Loop Products, Index Growth and Dynamics

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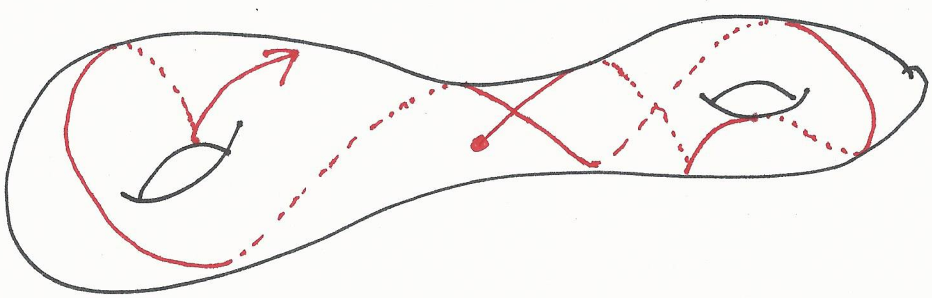
Recent joint work with Nathalie Wahl
on String Topology

First : Geometric Motivation

Relationship between Products and geodesics

Poincaré Duality in the Free Loop Space

M compact, oriented Riemannian manifold
dim n



Seach for closed geodesics on M.

Poincaré, Birkhoff, Morse, ...

We know (assume $\pi_1 M = 0$)

Gromoll-Meyer, Sullivan-Vigué-Poirrrier:

For most manifolds, every metric on M has infinitely many closed geodesics.

Rademacher: For most metrics on all manifolds there are infinity many closed geodesics.

Birkhoff-Bangert-Franks-Grayson

There are infinitely many closed geodesics for every metric on S^2.

BUT...

$M = S^n, n > 2 \quad M = \mathbb{C}P^k \quad k > 1$

min # ≥ 1

$\exists$ nondegenerate Finsler metrics with finitely many (Katok, Ziller)

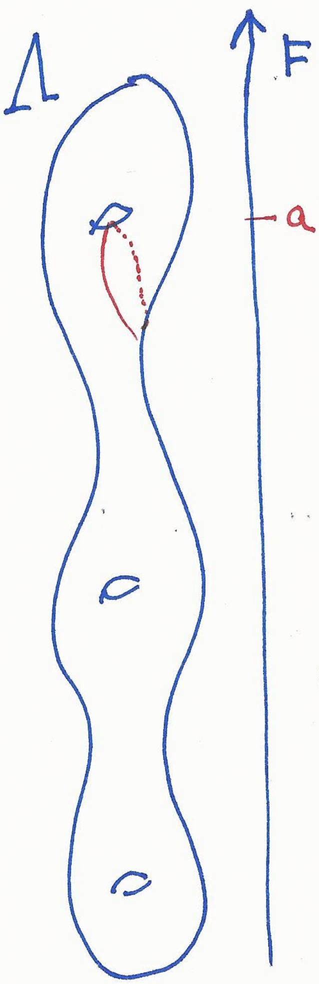
Method (Morse, ...) Fix a metric on M.

$\Lambda M = \{ \gamma : S^1 \to M \}$ Free Loop Space

$E : \Lambda M \to \mathbb{R}$ $\quad E(\gamma) = \int_{S^1} |\dot{\gamma}|^2 \, dt$

Better function $F = \sqrt{E} \approx$ length

Critical points $\equiv$ Closed geodesics on M



$H_k(\Lambda) \simeq$ Critical points index k

Given $X \in H_*(\Lambda)$

$$Cr(X) \equiv \inf \{ a \in \mathbb{R} \mid X \text{ supported on } \Lambda^{\leq a} \}$$

is a critical value of F

Q Can we use this correspondence to "count" closed geodesics on M?

Difficulty: Iterates

Inside the free Loop space ΛM,

one closed geodesic γ appears as an infinite sequence $\gamma, \gamma^2, \gamma^3, \ldots$ different $\begin{cases} \text{length} \\ \text{index.} \end{cases}$

$$H_k(\Lambda) \approx \begin{matrix} \text{Critical points} \\ \text{index } k \end{matrix}$$

Q. Is there an algebraic structure on $H_*(\Lambda)$ (e.g. product) that corresponds to iteration of closed geodesics?

A. In many critical cases, YES.

Products on Loop Spaces

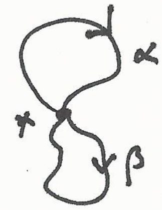
(topology)

Based loop space $\Omega M = \{\gamma \in \Lambda M : \gamma(0) = *\}$

→ Pontryagin product on $H_*(\Omega M)$:

$A, B \subseteq \Omega$ cycles $\quad [A] \in H_i(\Omega)$

$\quad [B] \in H_j(\Omega)$



$$[A] \cdot_{pp} [B] = [\{\alpha \cdot \beta \mid \alpha \in A, \beta \in B\}] \in H_{i+j}\Omega$$

Concatenation

→ Coproduct $\vee$ on $H_*(\Omega)$ (Sullivan, Goresky·H)

$A_{cycle} \subseteq \Omega \quad [A] \in H_k \Omega$

$$A \times I \subseteq \Omega \times I \supseteq \mathcal{F}_\Omega = \{(\gamma, t) \mid \gamma(0) = \gamma(t)\}$$

$$A \times I \cap \mathcal{F}_\Omega \subseteq \mathcal{F}_\Omega \xrightarrow{\text{cut}} \Omega \times \Omega$$

$$(\gamma, t) \longmapsto (\gamma|_{[0,t]}, \gamma|_{[t,1]})$$

$$\underset{\substack{\Psi \\ [A]}}{H_k(\Omega)} \to H_{k+1-n}(\Omega \times \Omega) \cong \bigoplus_{i+j=k+1-n} \underset{\substack{\Psi \\ \vee[A]}}{H_i(\Omega) \otimes H_j(\Omega)}$$

..... well not quite

- Definition of coproduct not rigorous
- Output in $H_*(\Omega \times \Omega, \Omega \times \{*\} \cup \{*\} \times \Omega)$

Products on ΛM

$\longrightarrow$ Chas-Sullivan 1999 $\quad A, B \subseteq \Lambda$ cycles

$$[A] \bullet_{cs} [B] = \left[\left\{ \alpha \cdot \beta \;\middle|\; \begin{array}{c} \alpha \in A \\ \beta \in B \\ \underline{\text{and}} \\ \alpha(0) = \beta(0) \end{array} \right\} \right]$$

$$H_i(\Lambda) \otimes H_j(\Lambda) \longrightarrow H_{i+j-n}(\Lambda)$$

$\longrightarrow$ Coproduct on $H_*(\Lambda)$:

$$H_k(\Lambda) \to H_{k+1-n}(\Lambda \times \Lambda, \Lambda \times \Lambda^0 \cup \Lambda^0 \times \Lambda)$$

Λ^0 = trivial loops

Definitions – Cohen-Jones, H-Wahl

(algebra)

Coproduct on homology $\longleftrightarrow$ Product on cohomology

$$\langle \vee A, x \otimes y \rangle = \langle A, x \cdot y \rangle$$

$\uparrow$ Coproduct on homology

$\uparrow$ GH product on cohomology

if $A \in H_k(\Lambda)$

$x \in H^i(\Lambda)$, $y \in H^j(\Lambda)$

$i + j = k + 1 - n$

(geometry)

G-H product on cohomology was discovered using

<u>Poincaré Duality</u> on Λ

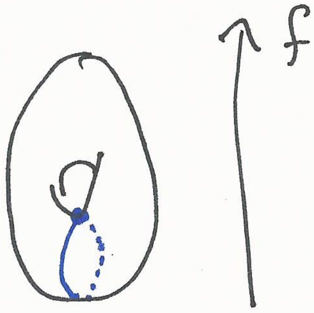
Recall.... X closed, oriented manifold of dim n

$$\text{P.D.}: H^k(X) \cong H_{n-k}(X)$$

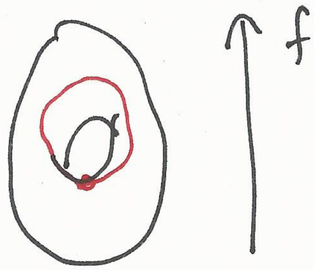
Poincare Duality via Morse Theory

$f: X \to \mathbb{R}$ Morse

$H_k(X) \leftrightarrow$ k-dim cycles hanging down



$H^{n-k}(X) \leftrightarrow$ $(n-k)$ dim cycles hanging up



P.D : $f \leftrightarrow -f$

P.D in Ω, Λ Wahl, Cohen, H. working on giving a correct statement.

... in the mean time

Powerful principle that works every time!

Example: (morality)

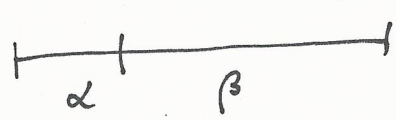
Homology coproduct on Ω $\xleftrightarrow{PD}$ Pontryagin product on cochains on Ω

Homology coproduct on Λ $\xleftrightarrow{PD}$ C.S product on cochains on Λ

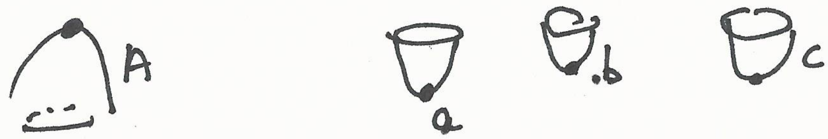
"Poincaré Duality in the Shower"

— space of constant speed loops —

Concatenate at optimal time



$A \in C_*(\Omega)$ $\quad$ $a, b, c \in C^*(\Omega)$



Assume/pretend transversality

$\langle A, a \rangle =$ # points in $A \cap a$

$$\langle \vee_{\Omega} A, b \otimes c \rangle = \langle A, b \bullet c \rangle$$

coproduct on chains

Pontryagin product on cochains

$= \# (\alpha, \beta, \gamma) \in \Omega^3 : \ \beta \in b, \gamma \in c, a \in A$ and $\alpha = \beta \cdot \gamma$

Similarly

$$\langle \vee_{\Lambda} A, b \otimes c \rangle = \langle A, b \bullet c \rangle$$

Coproduct on chains

CS product on cochains

What if we don't use CS loops?

If a is a cocycle  a

A, A' cycles with A' a reparam. of A

then $\langle A, a\rangle = \langle A', a\rangle$

$\Rightarrow$ The "set" of loops in a cocycle a is invariant under reparam.

If $\alpha = \beta\cdot\gamma \in A$, a cocycle dual to VA would have to include all loops of the form

$$\alpha' = \beta' \cdot_t \gamma'$$

β' reparam of β
γ' . . . . γ
$t \in (0,1)$

In above equations

$b\cdot c \rightsquigarrow 1$ param family of repar's of $b\cdot c$

Geometry, Iteration and Products

Examples of $\begin{cases} \text{Poincaré Duality in } \Lambda, \Omega \\ \text{Relations b/n products and geometry} \end{cases}$

- Basic Inequalities

$$X, Y \in H_*(\Lambda M) \Rightarrow Cr(X \cdot Y) \leq Cr(X) + Cr(Y)$$

$$x, y \in H^*(\Lambda M) \Rightarrow Cr(x \cdot y) \geq cr(x) + Cr(y)$$

- Old theorems Restated

Bott (56) Suppose all closed geodesics on M are nondegenerate. Then

$\forall X \in H_*(\Lambda),\ Cr(X^m) < m\,Cr(X)$ for large m

$\forall x \in H^*(\Lambda)\ \ Cr(x^m) > m\,Cr(x)$ for large m

H. (93) If $\exists\, X \in H_*(\Lambda) : Cr(X^m) = m\,Cr(X)\ \forall m$

$\Rightarrow$ M has infinitely many closed geodesics

(97) If $\exists\, x \in H_*(\Lambda) : Cr(x^m) = m\,Cr(x)\ \forall m$

$\Rightarrow$ M has infinitely many closed geodesics

- Index Growth

Bott (56) γ a closed geodesic on M.

$$m \cdot \mathrm{Index}(\gamma) - (m-1)(n-1) \leq \mathrm{Index}(\gamma^m) \leq m \cdot \mathrm{Index}(\gamma) + (m-1)(n-1)$$

$$m \cdot \mathrm{Index}_\Omega(\gamma) \leq \mathrm{Index}_\Omega(\gamma^m) \leq m \cdot \mathrm{Index}_\Omega(\gamma) + (m-1)(n-1)$$

4 inequalities

4 products

Equality $\Longleftrightarrow$ Nontrivial Products

Ex. Spheres, projective spaces

all geodesics closed

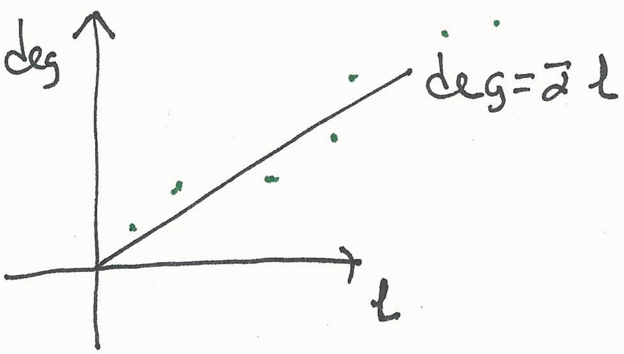
$\Rightarrow$ index growth minimal and maximal

"self-dual"

$\Rightarrow$ All products highly nontrivial

• Resonance [Rademacher, H] ('13)

Fix a metric on S^n. The points $(Cr(x), \deg(x))$ lie at bounded distance from a line



of slope $\bar{\alpha}$ $\quad \dfrac{\#\text{ conj. pts}}{\text{length}}$

Proof uses homology <u>and</u> cohomology products.

Recent work on loop products
aka string topology
w/ Nathalie Wahl

(1) $\exists!$ lift $\hat{V} : H_k(\Lambda) \to H_{k+n-1}(\Lambda \times \Lambda)$

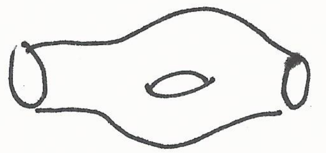
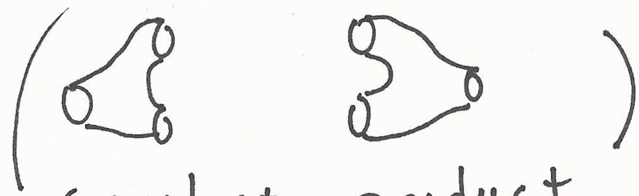
of the homology coproduct satisfying....

The lift is associative, commutative,
satisfies basic inequalities.

Should be true: If $[A]$ has a rep
consisting of simple loops, then

$$V[A] = 0.$$

(2) The (signed) incestuous product is trivial:

 $\equiv 0$ 

coproduct product

If n is odd, $\bullet(\vee) \equiv 0 \pmod{\Lambda^0}$

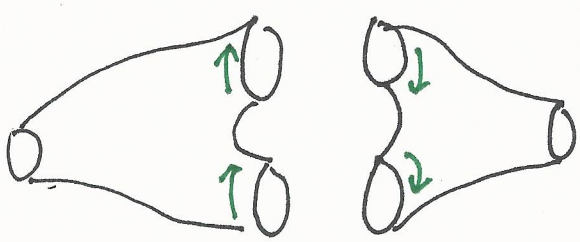
If n is even, $\bullet(\vee) \equiv 0 \left(\begin{matrix}\text{mod } 2 \\ \text{mod } \Lambda^0\end{matrix}\right)$

However e.g. $M = S^3$ $k \geq 14$ even

$\frac{k}{2} \equiv 3 \ (4)$

$\Rightarrow \exists\ X \in H_k(\Lambda S^3)$:

$\bullet(\Delta_1, \Delta_2) \vee X \neq 0$

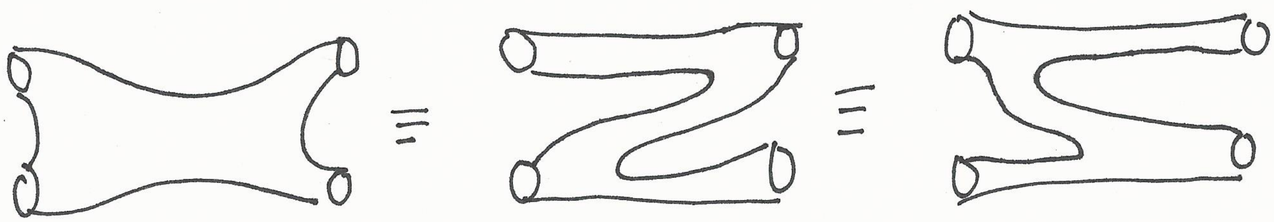


(3) Sullivan formula

What about $V(\cdot)$?

For a field theory we would have

Frobenius: $V(A\cdot B) = (VA)\cdot B = A\cdot(VB)$



This seems to be true only for trivial coproducts.

Sullivan formula

$$V(A\cdot B) = (VA\cdot B) + A\cdot(VB)$$

Supp: The self-intersections in $A\cdot B$ come from the self-intersections in A and the self-intersections of B.

— turns out to be FALSE. Examples

$$LHS \neq RHS$$

Geometric Interpretation of $LHS - RHS$ in the finite-dimensional approximation to ΛM:

$$V(A\cdot B) - \big((VA\cdot B) + A\cdot(VB)\big)$$

is picking up 2^{nd} order intersections of A, B.

WIP (w N. Wahl)

Poincaré Duality in Λ

Higher order Products

- how do they fit together?

Applications?