Mass in

Kähler Geometry

Claude LeBrun
Stony Brook University

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Joint work with

Joint work with

Hans-Joachim Hein
University of Maryland

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e-print: arXiv:1507.08885 [math.DG]

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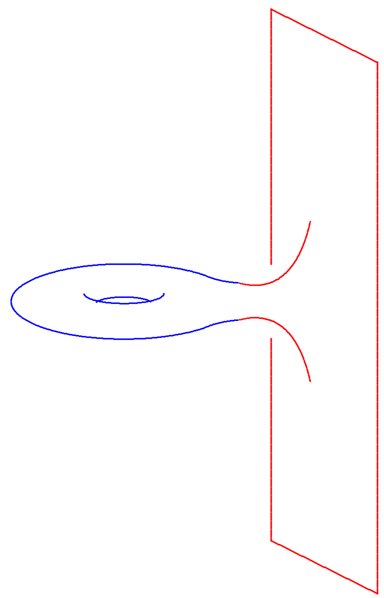
To appear in Comm. Math. Phys.

Definition. *A complete, non-compact Riemannian* n*-manifold* (M^n, g)



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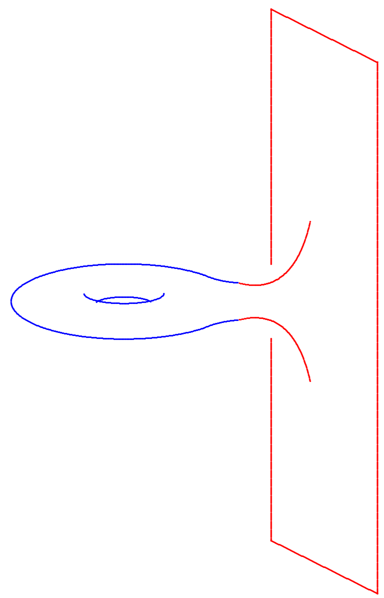

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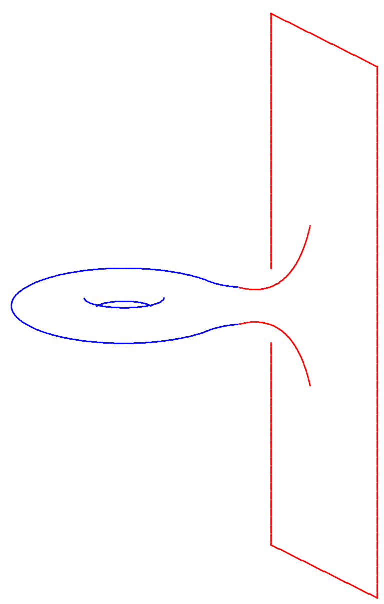

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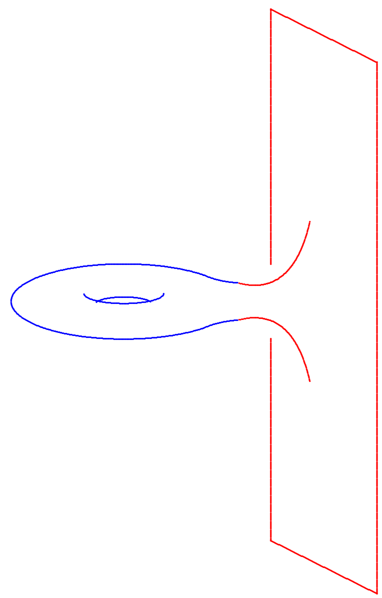

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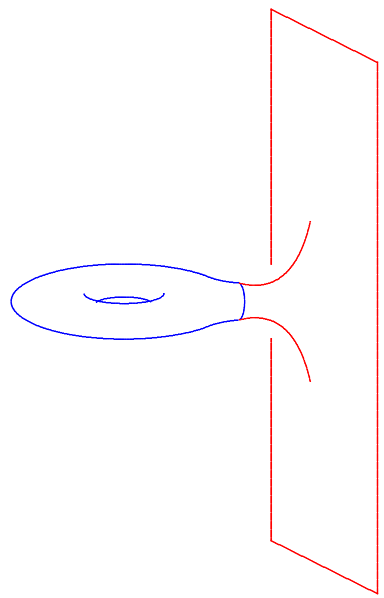

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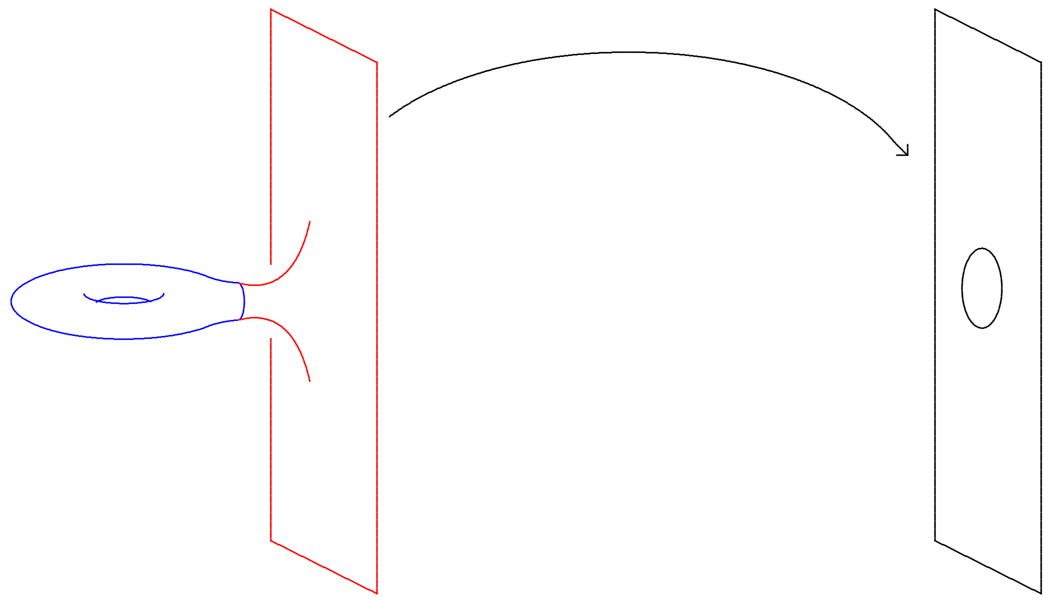

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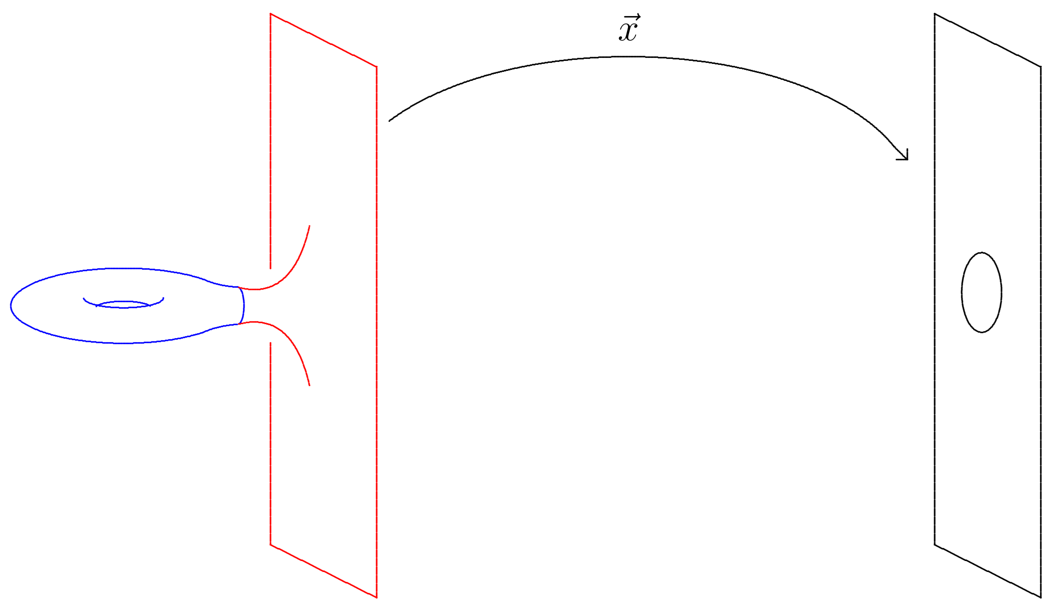

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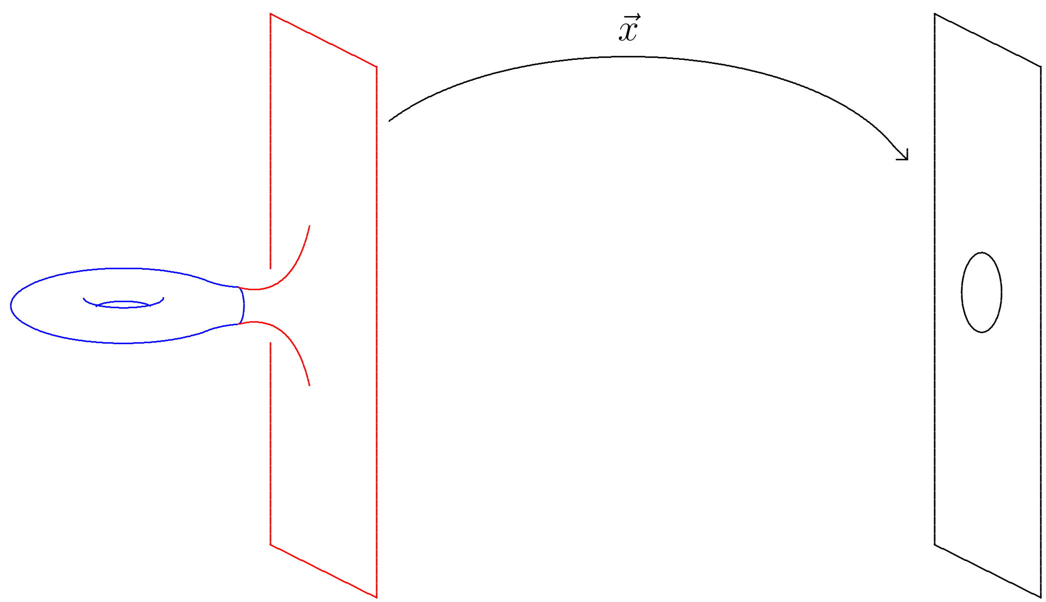

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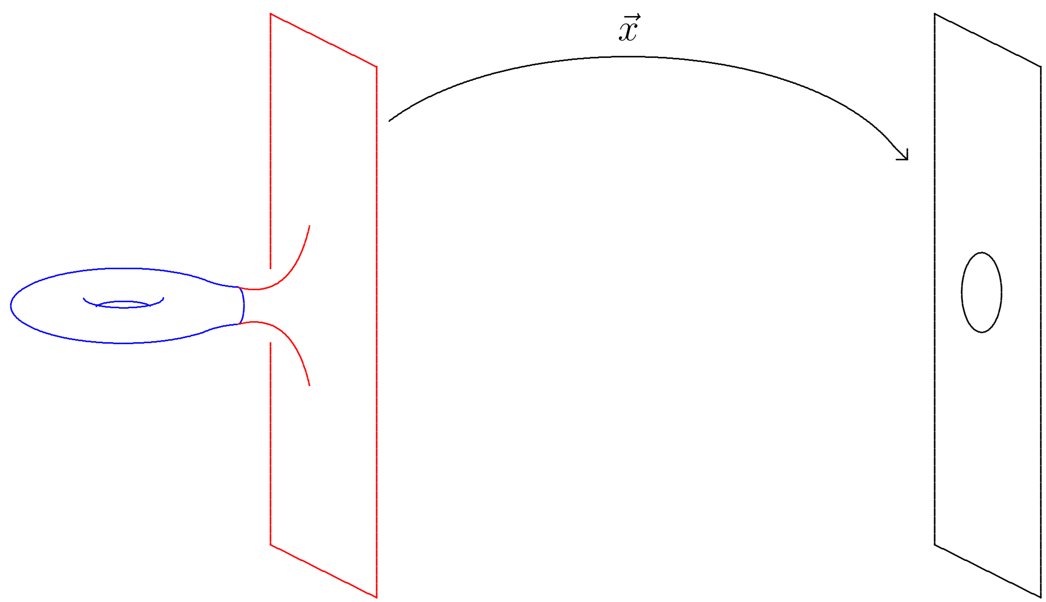

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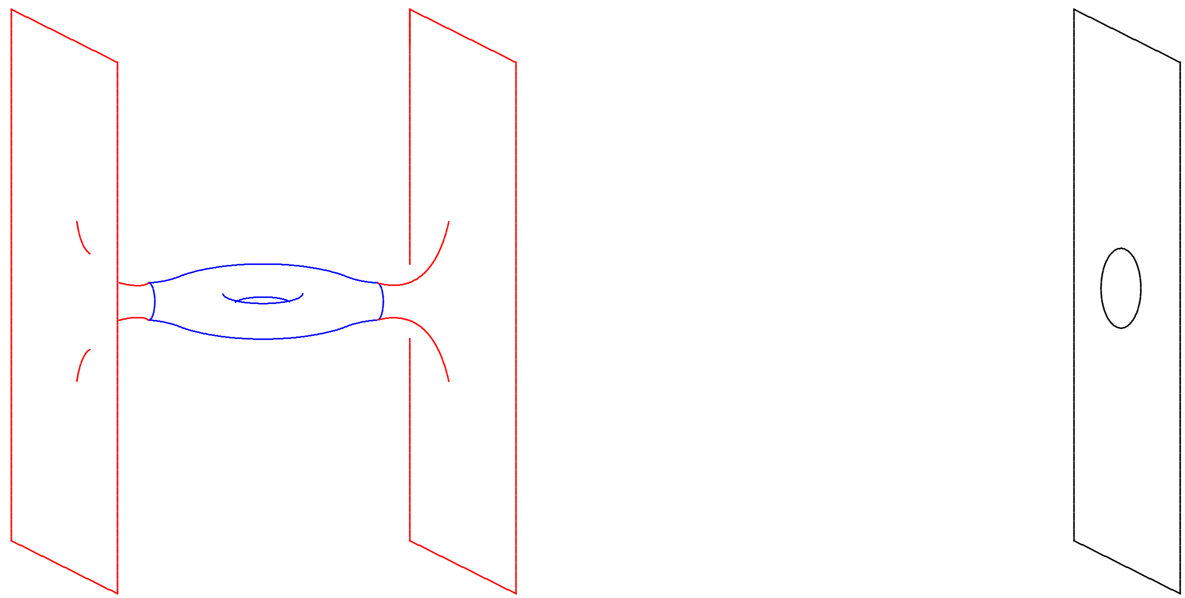

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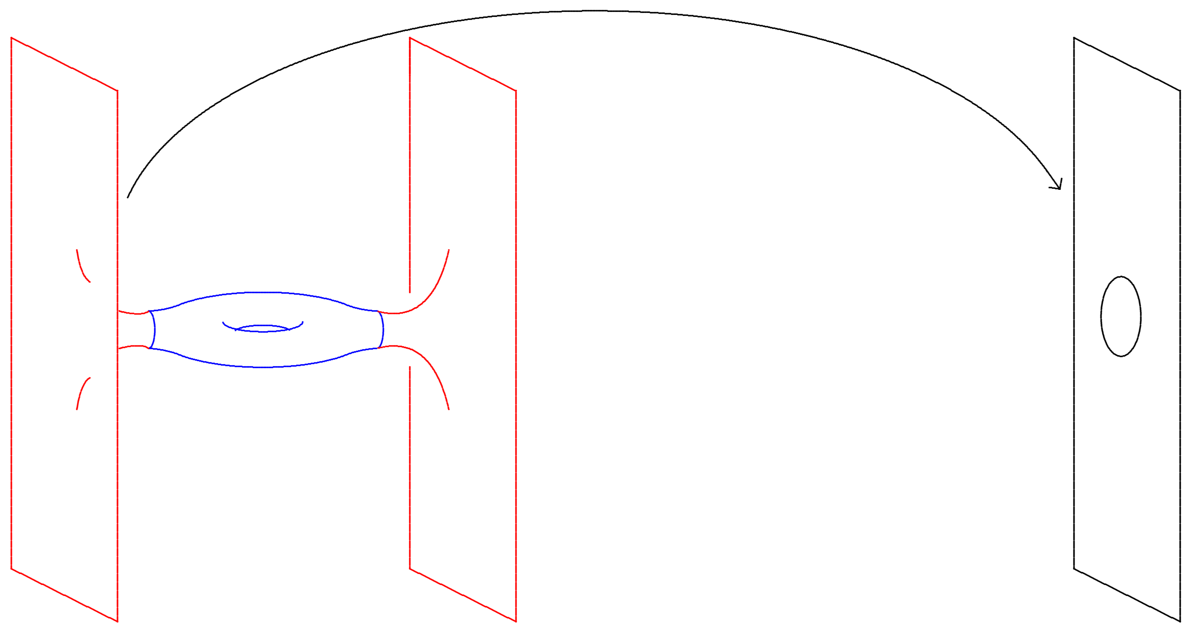

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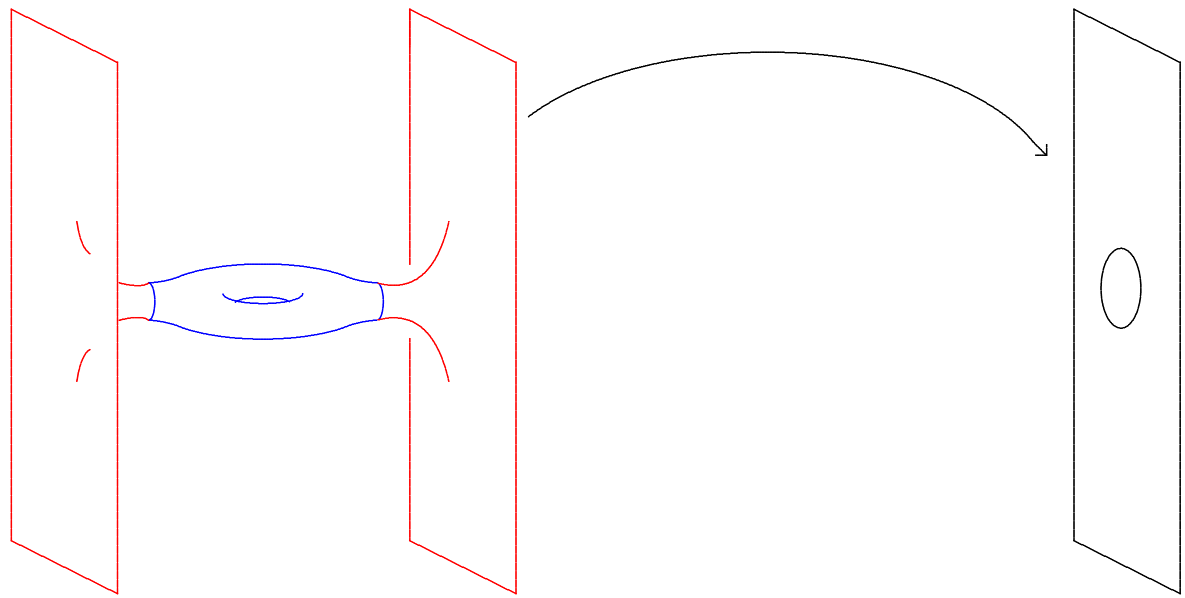

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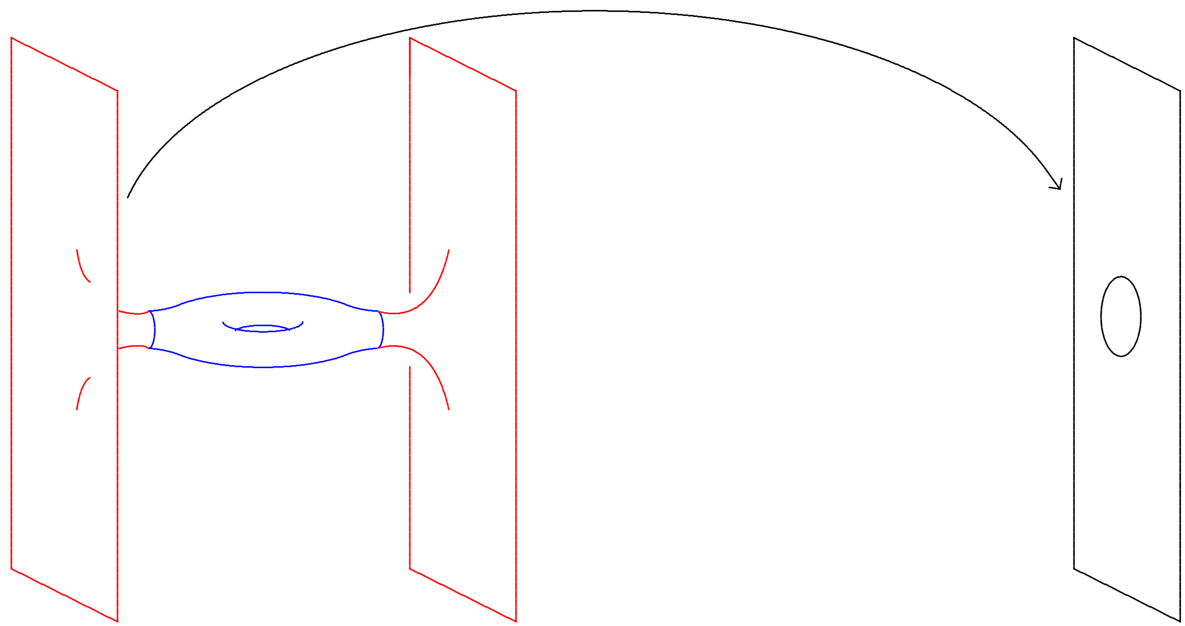

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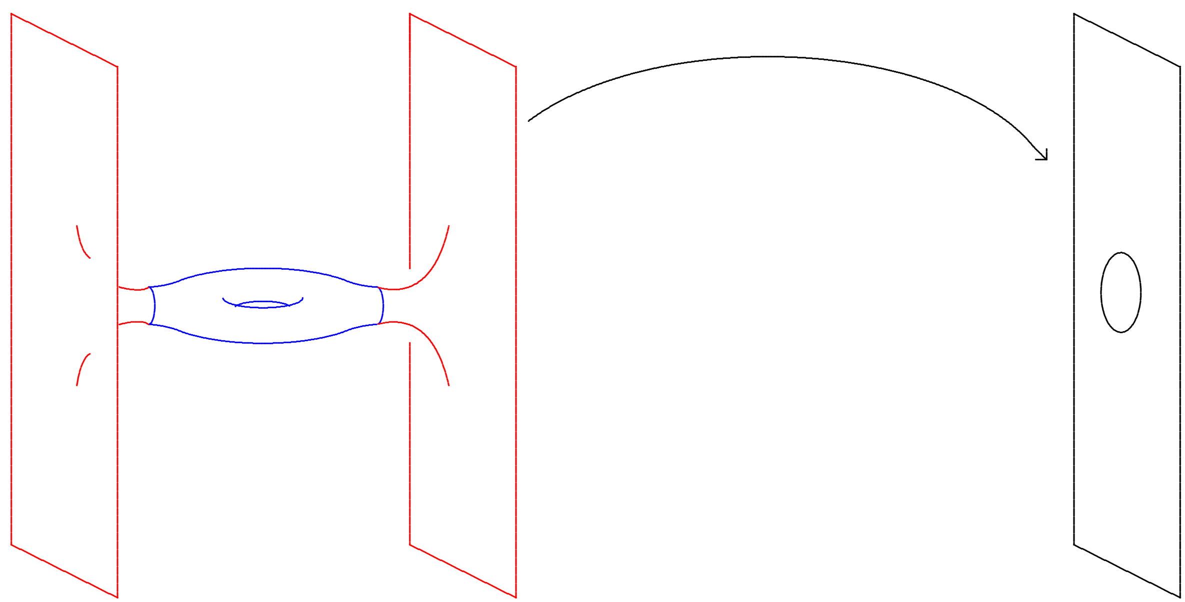

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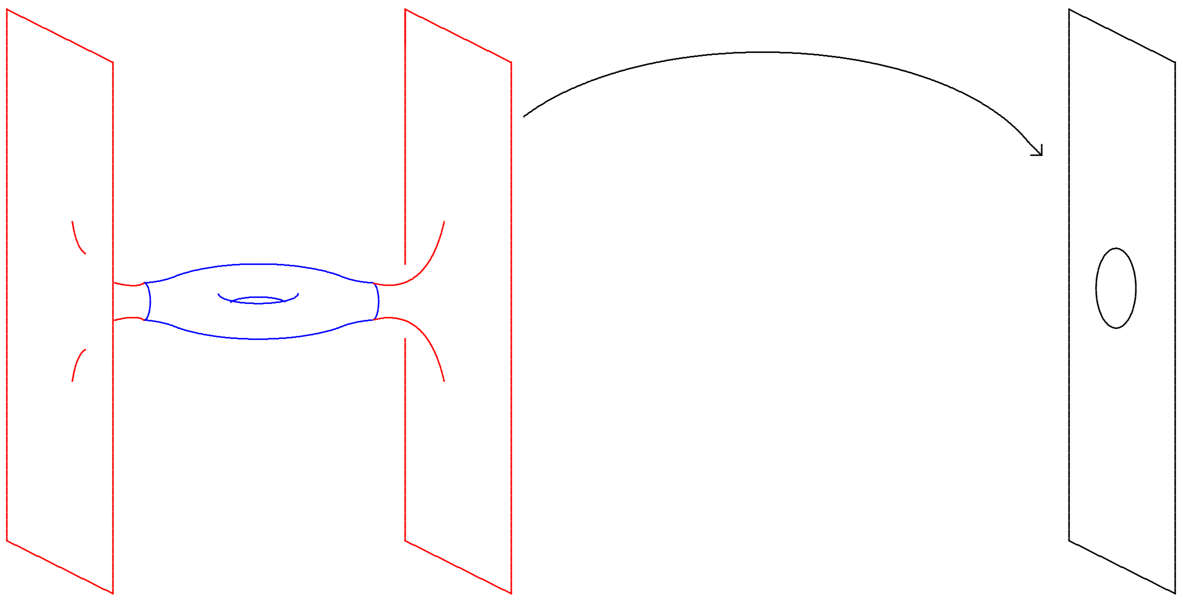

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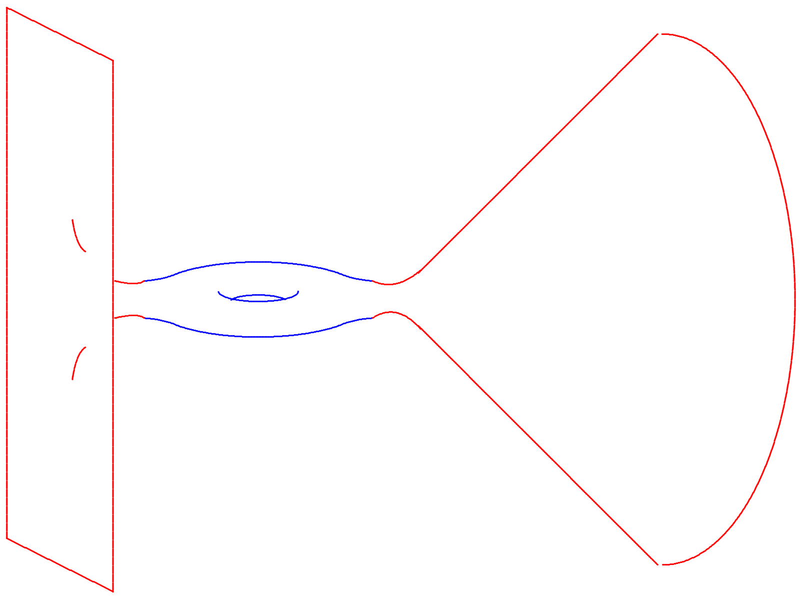

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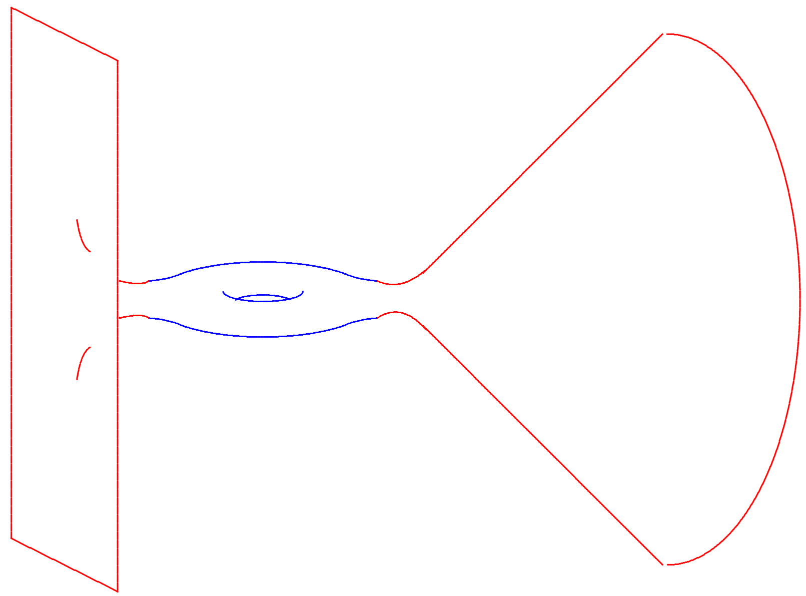

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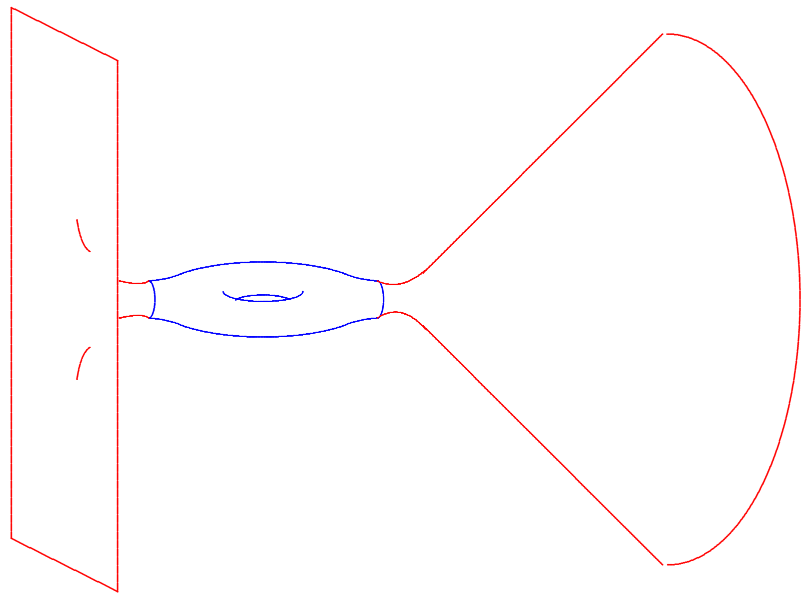

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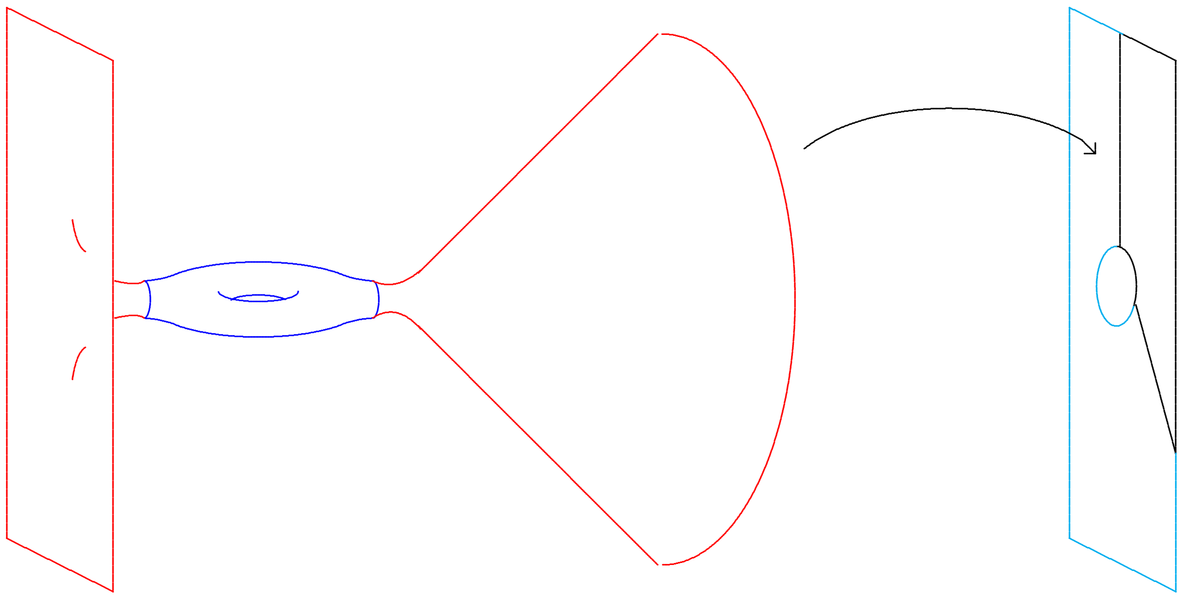

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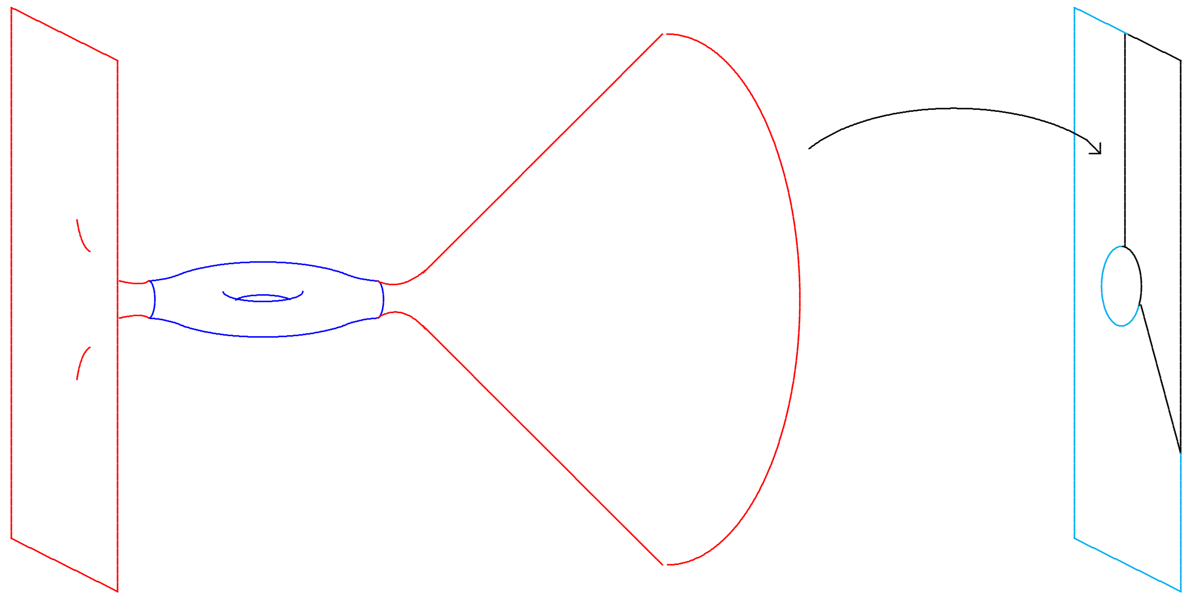

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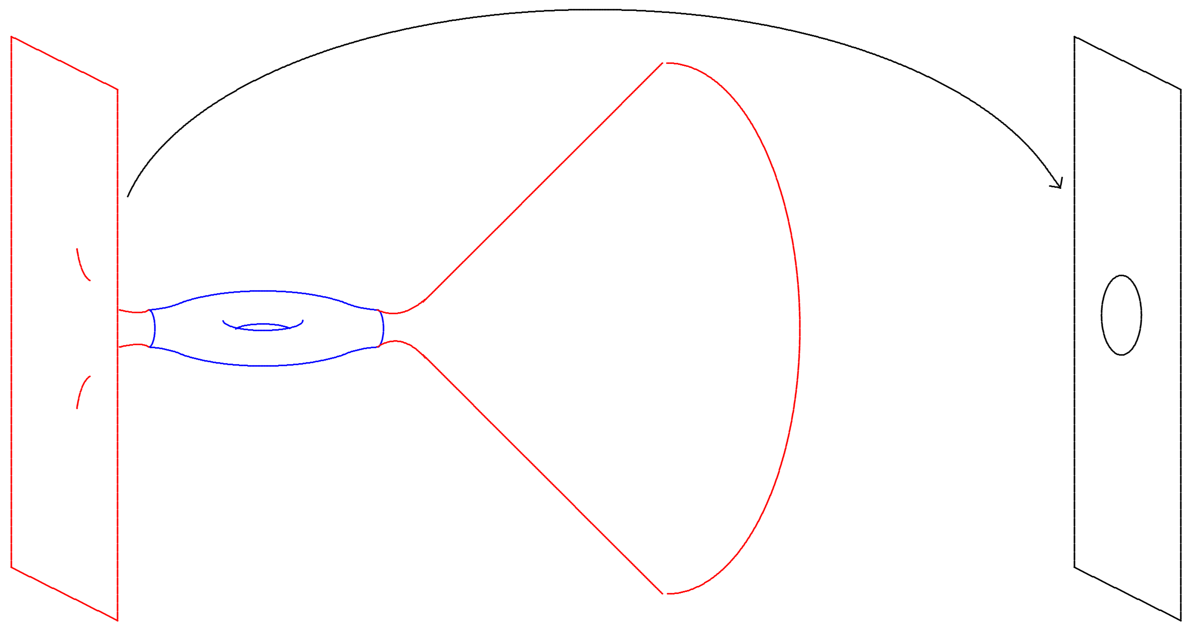

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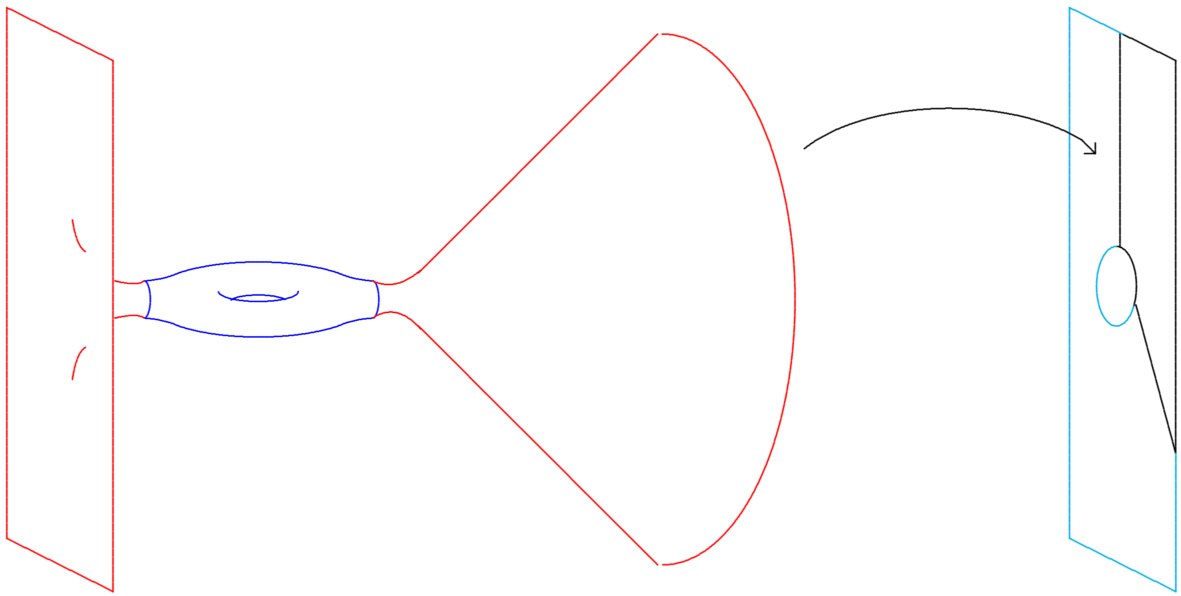

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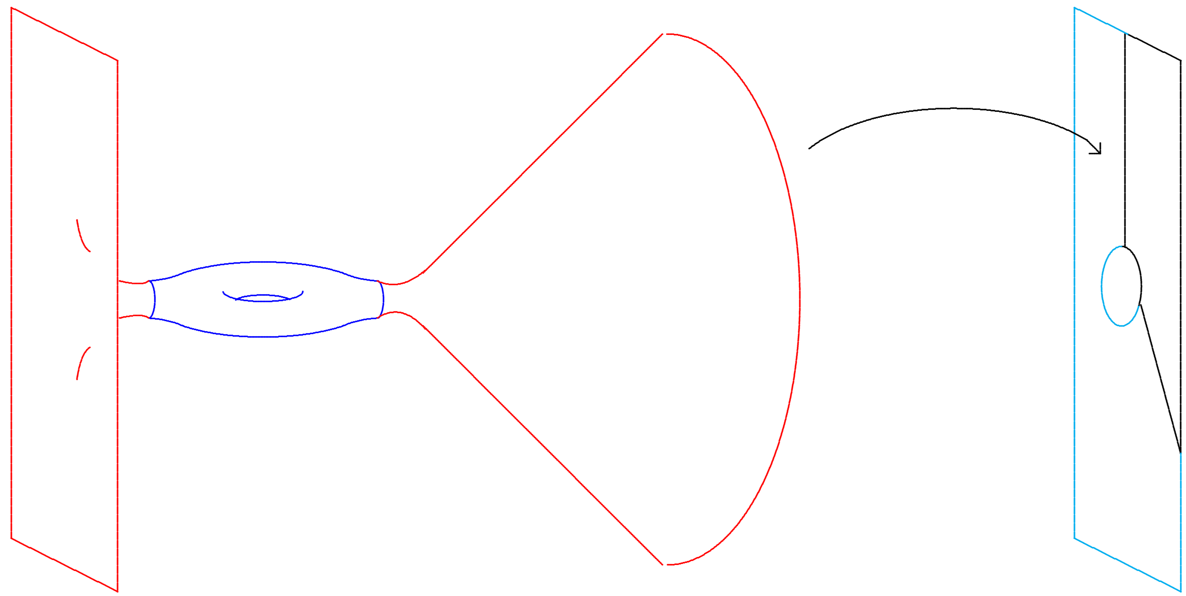

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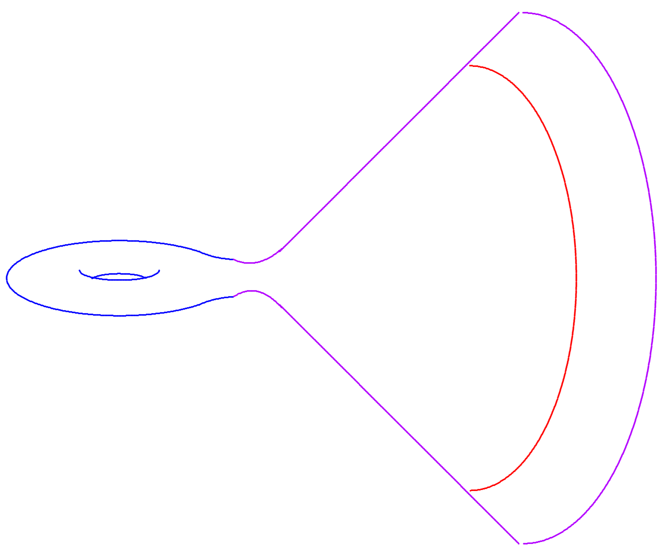

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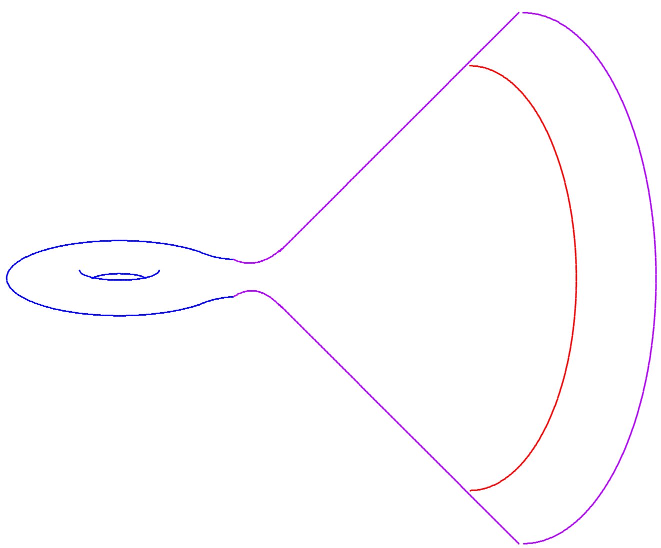

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Seems to depend on choice of coordinates!

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Chruściel-type fall-off:

$$g_{jk} - \delta_{jk} \in C^1_{-\tau}, \quad \tau > \frac{n-2}{2}$$

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Bartnik/Chruściel (1986): With weak fall-off conditions, the mass is well-defined & coordinate independent.

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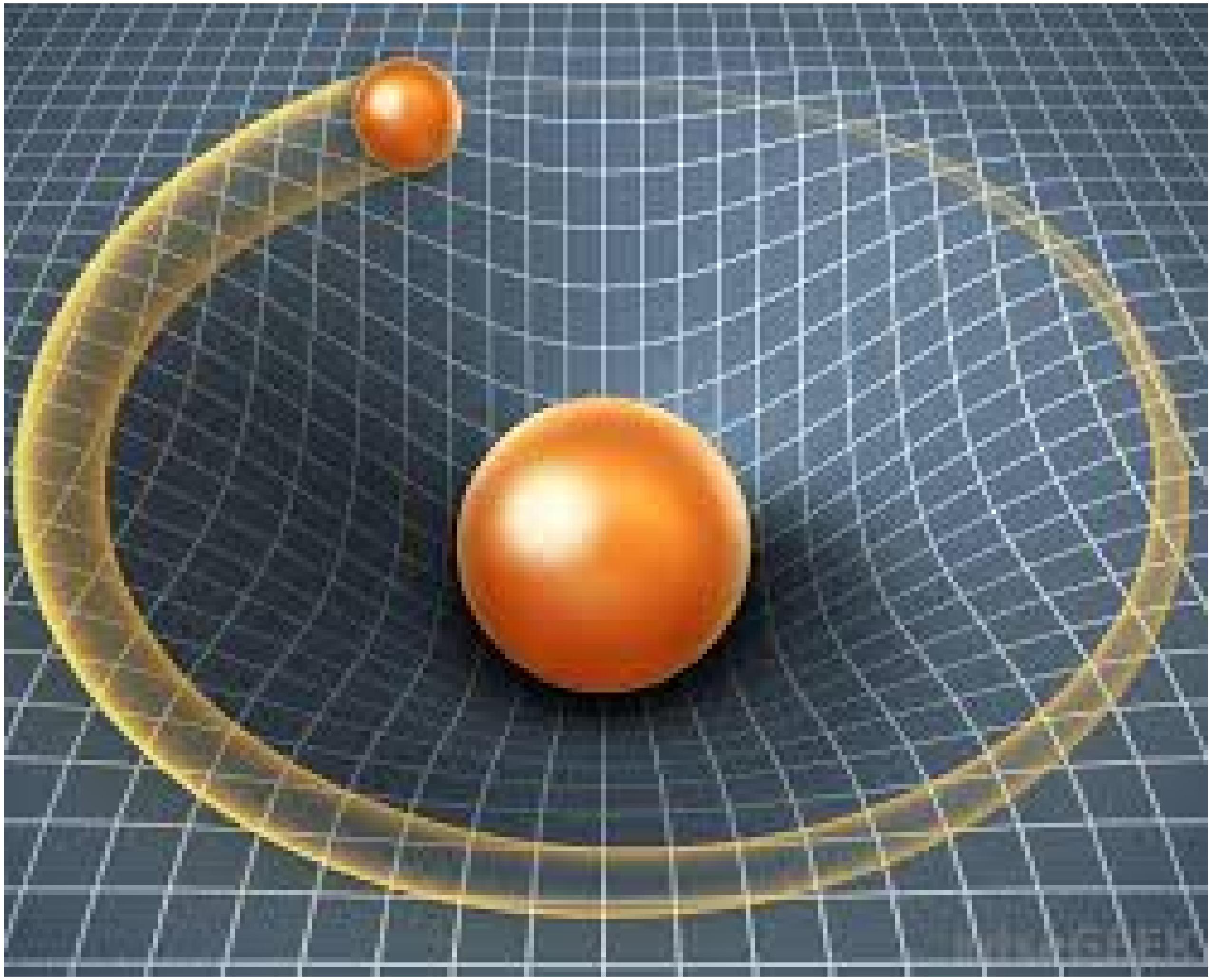

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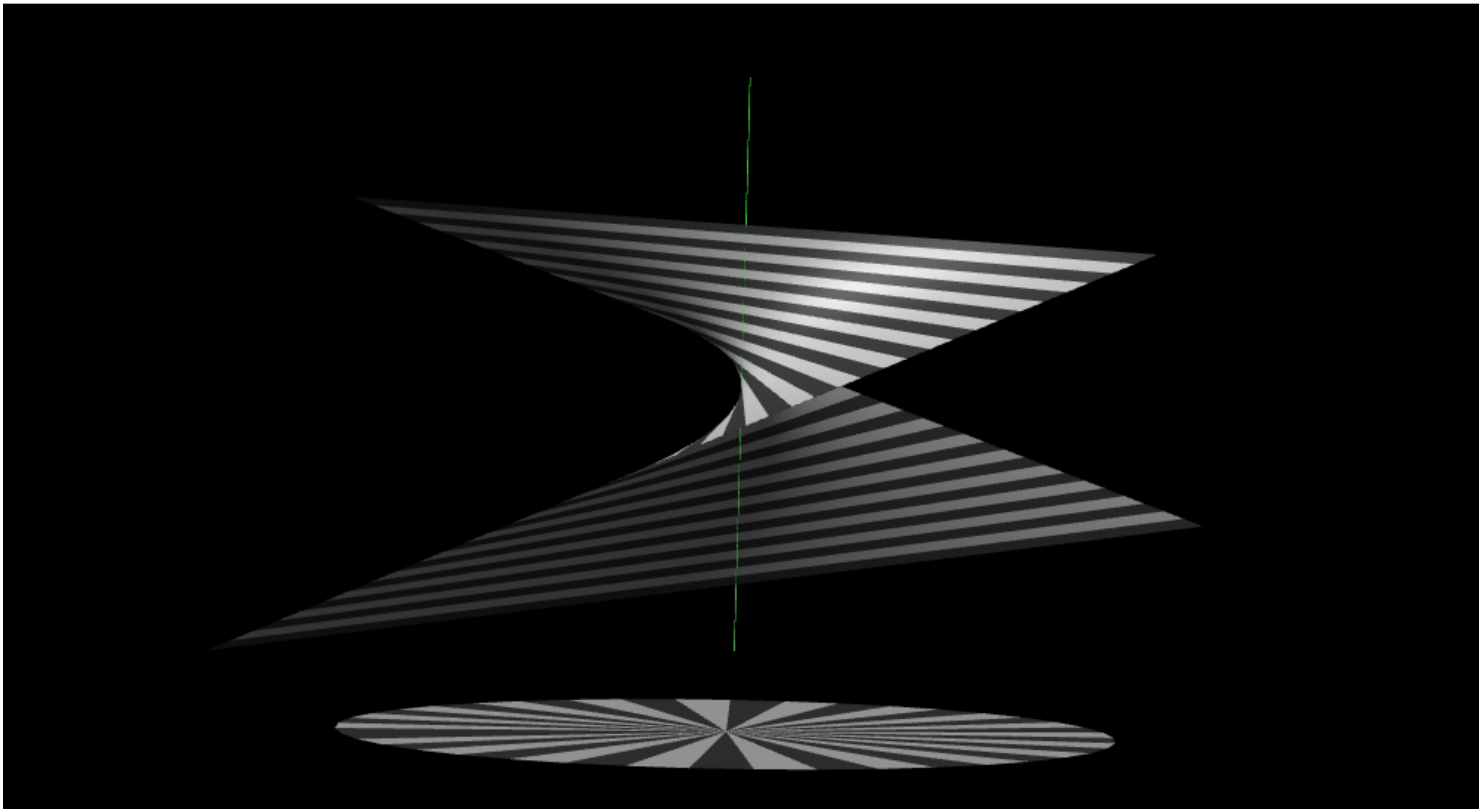

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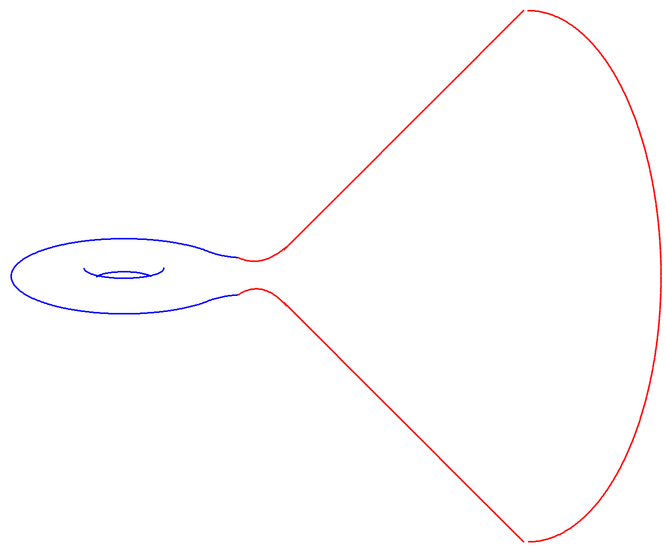

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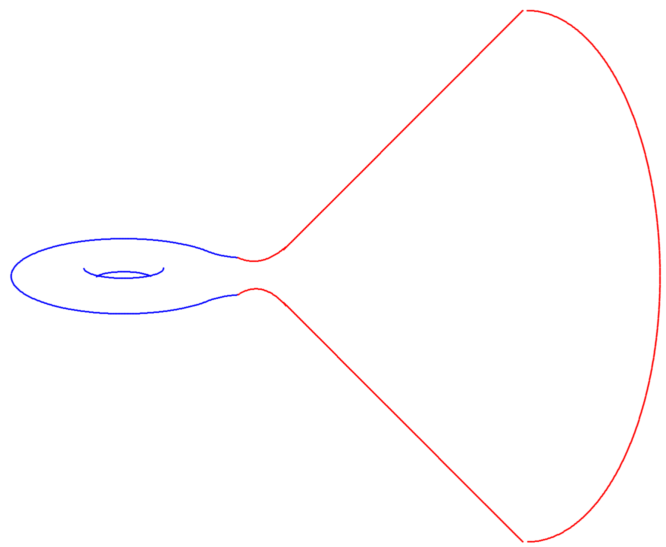

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Does not depend on the choice of an end!

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Theorem A.

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In fact, we will see that there is an explicit formula for the mass in terms of these data!

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(Discovered independently by Rollin, Singer, & Şuvaina, using different methods.)

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- $\langle \ , \ \rangle$ *is pairing between* $H^2_c(M)$ *and* $H^{2m-2}(M)$.

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$$\frac{4\pi^{m}(2m-1)}{(m-1)!}\mathfrak{m}(M,g) = -\frac{4\pi}{(m-1)!}\langle \clubsuit(c_1), [\omega]^{m-1}\rangle + \int_M s_g d\mu_g$$

For a compact Kähler manifold (M^{2m}, g, J),

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So the mass is a "boundary correction" to the topological formula for the total scalar curvature.

Theorem C. *Any ALE Kähler manifold* (M, g, J) *of complex dimension* m *has mass given by*

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Corollary. *Any ALE scalar-flat Kähler manifold* (M, g, J) *of complex dimension* m *has mass given by*

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So **Theorem A** is an immediate consequence!

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$$g^{jk}\left(g_{j\ell,k} - g_{jk,\ell}\right)\nu^\ell \alpha_E = -\star d \log\left(\sqrt{\det g}\right) + O(\varrho^{-3-\varepsilon}).$$

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$$d(\theta\wedge\omega) = \rho\wedge\omega = \frac{s}{4}\omega^2 = 0.$$

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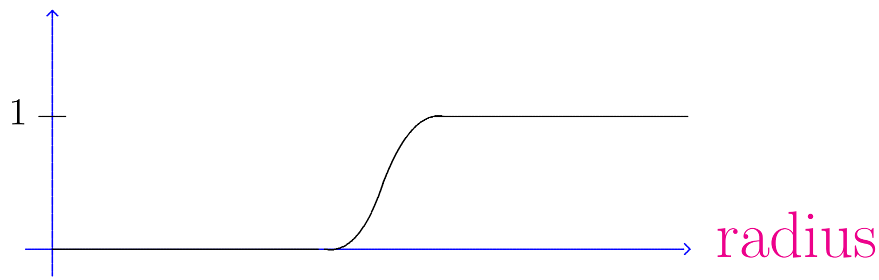

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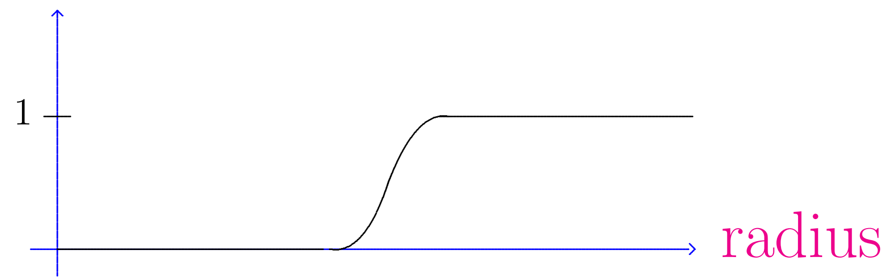

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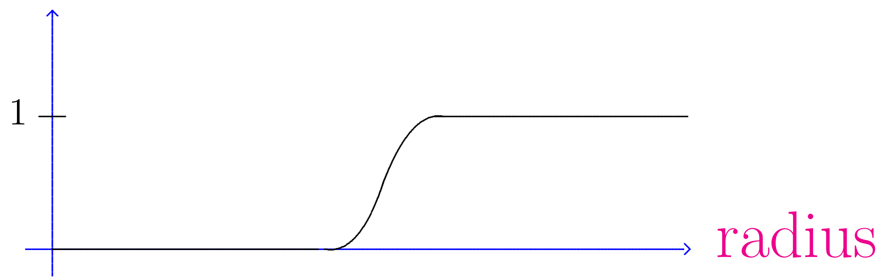

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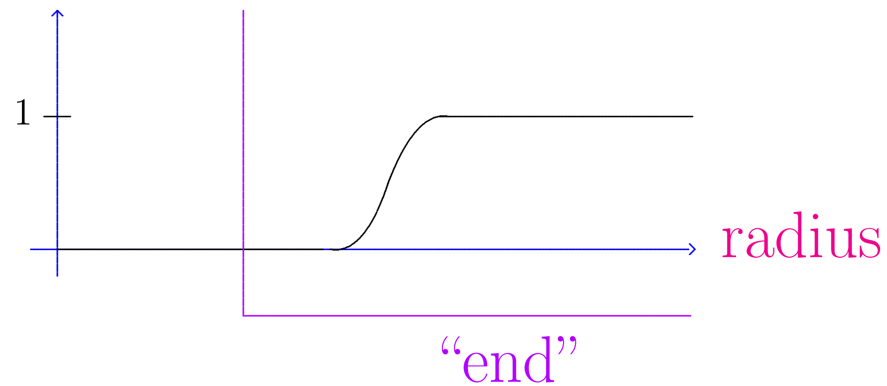

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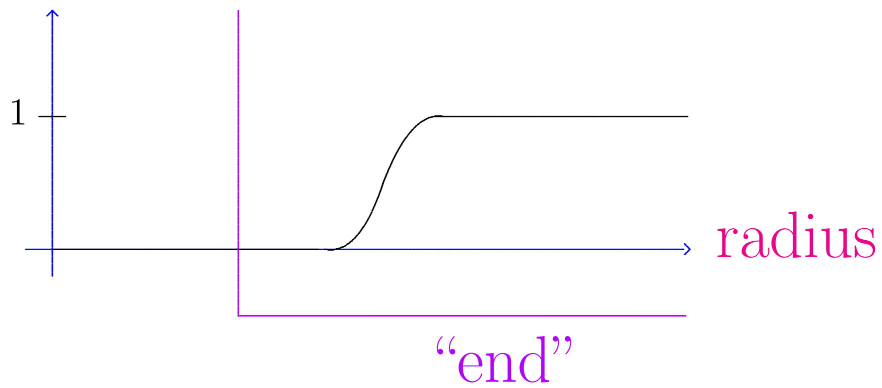

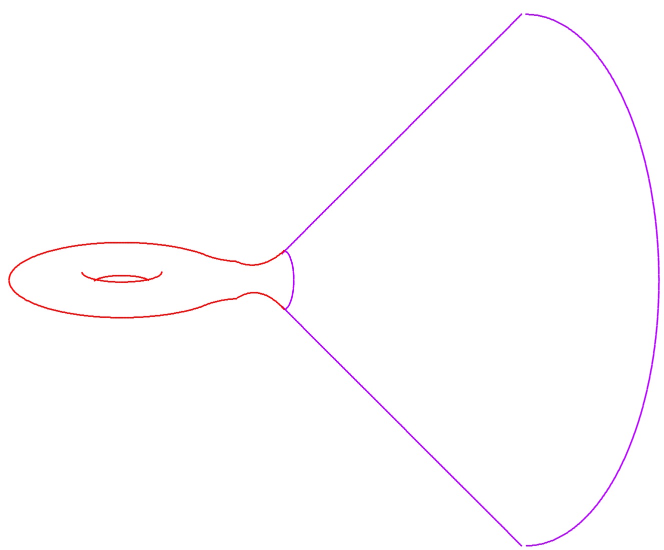

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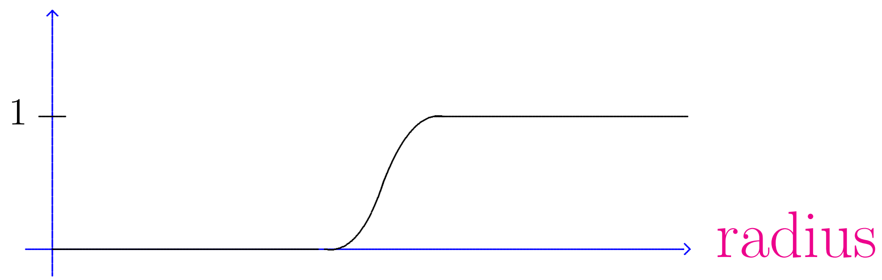

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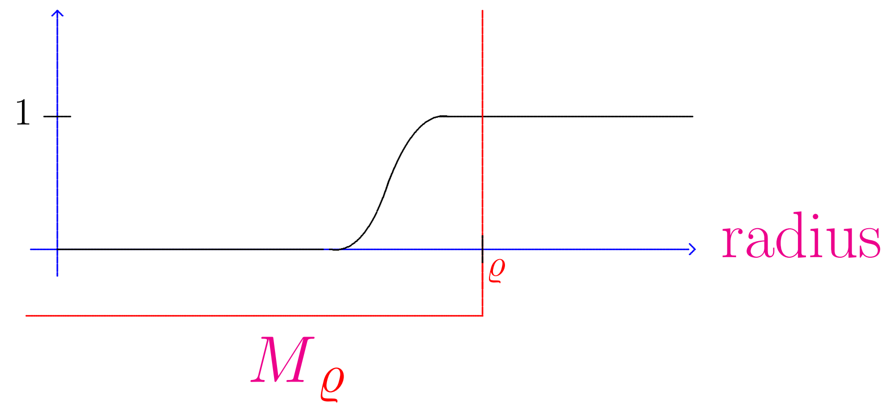

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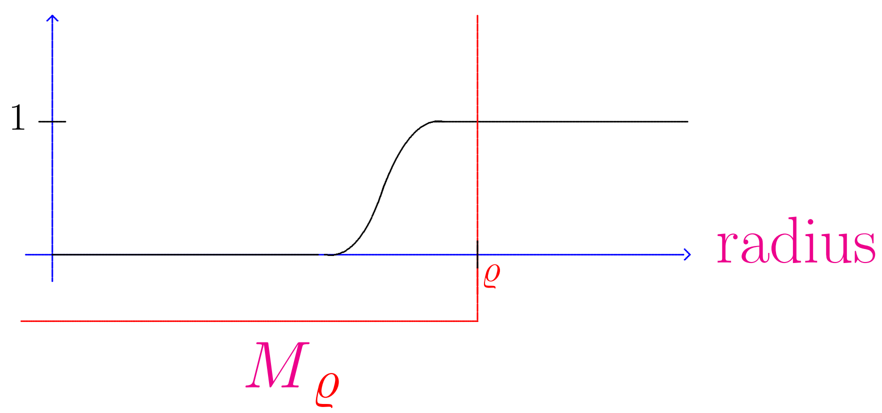

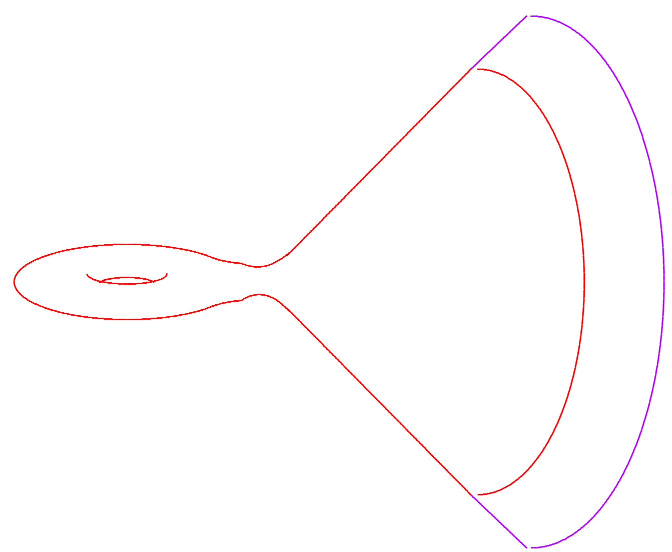

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Compactly supported, because $d\theta = \rho$ near infinity.

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where M_ϱ defined by radius $\leq \varrho$.

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because scalar-flat $\Longrightarrow$ $\rho \wedge \omega = 0$.

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by Stokes' theorem.

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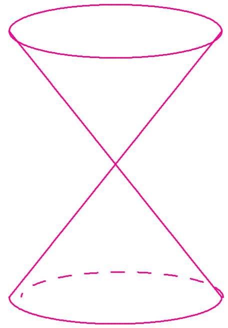

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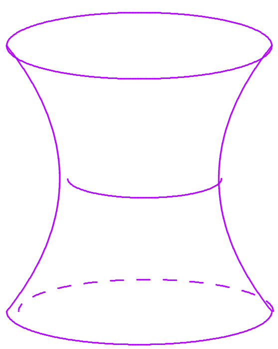

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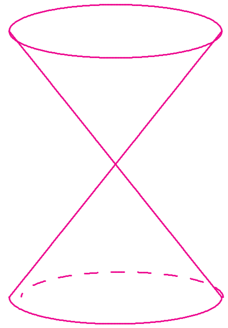

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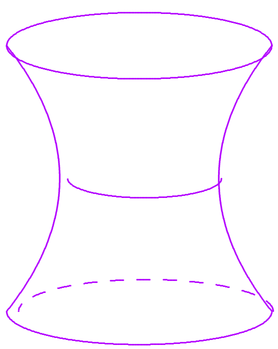

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One argument proceeds by osculation:

$$J = J_0 + O(\varrho^{-3}), \qquad \nabla J = O(\varrho^{-4})$$

in suitable asymptotic coordinates adapted to g.

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This has some interesting consequences. . .

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Proof actually shows something stronger!

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with $=$ $\Longleftrightarrow$ (M, g, J) is scalar-flat Kähler.

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so the mass formula implies the claim.

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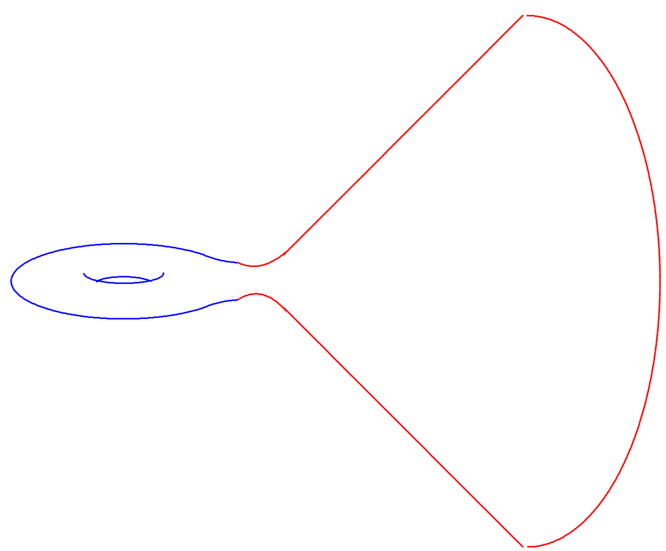

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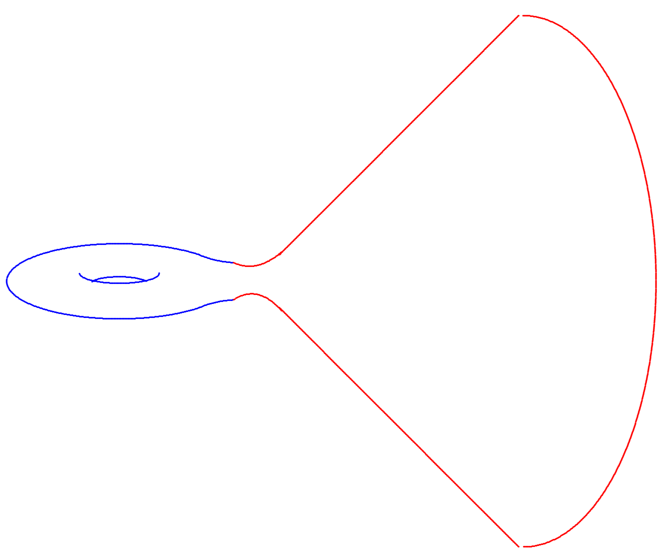

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