic properties of the $\overset{+}{P}{}_{\alpha}^{y'}$ are as follows.

Lemma 12.3. *We have*

$$\partial^{\beta}\overset{+}{P}{}_{\bar\beta}^{y'}(y') = \delta_{\beta\bar\beta} \qquad \textit{for all}\quad \beta \in \bar{\mathcal{A}}^{y'} \tag{39}$$

$$\partial^{\beta}\overset{+}{P}{}_{\alpha}^{y'}(y') = \delta_{\beta\alpha} \qquad \textit{for all}\quad \beta,\alpha \in \bar{\mathcal{A}}^{y'} \smallsetminus \{\bar\beta\}. \tag{40}$$

$$\delta_Q^{|\beta|-|\alpha|}\big|\partial^{\beta}\overset{+}{P}{}_{\alpha}^{y'}(y')\big| \le C\cdot(a_1)^{-m} \qquad \textit{for all}\quad \alpha \in \bar{\mathcal{A}}^{y'},\ \beta \in \mathcal{M}. \tag{41}$$

$$\delta_Q^{|\beta|-|\bar\beta|}\big|\partial^{\beta}\overset{+}{P}{}_{\bar\beta}^{y'}(y')\big| \le 1 \qquad \textit{for all}\ \ \beta \in \mathcal{M}. \tag{41a}$$

(42) *Given $\alpha \in \bar{\mathcal{A}}^{y'}$ and $S \subset E$ with $\#(S) \le k^{\#}_{\text{old}}$, there exists $\overset{+}{\varphi}{}_{\alpha}^{S} \in C^m(\mathbb{R}^n)$, with*

(a) $\delta_Q^{m-|\alpha|}\big\|\partial^m\overset{+}{\varphi}{}_{\alpha}^{S}\big\|_{C^0(\mathbb{R}^n)} \le Ca_1$,

(b) $\delta_Q^{m-|\alpha|}\big|\overset{+}{\varphi}{}_{\alpha}^{S}(x)\big| \le C\sigma(x)$ *for all* $x \in S$,

(c) $J_{y'}(\overset{+}{\varphi}{}_{\alpha}^{S}) = \overset{+}{P}{}_{\alpha}^{y'}$.

Proof. To check (39), we note that for $\beta \in \bar{\mathcal{A}}^{y'} \setminus \{\bar{\beta}\}$, we have $\partial^\beta \overset{+}{P}{}^{y'}_{\bar{\beta}}(y') = \partial^\beta \check{P}^{y'}_{\bar{\alpha}}(y') / \big(\partial^{\bar{\beta}} \check{P}^{y'}_{\bar{\alpha}}(y')\big) = 0$, thanks to (37) and (23). (Note that (23) applies; see (32) and (35).) On the other hand, for $\beta = \bar{\beta}$, we have $\partial^\beta \overset{+}{P}{}^{y'}_{\bar{\beta}}(y') = \partial^{\bar{\beta}} \check{P}^{y'}_{\bar{\alpha}}(y') / \big(\partial^{\bar{\beta}} \check{P}^{y'}_{\bar{\alpha}}(y')\big) = 1$, by (37). This proves conclusion (39).

Conclusion (40) is immediate from (23) and (36), since $\bar{\mathcal{A}}^{y'} \setminus \{\bar{\beta}\} = \mathcal{A} \setminus \{\bar{\alpha}\}$.

Similarly, for $\alpha \in \bar{\mathcal{A}}^{y'} \setminus \{\bar{\beta}\}$, conclusion (41) is immediate from (24) and (36). On the other hand, for $\alpha = \bar{\beta}$, (31) and (37) give

$$\delta_Q^{|\beta|-|\bar{\beta}|} |\partial^\beta \overset{+}{P}{}^{y'}_{\bar{\beta}}(y')| = \big[\delta_Q^{|\beta|-|\bar{\alpha}|} |\partial^\beta \check{P}^{y'}_{\bar{\alpha}}(y')|\big] / \big[\delta_Q^{|\bar{\beta}|-|\bar{\alpha}|} |\partial^{\bar{\beta}} \check{P}^{y'}_{\bar{\alpha}}(y')|\big] \le 1$$

for all $\beta \in \mathcal{M}$. This proves conclusion (41a), and completes the proof of conclusion (41).

It remains to check conclusion (42). For $\alpha \in \bar{\mathcal{A}}^{y'} \setminus \{\bar{\beta}\} = \mathcal{A} \setminus \{\bar{\alpha}\}$, conclusion (42) is immediate from (26). Suppose $\alpha = \bar{\beta}$, and let $S \subset E$, with $\#(S) \le k_{\mathrm{old}}$. Let $\check{\varphi}^S_{\bar{\alpha}}$ be as in (26), and define

$$\overset{+}{\varphi}{}^S_{\bar{\beta}} = \check{\varphi}^S_{\bar{\alpha}} / (\partial^{\bar{\beta}} \check{P}^{y'}_{\bar{\alpha}}(y')). \tag{43}$$

From (26a) and (30), we have

$$\begin{aligned} \delta_Q^{m-|\bar{\beta}|} \|\partial^m \overset{+}{\varphi}{}^S_{\bar{\beta}}\|_{C^0(\mathbb{R}^n)} &= \big[\delta_Q^{m-|\bar{\alpha}|} \|\partial^m \check{\varphi}^S_{\bar{\alpha}}\|_{C^0(\mathbb{R}^n)}\big] / \big[\delta_Q^{|\bar{\beta}|-|\bar{\alpha}|} |\partial^{\bar{\beta}} \check{P}^{y'}_{\bar{\alpha}}(y')|\big] \\ &\le [Ca_1]/[ca_1^{-m}] < Ca_1. \end{aligned}$$

This proves conclusion (42(a)) for $\alpha = \bar{\beta}$.

Also, (26b), (30), (43) show that, for $x \in S$, we have

$$\begin{aligned} \delta_Q^{m-|\bar{\beta}|} |\overset{+}{\varphi}{}^S_{\bar{\beta}}(x)| &= \big[\delta_Q^{m-|\bar{\alpha}|} |\check{\varphi}^S_{\bar{\alpha}}(x)|\big] / \big[\delta_Q^{|\bar{\beta}|-|\bar{\alpha}|} |\partial^{\bar{\beta}} \check{P}^{y'}_{\bar{\alpha}}(y')|\big] \\ &\le [C\sigma(x)]/[ca_1^{-m}] < C\sigma(x). \end{aligned}$$

This proves conclusion (42(b)) for $\alpha = \bar{\beta}$.

Finally, comparing (37) with (43), and applying (26(c)), we obtain conclusion (42(c)) for $\alpha = \bar{\beta}$.

Thus, conclusion (42) holds also for $\alpha = \bar{\beta}$. The proof of Lemma 12.3 is complete. ∎

Next, we define polynomials $\bar{P}_\alpha^{y'}(\alpha \in \bar{\mathcal{A}}^{y'})$, by setting

$$\bar{P}_{\bar{\beta}}^{y'} = \overset{+}{P}{}_{\bar{\beta}}^{y'} \tag{44}$$

and

$$\bar{P}_\alpha^{y'} = \overset{+}{P}{}_\alpha^{y'} - [\partial^{\bar{\beta}} \overset{+}{P}{}_\alpha^{y'}(y')] \cdot \overset{+}{P}{}_{\bar{\beta}}^{y'} \qquad \text{for all } \alpha \in \mathcal{A} \setminus \{\bar{\alpha}\}. \tag{45}$$

The basic properties of these polynomials are as follows.

Lemma 12.4. *We have*

$$\partial^\beta \bar{P}_\alpha^{y'}(y') = \delta_{\beta\alpha} \qquad \textit{for all } \alpha, \beta \in \bar{\mathcal{A}}^{y'}. \tag{46}$$

$$\delta_Q^{|\beta|-|\alpha|} |\partial^\beta \bar{P}_\alpha^{y'}(y')| \leq C(a_1)^{-m} \quad \textit{for all } \beta \in \mathcal{M},\ \alpha \in \bar{\mathcal{A}}^{y'}. \tag{47}$$

(48) *Given $\alpha \in \bar{\mathcal{A}}^{y'}$ and $S \subset E$ with $\#(S) \leq k^{\#}_{\text{old}}$, there exists $\bar{\varphi}_\alpha^S \in C^m(\mathbb{R}^n)$, with*

(a) $\delta_Q^{m-|\alpha|} \| \partial^m \bar{\varphi}_\alpha^S \|_{C^0(\mathbb{R}^n)} \leq C \cdot (a_1)^{-(m-1)}$,

(b) $\delta_Q^{m-|\alpha|} |\bar{\varphi}_\alpha^S(x)| \leq C \cdot (a_1)^{-m} \sigma(x)$ *for all $x \in S$.*

(c) $J_{y'}(\bar{\varphi}_\alpha^S) = \bar{P}_\alpha^{y'}$.

Proof. For $\alpha = \bar{\beta}$, conclusion (46) is immediate from (39) and (44). For $\alpha \in \bar{\mathcal{A}}^{y'} \setminus \{\bar{\beta}\}$ $= \mathcal{A} \setminus \{\bar{\alpha}\}$, (45) gives

$$\partial^\beta \bar{P}_\alpha^{y'}(y') = \partial^\beta \overset{+}{P}{}_\alpha^{y'}(y') - \left[\partial^{\bar{\beta}} \overset{+}{P}{}_\alpha^{y'}(y')\right] \cdot \partial^\beta \overset{+}{P}{}_{\bar{\beta}}^{y'}(y') \qquad \text{for all } \beta \in \mathcal{M}. \tag{49}$$

If $\beta \in \mathcal{A} \setminus \{\bar{\alpha}\}$, then $\partial^\beta \overset{+}{P}{}_\alpha^{y'}(y') = \delta_{\beta\alpha}$ (see (40)), and $\partial^\beta \overset{+}{P}{}_{\bar{\beta}}^{y'}(y') = 0$ (see (39)). Hence, (49) implies conclusion (46) for the case, $\alpha \in \bar{\mathcal{A}}^{y'} \setminus \{\bar{\beta}\}$, $\beta \in \bar{\mathcal{A}}^{y'} \setminus \{\bar{\beta}\}$. If $\alpha \in \bar{\mathcal{A}}^{y'} \setminus \{\bar{\beta}\}$ and $\beta = \bar{\beta}$, then, since $\partial^{\bar{\beta}} \overset{+}{P}{}_{\bar{\beta}}^{y'}(y') = 1$ by (39), we see that (49) implies $\partial^{\bar{\beta}} \bar{P}_\alpha^{y'}(y') = \partial^{\bar{\beta}} \overset{+}{P}{}_\alpha^{y'}(y') - [\partial^{\bar{\beta}} \overset{+}{P}{}_\alpha^{y'}(y')] \cdot 1 = 0 = \delta_{\bar{\beta}\alpha}$. Hence, conclusion (46) holds also for $\alpha \in \bar{\mathcal{A}}^{y'} \setminus \{\bar{\beta}\}$, $\beta = \bar{\beta}$.

Thus, we have verified conclusion (46) in all cases.

Next, conclusion (47) holds for $\alpha = \bar{\beta}$, thanks to (41) and (44). Suppose $\alpha \in \bar{\mathcal{A}}^{y'} \setminus \{\bar{\beta}\}$ and $\beta \in \mathcal{M}$. Then (45), together with (41) and (41a), yields

$$\delta_Q^{|\beta|-|\alpha|} \left|\partial^\beta \bar{P}_\alpha^{y'}(y')\right| \leq \delta_Q^{|\beta|-|\alpha|} \left|\partial^\beta \overset{+}{P}{}_\alpha^{y'}(y')\right| + \left[\delta_Q^{|\bar{\beta}|-|\alpha|} \left|\partial^{\bar{\beta}} \overset{+}{P}{}_\alpha^{y'}(y')\right|\right] \cdot \left[\delta_Q^{|\beta|-|\bar{\beta}|} \left|\partial^\beta \overset{+}{P}{}_{\bar{\beta}}^{y'}(y')\right|\right]$$
$$\leq C \cdot (a_1)^{-m} + [C \cdot (a_1)^{-m}] \cdot [1] \leq C' \cdot (a_1)^{-m}.$$

Hence, conclusion (47) holds in all cases.

It remains to check conclusion (48). For $\alpha = \bar{\beta}$, conclusion (48) is immediate from (42) and (44), thanks to (SU4). Suppose $\alpha \in \bar{\mathcal{A}}^{y'} \setminus \{\bar{\beta}\}$, and let $S \subset E$ with $\#(S) \leq k^{\#}_{\text{old}}$. We apply (42), (for the given α, and for $\bar{\beta}$), and we define

$$\bar{\varphi}^S_\alpha = \overset{+}{\varphi}{}^S_\alpha - \left[\partial^{\bar{\beta}} \overset{+}{P}{}^{y'}_\alpha(y')\right] \cdot \overset{+}{\varphi}{}^S_{\bar{\beta}}. \tag{50}$$

From (42(a)) and (41), we find that

$$\delta_Q^{m-|\alpha|} \left\|\partial^m \bar{\varphi}^S_\alpha\right\|_{C^0(\mathbb{R}^n)} \leq \delta_Q^{m-|\alpha|} \left\|\partial^m \overset{+}{\varphi}{}^S_\alpha\right\|_{C^0(\mathbb{R}^n)} + \left[\delta_Q^{m-|\bar{\beta}|} \left\|\partial^m \overset{+}{\varphi}{}^S_{\bar{\beta}}\right\|_{C^0(\mathbb{R}^n)}\right] \cdot \left[\delta_Q^{|\bar{\beta}|-|\alpha|} \left|\partial^{\bar{\beta}} \overset{+}{P}{}^{y'}_\alpha(y')\right|\right]$$

$\leq (Ca_1) + [Ca_1] \cdot [C \cdot (a_1)^{-m}] \leq C' \cdot (a_1)^{-(m-1)}$, thanks to (SU4).

This proves conclusion (48(a)) for the given α.

Also, for all $x \in S$, we obtain from (41), (42(b)), (50) that

$$\delta_Q^{m-|\alpha|} |\bar{\varphi}^S_\alpha(x)| \leq \delta_Q^{m-|\alpha|} \left|\overset{+}{\varphi}{}^S_\alpha(x)\right| + \left[\delta_Q^{|\bar{\beta}|-|\alpha|} \left|\partial^{\bar{\beta}} \overset{+}{P}{}^{y'}_\alpha(y')\right|\right] \cdot \left[\delta_Q^{m-|\bar{\beta}|} \left|\overset{+}{\varphi}{}^S_{\bar{\beta}}(x)\right|\right]$$

$\leq C\sigma(x) + \left[C \cdot (a_1)^{-m}\right] \cdot \left[C\sigma(x)\right] \leq C' \cdot (a_1)^{-m} \cdot \sigma(x)$, thanks to (SU4).

This proves conclusion (48(b)) for the given α.

Finally, comparing (45) with (50), and applying (42(c)), we obtain conclusion (48(c)) for the given α.

Thus, conclusion (48) holds also for $\alpha \in \bar{\mathcal{A}}^{y'} \setminus \{\bar{\beta}\}$. The proof of Lemma 12.4 is complete. ∎

We are ready to give the main result of this section.

Lemma 12.5. *The cube Q^+ is OK.*

Proof. For every $y' \in (Q^+)^{\star\star}$, (see (1)), we have constructed $\bar{\mathcal{A}}^{y'} < \mathcal{A}$ (see (38)), and $\bar{P}^{y'}_\alpha (\alpha \in \bar{\mathcal{A}}^{y'})$ satisfying (46), (47), (48).

We will check that $\bar{\mathcal{A}}^{y'}$ and the $\bar{P}^{y'}_\alpha (\alpha \in \bar{\mathcal{A}}^{y'})$ satisfy (OK1, 2, 3) for the cube Q^+.

In fact, (OK1) for Q^+ is just (46).

Condition (OK2) for Q^+ says that

$$(2\delta_Q)^{|\beta|-|\alpha|}\left|\partial^\beta \bar{P}_\alpha^{y'}(y')\right| \leq (a_1)^{-(m+1)} \qquad \text{for } \alpha \in \bar{\mathcal{A}}^{y'} \text{ and } \beta \in \mathcal{M} \text{ with } \beta \geq \alpha.$$

This estimate, without the restriction to $\beta \geq \alpha$, is immediate from (47) and (SU4).

Condition (OK3) for Q^+ says that, given $\alpha \in \bar{\mathcal{A}}^{y'}$ and $S \subset E$ with $\#(S) \leq k^\#_{\text{old}}$, there exists $\bar{\varphi}_\alpha^S \in C^m(\mathbb{R}^n)$, with

(a) $(2\delta_Q)^{m-|\alpha|}\|\partial^m\ \bar{\varphi}_\alpha^S\|_{C^0(\mathbb{R}^n)} \leq (a_1)^{-(m+1)}$,
(b) $(2\delta_Q)^{m-|\alpha|}|\bar{\varphi}_\alpha^S\ (x)| \leq (a_1)^{-(m+1)} \cdot \sigma(x)$ for all $x \in S$,
(c) $J_{y'}(\bar{\varphi}_\alpha^S) = \bar{P}_\alpha^{y'}$.

This follows immediately from (48), thanks to (SU4).

We have shown that (OK1, 2, 3) hold for the cube Q^+ and arbitrary $y' \in (Q^+)^{\star\star}$. Thus, Q^+ is OK. The proof of Lemma 12.5 is complete. ■

13. Controlling Auxiliary Polynomials II

In this section, we are again in the setting of section 9, and we assume (SU0, . . . , 6). The result of this section is as follows.

Lemma 13.1. *Fix an integer $k_1^\#$, satisfying*

(1) $$k^\# \geq (D+1) \cdot k_1^\#, \quad k_1^\# \geq (D+1) \cdot k^\#_{\text{old}}.$$

Let Q be a CZ cube, and let

(2) $$y \in Q^{\star\star\star}$$

be given. Let $P_\alpha^y \in \mathcal{P}$ be a family of polynomials, indexed by $\alpha \in \mathcal{A}$.

Suppose that

(3) *The $P_\alpha^y (\alpha \in \mathcal{A})$ satisfy conditions (WL1)^y, (WL2)^y, (WL3)^y. (See Lemma 10.3.)*

Then we have the estimate

(4) $$\delta_Q^{|\beta|-|\alpha|}|\partial^\beta P_\alpha^y(y)| \leq (a_1)^{-m} \quad \textit{for all } \alpha \in \mathcal{A},\ \beta \in \mathcal{M}.$$

Proof. Suppose (4) to be false. There are finitely many dyadic cubes $\hat{Q}$ containing Q. (Recall that, by our definition, every dyadic cube is contained in Q°.) For each such $\hat{Q}$, define

$$\Phi(\hat{Q}) = \max_{\substack{\beta\in\mathcal{M}\\ \alpha\in\mathcal{A}}} \delta_{\hat{Q}}^{|\beta|-|\alpha|}|\partial^\beta P_\alpha^y(y)|. \tag{5}$$

Then $\Phi(Q) > (a_1)^{-m}$, since (4) is assumed false. Let $\bar{Q}$ be the maximal dyadic cube containing Q, with $\Phi(\bar{Q}) > (a_1)^{-m}$.

Thus,

$$\Phi(\bar{Q}) > (a_1)^{-m}, \tag{6}$$

$$Q \subset \bar{Q}, \quad \text{and} \tag{7}$$

$$\text{Either} \quad \bar{Q} = Q^\circ \quad \text{or else} \quad \Phi(\bar{Q}^+) \leq (a_1)^{-m}. \tag{8}$$

We can check easily that $\bar{Q} \neq Q^\circ$. In fact, (11.3), (WL2)^y and (SU4) show that

$$\delta_{Q^\circ}^{|\beta|-|\alpha|}|\partial^\beta P_\alpha^y(y)| \leq C\delta_{Q^\circ}^{|\beta|-|\alpha|} \leq C\delta_{Q^\circ}^{-(m-1)} \leq C'(a_1)^{-(m-1)} < a_1^{-m}$$

for all $\alpha \in \mathcal{A}$, $\beta \in \mathcal{M}$. (Recall, $\mathcal{A} \subset \mathcal{M}$, and $|\gamma| \leq m-1$ for all $\gamma \in \mathcal{M}$.)

Thus, $\Phi(Q^\circ) < (a_1)^{-m}$, and hence $Q^\circ \neq \bar{Q}$, by (6). From (8) we now obtain

$$\Phi(\bar{Q}^+) \leq (a_1)^{-m}. \tag{9}$$

A glance at the definition (5) shows that $\Phi(\bar{Q}^+)$ and $\Phi(\bar{Q})$ can differ at most by a factor of $2^{(m-1)}$. Hence, (9) implies $\Phi(\bar{Q}) \leq 2^{m-1} \cdot (a_1)^{-m}$. Together with (5) and (6), this shows that

$$(a_1)^{-m} \leq \max_{\substack{\beta\in\mathcal{M}\\ \alpha\in\mathcal{A}}} \delta_{\bar{Q}}^{|\beta|-|\alpha|}|\partial^\beta P_\alpha^y(y)| \leq 2^{m-1} \cdot (a_1)^{-m}. \tag{10}$$

Note also that

$$y \in \bar{Q}^{\star\star\star}, \tag{11}$$

thanks to (2) and (7).

We prepare to apply the results of section 12 to the cube $\bar{Q}$. Let us check that assumptions (CAP1, ... , 5) of that section are satisfied. In fact, (CAP1) is merely our present hypothesis (1); (CAP2) is (11); (CAP3) holds since $\bar{Q}$ is a dyadic cube not equal to Q°; (CAP4) is our present hypothesis (3); and (CAP5) is immediate from (10). Hence, the results of section 12 apply to the cube $\bar{Q}$. In particular, Lemma 12.5 tells us that the cube $\bar{Q}^+$ is OK. On the other hand, (7) shows that $\bar{Q}^+$ is a dyadic cube properly containing the CZ cube Q. By the definition of a CZ cube, it follows that $\bar{Q}^+$ cannot be OK. This contradiction proves that (4) must hold. The proof of Lemma 13.1 is complete. ■

14. Controlling the Main Polynomials

In this section, we again place ourselves in the setting of section 9, and assume (SU0, ..., 6). Our goal is to control the polynomials in $\mathcal{K}_f^{\#}(y; k_1^{\#}, M)$ in terms of the CZ cubes Q, for suitable $k_1^{\#}$ and M.

Lemma 14.1. *Let Q, Q' be CZ cubes that abut or coincide. Suppose we are given*

(1) $$y \in Q^{\star\star\star},\ y' \in (Q')^{\star\star\star}$$

and

(2) $$P \in \mathcal{K}_f^{\#}(y; k_1^{\#}, C),$$

with

(3) $$k^{\#} \geq (D+1) \cdot k_1^{\#}, \quad k_1^{\#} \geq (D+1) \cdot k_2^{\#},\ \textit{and } k_2^{\#} \geq k_{\text{old}}^{\#}.$$

Then there exists

(4) $$P' \in \mathcal{K}_f^{\#}(y'; k_2^{\#}, C'),$$

with

(5) $$|\partial^\beta (P' - P)(y')| \leq C'' \cdot (a_1)^{-m} \cdot \delta_Q^{m-|\beta|} \qquad \textit{for all } \beta \in \mathcal{M}.$$

Proof. By Lemma 10.2, there exists

(6) $$\tilde{P} \in \mathcal{K}_f(y'; k_2^{\#}, C'),$$

with

(7) $$|\partial^\beta (\tilde{P} - P)(y')| \leq C''|y - y'|^{m-|\beta|} \leq C_1'' \delta_Q^{m-|\beta|} \qquad \text{for all } \beta \in \mathcal{M}.$$

(Here we use Lemma 11.2 on the good geometry of the CZ cubes.) In view of (6) and the definition of $\mathcal{K}_f$, we know the following.

(8) Given $S \subset E$ with $\#(S) \leq k_2^{\#}$, there exists $\tilde{F}^S \in C^m(\mathbb{R}^n)$, with $\|\tilde{F}^S\|_{C^m(\mathbb{R}^n)} \leq C'$, $|\tilde{F}^S(x) - f(x)| \leq C'\sigma(x)$ on S, and $J_{y'}(\tilde{F}^S) = \tilde{P}$.

In particular, taking $S =$ empty set in (8), we learn that

(9) $$|\partial^\beta \tilde{P}(y')| \leq C' \qquad \text{for all } \beta \in \mathcal{M}.$$

Also, (2) and the definition of $\mathcal{K}_f^{\#}$ give $\partial^\beta P(y) = 0$ for all $\beta \in \mathcal{A}$. Applying (SU0), we conclude that $\partial^{\gamma+\beta} P(y) = 0$ for all $\beta \in \mathcal{A}$, $|\gamma| \leq m - 1 - |\beta|$.

On the other hand, we have

$$\partial^\beta P(y') = \sum_{|\gamma| \leq m-1-|\beta|} \frac{1}{\gamma!} \left(\partial^{\gamma+\beta} P(y)\right) \cdot (y' - y)^\gamma,$$

since P is a polynomial of degree at most $(m-1)$. Hence, $\partial^\beta P(y') = 0$ for all $\beta \in \mathcal{A}$. Consequently, (7) implies

$$|\partial^\beta \tilde{P}(y')| \leq C_1'' \delta_Q^{m-|\beta|} \qquad \text{for all } \beta \in \mathcal{A}. \tag{10}$$

Next, note that $y' \in (Q')^{\star\star\star} \subset (Q^\circ)^{\star\star\star} \subset B(y^0, a_1)$, thanks to (1), the fact that Q' is a CZ cube, and (11.2). Hence, Lemma 10.3 applies, with y' in place of y. Thus, we obtain polynomials $P_\alpha^{y'} (\alpha \in \mathcal{A})$, satisfying $(\mathrm{WL1})^{y'}$, $(\mathrm{WL2})^{y'}$, $(\mathrm{WL3})^{y'}$. The hypotheses of Lemma 13.1 hold here, with Q', y' and $P_\alpha^{y'}$ in place of Q, y and P_α^y. In fact, hypothesis (1) in section 13 is immediate from our present assumption (3). Also, hypothesis (2) in section 13 is contained in our present assumption (1). Hypothesis (3) in Section 13 merely asserts that the $P_\alpha^{y'} (\alpha \in \mathcal{A})$ satisfy $(\mathrm{WL1})^{y'}$, $(\mathrm{WL2})^{y'}$, $(\mathrm{WL3})^{y'}$, which we have just noted above. Since also Q' is a CZ cube, we have shown that the hypotheses of Lemma 13.1 hold for $Q', y', P_\alpha^{y'} (\alpha \in \mathcal{A})$. Applying that Lemma, we conclude that

$$\delta_{Q'}^{|\beta|-|\alpha|} \cdot |\partial^\beta P_\alpha^{y'}(y')| \leq (a_1)^{-m} \qquad \text{for all } \alpha \in \mathcal{A},\ \beta \in \mathcal{M}. \tag{11}$$

Now define

$$P' = \tilde{P} - \sum_{\alpha \in \mathcal{A}} [\partial^\alpha \tilde{P}(y')] \cdot P_\alpha^{y'} \in \mathcal{P}. \tag{12}$$

For all $\beta \in \mathcal{A}$, we have

$$\partial^\beta P'(y') = \partial^\beta \tilde{P}(y') - \sum_{\alpha \in \mathcal{A}} [\partial^\alpha \tilde{P}(y')] \cdot \partial^\beta P_\alpha^{y'}(y') = 0 \tag{13}$$

since $\partial^\beta P_\alpha^{y'}(y') = \delta_{\beta\alpha}$ for all $\beta, \alpha \in \mathcal{A}$ (see $(\mathrm{WL1})^{y'}$).

Note also that, for any $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{M}$, we have

$$\left|\partial^\beta \{[\partial^\alpha \tilde{P}(y')] \cdot P_\alpha^{y'}\}(y')\right| = \left|\partial^\alpha \tilde{P}(y')\right| \cdot \left|\partial^\beta P_\alpha^{y'}(y')\right| \leq C_1'' \delta_Q^{m-|\alpha|} \cdot (a_1)^{-m} \delta_{Q'}^{|\alpha|-|\beta|}$$

by (10) and (11). Hence, (12) implies that

$$|\partial^\beta (P' - \tilde{P})(y')| \leq C_2'' \delta_Q^{m-|\beta|} \cdot (a_1)^{-m} \qquad \text{for all } \beta \in \mathcal{M}.$$

(Recall that δ_Q and $\delta_{Q'}$ are comparable by Lemma 11.2.) Together with (7) and (SU4), this yields

$$|\partial^\beta (P' - P)(y')| \leq C_3'' \cdot (a_1)^{-m} \cdot \delta_Q^{m-|\beta|} \qquad \text{for all } \beta \in \mathcal{M}, \text{ which is conclusion (5).}$$

Moreover, let $S \subset E$ be given, with $\#(S) \leq k_2^\#$. Let $\tilde{F}^S$ be as in (8), and, for each $\alpha \in \mathcal{A}$, let φ_α^S be as in $(\mathrm{WL3})^{y'}$. (Note that $(\mathrm{WL3})^{y'}$ applies, since $k_1^\# \geq k_2^\#$.) Introduce a cutoff function θ on $\mathbb{R}^n$, with

$$0 \leq \theta \leq 1 \text{ on } \mathbb{R}^n,\ \theta = 1 \text{ on } B(y', 1/2), \ \text{ supp } \theta \subset B(y', 1),\ \|\theta\|_{C^m(\mathbb{R}^n)} \leq C'''. \tag{14}$$

Then define

$$F^S = \tilde{F}^S - \sum_{\alpha \in \mathcal{A}} [\partial^\alpha \tilde{P}(y')] \cdot \varphi_\alpha^S \cdot \theta. \tag{15}$$

From $(\mathrm{WL2})^{y'}$, $(\mathrm{WL3})^{y'}$(a), $(\mathrm{WL3})^{y'}$(c), we conclude that $|\partial^\beta \varphi_\alpha^S| \leq C_4''$ on $B(y',1)$, $|\beta| \leq m$.

Hence, (9) and (14) imply $\|[\partial^\alpha \tilde{P}(y')] \cdot \varphi_\alpha^S \cdot \theta\|_{C^m(\mathbb{R}^n)} \leq C_5''$. Together with (8) and (15), this yields

$$\|F^S\|_{C^m(\mathbb{R}^n)} \leq C_6''. \tag{16}$$

Next, suppose $x \in S$. Then (8), (9), $(\mathrm{WL3})^{y'}$(b), (14) and (15) yield

$$|F^S(x) - f(x)| \leq C'\sigma(x) + \sum_{\alpha \in \mathcal{A}} C' \cdot C\sigma(x) \leq C_7'' \cdot \sigma(x). \tag{17}$$

Also, comparing (12) with (15), recalling (14), and applying (8) and $(\mathrm{WL3})^{y'}$(c), we find that

$$J_{y'}(F^S) = \tilde{P} - \sum_{\alpha \in \mathcal{A}} [\partial^\alpha \tilde{P}(y')] \cdot P_\alpha^{y'} = P'. \tag{18}$$

For every $S \subset E$ with $\#(S) \leq k_2^\#$, we have exhibited a function $F^S \in C^m(\mathbb{R}^n)$ that satisfies (16), (17), (18). Thus, by definition, P' belongs to $\mathcal{K}_f(y'; k_2^\#, C_8'')$. Recalling (13), we conclude that $P' \in \mathcal{K}_f^\#(y'; k_2^\#, C_8'')$, which is conclusion (4).

Thus, conclusions (4) and (5) hold for P'.

The proof of Lemma 14.1 is complete. ■

Lemma 14.2. *Fix $k_1^\#$, with*

$$k^\# \geq (D+1) \cdot k_1^\#, \quad k_1^\# \geq (D+1) \cdot k_{\mathrm{old}}^\#. \tag{19}$$

Suppose Q is a CZ cube, $y \in Q^{\star\star}$, and $P_1, P_2 \in \mathcal{K}_f^\#(y; k_1^\#, C)$. Then

$$|\partial^\beta (P_1 - P_2)(y)| \leq (a_1)^{-(m+1)} \cdot \delta_Q^{m-|\beta|} \qquad \textit{for all } \beta \in \mathcal{M}. \tag{20}$$

Proof. Suppose (20) fails. We will show that

(21) $\qquad Q$ is a proper subcube of Q°, and that

(22) $\qquad Q^+$ is OK.

This will lead to a contradiction, since Q^+ is a dyadic cube that properly contains the CZ cube Q; thus Q^+ cannot be OK, by the definition of a CZ cube. Consequently, the proof of Lemma 14.2 reduces to the proof of (21) and (22) under the assumption that (20) fails.

Since $P_1, P_2 \in \mathcal{K}_f^{\#}(y; k_1^{\#}, C)$, we know that

$$\text{Given } S \subset E \text{ with } \#(S) \leq k_1^{\#}, \text{ there exist } F_i^S \in C^m(\mathbb{R}^n)(i=1,2), \text{ with } \|F_i^S\|_{C^m(\mathbb{R}^n)} \leq C,\ |F_i^S(x) - f(x)| \leq C\sigma(x) \text{ on } S,\ J_y(F_i^S) = P_i. \tag{23}$$

In particular, taking $S =$ empty set in (23), we learn that

$$|\partial^\beta P_i(y)| \leq C \qquad \text{for } |\beta| \leq m-1 \text{ and } i = 1,2. \tag{24}$$

It is now easy to prove (21). Since Q is dyadic, it is enough to show that $Q \neq Q^\circ$. Since we are assuming that (20) fails for Q, it is enough to show that (20) holds for Q°. However, (24) and (11.3) show that

$$|\partial^\beta (P_1 - P_2)(y)| \leq C' \leq (a_1)^{-(m+1)} (\delta_{Q^\circ})^{m-|\beta|}, \text{ thanks to (SU4).}$$

Thus, (20) holds for Q°, completing the proof of (21).

We start the proof of (22). Let

$$y' \in (Q^+)^{\star\star} \tag{25}$$

be given. Then $y, y' \in Q^{\star\star\star}$, and $P_1, P_2 \in \mathcal{K}_f^{\#}(y; k_1^{\#}, C)$. Also, $k^{\#} \geq (D+1) \cdot k_1^{\#}$ and $k_1^{\#} \geq (D+1) \cdot k_{\text{old}}^{\#}$. Hence, Lemma 14.1 applies, with $k_2^{\#} = k_{\text{old}}^{\#}$. Consequently, there exist

$$\tilde{P}_1, \tilde{P}_2 \in \mathcal{K}_f^{\#}(y'; k_{\text{old}}^{\#}, C') \tag{26}$$

with

$$|\partial^\beta (\tilde{P}_i - P_i)(y')| \leq C'' \cdot (a_1)^{-m} \cdot \delta_Q^{m-|\beta|} \qquad \text{for all } \beta \in \mathcal{M},\ i = 1,2. \tag{27}$$

From (27), we see that

$$\max_{\beta \in \mathcal{M}} \delta_Q^{|\beta|-m} |\partial^\beta (P_1 - P_2)(y')| \leq 2C'' \cdot (a_1)^{-m} + \max_{\beta \in \mathcal{M}} \delta_Q^{|\beta|-m} |\partial^\beta (\tilde{P}_1 - \tilde{P}_2)(y')|. \tag{28}$$

Also, for $\beta \in \mathcal{M}$, we have

$$\begin{aligned} |\partial^\beta (P_1 - P_2)(y)| &= \left| \sum_{|\gamma| \leq m-1-|\beta|} \frac{1}{\gamma!} \left[\partial^{\gamma+\beta}(P_1 - P_2)(y')\right] \cdot (y - y')^\gamma \right| \\ &\leq C \max_{|\gamma| \leq m-1-|\beta|} |\partial^{\gamma+\beta}(P_1 - P_2)(y')| \delta_Q^{|\gamma|}; \end{aligned}$$

therefore,

$$\max_{\beta\in\mathcal{M}} \delta_Q^{|\beta|-m}|\partial^\beta(P_1-P_2)(y)| \le C \max_{\beta\in\mathcal{M}} \delta_Q^{|\beta|-m}|\partial^\beta(P_1-P_2)(y')|. \tag{29}$$

Moreover, since (20) fails, we have

$$a_1^{-(m+1)} \le \max_{\beta\in\mathcal{M}} \delta_Q^{|\beta|-m}|\partial^\beta(P_1-P_2)(y)|. \tag{30}$$

Combining (28), (29), (30), we find that

$$a_1^{-(m+1)} \le C''' a_1^{-m} + C \max_{\beta\in\mathcal{M}} \delta_Q^{|\beta|-m}|\partial^\beta(\tilde{P}_1-\tilde{P}_2)(y')|,$$

which implies

$$\max_{\beta\in\mathcal{M}} \delta_Q^{|\beta|-m}\big|\partial^\beta(\tilde{P}_1-\tilde{P}_2)(y')\big| \ge c\cdot a_1^{-(m+1)}, \tag{31}$$

thanks to (SU4).

From (26) and the definition of $\mathcal{K}_f^\#$, we know that

$$\partial^\beta\tilde{P}_1(y') = \partial^\beta\tilde{P}_2(y') = 0 \qquad \text{for all } \beta\in\mathcal{A}, \tag{32}$$

and that

(33) Given $S\subset E$ with $\#(S)\le k_{\text{old}}^\#$, there exist $\tilde{F}_1^S, \tilde{F}_2^S \in C^m(\mathbb{R}^n)$, with $\|\tilde{F}_i^S\|_{C^m(\mathbb{R}^n)} \le C'$, $|\tilde{F}_i^S(x)-f(x)| \le C'\sigma(x)$ on S, and $J_{y'}(\tilde{F}_i^S) = \tilde{P}_i$ for $i=1,2$.

Immediately from (33), we see that

(34) Given $S\in E$ with $\#(S)\le k_{\text{old}}^\#$, there exists $\tilde{F}^S\in C^m(\mathbb{R}^n)$, with $\|\tilde{F}^S\|_{C^m(\mathbb{R}^n)} \le C_1'$, $|\tilde{F}^S(x)| \le C_1'\sigma(x)$ on S, and $J_{y'}(\tilde{F}^S) = \tilde{P}_1-\tilde{P}_2$.

Now, pick $\bar{\beta}\in\mathcal{M}$ to maximize $\delta_Q^{|\bar{\beta}|-m}|\partial^{\bar{\beta}}(\tilde{P}_1-\tilde{P}_2)(y')|$, and define

$$\Omega = \partial^{\bar{\beta}}(\tilde{P}_1-\tilde{P}_2)(y'). \tag{35}$$

By (31) and the definitions of $\bar{\beta},\Omega$, we have

$$|\partial^\beta(\tilde{P}_1-\tilde{P}_2)(y')| \le |\Omega|\cdot\delta_Q^{|\bar{\beta}|-|\beta|} \qquad \text{for all } \beta\in\mathcal{M} \tag{36}$$

and

$$|\Omega| \ge c\cdot(a_1)^{-(m+1)}\cdot\delta_Q^{m-|\bar{\beta}|}. \tag{37}$$

In particular, $\Omega \neq 0$. We define

(38) $$\bar{P} = (\tilde{P}_1 - \tilde{P}_2)/\Omega \in \mathcal{P}.$$

From (32), we have

(39) $$\partial^\beta \bar{P}(y') = 0 \qquad \text{for all } \beta \in \mathcal{A}.$$

From (35) and (36), we have

(40) $$\partial^{\bar{\beta}} \bar{P}(y') = 1, \quad \text{and}$$

(41) $$|\partial^\beta \bar{P}(y')| \leq \delta_Q^{|\bar{\beta}|-|\beta|} \qquad \text{for all } \beta \in \mathcal{M}.$$

Also, from (34), (37), (38) and (SU4), we learn the following:

(42)

Given $S \subset E$ with $\#(S) \leq k^{\#}_{\text{old}}$, there exists $\bar{F}^S \in C^m(\mathbb{R}^n)$, with

(a) $\|\bar{F}^S\|_{C^m(\mathbb{R}^n)} \leq C'' \cdot (a_1)^{m+1} \cdot \delta_Q^{|\bar{\beta}|-m} \leq \delta_Q^{|\bar{\beta}|-m}$,

(b) $|\bar{F}^S(x)| \leq \dfrac{C_1'\sigma(x)}{|\Omega|} \leq C'' \cdot (a_1)^{m+1} \cdot \delta_Q^{|\bar{\beta}|-m} \cdot \sigma(x) \leq \delta_Q^{|\bar{\beta}|-m} \cdot \sigma(x)$ on S,

(c) $J_{y'}(\bar{F}^S) = \bar{P}$.

Note that

(43) $$\bar{\beta} \notin \mathcal{A},$$

as we see at once from (39), (40).

Next, recall that $y' \in (Q^+)^{\star\star} \subset Q^{\star\star\star} \subset (Q^\circ)^{\star\star\star} \subset B(y^0, a_1)$ (see (11.2)). Hence, Lemma 10.3 shows that there exist polynomials $P_\alpha^{y'} (\alpha \in \mathcal{A})$, with properties $(\text{WL1})^{y'}$, $(\text{WL2})^{y'}$, $(\text{WL3})^{y'}$. We now define

(44) $$\bar{\mathcal{A}}^{y'} = \mathcal{A} \cup \{\bar{\beta}\},$$

(45) $$\bar{P}_{\bar{\beta}} = \bar{P}$$

(46) $$\bar{P}_\alpha = P_\alpha^{y'} - [\partial^{\bar{\beta}} P_\alpha^{y'}(y')] \cdot \bar{P} \qquad \text{for } \alpha \in \mathcal{A}.$$

Thus, we have defined $\bar{P}_\beta$ for all $\beta \in \bar{\mathcal{A}}^{y'}$. Note that $\mathcal{A}$ is a proper subset of $\bar{\mathcal{A}}^{y'}$, by (43). Hence, Lemma 3.2 shows that

(47) $$\bar{\mathcal{A}}^{y'} < \mathcal{A}.$$

We will check that

$$\partial^\beta \bar{P}_\alpha(y') = \delta_{\beta\alpha} \qquad \text{for all } \beta, \alpha \in \bar{\mathcal{A}}^{y'}. \tag{48}$$

In fact, (48) holds for $\alpha = \bar{\beta}$, thanks to (39), (40), (45).

For $\alpha, \beta \in \mathcal{A}$, we have

$$\partial^\beta \bar{P}_\alpha(y') = \partial^\beta P_\alpha^{y'}(y') - [\partial^{\bar{\beta}} P_\alpha^{y'}(y')] \cdot \partial^\beta \bar{P}(y') = \delta_{\beta\alpha}$$

by $(\mathrm{WL1})^{y'}$ and (39), hence again (48) holds. For $\alpha \in \mathcal{A}$, $\beta = \bar{\beta}$, we have

$$\partial^{\bar{\beta}} \bar{P}_\alpha(y') = \partial^{\bar{\beta}} P_\alpha^{y'}(y') - [\partial^{\bar{\beta}} P_\alpha^{y'}(y')] \cdot \partial^{\bar{\beta}} \bar{P}(y') = 0$$

by (40), hence again (48) holds. Thus, (48) holds in all cases.

Next, we apply Lemma 13.1, with y' in place of y. Note that the hypotheses of Lemma 13.1 are satisfied, since: $y' \in (Q^+)^{\star\star} \subset Q^{\star\star\star}$, with Q a CZ cube; the $P_\alpha^{y'} (\alpha \in \mathcal{A})$ satisfy $(\mathrm{WL1})^{y'}$, $(\mathrm{WL2})^{y'}$, $(\mathrm{WL3})^{y'}$; $k^\# \geq (D+1)k_1^\#$ and $k_1^\# \geq (D+1) \cdot k_{\mathrm{old}}^\#$. From Lemma 13.1, we learn that

$$\delta_Q^{|\beta|-|\alpha|} |\partial^\beta P_\alpha^{y'}(y')| \leq (a_1)^{-m} \qquad \text{for all } \alpha \in \mathcal{A}, \beta \in \mathcal{M}. \tag{49}$$

Using (49), we can check that

$$|\partial^\beta \bar{P}_\alpha(y')| \leq C \cdot (a_1)^{-m} \cdot \delta_Q^{|\alpha|-|\beta|} \qquad \text{for all } \alpha \in \bar{\mathcal{A}}^{y'} \text{ and } \beta \in \mathcal{M}. \tag{50}$$

In fact, for $\alpha = \bar{\beta}$, (50) is immediate from (41), (45) and (SU4). For $\alpha \in \mathcal{A}, \beta \in \mathcal{M}$, we have

$$\begin{aligned} |\partial^\beta \bar{P}_\alpha(y')| \leq |\partial^\beta P_\alpha^{y'}(y')| + |\partial^{\bar{\beta}} P_\alpha^{y'}(y')| \cdot |\partial^\beta \bar{P}(y')| \leq (a_1)^{-m} \cdot [\delta_Q^{|\alpha|-|\beta|}] + \\ + [\delta_Q^{|\bar{\beta}|-|\beta|}] \cdot [(a_1)^{-m} \delta_Q^{|\alpha|-|\bar{\beta}|}] \leq C \cdot (a_1)^{-m} \delta_Q^{|\alpha|-|\beta|}, \end{aligned}$$

thanks to (41) and (49). Thus, (50) holds in all cases.

Let $S \subset E$ be given, with $\#(S) \leq k_{\mathrm{old}}^\#$. Let $\bar{F}^S$ be as in (42), and let $\varphi_\alpha^S (\alpha \in \mathcal{A})$ be as in $(\mathrm{WL3})^{y'}$. (Note that $(\mathrm{WL3})^{y'}$ applies, since $k_1^\# \geq k_{\mathrm{old}}^\#$.) We define

$$\bar{\varphi}_{\bar{\beta}}^S = \bar{F}^S \tag{51}$$

and

$$\bar{\varphi}_\alpha^S = \varphi_\alpha^S - [\partial^{\bar{\beta}} P_\alpha^{y'}(y')] \cdot \bar{F}^S \qquad \text{for all } \alpha \in \mathcal{A}. \tag{52}$$

Thus, $\varphi_\alpha^S \in C^m(\mathbb{R}^n)$ for all $\alpha \in \bar{\mathcal{A}}^{y'}$. We will check that

$$\|\partial^m \bar{\varphi}_\alpha^S\|_{C^0(\mathbb{R}^n)} \leq C \cdot (a_1)^{-m} \cdot \delta_Q^{|\alpha|-m} \qquad \text{for all } \alpha \in \bar{\mathcal{A}}^{y'}. \tag{53}$$

In fact, for $\alpha = \bar{\beta}$, (53) is immediate from (42(a)), (51), and (SU4).

For $\alpha \in \mathcal{A}$, we have

$$\begin{aligned}\|\partial^m \bar{\varphi}^S_\alpha\|_{C^0(\mathbb{R}^n)} &\leq \|\partial^m \varphi^S_\alpha\|_{C^0(\mathbb{R}^n)} + |\partial^{\bar{\beta}} P^{y'}_\alpha(y')| \cdot \|\partial^m \bar{F}^S\|_{C^0(\mathbb{R}^n)} \\ &\leq Ca_1 + [(a_1)^{-m} \cdot \delta_Q^{|\alpha|-|\bar{\beta}|}] \cdot [\delta_Q^{|\bar{\beta}|-m}] \leq C \cdot (a_1)^{-m} \cdot \delta_Q^{|\alpha|-m},\end{aligned}$$

thanks to $(\mathrm{WL3})^{y'}$(a), (49), (42(a)), (SU4). (Recall that $|\alpha| \leq m-1$ and $\delta_Q \leq \delta_{Q^\circ} \leq a_1$ by (11.3).) Thus, (53) holds in all cases.

Next, we check that

$$(54) \qquad |\bar{\varphi}^S_\alpha(x)| \leq C \cdot (a_1)^{-m} \cdot \delta_Q^{|\alpha|-m} \cdot \sigma(x) \qquad \text{for all } \ x \in S, \ \alpha \in \bar{\mathcal{A}}^{y'}.$$

In fact, for $\alpha = \bar{\beta}$, (54) is immediate from (42(b)), (51), and (SU4). For $\alpha \in \mathcal{A}$ and $x \in S$, we have

$$\begin{aligned}|\bar{\varphi}^S_\alpha(x)| &\leq |\varphi^S_\alpha(x)| + |\partial^{\bar{\beta}} P^{y'}_\alpha(y')| \cdot |\bar{F}^S(x)| && \text{(see (52))} \\ &\leq C\sigma(x) + \left[(a_1)^{-m}\delta_Q^{|\alpha|-|\bar{\beta}|}\right] \cdot \left[\delta_Q^{|\bar{\beta}|-m}\sigma(x)\right] && \text{(thanks to (WL3)}^{y'}\text{(b), (49), (42(b))} \\ &\leq C \cdot (a_1)^{-m}\delta_Q^{|\alpha|-m}\sigma(x) && \text{(thanks to (SU4)).}\end{aligned}$$

(Again, recall that $|\alpha| \leq m-1$ and $\delta_Q \leq \delta_{Q^\circ} \leq a_1$.) Thus, (54) holds in all cases.

We check also that

$$(55) \qquad J_{y'}(\bar{\varphi}^S_\alpha) = \bar{P}_\alpha \qquad \text{for all } \ \alpha \in \bar{\mathcal{A}}^{y'}.$$

In fact, for $\alpha = \bar{\beta}$, (55) is immediate from (51), (45), (42(c)). For $\alpha \in \mathcal{A}$, (55) follows from (46), (52), $(\mathrm{WL3})^{y'}$(c), and (42(c)). Thus, (55) holds in all cases.

Given $y' \in (Q^+)^{\star\star}$ (see (25)), we have constructed $\bar{\mathcal{A}}^{y'} < \mathcal{A}$ (see (47)) and $\bar{P}_\alpha(\alpha \in \bar{\mathcal{A}}^{y'})$ satisfying (48) and (50). Moreover, given $S \subset E$ with $\#(S) \leq k^{\#}_{\text{old}}$, we have constructed $\bar{\varphi}^S_\alpha(\alpha \in \bar{\mathcal{A}}^{y'})$, satisfying (53), (54), (55). We will check that $\bar{\mathcal{A}}^{y'}$ and the $\bar{P}_\alpha(\alpha \in \bar{\mathcal{A}}^{y'})$ satisfy conditions (OK1), (OK2), (OK3) for the cube Q^+ and the point y'.

In fact, (OK1) for Q^+, y' says that $\partial^\beta \bar{P}_\alpha(y') = \delta_{\beta\alpha}$ for all $\beta, \alpha \in \bar{\mathcal{A}}^{y'}$. That's just (48).

Condition (OK2) for Q^+, y' says that

$$(2\delta_Q)^{|\beta|-|\alpha|} \cdot |\partial^\beta \bar{P}_\alpha(y')| \leq (a_1)^{-(m+1)} \text{ for all } \alpha \in \bar{\mathcal{A}}^{y'} \text{ and } \beta \in \mathcal{M} \text{ with } \beta \geq \alpha.$$

This follows at once from (50) and (SU4), without the restriction to $\beta \geq \alpha$.

Condition (OK3) for Q^+, y' says that, given $\alpha \in \bar{\mathcal{A}}^{y'}$ and $S \subset E$ with $\#(S) \leq k^{\#}_{\text{old}}$, there exists $\bar{\varphi}^S_\alpha \in C^m(\mathbb{R}^n)$, with

$$(2\delta_Q)^{m-|\alpha|} \| \partial^m \bar{\varphi}^S_\alpha \|_{C^0(\mathbb{R}^n)} \leq (a_1)^{-(m+1)},$$
$$(2\delta_Q)^{m-|\alpha|} |\bar{\varphi}^S_\alpha(x)| \leq (a_1)^{-(m+1)} \cdot \sigma(x) \quad \text{on } S,$$

and

$$J_{y'}(\bar{\varphi}^S_\alpha) = \bar{P}_\alpha.$$

These assertions follow at once from (53), (54), (55) and (SU4).

Thus, conditions (OK1), (OK2), (OK3) hold (with $\bar{\mathcal{A}}^{y'} < \mathcal{A}$) for Q^+, y', for arbitrary $y' \in (Q^+)^{\star\star}$. By definition, this means that Q^+ is OK. This completes the proof of (22), and hence also that of Lemma 14.2. ■

The main result of this section is as follows.

Lemma 14.3. *Let $y \in Q^{\star\star}$ and $y' \in (Q')^{\star\star}$, where Q and Q' are CZ cubes. Let $P \in \mathcal{K}^{\#}_f(y; k^{\#}_A, C)$ and $P' \in \mathcal{K}^{\#}_f(y'; k^{\#}_A, C)$ be given, where*

$$k^{\#} \geq (D+1) \cdot k^{\#}_A \qquad \text{and} \quad k^{\#}_A \geq (D+1)^2 \cdot k^{\#}_{\text{old}}. \tag{56}$$

If the cubes Q and Q' abut, then we have

$$|\partial^\beta (P' - P)(y')| \leq C' \cdot (a_1)^{-(m+1)} \cdot \delta_Q^{m-|\beta|} \qquad \text{for all } \beta \in \mathcal{M}. \tag{57}$$

Proof. Let $k^{\#}_B = (D+1) \cdot k^{\#}_{\text{old}}$. Then, by Lemma 14.1, there exists

$$\tilde{P} \in \mathcal{K}^{\#}_f(y'; k^{\#}_B, C'), \tag{58}$$

with

$$|\partial^\beta (\tilde{P} - P)(y')| \leq C'' \cdot (a_1)^{-m} \cdot \delta_Q^{m-|\beta|}, \qquad \text{for all } \beta \in \mathcal{M}. \tag{59}$$

By hypothesis, and by (58), both P' and $\tilde{P}$ belong to $\mathcal{K}^{\#}_f(y'; k^{\#}_B, \widetilde{C})$. Hence, Lemma 14.2 applies to Q', y', and shows that

$$|\partial^\beta (P' - \tilde{P})(y')| \leq (a_1)^{-(m+1)} \cdot \delta_{Q'}^{m-|\beta|}, \qquad \text{for all } \beta \in \mathcal{M}. \tag{60}$$

Conclusion (57) is immediate from (59), (60), (SU4), and Lemma 11.2. ■

15. Proof of Lemmas 9.1 and 5.2

In this section, we complete the proof of Lemma 9.1. Thanks to Lemma 9.2, this will also establish Lemma 5.2. We place ourselves in the setting of section 9, and assume (SU0, ... , 6). In particular,

(1) E is a given finite subset of $\mathbb{R}^n$,

(2) $\sigma : E \to (0,\infty)$ and $f : E \to \mathbb{R}$ are given, and

(3) $\mathcal{A} \subset \mathcal{M}$ is given.

We use the Calderón–Zygmund decomposition from section 11. Let $Q_\nu (1 \le \nu \le \nu_{\max})$ be the CZ cubes, and let $\delta_\nu = \delta_{Q_\nu}$ = diameter of Q_ν, y_ν = center of Q_ν. Recall that

(4) $$\delta_\nu \le a_1 \le 1 \qquad \text{for each } \nu,$$

thanks to (11.3).

We take

(5) $$k^\# = (D+1)^3 \cdot k^\#_{\text{old}}.$$

Lemma 10.5 shows that $\mathcal{K}^\#_f(y_\nu; (D+1)^2 \cdot k^\#_{\text{old}}, C)$ is non–empty for each ν, where C is a large enough controlled constant.

For each ν, fix

(6) $$P_\nu \in \mathcal{K}^\#_f(y_\nu; (D+1)^2 \cdot k^\#_{\text{old}}, C).$$

Applying Lemma 14.3, we see that, whenever Q_μ and Q_ν abut, we have

(7) $$|\partial^\beta (P_\mu - P_\nu)(y_\nu)| \le C'(a_1)^{-(m+1)} \cdot \delta_\nu^{m-|\beta|} \qquad \text{for all } \beta \in \mathcal{M}.$$

Since $\partial^\beta (P_\mu - P_\nu)(x) = \sum_{|\gamma| \le m-1-|\beta|} \frac{1}{\gamma!} (\partial^{\gamma+\beta} (P_\mu - P_\nu)(y_\nu)) \cdot (x - y_\nu)^\gamma$ with $|x - y_\nu| \le C_1 \delta_\nu$ for any $x \in Q^\star_\nu$, estimate (7) implies

(8) $$|\partial^\beta (P_\mu - P_\nu)(x)| \le C_2 \cdot (a_1)^{-(m+1)} \cdot \delta_\nu^{m-|\beta|} \qquad \text{for all } x \in Q^\star_\nu \text{ and all } \beta \in \mathcal{M}.$$

Let $\tilde{\theta}_\nu (1 \le \nu \le \nu_{\max})$ be a cutoff function, with the following properties.

(9) $$0 \le \tilde{\theta}_\nu \le 1 \text{ on } \mathbb{R}^n, \tilde{\theta}_\nu = 1 \text{ on } Q^\star_\nu,\ \text{supp}\tilde{\theta}_\nu \subset Q^{\star\star}_\nu,$$

(10) $$|\partial^\beta \tilde{\theta}_\nu| \le C_3 \delta_\nu^{-|\beta|} \text{ for } \beta \in \mathcal{M}.$$

Fix $\nu (1 \leq \nu \leq \nu_{\max})$, and define

$$\hat{f}_\nu(x) = \tilde{\theta}_\nu(x) \cdot [f(x) - P_\nu(x)] \text{ for all } \quad x \in E. \tag{11}$$

Note that

$$f(x) = \hat{f}_\nu(x) + P_\nu(x) \text{ for all } \qquad x \in E \cap Q_\nu^\star. \tag{12}$$

Our plan is to apply Lemma 8.1 to the function $\hat{f}_\nu$ and the cube Q_ν. Recall that, since Q_ν is a CZ cube, it is OK. Thus,

(13) For each $y \in Q_\nu^{\star\star}$, we are given $\bar{\mathcal{A}}^y < \mathcal{A}$, and polynomials $\bar{P}_\alpha^y (\alpha \in \bar{\mathcal{A}}^y)$, satisfying (OK1), (OK2), (OK3).

We will check the following straightforward result.

Lemma 15.1. *The hypotheses of Lemma 8.1 hold, with $A = (a_1)^{-(m+1)}$, for the set E, the functions $\hat{f}_\nu$ and σ on E, the cube Q_ν, the sets of multi–indices $\mathcal{A}, \bar{\mathcal{A}}^y (y \in Q_\nu^{\star\star})$, and the polynomials $\bar{P}_\alpha^y (y \in Q_\nu^{\star\star}, \alpha \in \bar{\mathcal{A}}^y)$.*

Proof. The hypotheses of Lemma 8.1 are as follows:

- The STRONG MAIN LEMMA holds for all $\bar{\mathcal{A}} < \mathcal{A}$. (That's just (SU1), which we are assuming here.)
- $E \subset \mathbb{R}^n$ is finite, $\hat{f}_\nu : E \to \mathbb{R}$ and $\sigma : E \to (0, \infty)$.
- For each $y \in Q_\nu^{\star\star}$, we are given $\bar{\mathcal{A}}^y < \mathcal{A}$ and $\bar{P}_\alpha^y (\alpha \in \bar{\mathcal{A}}^y)$. (That's immediate from (13).)
- Conditions (G1), (G2), (G3) hold, with $A = a_1^{-(m+1)}$. (That's immediate from (OK1), (OK2), (OK3) for Q_ν; these conditions hold, thanks to (13).)
- Condition (G4) holds, with $A = a_1^{-(m+1)}$.

To check this last hypothesis, we use (6). From (6) and the definitions of $\mathcal{K}_f^\#$ and $\mathcal{K}_f$, we learn the following.

(14) Given $S \subset E$ with $\#(S) \leq (D+1)^2 \cdot k^\#_{\text{old}}$, there exists $F_\nu^S \in C^m(\mathbb{R}^n)$, with $\|F_\nu^S\|_{C^m(\mathbb{R}^n)} \leq C$, $|F_\nu^S(x) - f(x)| \leq C\sigma(x)$ on S, and $J_{y_\nu}(F_\nu^S) = P_\nu$.

With F_ν^S as in (14), and with $\tilde{\theta}_\nu$ as in (9), (10), (11), we define

$$\hat{F}_\nu^S = \tilde{\theta}_\nu \cdot [F_\nu^S - P_\nu]. \tag{16}$$

Note that

(17) $|\hat{F}_\nu^S(x) - \hat{f}_\nu(x)| = \tilde{\theta}_\nu(x) \cdot |F_\nu^S(x) - f(x)| \leq C\sigma(x)$ on S, thanks to (9), (11), (14), (16).

From (14) and Taylor's theorem, we have

$$|\partial^\beta(F_\nu^S - P_\nu)| \leq C'\delta_\nu^{m-|\beta|} \text{ on } Q_\nu^{\star\star}, \quad \text{for } |\beta| \leq m.$$

Together with (9), (10) and (16), this implies

$$|\partial^\beta \hat{F}_\nu^S| \leq C''\delta_\nu^{m-|\beta|} \quad \text{on } \mathbb{R}^n, \text{ for } |\beta| \leq m.$$

Thus,

(18) Given $S \subset E$ with $\#(S) \leq (D+1)^2 \cdot k_{\text{old}}^\#$, there exists $\hat{F}_\nu^S \in C^m(\mathbb{R}^n)$, with

(a) $|\partial^\beta \hat{F}_\nu^S(x)| \leq C''\delta_\nu^{m-|\beta|}$ for all $x \in \mathbb{R}^n, |\beta| \leq m$; and

(b) $|\hat{F}_\nu^S(x) - \hat{f}_\nu(x)| \leq C\sigma(x)$ for all $x \in S$.

Condition (G4) for $Q_\nu, \hat{f}_\nu$, etc., with $A = (a_1)^{-(m+1)}$, follows at once from (18), thanks to (SU4).

The proof of Lemma 15.1 is complete. ∎

Applying Lemmas 15.1 and 8.1, we obtain a function $F_\nu \in C^m(\mathbb{R}^n)$, for each $\nu(1 \leq \nu \leq \nu_{\max})$, satisfying

(19) $$|\partial^\beta F_\nu(x)| \leq A'\delta_\nu^{m-|\beta|} \text{ for all } \quad x \in \mathbb{R}^n, |\beta| \leq m$$

and

(20) $$|F_\nu(x) - \hat{f}_\nu(x)| \leq A'\sigma(x) \text{ for all } \quad x \in E \cap Q_\nu^\star.$$

Here, A' is determined by a_1, m, n. For the rest of this section, we write A, A', A'', A_1, etc., to denote constants determined by a_1, m, n.

From (12) and (20), we see that

(21) $$|f(x) - (P_\nu(x) + F_\nu(x))| \leq A'\sigma(x) \quad \text{for all } x \in E \cap Q_\nu^\star.$$

Our plan is to patch together the "local solutions" $P_\nu(x) + F_\nu(x)(\nu = 1, \dots, \nu_{\max})$, using a partition of unity.

For each $\nu(1 \leq \nu \leq \nu_{\max})$, we introduce a cutoff function $\hat{\theta}_\nu$, satisfying

(22) $0 \leq \hat{\theta}_\nu \leq 1$ on $\mathbb{R}^n$, $\hat{\theta}_\nu = 1$ on Q_ν, $\hat{\theta}_\nu(x) = 0$ for dist $(x, Q_\nu) > \hat{c}\delta_\nu$,

and

(23) $|\partial^\beta \hat{\theta}_\nu| \leq C\delta_\nu^{-|\beta|}$ for $|\beta| \leq m$.

Taking $\hat{c}$ small enough in (22), and recalling Lemma 11.2, we obtain the following.

(24) If Q_μ contains a point of supp$\hat{\theta}_\nu$, then Q_μ and Q_ν coincide or abut.

Define $\theta_\nu = \hat{\theta}_\nu / (\sum_\mu \hat{\theta}_\mu)$ on Q°. From (22), ... , (24), the Cor. to Lemma 11.1, and Lemma 11.2, we obtain:

(25) $\sum_{1 \leq \nu \leq \nu_{\max}} \theta_\nu = 1$ on Q°.

(26) $0 \leq \theta_\nu \leq 1$ on Q°.

(27) $|\partial^\beta \theta_\nu| \leq C\delta_\nu^{-|\beta|}$ for $|\beta| \leq m$.

(28) $\theta_\nu = 0$ outside $Q_\nu^\star$.

(29) If $x \in Q_\mu$, then $\theta_\nu = 0$ in a neighborhood of x, unless Q_μ and Q_ν coincide or abut.

We define

$$\tilde{F}(x) = \sum_{1 \leq \nu \leq \nu_{\max}} \theta_\nu(x) \cdot (P_\nu(x) + F_\nu(x)) \text{ for } x \in Q^\circ. \tag{30}$$

Note that θ_ν and $\tilde{F}$ are defined only on Q°.

Given $x \in E \cap Q^\circ$, we see from (21), (26), (28) that $|\theta_\nu(x) \cdot (P_\nu(x) + F_\nu(x)) - \theta_\nu(x) \cdot f(x)| \leq A'\sigma(x) \cdot \theta_\nu(x)$. Summing over ν, and using (25) and (30), we obtain

$$\left|\tilde{F}(x) - f(x)\right| \leq A'\sigma(x) \text{ for all } x \in E \cap Q^\circ. \tag{31}$$

We prepare to estimate the derivatives of $\tilde{F}$. Fix $x \in Q^\circ$, and let Q_μ be a CZ cube containing x. Differentiating (30), we obtain

$$\partial^\beta \tilde{F}(x) = \sum_{\beta'+\beta''=\beta} c(\beta',\beta'') \sum_{1 \leq \nu \leq \nu_{\max}} (\partial^{\beta'}\theta_\nu(x)) \cdot [\partial^{\beta''}P_\nu(x) + \partial^{\beta''}F_\nu(x)]. \tag{32}$$

We look separately at the cases $\beta' = 0$, $\beta' \neq 0$. We will need an estimate for $\partial^\beta P_\nu(x)$. Recalling (14), and taking $S =$ empty set, we find that $\left|\partial^\beta P_\nu(y_\nu)\right| \leq C$ for $|\beta| \leq m-1$, $1 \leq \nu \leq \nu_{\max}$.

For $\tilde{x} \in Q_\nu^\star$, we have $|\tilde{x} - y_\nu| \leq C\delta_\nu \leq C$ (see(4)), hence

$$\left|\partial^\beta P_\nu(\tilde{x})\right| = \left| \sum_{|\gamma| \leq m-1-|\beta|} \frac{1}{\gamma!} \left(\partial^{\gamma+\beta} P_\nu(y_\nu)\right) \cdot (\tilde{x} - y_\nu)^\gamma \right| \leq C' \text{ for } |\beta| \leq m-1.$$

For $|\beta| = m$, we have $\partial^\beta P_\nu \equiv 0$, since P_ν is a polynomial of degree at most $m-1$. Thus,

$$\left|\partial^\beta P_\nu(\tilde{x})\right| \leq C' \text{ for all } \tilde{x} \in Q_\nu^\star,\ |\beta| \leq m. \tag{33}$$

Now, combining (19), (33) with (26), (28), we see that

$$\left|\theta_\nu(x) \cdot \left[\partial^\beta P_\nu(x) + \partial^\beta F_\nu(x)\right]\right| \leq A''\theta_\nu(x) \text{ for } 1 \leq \nu \leq \nu_{\max},\ |\beta| \leq m.$$

(Here we again use (4).)

Summing over ν, and recalling (25), we obtain the estimate

$$\left|\sum_\nu \theta_\nu(x) \cdot \left[\partial^\beta P_\nu(x) + \partial^\beta F_\nu(x)\right]\right| \leq A'', \tag{34}$$

which controls the term $\beta' = 0$ in (32).

For $\beta' \neq 0$, we have $\sum_\nu \partial^{\beta'} \theta_\nu(x) = 0$ by (25), hence

(35)
$$\sum_\nu \left(\partial^{\beta'}\theta_\nu(x)\right) \cdot \left[\partial^{\beta''}P_\nu(x) + \partial^{\beta''}F_\nu(x)\right] = \sum_\nu \left(\partial^{\beta'}\theta_\nu(x)\right) \cdot \left(\partial^{\beta''}P_\nu(x) - \partial^{\beta''}P_\mu(x)\right) + \sum_\nu \left(\partial^{\beta'}\theta_\nu(x)\right) \cdot \partial^{\beta''}F_\nu(x).$$

Suppose $\beta' \neq 0$, $|\beta'| + |\beta''| \leq m$.

We will check that

$$\left|\left(\partial^{\beta'}\theta_\nu(x)\right) \cdot \left(\partial^{\beta''}P_\nu(x) - \partial^{\beta''}P_\mu(x)\right)\right| \leq \tilde{A} \cdot \delta_\mu^{m-|\beta'|-|\beta''|}. \tag{36}$$

In the fact, the left–hand side of (36) is equal to zero in the following cases: $x \notin Q_\nu^\star$ (see (28)); Q_μ and Q_ν neither coincide nor abut (see (29)); $Q_\mu = Q_\nu$ (see (36)). Hence, in checking (36), we may suppose that $x \in Q_\nu^\star$ and that Q_μ and Q_ν abut. In this case (36) follows from (8), (27) and Lemma 11.2. Thus, (36) holds in all cases.

We sum (36) over all ν. We obtain a non–zero term on the left only when Q_μ and Q_ν abut, which occurs for at most $\widetilde{C}$ distinct ν, thanks to Lemma 11.2. Consequently,

$$\left|\sum_\nu \left(\partial^{\beta'}\theta_\nu(x)\right) \cdot \left(\partial^{\beta''}P_\nu(x) - \partial^{\beta''}P_\mu(x)\right)\right| \leq A_1\delta_\mu^{m-|\beta'|-|\beta''|} \leq A_1, \tag{37}$$

thanks to (4).

This controls the first term on the right in (35). We turn to the second term. Estimates (19) and (27) show that

$$\left|\left(\partial^{\beta'}\theta_\nu(x)\right)\cdot\left(\partial^{\beta''}F_\nu(x)\right)\right| \le A_2\delta_\nu^{m-|\beta''|-|\beta'|} \text{ for } |\beta'|+|\beta''|\le m. \tag{38}$$

Moreover, the left–hand side of (38) is non–zero only when Q_ν and Q_μ coincide or abut. There are at most $\widetilde{C}$ distinct ν for which this occurs, since we have fixed Q_μ. Together with Lemma 11.2, these remarks imply the estimate

$$\left|\sum_\nu \left(\partial^{\beta'}\theta_\nu(x)\right)\cdot\left(\partial^{\beta''}F_\nu(x)\right)\right| \le A_3\delta_\mu^{m-|\beta'|-|\beta''|}\le A_3, \tag{39}$$

thanks to (4).

Now, from (35), (37), (39), we obtain

$$\left|\sum_\nu \left(\partial^{\beta'}\theta_\nu(x)\right)\cdot\left[\partial^{\beta''}P_\nu(x)+\partial^{\beta''}F_\nu(x)\right]\right| \le A_4 \text{ for } |\beta'|+|\beta''|\le m, \beta'\ne 0.$$

Together with (34) and (32), this shows that

$$\left|\partial^\beta \tilde{F}(x)\right| \le A_5 \text{ for all } x\in Q^\circ, |\beta|\le m. \tag{40}$$

Our function $\tilde{F}$ satisfies the good properties (31) and (40), but it is defined only on Q°. Recall that Q° is centered at y^0, and has diameter $ca_1<\delta_{Q^\circ}<a_1$. (See (11.1) and (11.3).) Hence, we may find a cutoff function θ^0 on $\mathbb{R}^n$, with $\theta^0=1$ on $B(y^0,c'a_1)$, $\operatorname{supp}\theta^0\subset Q^\circ$, $0\le\theta^0\le 1$ on $\mathbb{R}^n$, and

$$\left|\partial^\beta\theta^0\right|\le Ca_1^{-|\beta|} \text{ for } |\beta|\le m.$$

Setting $F=\tilde{F}\cdot\theta^0$, we obtain a function on all of $\mathbb{R}^n$. From (31), (40) and the properties of θ^0, we have at once

$$\|F\|_{C^m(\mathbb{R}^n)}\le A_6 \tag{41}$$

and

$$|F(x)-f(x)|\le A_6\sigma(x) \text{ for all } x\in E\cap B(y^0,c'a_1). \tag{42}$$

Since A_6 and $c'a_1$ are both determined by a_1, m and n, estimates (41) and (42) immediately imply the conclusions of Lemma 9.1.

This completes the proofs of Lemma 9.1 and Lemma 5.2. ■

16. A rescaling Lemma

Recall that $\mathcal{M}^+$ denotes the set of multi–indices β with $|\beta| \leq m$. The following result will be used in the next section, to prove Lemma 5.3.

Lemma 16.1. *Let $\mathcal{A} \subset \mathcal{M}$ be given, and let $C_1, \bar{a}$ be positive numbers. Suppose we are given real numbers $F_{\alpha,\beta}$, indexed by $\alpha \in \mathcal{A}, \beta \in \mathcal{M}^+$. Assume that the following conditions are satisfied.*

(0) $\quad F_{\alpha,\alpha} \neq 0$ *for all* $\quad \alpha \in \mathcal{A}$.

(1) $\quad |F_{\alpha,\beta}| \leq C_1 |F_{\alpha,\alpha}|$ *for all* $\quad \alpha \in \mathcal{A}, \beta \in \mathcal{M}^+ \quad$ *with* $\beta > \alpha$.

(2) $\quad F_{\alpha,\beta} = 0$ *for all* $\quad \alpha, \beta \in \mathcal{A} \quad$ *with* $\alpha \neq \beta$.

Then there exist positive numbers $\lambda_1, \ldots, \lambda_n$, and a map $\phi : \mathcal{A} \to \mathcal{M}$, with the following properties:

(3) $\quad c < \lambda_i \leq 1$ *for all* $i = 1, \ldots, n$, *where c is a positive constant determined by* $C_1, \bar{a}, m, n$.

(4) $\quad \phi(\alpha) \leq \alpha$ *for all* $\alpha \in \mathcal{A}$.

(5) $\quad$ *For each* $\alpha \in \mathcal{A}$, *either* $\phi(\alpha) = \alpha$ *or* $\phi(\alpha) \notin \mathcal{A}$.

(6) $\quad$ *Suppose we define* $\hat{F}_{\alpha,\beta}$ *for* $\alpha \in \mathcal{A}, \beta \in \mathcal{M}^+$, *by*

(a) $\quad \hat{F}_{\alpha,\beta} = \lambda_1^{\beta_1} \cdots \lambda_n^{\beta_n} F_{\alpha,\beta} \quad (\beta = (\beta_1, \ldots, \beta_n))$.

Then we have

(b) $\quad |\hat{F}_{\alpha,\beta}| \leq \bar{a} \cdot |\hat{F}_{\alpha,\phi(\alpha)}|$ *for all* $\alpha \in \mathcal{A}, \beta \in \mathcal{M}^+$ *with* $\beta \neq \phi(\alpha)$.

Proof. By possibly making C_1 larger, we may assume that

$$C_1 > 1. \tag{7}$$

By possibly making $\bar{a}$ smaller, we may assume that

$$C_1 \bar{a} < 1. \tag{8}$$

The main point of our proof is to show that we can pick $\lambda_1, \ldots, \lambda_n$ satisfying (3), and satisfying also the following conditions, where $\hat{F}_{\alpha,\beta}$ is defined by (6(a)):

$$|\hat{F}_{\alpha,\beta} / \hat{F}_{\alpha,\alpha}| \leq |F_{\alpha,\beta} / F_{\alpha,\alpha}| \text{ for all } \alpha \in \mathcal{A},\ \beta \in \mathcal{M}^+ \text{ with } \beta > \alpha. \tag{9}$$

(10) $|\hat{F}_{\alpha,\beta}/\hat{F}_{\alpha,\beta'}| \notin [\bar{a},\bar{a}^{-1}]$ whenever $\alpha \in \mathcal{A}$, $\beta,\beta' \in \mathcal{M}^+$, $\beta \neq \beta'$, and $F_{\alpha,\beta'} \neq 0$.

We first show that if $\lambda_1,\dots,\lambda_n$ can be picked to satisfy (3), (9) and (10), then we can find ϕ so that all the conclusions (3),$\dots$, (6) of Lemma 16.1 are satisfied. Then we return to the task of finding $\lambda_1,\dots,\lambda_n$ satisfying (3), (9), (10).

Suppose $\lambda_1,\dots,\lambda_n$ satisfy (3), (9), (10). Define a map $\phi : \mathcal{A} \to \mathcal{M}^+$, by taking $\phi(\alpha)$ to be a value of β that maximizes $|\hat{F}_{\alpha,\beta}|$ for the given α. Thus,

(11) $$|\hat{F}_{\alpha,\phi(\alpha)}| \geq |\hat{F}_{\alpha,\beta}| \quad \text{for all} \quad \beta \in \mathcal{M}^+, \alpha \in \mathcal{A}.$$

In particular, taking $\beta = \alpha$ in (11), and recalling (0) and (6(a)), we see that

(12) $$\hat{F}_{\alpha,\phi(\alpha)} \neq 0, \quad \text{for all } \alpha \in \mathcal{A}.$$

Together with (2), this implies conclusion (5).

Also, (11), (12), and (10) with $\beta' = \phi(\alpha)$, together imply conclusion (6).

Next, suppose $\alpha \in \mathcal{A}$ and $\phi(\alpha) > \alpha$. From (9) and (1), we then have $\left|\hat{F}_{\alpha,\phi(\alpha)}/\hat{F}_{\alpha,\alpha}\right| \leq C_1 < \bar{a}^{-1}$, by (8).

Hence, (0) and (10) show that $\left|\hat{F}_{\alpha,\phi(\alpha)}\right| \leq \bar{a}\left|\ \hat{F}_{\alpha,\alpha}\right|$, which in turn implies

(13) $$\left|\hat{F}_{\alpha,\phi(\alpha)}\right| < \left|\hat{F}_{\alpha,\alpha}\right|, \qquad \text{thanks to (0), (7), (8).}$$

However, (13) contradicts (11). Therefore, we cannot have $\phi(\alpha) > \alpha$, which proves conclusion (4). Moreover, since $\phi(\alpha) \leq \alpha$, we have $\phi(\alpha) \in \mathcal{M}$ for all $\alpha \in \mathcal{A}$. (Recall that we knew at first merely that $\phi(\alpha) \in \mathcal{M}^+$.) Thus, $\phi : \mathcal{A} \to \mathcal{M}$, and conclusions (3),$\dots$,(6) are satisfied by $\phi, \lambda_1,\dots,\lambda_n$.

This completes the reduction of Lemma 16.1 to the task of finding $\lambda_1,\dots,\lambda_n$ that satisfy (3), (9), (10).

We take

(14) $$\lambda_k = \exp(-[\tau_k + \cdots + \tau_n]), \quad k = 1,\dots,n.$$

for $\tau_1,\tau_2,\dots,\tau_n > 0$ to be picked below. Evidently, (3) holds, provided $\tau_1,\dots,\tau_n$ are bounded above by a constant determined by $C_1, \bar{a}, m, n$. Regarding (9) and (10), we note first that, for $\alpha \in \mathcal{A}$, $\beta = (\beta_1,\dots,\beta_n) \in \mathcal{M}^+$, $\beta' = (\beta_1',\dots,\beta_n') \in \mathcal{M}^+$, with $F_{\alpha,\beta'} \neq 0$, we have

(15) $$\left|\hat{F}_{\alpha,\beta}/\hat{F}_{\alpha,\beta'}\right| = \left|F_{\alpha,\beta}/F_{\alpha,\beta'}\right| \cdot \exp(-[p_1\tau_1 + \cdots + p_n\tau_n]),$$

with

$$p_k = (\beta_1 + \cdots + \beta_k) - (\beta'_1 + \cdots + \beta'_k). \tag{16}$$

Formulas (15), (16) are immediate from definitions (6(a)) and (14). Since $\beta, \beta' \in \mathcal{M}^+$, each p_k is an integer, and

$$-m \leq p_k \leq +m, \qquad k = 1, \ldots, n. \tag{17}$$

If $\beta \neq \beta'$, then the p_k are not all zero, thanks to (16). Suppose $\beta > \beta'$. Then, by definition of the order relation $>$, there exists $\bar{k}$, for which we have $p_{\bar{k}} \geq 1$, and $p_k = 0$ for $k > \bar{k}$. Hence, in this case (15) and (17) show that

$$\begin{aligned} \left|\hat{F}_{\alpha,\beta}/\hat{F}_{\alpha,\beta'}\right| &= \left|F_{\alpha,\beta}/F_{\alpha,\beta'}\right| \cdot \exp(-[p_1\tau_1 + \cdots + p_{\bar{k}}\tau_{\bar{k}}]) \\ &\leq \left|F_{\alpha,\beta}/F_{\alpha,\beta'}\right| \cdot \exp(-\tau_{\bar{k}} + m \sum_{1 \leq k < \bar{k}} \tau_k). \end{aligned} \tag{17a}$$

Estimates (17a) hold whenever $F_{\alpha,\beta'} \neq 0$ and $\beta > \beta'$. In particular, taking $\beta' = \alpha$, and recalling (0), we see that (9) holds, provided we have

$$\tau_{\bar{k}} > m \sum_{1 \leq k < \bar{k}} \tau_k \qquad \text{for all} \quad \bar{k} = 1, \ldots, n. \tag{18}$$

To ensure that (18) holds, we introduce new variables $t_1, \ldots, t_n$, and define $\tau_1, \ldots, \tau_n$ inductively by setting

$$\tau_{\bar{k}} = m \cdot \sum_{1 \leq k < \bar{k}} \tau_k + t_{\bar{k}} \qquad \text{for} \quad \bar{k} = 1, \ldots, n. \tag{19}$$

If $t_1, \cdots, t_n > 0$, then (18) holds, hence $\lambda_1, \ldots, \lambda_n$ satisfy (9). Note that (19) shows that

$$(\tau_1, \ldots, \tau_n) = (t_1, \ldots, t_n)M, \tag{20}$$

where M is a triangular $n \times n$ matrix, with integer entries, and with 1's on the main diagonal. The matrix M is determined by m and n. Hence, if $t_1, \ldots, t_n$ are bounded above by a constant determined by $C_1, \bar{a}, m, n$, then so are $\tau_1, \ldots, \tau_n$, and therefore (3) will hold.

Thus, to complete the proof of Lemma 16.1, it is enough to find $t_1, \ldots, t_n > 0$, bounded above by a constant determined by $C_1, \bar{a}, m, n$, for which $\lambda_1, \ldots, \lambda_n$ satisfy (10). We return to (15), which we write in the form

$$\left|\hat{F}_{\alpha\beta}/\hat{F}_{\alpha,\beta'}\right| = \left|F_{\alpha,\beta}/F_{\alpha,\beta'}\right| \cdot \exp(-\vec{\tau}\vec{p}^{\dagger}) \qquad \text{for} \quad \beta \neq \beta', F_{\alpha,\beta'} \neq 0. \tag{21}$$

Here, $\vec{\tau} = (\tau_1, \dots, \tau_n)$, and $\vec{p} = (p_1, \dots, p_n)$ is a non–zero lattice point determined by β and β'.

From (20) and (21), we obtain

$$\left|\hat{F}_{\alpha,\beta}/\hat{F}_{\alpha,\beta'}\right| = \left|F_{\alpha,\beta}/F_{\alpha,\beta'}\right| \cdot \exp(-\vec{t}\vec{q}^{\dagger}) \qquad \text{for} \quad \beta \neq \beta',\ F_{\alpha,\beta'} \neq 0, \tag{22}$$

with $\vec{t} = (t_1, \dots, t_n)$, and with $\vec{q} = (q_1, \dots, q_n) = (p_1, \dots, p_n)M^{\dagger}$ a non–zero lattice point determined by β, β', m, n. In particular, (22) shows that $|\hat{F}_{\alpha,\beta}/\hat{F}_{\alpha,\beta'}| \notin [\bar{a}, \bar{a}^{-1}]$, unless we have $F_{\alpha,\beta} \neq 0$, and

$$\left|q_1 t_1 + \cdots + q_n t_n - \ln|F_{\alpha,\beta}/F_{\alpha,\beta'}|\right| \leq |\ln \bar{a}|. \tag{23}$$

Hence, to prove Lemma 16.1, it is enough to show that there exist positive $t_1, \dots, t_n$, bounded by a constant determined by $C_1, \bar{a}, m, n$, for which (23) fails whenever $\alpha \in \mathcal{A}$, β and $\beta' \in \mathcal{M}^+$, $\beta \neq \beta'$, $F_{\alpha,\beta'} \neq 0$, $F_{\alpha,\beta} \neq 0$.

Let T be a large positive number to be fixed later, and let $Q_T = \{(t_1, \dots, t_n) \in \mathbb{R}^n\colon$ Each t_i belongs to $(0, T)\}$

Thus, Q_T is a cube of volume T^n. On the other hand, suppose we fix $\alpha \in \mathcal{A}$, $\beta, \beta' \in \mathcal{M}^+$ with $\beta \neq \beta'$ and $F_{\alpha,\beta}$, $F_{\alpha,\beta'} \neq 0$. Let $(q_1, \dots, q_n)$ be the non–zero lattice point in (23). Say, $q_\ell \neq 0$. Then, for each fixed $(t_1, \dots, t_{\ell-1}, t_{\ell+1}, \dots, t_n)$, the set of all t_ℓ for which (23) holds is an interval of length $2|\ln \bar{a}|/|q_\ell| \leq 2|\ln \bar{a}|$. Consequently, the volume of the set of all $(t_1, \dots, t_n) \in Q_T$ for which (23) holds is at most $2|\ln \bar{a}| \cdot T^{n-1}$.

It follows that the set $\Omega_T = \{(t_1, \dots, t_n) \in Q_T\colon$ (23) holds for some $\alpha \in \mathcal{A}, \beta, \beta' \in \mathcal{M}^+$ with $\beta \neq \beta', F_{\alpha,\beta} \neq 0,\ F_{\alpha,\beta'} \neq 0\}$ has volume at most $N \cdot 2|\ln \bar{a}| \cdot T^{n-1}$, where N is the number of triples $(\alpha, \beta, \beta') \in \mathcal{M} \times \mathcal{M}^+ \times \mathcal{M}^+$ with $\beta \neq \beta'$. Note that N is determined by m and n.

We now take T to be a constant, determined by $\bar{a}, m, n$, large enough to satisfy $T^n > N \cdot 2|\ln \bar{a}| \cdot T^{n-1}$.

Then the set $Q_T \smallsetminus \Omega_T$ has positive volume. Picking $(t_1, \dots, t_n) \in Q_T \smallsetminus \Omega_T$, we see that the t_i are positive and bounded above by a constant determined by $\bar{a}, m$ and n; and that (23) fails, whenever $\alpha \in \mathcal{A}$, $\beta, \beta' \in \mathcal{M}^+$, $\beta \neq \beta'$, $F_{\alpha,\beta} \neq 0$, $F_{\alpha,\beta'} \neq 0$. The proof of Lemma 16.1 is complete. ■

17. Proof of Lemma 5.3

In this section, we give the proof of Lemma 5.3. We fix $\mathcal{A} \subset \mathcal{M}$, and assume that the WEAK MAIN LEMMA holds for all $\bar{\mathcal{A}} \leq \mathcal{A}$. We must show that the STRONG

MAIN LEMMA holds for $\mathcal{A}$. We may assume that the WEAK MAIN LEMMA holds for all $\bar{\mathcal{A}} \leq \mathcal{A}$, with $k^{\#}$ and a_0 independent of $\bar{\mathcal{A}}$. (Although each $\bar{\mathcal{A}} \leq \mathcal{A}$ gives rise to its own $k^{\#}$ and a_0 , we may simply use the maximum of all the $k^{\#}$, and the minimum of all the a_0, arising in the WEAK MAIN LEMMA for all $\bar{\mathcal{A}} \leq \mathcal{A}$.) Fix $k^{\#}$ and a_0 as in the WEAK MAIN LEMMA for $\bar{\mathcal{A}} \leq \mathcal{A}$.

Let $E, f, \sigma, y^0, P_\alpha(\alpha \in \mathcal{A})$ satisfy the hypotheses of the STRONG MAIN LEMMA for $\mathcal{A}$. Without loss of generality, we may suppose

(1) $$y^0 = 0.$$

We want to show that there exists an $F \in C^m(\mathbb{R}^n)$, satisfying the conclusions (SL5, 6) of the STRONG MAIN LEMMA for $\mathcal{A}$.

In this section, we say that a constant is *controlled* if it is determined by C, m, n in the hypotheses (SL1, ... , 4) of the STRONG MAIN LEMMA for $\mathcal{A}$ We write c, C', C'', C_1, etc., to denote controlled constants. Also, we introduce a small constant $\bar{a}$ to be picked later. Initially, we do not assume that $\bar{a}$ is a controlled constant. We say that a constant is *weakly controlled* if it is determined by $\bar{a}$ together with C, m, n in (SL1, ... , 4). We write $c(\bar{a}), C(\bar{a}), C'(\bar{a})$, etc., to denote weakly controlled constants. Note that the constants $k^{\#}$ and a_0 are controlled. We assume that

(2) $\bar{a}$ is less than a small enough controlled constant.

Our plan is simply to rescale E, f, σ, P_α using the linear map $T : \mathbb{R}^n \to \mathbb{R}^n$, defined by

(3) $$T : (\hat{x}_1, \dots , \hat{x}_n) \mapsto (\lambda_1\hat{x}_1, \dots , \lambda_n\hat{x}_n),$$

for $\lambda_1, \dots , \lambda_n > 0$ to be picked below. We define

(4) $$\hat{E} = T^{-1}(E), \hat{f} = f \circ T, \hat{\sigma} = \sigma \circ T, \hat{P}_\alpha = P_\alpha \circ T.$$

Thus, $\hat{E} \subset \mathbb{R}^n$ is a finite set, $\hat{f} : \hat{E} \to \mathbb{R}, \hat{\sigma} : \hat{E} \to (0, \infty)$, and $\hat{P}_\alpha \in \mathcal{P}$ for each $\alpha \in \mathcal{A}$. Evidently,

(4a) $$\partial^\beta \hat{P}_\alpha(0) = \lambda_1^{\beta_1} \cdots \lambda_n^{\beta_n} \partial^\beta P_\alpha(0) \qquad \text{for} \quad \alpha \in \mathcal{A},\ \beta = (\beta_1, \dots , \beta_n).$$

To pick $\lambda_1, \dots , \lambda_n$, we appeal to Lemma 16.1, with

(5) $$F_{\alpha,\beta} = \partial^\beta P_\alpha(0) \qquad \text{for} \quad \alpha \in \mathcal{A},\ |\beta| \leq m - 1,$$

and

(6) $$F_{\alpha,\beta} = 1 \qquad \text{for } \alpha \in \mathcal{A},\ |\beta| = m.$$

Note that the hypotheses (16.0), (16.1), (16.2) of Lemma 16.1 hold, with C_1 a controlled constant, thanks to (SL1), (SL2), and (1). Hence, Lemma 16.1 produces numbers $\lambda_1, \dots, \lambda_n$, and a map $\phi : \mathcal{A} \to \mathcal{M}$, with the following properties:

(7) $c(\bar{a}) < \lambda_i \le 1$ for all $i = 1, \dots, n$.

(8) $\phi(\alpha) \le \alpha$ for each $\alpha \in \mathcal{A}$.

(9) For each $\alpha \in \mathcal{A}$, either $\phi(\alpha) = \alpha$ or $\phi(\alpha) \notin \mathcal{A}$.

(10) For any $\alpha \in \mathcal{A}, \beta \in \mathcal{M}$ with $\beta \neq \phi(\alpha)$, we have $|\partial^\beta \hat{P}_\alpha(0)| \le \bar{a} |\partial^{\phi(\alpha)} \hat{P}_\alpha(0)|$.

(11) For any $\alpha \in \mathcal{A}$, we have $\lambda_1^{\beta_1} \cdots \lambda_n^{\beta_n} \le \bar{a}\ |\partial^{\phi(\alpha)} \hat{P}_\alpha(0)|$ for $\beta_1 + \cdots + \beta_n = m$.

Here, conclusions (10) and (11) follow from (16.6) and (4a), (5), (6). We fix $\lambda_1, \dots, \lambda_n$ and ϕ, satisfying (7), ... , (11).

Let $\hat{S} \subset \hat{E}$ be given, with $\#(\hat{S}) \le k^\#$. Set $S = T(\hat{S}) \subset E$, and apply (SL3). Let $\varphi_\alpha^S (\alpha \in \mathcal{A})$ be as in (SL3), and define $\hat{\varphi}_\alpha^{\hat{S}} = \varphi_\alpha^S \circ T$. For $\beta = (\beta_1, \dots, \beta_n)$ with $|\beta| = m$, we learn from (SL3)(a) and from (11) that

$$\|\partial^\beta \hat{\varphi}_\alpha^{\hat{S}}\|_{C^0(\mathbb{R}^n)} = \lambda_1^{\beta_1} \cdots \lambda_n^{\beta_n} \|\partial^\beta \varphi_\alpha^S\|_{C^0(\mathbb{R}^n)} \le C\bar{a} |\partial^{\phi(\alpha)} \hat{P}_\alpha(0)|.$$

Also, (SL3)(b) and (c), together with (1) and (4), show that

$$|\hat{\varphi}_\alpha^{\hat{S}}(\hat{x})| \le C\hat{\sigma}(\hat{x}) \text{ on } \hat{S}, \text{ and } J_0(\hat{\varphi}_\alpha^{\hat{S}}) = \hat{P}_\alpha.$$

Thus,

(12) Given $\hat{S} \subset \hat{E}$ with $\#(\hat{S}) \le k^\#$, and given $\alpha \in \mathcal{A}$, there exists $\hat{\varphi}_\alpha^{\hat{S}} \in C^m(\mathbb{R}^n)$, with

(a) $\|\partial^m \hat{\varphi}_\alpha^{\hat{S}}\|_{C^0(\mathbb{R}^n)} \le C\bar{a} |\partial^{\phi(\alpha)} \hat{P}_\alpha(0)|$,

(b) $|\hat{\varphi}_\alpha^{\hat{S}}(\hat{x})| \le C\hat{\sigma}(\hat{x})$ for all $\hat{x} \in \hat{S}$,

and (c) $J_0(\hat{\varphi}_\alpha^{\hat{S}}) = \hat{P}_\alpha$.

Similarly, let $\hat{S} \subset \hat{E}$ be given, with $\#(\hat{S}) \le k^\#$. Set $S = T(\hat{S}) \subset E$, and let F^S be as in (SL4). Then define $\hat{F}^{\hat{S}} = F^S \circ T$. For $\beta = (\beta_1, \dots, \beta_n)$ with $|\beta| \le m$, we have

$$\begin{aligned}\|\partial^\beta \hat{F}^{\hat{S}}\|_{C^0(\mathbb{R}^n)} &= \lambda_1^{\beta_1} \cdots \lambda_n^{\beta_n} \|\partial^\beta F^S\|_{C^0(\mathbb{R}^n)} \le C\lambda_1^{\beta_1} \cdots \lambda_n^{\beta_n} \quad \text{(by (SL4)(a))}\\ &\le C \quad \text{(since each } \lambda_j \le 1 \text{, by (7)).}\end{aligned}$$

Also, for $\hat{x} \in \hat{S}$, we have $|\hat{F}^{\hat{S}}(\hat{x}) - \hat{f}(\hat{x})| \le C\hat{\sigma}(\hat{x})$, thanks to (SL4)(b) and (4). Thus,

(13) Given $\hat{S} \subset \hat{E}$ with $\#(\hat{S}) \le k^\#$, there exists $\hat{F}^{\hat{S}} \in C^m(\mathbb{R}^n)$, with $\|\hat{F}^{\hat{S}}\|_{C^m(\mathbb{R}^n)} \le C$, and $|\hat{F}^{\hat{S}}(\hat{x}) - \hat{f}(\hat{x})| \le C\hat{\sigma}(\hat{x})$ on $\hat{S}$.

Now define

$$\bar{\mathcal{A}} = \phi(\mathcal{A}), \tag{14}$$

and let $\psi : \bar{\mathcal{A}} \to \mathcal{A}$ satisfy

$$\phi(\psi(\bar{\alpha})) = \bar{\alpha} \quad \text{for} \quad \bar{\alpha} \in \bar{\mathcal{A}}. \tag{15}$$

Note that

$$\bar{\mathcal{A}} \leq \mathcal{A}, \tag{16}$$

by (8), (9), (14), and Lemma 3.3. For $\bar{\alpha} \in \bar{\mathcal{A}}$, define

$$\tilde{P}_{\bar{\alpha}} = \hat{P}_{\psi(\bar{\alpha})} / (\partial^{\bar{\alpha}} \hat{P}_{\psi(\bar{\alpha})}(0)). \tag{17}$$

We check that the denominator in (17) is non–zero. In fact, (10) shows that $|\partial^\beta \hat{P}_\alpha(0)| \leq |\partial^{\phi(\alpha)} \ \hat{P}_\alpha(0)|$ for any $\alpha \in \mathcal{A}$, $\beta \in \mathcal{M}$. Taking $\beta = \alpha = \psi(\bar{\alpha})$, and recalling that $\partial^\alpha P_\alpha(0) = 1$ by (SL1), we see that

$$|\partial^{\bar{\alpha}} \hat{P}_{\psi(\bar{\alpha})}(0)| = |\partial^{\phi(\alpha)} \hat{P}_\alpha(0)| \geq |\partial^\alpha \hat{P}_\alpha(0)| = \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n} |\partial^\alpha P_\alpha(0)| = \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n}.$$

Hence,

$$|\partial^{\bar{\alpha}} \hat{P}_{\psi(\bar{\alpha})}(0)| \geq c'(\bar{a}) \quad \text{for all } \ \bar{\alpha} \in \bar{\mathcal{A}} \tag{18}$$

thanks to (7). In particular, $\partial^{\bar{\alpha}} \hat{P}_{\psi(\bar{\alpha})}(0) \neq 0$.

We derive the basic properties of the $\tilde{P}_{\bar{\alpha}}$. From (10), with $\alpha = \psi(\bar{\alpha})$, we see that $|\partial^\beta \hat{P}_{\psi(\bar{\alpha})}(0)| \leq \bar{a} \ |\partial^{\bar{\alpha}} \hat{P}_{\psi(\bar{\alpha})}(0)|$ for $\bar{\alpha} \in \bar{\mathcal{A}}$, $\beta \neq \bar{\alpha}$, $\beta \in \mathcal{M}$. Hence, (17) gives

$$|\partial^\beta \tilde{P}_{\bar{\alpha}}(0) - \delta_{\beta\bar{\alpha}}| \leq \bar{a} \quad \text{for all } \ \bar{\alpha} \in \bar{\mathcal{A}}, \ \beta \in \mathcal{M}. \tag{19}$$

Also, from (12), (17), (18), we see that

(20) Given $\hat{S} \subset \hat{E}$ with $\#(\hat{S}) \leq k^\#$, and given $\bar{\alpha} \in \bar{\mathcal{A}}$, there exists $\tilde{\varphi}_{\bar{\alpha}}^{\hat{S}} \in C^m(\mathbb{R}^n)$, with

- (a) $\|\partial^m \tilde{\varphi}_{\bar{\alpha}}^{\hat{S}}\|_{C^0(\mathbb{R}^n)} \leq C\bar{a}$,
- (b) $|\tilde{\varphi}_{\bar{\alpha}}^{\hat{S}}(\hat{x})| \leq C(\bar{a})\hat{\sigma}(\hat{x})$ on $\hat{S}$,

and

- (c) $J_0(\tilde{\varphi}_{\bar{\alpha}}^{\hat{S}}) = \tilde{P}_{\bar{\alpha}}$.

(In fact, we just apply (12), with $\alpha = \psi(\bar{\alpha}) \in \mathcal{A}$, and put $\tilde{\varphi}_{\bar{\alpha}}^{\hat{S}} = \hat{\varphi}_\alpha^{\hat{S}} \ / (\partial^{\bar{\alpha}} \hat{P}_\alpha(0))$.)

Thanks to (19), the matrix $(\partial^\beta \tilde{P}_{\bar{\alpha}}(0))_{\beta, \bar{\alpha} \in \bar{\mathcal{A}}}$ has an inverse $M_{\alpha'\bar{\alpha}}$, with

$$|M_{\alpha'\bar{\alpha}} - \delta_{\alpha'\bar{\alpha}}| \leq C'\bar{a} \qquad \text{for all } \ \alpha', \bar{\alpha} \in \bar{\mathcal{A}}. \tag{21}$$

By definition, we have

$$\sum_{\alpha' \in \bar{\mathcal{A}}} \partial^\beta \tilde{P}_{\alpha'}(0) \cdot M_{\alpha'\bar{\alpha}} = \delta_{\beta\bar{\alpha}} \qquad \text{for all } \beta, \bar{\alpha} \in \bar{\mathcal{A}}. \tag{22}$$

Now define

$$\bar{P}_{\bar{\alpha}} = \sum_{\alpha' \in \bar{\mathcal{A}}} \tilde{P}_{\alpha'} M_{\alpha'\bar{\alpha}} \quad \text{for all } \bar{\alpha} \in \bar{\mathcal{A}}. \tag{23}$$

Given $\hat{S} \subset \hat{E}$ with $\#(S) \le k^\#$, we let $\tilde{\varphi}^{\hat{S}}_{\bar{\alpha}}$ be as in (20) (all $\bar{\alpha} \in \bar{\mathcal{A}}$), and define

$$\bar{\varphi}^{\hat{S}}_{\bar{\alpha}} = \sum_{\alpha' \in \bar{\mathcal{A}}} \tilde{\varphi}^{\hat{S}}_{\alpha'} M_{\alpha'\bar{\alpha}} \qquad \text{for all } \bar{\alpha} \in \bar{\mathcal{A}}. \tag{24}$$

From (22) and (23), we have

$$\partial^\beta \bar{P}_{\bar{\alpha}}(0) = \delta_{\beta\bar{\alpha}} \qquad \text{for all } \beta, \bar{\alpha} \in \bar{\mathcal{A}}. \tag{25}$$

Also, (19), (21), (23) and (2) imply

$$|\partial^\beta \bar{P}_{\bar{\alpha}}(0) - \delta_{\beta\bar{\alpha}}| \le C''\bar{a} \qquad \text{for all } \bar{\alpha} \in \bar{\mathcal{A}}, \beta \in \mathcal{M}. \tag{26}$$

Given $\hat{S} \subset \hat{E}$ with $\#(\hat{S}) \le k^\#$, and given $\bar{\alpha} \in \bar{\mathcal{A}}$, we conclude from (20(a)), (21), (24), and (2), that

$$\|\partial^m \bar{\varphi}^{\hat{S}}_{\bar{\alpha}}\|_{C^0(\mathbb{R}^n)} \le C'''\bar{a}.$$

From (20(b)), (21), (24), and (2), we obtain

$$|\bar{\varphi}^{\hat{S}}_{\bar{\alpha}}(\hat{x})| \le C'(\bar{a}) \cdot \hat{\sigma}(\hat{x}) \text{ on } \hat{S}.$$

Comparing (23) with (24), and recalling (20(c)), we obtain

$$J_0(\bar{\varphi}^{\hat{S}}_{\bar{\alpha}}) = \bar{P}_{\bar{\alpha}}.$$

Thus,

(27) Given $\bar{\alpha} \in \bar{\mathcal{A}}$ and $\hat{S} \subset \hat{E}$ with $\#(\hat{S}) \le k^\#$, there exists $\bar{\varphi}^{\hat{S}}_{\bar{\alpha}} \in C^m(\mathbb{R}^n)$, with

(a) $\|\partial^m \bar{\varphi}^{\hat{S}}_{\bar{\alpha}}\|_{C^0(\mathbb{R}^n)} \le C'''\bar{a}$,

(b) $|\bar{\varphi}^{\hat{S}}_{\bar{\alpha}}(\hat{x})| \le C'(\bar{a}) \cdot \hat{\sigma}(\hat{x})$ on $\hat{S}$,

and

(c) $J_0(\bar{\varphi}^{\hat{S}}_{\alpha}) = \bar{P}_{\bar{\alpha}}$.

We prepare to apply the WEAK MAIN LEMMA for $\bar{\mathcal{A}}$ to the set $\hat{E}$, the functions $\hat{f}$, $\hat{\sigma}$, the set $\bar{\mathcal{A}}$ of multi–indices, the base point $y^0 = 0$, and the family of polynomials $(\bar{P}_{\bar{\alpha}})_{\bar{\alpha}\in\bar{\mathcal{A}}}$. We will check that the hypotheses of the WEAK MAIN LEMMA hold, and that the constant called C in hypotheses (WL3), (WL4) is weakly controlled. In fact, (WL1) is just (25); (WL2) is immediate from (2) and (26), since a_0 is a controlled constant; (WL3) (with a weakly controlled constant) is immediate from (2) and (27) since a_0 is controlled; and (WL4) (with a controlled constant) is immediate from (13). Thus, the hypotheses of the WEAK MAIN LEMMA are satisfied. Since we are assuming the WEAK MAIN LEMMA for $\bar{\mathcal{A}} \le \mathcal{A}$, and since we know that $\bar{\mathcal{A}} \le \mathcal{A}$ (see (16)), we conclude that there exists $\hat{F} \in C^m(\mathbb{R}^n)$, with

$$\|\hat{F}\|_{C^m(\mathbb{R}^n)} \le C_1(\bar{a}), \text{ and} \tag{28}$$

$$|\hat{F}(\hat{x}) - \hat{f}(\hat{x})| \le C_1(\bar{a}) \cdot \hat{\sigma}(\hat{x}) \text{ for all } \hat{x} \in \hat{E} \cap B(0, c_1(\bar{a})). \tag{29}$$

Now define $F = \hat{F} \circ T^{-1}$ on $\mathbb{R}^n$. Since

$$\|\partial^\beta F\|_{C^0(\mathbb{R}^n)} = \lambda_1^{-\beta_1} \cdots \lambda_n^{-\beta_n} \|\partial^\beta \hat{F}\|_{C^0(\mathbb{R}^n)} \text{ for } \beta = (\beta_1, \dots, \beta_n),$$

estimates (28) and (7) imply

$$\|F\|_{C^m(\mathbb{R}^n)} \le C_2(\bar{a}). \tag{30}$$

Also from (7), we learn that $x \in B(0, c_2(\bar{a}))$ for small enough $c_2(\bar{a})$ implies $T^{-1}x \in B(0, c_1(\bar{a}))$, with $c_1(\bar{a})$ as in (29). Hence, (4) and (29) imply

$$|F(x) - f(x)| \le C_2(\bar{a}) \cdot \sigma(x) \text{ for all } x \in E \cap B(0, c_2(\bar{a})). \tag{31}$$

Finally, let us fix $\bar{a}$ to be a controlled constant, small enough to satisfy (2). Then the constants $c_2(\bar{a})$ and $C_2(\bar{a})$ are determined entirely by C, m, n in (SL1, ..., 4). Hence, (30) and (31) are the conclusions of the STRONG MAIN LEMMA for $\mathcal{A}$.

Thus, the STRONG MAIN LEMMA holds for $\mathcal{A}$. The proof of Lemma 5.3 is complete. ■

18. Proofs of the Theorems

We have now proven Lemmas 5.1, 5.2, and 5.3. As explained in section 5, these lemmas imply the WEAK and STRONG MAIN LEMMA for all $\mathcal{A} \subset \mathcal{M}$, as well as the Local Theorem 1. In this section, we show that the Local Theorem 1 implies Theorems 1, 2, 3, which in turn trivially imply Theorems A, B, C. The first step is as follows.

Lemma 18.1. *Let $m, n \geq 1$ be given. Then there exist constants $k^{\#}, C_1, c_0$, depending only on m and n, for which the following holds: Suppose we are given a finite set $E \subset \mathbb{R}^n$, and functions $f : E \to \mathbb{R}$ and $\sigma : E \to [0, \infty)$.*

Assume that, for any $S \subset E$ with $\#(S) \leq k^{\#}$, there exists $F^S \in C^m(\mathbb{R}^n)$, with

$$\|F^S\|_{C^m(\mathbb{R}^n)} \leq 1 \text{ and } |F^S(x) - f(x)| \leq \sigma(x) \text{ on } S.$$

Then, for each $y^0 \in \mathbb{R}^n$, there exists $F \in C^m(\mathbb{R}^n)$, with

$$\|F\|_{C^m(\mathbb{R}^n)} \leq C_1 \text{ and } |F(x) - f(x)| \leq C_1 \sigma(x) \text{ on } E \cap B(y^0, c_0).$$

(This result differs from the Local Theorem 1 of section 5 in that we assume merely that $\sigma : E \to [0, \infty)$, not $\sigma : E \to (0, \infty)$.)

Proof. Let $k^{\#}, A, c'$ be as in the Local Theorem 1, and let E, f, σ satisfy the hypotheses of Lemma 18.1. Let $y^0 \in \mathbb{R}^n$ be given. For each $\varepsilon > 0$, set $\sigma_\varepsilon(x) = \sigma(x) + \varepsilon$ for all $x \in E$. Then $\sigma_\varepsilon : E \to (0, \infty)$, and one checks trivially that E, f, σ_ε satisfy the hypotheses of the Local Theorem 1. Hence, for each $\varepsilon > 0$, there exists $F_\varepsilon \in C^m(\mathbb{R}^n)$, with $\|F_\varepsilon\|_{C^m(\mathbb{R}^n)} \leq A$, and $|F_\varepsilon(x) - f(x)| \leq A\sigma(x) + A\varepsilon$ for all $x \in E \cap B(y^0, c')$.

For $x \in E \cap B(y^0, c')$, define

$$g_\varepsilon(x) = \left\{ \begin{array}{c} (F_\varepsilon(x) - f(x) - A\sigma(x)) \text{ if } F_\varepsilon(x) > f(x) + A\sigma(x) \\ (F_\varepsilon(x) - f(x) + A\sigma(x)) \text{ if } F_\varepsilon(x) < f(x) - A\sigma(x) \\ 0 \quad \text{otherwise} \end{array} \right\}.$$

For $x \in E \setminus B(y^0, c')$ set $g_\varepsilon(x) = 0$. Then we have $|g_\varepsilon(x)| \leq A\varepsilon$ for all $x \in E$, and $|F_\varepsilon(x) - f(x) - g_\varepsilon(x)| \leq A\sigma(x)$ for all $x \subset E \cap B(y^0, c')$. On the other hand, since E is finite, there exists a constant $\Gamma(E)$ with the following property.

Given a function $g : E \to \mathbb{R}^n$, there exists $G \in C^m(\mathbb{R}^n)$, with $\|G\|_{C^m(\mathbb{R}^n)} \leq \Gamma(E) \cdot \max_{x \in E} |g(x)|$, and $G = g$ on E.

Hence, there exists $G_\varepsilon \in C^m(\mathbb{R}^n)$, with $\|G_\varepsilon\|_{C^m(\mathbb{R}^n)} \leq \Gamma(E) \cdot A\varepsilon$, and $G_\varepsilon = g_\varepsilon$ on E. Taking $\varepsilon < 1/\Gamma(E)$, and setting $F = F_\varepsilon - G_\varepsilon$, we find that $\|F\|_{C^m(\mathbb{R}^n)} \leq A + \Gamma(E) \cdot A\varepsilon \leq 2A$, and

$$|F(x) - f(x)| = |F_\varepsilon(x) - f(x) - g_\varepsilon(x)| \leq A\sigma(x) \text{ on } E \cap B(y^0, c').$$

Thus, Lemma 18.1 holds, with $C_1 = 2A$ and $c_0 = c'$. ∎

Next, we pass from finite E to arbitrary E, and from C^m to $C^{m-1,1}$.

Lemma 18.2. *Let $m, n \geq 1$ be given. Then there exist constants $k^{\#}, C_2, c_2$, depending only on m and n, for which the following holds.*

Suppose we are given an arbitrary set $E \subset \mathbb{R}^n$ and functions $f : E \to \mathbb{R}$ and $\sigma : E \to [0, \infty)$. Let $y^0 \in \mathbb{R}^n$. Assume that, for any $S \subset E$ with $\#(S) \leq k^{\#}$, there exists $F^S \in C^{m-1,1}(\mathbb{R}^n)$, with

$$\|F^S\|_{C^{m-1,1}(\mathbb{R}^n)} \leq 1, \quad \textit{and} \quad |F^S(x) - f(x)| \leq \sigma(x) \;\; \textit{on} \;\; S. \tag{1}$$

Then there exists $F \in C^{m-1,1}(\mathbb{R}^n)$, with

$$\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \leq C_2, \quad \textit{and} \quad |F(x) - f(x)| \leq C_2\sigma(x) \;\; \textit{on} \;\; E \cap B(y^0, c_2).$$

Proof. Let $k^{\#}$ be as in Lemma 18.1, and let $S \subset E$ be given, with $\#(S) \leq k^{\#}$. Then there exists a constant $\Gamma(S)$, for which the following holds:

(2) Given $g : S \to \mathbb{R}$, there exists $G \in C^m(\mathbb{R}^n)$, with $\|G\|_{C^m(\mathbb{R}^n)} \leq \Gamma(S) \cdot \max_{x \in S} |g(x)|$, and $G = g$ on S.

Let $F^S \in C^{m-1,1}(\mathbb{R}^n)$ be as in (1), and let $\varepsilon = 1/\Gamma(S)$. By convolving F^S with an approximate identity, we obtain a function $F^S_\varepsilon \in C^m(\mathbb{R}^n)$, with

$$\|F^S_\varepsilon\|_{C^m(\mathbb{R}^n)} \leq \widetilde{C}\|F^S\|_{C^{m-1,1}(\mathbb{R}^n)} \quad \text{and} \quad \|F^S_\varepsilon - F^S\|_{C^0(\mathbb{R}^n)} < \varepsilon. \tag{3}$$

(Here, $\widetilde{C}$ depends only on m and n.) From (1) and (3), we obtain

$$\|F^S_\varepsilon\|_{C^m(\mathbb{R}^n)} \leq \widetilde{C} \quad \text{and} \quad |F^S_\varepsilon(x) - f(x)| \leq \sigma(x) + \varepsilon \;\; \text{on} \;\; S. \tag{4}$$

Now define g^S_ε on S by setting

$$g^S_\varepsilon(x) = \left\{ \begin{array}{ll} F^S_\varepsilon(x) - f(x) - \sigma(x) & \text{if } \; F^S_\varepsilon(x) - f(x) > \sigma(x) \\ F^S_\varepsilon(x) - f(x) + \sigma(x) & \text{if } \; F^S_\varepsilon(x) - f(x) < -\sigma(x) \\ \qquad 0 \quad \text{otherwise} & \end{array} \right\}$$

Thus,

$$\max_{x \in S} |g^S_\varepsilon(x)| \leq \varepsilon, \quad \text{and} \quad |F^S_\varepsilon(x) - f(x) - g^S_\varepsilon(x)| \leq \sigma(x) \;\; \text{on} \;\; S,$$

thanks to (4). Applying (2) to g^S_ε, we obtain a function G^S_ε, with

$$\|G^S_\varepsilon\|_{C^m(\mathbb{R}^n)} \leq \Gamma(S) \cdot \varepsilon = 1, \quad \text{and} \quad |F^S_\varepsilon - G^S_\varepsilon - f| \leq \sigma \;\; \text{on} \;\; S.$$

Setting $\tilde{F}^S = F^S_\varepsilon - G^S_\varepsilon$, we learn the following:

(5) Given $S \subset E$ with $\#(S) \le k^{\#}$, there exists $\tilde{F}^S \in C^m(\mathbb{R}^n)$, with $\|\tilde{F}^S\|_{C^m(\mathbb{R}^n)} \le C'$, and $|\tilde{F}^S(x) - f(x)| \le \sigma(x)$ on S.

Here, C' depends only on m and n. In view of (5), we may apply Lemma 18.1 to any finite subset $E_1 \subset E$. Thus, we obtain the following result.

(6) Let E_1 be any finite subset of E. Then there exists $F_{E_1} \in C^m(\mathbb{R}^n)$, with $\|F_{E_1}\|_{C^m(\mathbb{R}^n)} \le C''$, and $|F_{E_1}(x) - f(x)| \le C''\sigma(x)$ on $E_1 \cap B(y^0, c_0)$.

Here, C'' and c_0 depend only on m and n. Let $B = B(y^0, c_0)$, and let

(7) $\mathcal{B} = \{F \in C^{m-1,1}(B) : \|F\|_{C^{m-1,1}(B)} \le C'''\}$, equipped with the $C^{m-1}(B)$–topology.

In (7), we take C''' to be a large enough constant determined by m and n. Hence, if we define

$$\mathcal{B}(x) = \{F \in \mathcal{B} : |F(x) - f(x)| \le C''\sigma(x)\} \quad \text{for each} \quad x \in E \cap B, \tag{8}$$

then (6) shows that $\bigcap_{x \in E_1} \mathcal{B}(x)$ is non–empty, for any finite subset $E_1 \subset E \cap B$.

On the other hand, each $\mathcal{B}(x)$ is a closed subset of $\mathcal{B}$, and $\mathcal{B}$ is compact, by Ascoli's theorem. Therefore, the intersection of $\mathcal{B}(x)$ over all $x \in E \cap B$ is non–empty. Thus, there exists $\tilde{F} \in C^{m-1,1}(B)$, with

$$\|\tilde{F}\|_{C^{m-1,1}(B)} \le C''', \quad \text{and} \quad |\tilde{F}(x) - f(x)| \le C''\sigma(x) \quad \text{for all} \quad x \in E \cap B. \tag{9}$$

The function $\tilde{F}$ is defined only on $B = B(y^0, c_0)$. Therefore, we introduce a cutoff function θ on $\mathbb{R}^n$, satisfying

$$\|\theta\|_{C^m(\mathbb{R}^n)} \le C^{\#}, 0 \le \theta \le 1 \ \text{ on } \ \mathbb{R}^n, \ \theta = 1 \ \text{ on } \ B(y^0, \tfrac{1}{2}c_0), \ \ \text{supp } \theta \subset B, \tag{10}$$

with $C^{\#}$ determined by m and n. Defining $F = \theta\tilde{F} \in C^{m-1,1}(\mathbb{R}^n)$, we learn from (9) and (10) that

$$\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \le C_2, \quad \text{and} \quad |F(x) - f(x)| \le C_2\sigma(x) \ \text{ on } \ E \cap B(y^0, \tfrac{1}{2}c_0), \tag{11}$$

with C_2 and c_0 depending only on m and n. However, (11) is the conclusion of Lemma 18.2. ∎

Proof of Theorem 1. Let E, f, σ be as in the hypotheses of Theorem 1, and let C_1, c_0 be as in Lemma 18.1. We introduce a partition of unity.

$$1 = \sum_\nu \theta_\nu \ \text{ on } \ \mathbb{R}^n, \ \text{ with} \tag{12}$$

$$(13) \qquad 0 \le \theta_\nu \le 1, \;\; \operatorname{supp} \theta_\nu \subset B(y_\nu, c_0), \; \|\theta_\nu\|_{C^m(\mathbb{R}^n)} \le C,$$

and with

(14) any given $x \in \mathbb{R}^n$ belonging to at most C of the balls $B(y_\nu, c_0)$.

In (13) and (14), C denotes a constant depending only on m and n. Applying Lemma 18.1, we obtain, for each ν, a function $F_\nu \in C^m(\mathbb{R}^n)$, with

$$(15) \qquad \|F_\nu\|_{C^m(\mathbb{R}^n)} \le C_1, \;\; \text{and} \;\; |F_\nu(x) - f(x)| \le C_1 \sigma(x) \;\; \text{on } E \cap B(y_\nu, c_0).$$

Define $F = \sum_\nu \theta_\nu F_\nu$. From (12),, ... , (15), we obtain

$$(16) \qquad \|F\|_{C^m(\mathbb{R}^n)} \le C', \;\; \text{and}$$

$$(17) \qquad |F(x) - f(x)| = |\sum_\nu \theta_\nu(x)[F_\nu(x) - f(x)]| \le \sum_\nu \theta_\nu(x)|F_\nu(x) - f(x)|$$
$$\le \sum_\nu \theta_\nu(x) \cdot C_1 \sigma(x) = C_1 \sigma(x) \;\; \text{on} \;\; E$$

(The constant C' in (16) depends only on m and n.)

The proof of Theorem 1 is complete. ■

Proof of Theorem 2. Let E, f, σ be as in the hypotheses of Theorem 2, and let C_2, c_2 be as in Lemma 18.2. We introduce a partition of unity

$$(18) \qquad 1 = \sum_\nu \theta_\nu \;\; \text{on} \;\; \mathbb{R}^n, \;\; \text{with}$$

$$(19) \qquad 0 \le \theta_\nu \le 1, \; \operatorname{supp} \theta_\nu \subset B(y_\nu, c_2), \|\theta_\nu\|_{C^m(\mathbb{R}^n)} \le C, \;\; \text{and with}$$

(20) any given $x \in \mathbb{R}^n$ belonging to at most C of the balls $B(y_\nu, c_2)$.

In (19) and (20), C denotes a constant depending only on m and n. Applying Lemma 18.2, we obtain, for each ν, a function $F_\nu \in C^{m-1,1}(\mathbb{R}^n)$ with

$$(21) \qquad \|F_\nu\|_{C^{m-1,1}(\mathbb{R}^n)} \le C_2, \;\; \text{and} \;\; |F_\nu(x) - f(x)| \le C_2 \sigma(x) \;\; \text{on} \;\; E \cap B(y_\nu, c_2).$$

Define $F = \sum_\nu \theta_\nu F_\nu$. From (18), ... , (21), we obtain

$$(22) \qquad \|F\|_{C^{m-1,1}(\mathbb{R}^n)} \le C', \;\; \text{and}$$

$$(23) \qquad |F(x) - f(x)| = \left|\sum_\nu \theta_\nu(x)[F_\nu(x) - f(x)]\right| \le \sum_\nu \theta_\nu(x)\big|F_\nu(x) - f(x)\big|$$
$$\le \sum_\nu \theta_\nu(x) \cdot C_2 \sigma(x) = C_2 \sigma(x) \;\; \text{on} \;\; E.$$

(The constant C' in (22) depends only on m and n.)

The proof of Theorem 2 is complete. $\blacksquare$

Proof of Theorem 3. Suppose we are given $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{R}^n$.

Assume that $\sup_{\vec{x}} \|f\|_{C^m(\vec{x})} < \infty$. Then, for any subset $S = \{x_1, \ldots, x_k\} \subset E$, with $k \leq k^{\#}$, we can assign polynomials $P_1^S, \ldots, P_k^S$ of degree at most $(m-1)$, satisfying $P_i^S(x_i) = f(x_i)$,

$$|\partial^\beta P_j^S(x_j)| \leq C \text{ and } |\partial^\beta (P_i^S - P_j^S)(x_j)| \leq C|x_i - x_j|^{m-|\beta|}$$

for $|\beta| \leq m-1, i, j = 1, \ldots, k$, with C independent of the $x_1, \ldots, x_k$.

Applying the Whitney extension theorem for $C^{m-1,1}$ (see [8, 9]) to $S, P_1^S, \ldots, P_k^S$, we conclude that there exists a function $F^S \in C^{m-1,1}(\mathbb{R}^n)$, with

$$J_{x_i}(F^S) = P_i^S \ (i = 1, \ldots, k), \text{ and } \|F^S\|_{C^{m-1,1}(\mathbb{R}^n)} \leq C',$$

with C' independent of the $x_1, \ldots, x_k$. In particular,

$$F^S = f \text{ on } S, \text{ and } \|F^S\|_{C^{m-1,1}(\mathbb{R}^n)} \leq C'. \tag{24}$$

We have achieved (24) for all $S \subset E$ with $\#(S) \leq k^{\#}$. Hence, Theorem 2, with $\sigma \equiv 0$, implies that there exists $F \in C^{m-1,1}(\mathbb{R}^n)$, with $F = f$ on E.

On the other hand, suppose we are given $E \subset \mathbb{R}^n$ and and $f : E \to \mathbb{R}$, and assume that f extends to a function $F \in C^{m-1,1}(\mathbb{R}^n)$. Then, for any subset $S = \{x_1, \ldots, x_k\} \subset E$, with $k \leq k^{\#}$, we may simply set $P_i = J_{x_i}(F)$ for $i = 1, \ldots, k$, and we have

$$P_i(x_i) = f(x_i), |\partial^\beta P_i(x_i)| \leq C, \ |\partial^\beta (P_i - P_j)(x_i)| \leq C|x_i - x_j|^{m-|\beta|} \tag{25}$$

for $|\beta| \leq m-1, i, j = 1, \ldots, k$, with C independent of $x_1, \ldots, x_k$.

Comparing (25) with the definition of $\|f\|_{C^m(\vec{x})}$, we conclude that $\|f\|_{C^m(\vec{x})} \leq C'$, with C' independent of $\vec{x}$.

Thus, f extends to a $C^{m-1,1}$ function on $\mathbb{R}^n$ if and only if $\sup_{\vec{x}} \|f\|_{C^m(\vec{x})} < \infty$.

The proof of Theorem 3 is complete. $\blacksquare$

There is an analogue of Theorem 3 without taking $\sigma \equiv 0$. Also, Theorems 1, 2, 3 and the standard Whitney extension theorem trivially imply Theorems A, B, C in the introduction. Details may be left to the reader.

19. A Bound for $k^\#$

Our proof of Theorems 1, 2, 3 gives an explicit (wasteful) bound for $k^\#$. In fact, when we start the main induction by proving Lemma 5.1, we take $k^\# = 1$. Every time we apply Lemma 5.2 for monotone $\mathcal{A}$, the constant $k^\#$ grows by a factor of $(D+1)^3$. (See equation (15.5)). When we apply Lemma 5.2 for non–monotone $\mathcal{A}$, and when we apply Lemma 5.3 for arbitrary $\mathcal{A}$, the constant $k^\#$ does not grow. Consequently, Theorems 1, 2, 3 hold, with

$$k^\# \leq [(D+1)^3]^N,$$

where N is the number of monotone subsets $\mathcal{A} \subset \mathcal{M}$. A trivial bound for N is $N \leq 2^D$, since D is the number of elements of $\mathcal{M}$. Thus,

$$k^\# \leq (D+1)^{3 \cdot 2^D}.$$

Recall that D is the number of multi–indices $(\beta_1, \dots, \beta_n)$, with $\beta_1 + \cdots + \beta_n \leq m-1$.

It would be interesting to determine the best possible values of $k^\#$ in Theorems 1, 2, 3.

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