Fourth order Paneitz operator and Q curvature equation

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1 What is Q curvature?

Given (M^n, g), $n \geq 3$,

$$\begin{aligned} Q &= -\frac{1}{2(n-1)}\Delta R - \frac{2}{(n-2)^2}|Rc|^2 \\ &\quad + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2}R^2. \end{aligned}$$

To simplify the formulas, in conformal geometry people use

$$J = \frac{R}{2(n-1)}, \quad A = \frac{1}{n-2}(Rc - Jg).$$

Then

$$Q = -\Delta J - 2|A|^2 + \frac{n}{2}J^2.$$

The Paneitz operator [Paneitz 1983]

$$P\varphi = \Delta^2\varphi + \operatorname{div}(4A(\nabla\varphi, e_i)e_i - (n-2)J\nabla\varphi) + \frac{n-4}{2}Q\varphi.$$

Here $e_1, \cdots, e_n$ is a local orthonormal frame.

For $n \neq 4$,

$$P_{\rho^{\frac{4}{n-4}}g}\varphi = \rho^{-\frac{n+4}{n-4}} P_g\left(\rho\varphi\right).$$

In particular

$$Q_{\rho^{\frac{4}{n-4}}g} = \frac{2}{n-4}\rho^{-\frac{n+4}{n-4}} P_g\rho.$$

This can be compared to

$$L = -\frac{4\left(n-1\right)}{n-2}\Delta + R,$$

which satisfies

$$L_{\rho^{\frac{4}{n-2}}g}\varphi = \rho^{-\frac{n+2}{n-2}} L_g(\rho\varphi)$$

and

$$R_{\rho^{\frac{4}{n-2}}g} = \rho^{-\frac{n+2}{n-2}} L_g\rho.$$

For $n = 4$,

$$P_{e^{2w}g}\varphi = e^{-4w} P_g\varphi$$

and

$$Q_{e^{2w}g} = e^{-4w} \left(P_g w + Q_g \right).$$

Moreover

$$\int_M Q d\mu + \frac{1}{4} \int_M |W|^2 \, d\mu = 8\pi^2 \chi(M).$$

This can be compared to Laplacian on surface

$$\Delta_{e^{2w}g} \varphi = e^{-2w} \Delta_g \varphi$$

and

$$K_{e^{2w}g} = e^{-2w} \left(K_g - \Delta_g w \right).$$

There are higher order analogue called GJMS operator with $(-\Delta)^m$ as leading term. There are even fractional order GJMS operators by scattering theory (Branson, Fefferman-Graham, Graham-Zworski, Juhl ...).

2 Q curvature equation in dimension 4

Problems in spectral geometry motivate the analytical study of Paneitz operator in dimension 4 [Branson-Chang-Yang, 1992], [Chang-Yang 1995]. Paneitz operator appears in the log determinant of conformal covariant operators. One typical breakthrough made along the way is

Theorem 1 (Chang-Gursky-Yang 2002) *If $\left(M^4, g\right)$ is compact with $Y\left(g\right) > 0$ and $\int_M Q d\mu > 0$, then there exists a $\widetilde{g} \in [g]$ such that $\widetilde{Rc} > 0$.*

Both $Y\left(g\right)$ and $\int_M Q d\mu$ are conformal invariants. The study also reveals the usefulness of $\sigma_2\left(A\right)$ equation.

[Djadli-Hebey-Ledoux 2000] started the question of finding constant Q curvature in a fixed conformal class (or closely related prescribing Q curvature problem) in dimension $n \geq 5$. [Xu-Yang 2002] started similar problem for dimension 3.

Theorem 2 (Ge-Lin-Wang, Catino-Djadli 2010) *Assume* $\left(M^3, g\right)$ *is compact with* $R > 0, Q > 0$*, then there exists a* $\widetilde{g} \in [g]$ *such that* $\widetilde{Rc} > 0$*.*

3 Q curvature equation in dimension $n \geq 5$

Let (M^n, g) be smooth and compact, $K : M \times M \to \mathbb{R}$, $\varphi : M \to \mathbb{R}$,

$$(T_K \varphi)(p) = \int_M K(p,q)\,\varphi(q)\,d\mu(q).$$

Given another $K' : M \times M \to \mathbb{R}$,

$$\left(K * K'\right)(p,q) = \int_M K(p,s)\,K'(s,q)\,d\mu(s).$$

This makes

$$T_K T_{K'} = T_{K*K'}.$$

Denote

$$Y(g) = \inf_{\widetilde{g} \in [g]} \widetilde{\mu}(M)^{-\frac{n-2}{n}} \int_M \widetilde{R}\,d\widetilde{\mu}.$$

Assume $Y(g) > 0$, G_L is the Green's function of L, $G_{L,p}(q) = G_L(p,q)$. Denote

$$H(p,q) = \frac{2^{\frac{n-6}{n-2}}(n-1)^{\frac{n-4}{n-2}}}{n^{\frac{2}{n-2}}(n-2)(n-4)\,\omega_n^{\frac{2}{n-2}}} G_L(p,q)^{\frac{n-4}{n-2}}.$$

and

$$\Gamma_1(p,q) = \frac{2^{\frac{n-6}{n-2}}(n-1)^{\frac{n-4}{n-2}}}{n^{\frac{2}{n-2}}(n-2)^3\,\omega_n^{\frac{2}{n-2}}} G_L(p,q)^{\frac{n-4}{n-2}} \left| Rc_{G_{L,p}^{\frac{4}{n-2}}g} \right|_g^2 (q).$$

ω_n is the volume of unit ball in $\mathbb{R}^n$. Note $G_{L,p}^{\frac{4}{n-2}} g$ is the stereographic projection at p. Then

$$\Gamma_1(p,q) = O\left(\overline{pq}^{4-n}\right).$$

If $\widetilde{g} = \rho^{\frac{4}{n-4}} g$, then

$$T_{\widetilde{\Gamma}_1}(\varphi) = \rho^{-1} T_{\Gamma_1}(\rho\varphi).$$

They have the same spectrum and spectral radius i.e. $\sigma\left(T_{\widetilde{\Gamma}_1}\right) = \sigma\left(T_{\Gamma_1}\right)$ and $r_\sigma\left(T_{\widetilde{\Gamma}_1}\right) = r_\sigma\left(T_{\Gamma_1}\right)$.

Theorem 3 (Hang-Yang 2015) *Assume* $Y(g) > 0$*, then the following statements are equivalent*

1. *there exists a* $\widetilde{g} \in [g]$ *with* $\widetilde{Q} > 0$.

2. $\ker P = 0$ *and the Green's function* $G_P(p,q) > 0$ *for* $p \neq q$.

3. $\ker P = 0$ *and there exists* $p \in M$ *such that* $G_P(p,q) > 0$ *for* $q \neq p$.

4. $r_\sigma\left(T_{\Gamma_1}\right) < 1$.

Moreover if $r_\sigma\left(T_{\Gamma_1}\right) < 1$*, then*

$$G_P = H + \sum_{k=1}^{\infty} \Gamma_k * H,$$

here $\Gamma_k = \Gamma_1 * \cdots * \Gamma_1$ *(*k *times). In particular,* $G_P \geq H$*, moreover if* $G_P(p,q) = H(p,q)$ *for some* $p \neq q$*, then* (M,g) *is conformal equivalent to the standard* S^n.

If ker $P = 0$, $\widetilde{g} = \rho^{\frac{4}{n-4}} g$, then

$$G_{\widetilde{P}}(p,q) = \rho(p)^{-1} \rho(q)^{-1} G_P(p,q).$$

The above statement can be compared to

$$\exists \widetilde{g} \in [g] \text{ with } \widetilde{R} > 0 \Leftrightarrow \lambda_1(L_g) > 0 \Leftrightarrow Y(g) > 0,$$

an observation of [Kazdan-Warner 1975]; and [Aubin 1974] about Green's function of the Laplacian. For relations between L and P, [Hijazi-Raulot 2007] shows if $Y(g) > 0$, then

$$\lambda_1(L_g)^2 \geq \frac{16n(n-1)^2}{(n+2)(n-2)(n-4)} \lambda_1(P_g),$$

equality holds iff g is Einstein metric.

Theorem 4 ([Gursky-Malchiodi, Hang-Yang 2015]) *If $Y(g) > 0$, $r_\sigma\left(T_{\Gamma_1}\right) < 1$, then there exists a $\widetilde{g} \in [g]$ with $\widetilde{Q} = 1$. Such $\widetilde{g}$ can be found as extremal metric of some functionals. If in addition we know $R > 0$ and $Q > 0$, then $\widetilde{R} > 0$.*

To find metrics with both positive scalar and Q curvature,

Theorem 5 (Gursky-Hang-Lin 2015) *For* $n \geq 6$,

$\exists \widetilde{g} \in [g]$ *with* $\widetilde{R} > 0$ *and* $\widetilde{Q} > 0 \Leftrightarrow L_g > 0$ *and* $P_g > 0$.

For some reason the approach does not work for $n = 5$. But it seems the statement should valid for $n = 5$ too.

Recall

$$\begin{aligned} Y(g) &= \inf_{\substack{u \in C^\infty(M) \\ u > 0}} \frac{\int_M \left(\frac{4(n-1)}{n-2} |\nabla u|^2 + R u^2 \right) d\mu}{\|u\|^2_{L^{\frac{2n}{n-2}}}} \\ &= \inf_{u \in H^1(M) \setminus \{0\}} \frac{\int_M \left(\frac{4(n-1)}{n-2} |\nabla u|^2 + R u^2 \right) d\mu}{\|u\|^2_{L^{\frac{2n}{n-2}}}}, \end{aligned}$$

by the fact $u \in H^1(M) \Rightarrow |u| \in H^1(M)$ (this simple fact is the reason why the first eigenfunction of L_g is strictly positive or negative, it is also a basic block for

DeGiorgi-Nash-Moser theory for second order scalar elliptic equations). In particular if the minimizer exists, it must be strictly positive or negative.

[Aubin 1976] shows $Y(g) \le Y(S^n)$; if $Y(g) < Y(S^n)$, then $Y(g)$ is achieved; for $n \ge 6$, g not locally conformally flat, then $Y(g) < Y(S^n)$. [Schoen 1984] solves all the remain case by applying positive mass theorem on stereographic projection.

To solve

$$Pu = const \cdot u^{\frac{n+4}{n-4}}, \quad u > 0;$$

let

$$Y_4(g) = \inf_{u \in H^2(M) \setminus \{0\}} \frac{\int_M Pu \cdot u d\mu}{\|u\|^2_{L^{\frac{2n}{n-4}}}},$$

then $Y_4(g) \le Y_4(S^n)$; if $Y_4(g) < Y_4(S^n)$, then $Y_4(g)$ is achieved; for $n \ge 8$, g not locally conformally flat, then $Y_4(g) < Y_4(S^n)$. Unfortunately $u \in H^2(M) \not\Rightarrow |u| \in H^2(M)$, how can we know the minimizer is strictly positive or negative? It needs positivity to qualify for being conformal factors.

[Robert 2009] made an observation in an unpublished lecture notes: if $P>0$, $G_P>0$ and $Y_4(g)$ is achieved, then the minimizer must be strictly positive or negative.

In fact u is a minimizer with $\|u\|_{L^{\frac{2n}{n-4}}}=1$ and $u^+\neq 0$, then

$$Pu=Y_4(g)\,|u|^{\frac{8}{n-4}}u.$$

Let $Pv=|Pu|$, then $v>0$ and $|u|\leq v$.

$$\begin{aligned} Y_4(g) &\leq \frac{\int_M Pv\cdot v\,d\mu}{\|v\|^2_{L^{\frac{2n}{n-4}}}}=Y_4(g)\frac{\int_M |u|^{\frac{n+4}{n-4}}v\,d\mu}{\|v\|^2_{L^{\frac{2n}{n-4}}}} \\ &\leq Y_4(g)\,\|v\|^{-1}_{L^{\frac{2n}{n-4}}}\leq Y_4(g)\,. \end{aligned}$$

Hence $\|v\|_{L^{\frac{2n}{n-4}}}=1=\|u\|_{L^{\frac{2n}{n-4}}}$ and $u=v$.

[Humbert-Raulot 2009] If (M,g) is locally conformally flat or $n=5,6,7$, $\ker P=0$, then under the conformal normal coordinate at p,

$$2n(n-2)(n-4)\omega_n G_{P,p} = r^{4-n} + A + O^{(4)}(r).$$

If $Y(g) > 0$, $G_{P,p} > 0$, then

$$A = c(n)\int_M G_{P,p} G_{L,p}^{\frac{n-4}{n-2}} \left| Rc_{G_{L,p}^{\frac{4}{n-2}} g} \right|_g^2 d\mu \geq 0.$$

$A = 0$ iff (M, g) is conformal equivalent to the standard S^n. Originally Humbert-Raulot only considered locally conformally flat manifolds, Gursky-Malchiodi pointed out their argument works in dimension $5, 6, 7$.

Indeed on $M \backslash \{p\}$,

$$P_g G_{P,p} = 0.$$

Use conformal covariant property,

$$P_{G_{L,p}^{\frac{4}{n-2}} g}\left(G_{L,p}^{-\frac{n-4}{n-2}} G_{P,p}\right) = 0.$$

Integrate the equation on $M \backslash B_{g,\varepsilon}(p)$ with respect to $d\mu_{G_{L,p}^{\frac{4}{n-2}} g}$ and let $\varepsilon \to 0^+$ we get the needed identity.

How can we know $G_P > 0$? Green's function for fourth order elliptic operator is not as simple as second order ones.

Example 6 *On S^1 consider the operator $u \mapsto u^{(4)} + \lambda u$, $\lambda > 0$, we have*

$$\begin{aligned} & G(x,y) \\ = & -\frac{1}{2\sqrt{\lambda}} \operatorname{Im}\left(\frac{\cosh\alpha(\pi - \overline{xy})}{\alpha \sinh \pi\alpha}\right) \\ = & \frac{\mu}{2\lambda(\cosh 2\mu\pi - \cos 2\mu\pi)} [\cosh \mu\overline{xy} \sin\mu(2\pi - \overline{xy}) \\ & + \cosh\mu(2\pi - \overline{xy}) \sin\mu\overline{xy} + \sinh\mu\overline{xy}\cos\mu(2\pi - \overline{xy}) \\ & + \sinh\mu(2\pi - \overline{xy})\cos\mu\overline{xy}] . \end{aligned}$$

Here $\mu = \sqrt[4]{\lambda/4}$, $\alpha = \mu + \mu i$. Note

$$G(1,-1) = \frac{\mu(\cosh\mu\pi \sin\mu\pi + \sinh\mu\pi\cos\mu\pi)}{\lambda(\cosh 2\mu\pi - \cos 2\mu\pi)}.$$

If $\mu = 2k+1$, $k \in \mathbb{Z}_+$, then $G(1,-1) < 0$. Indeed careful study shows if $\lambda \geq 4$, G is negative somewhere.

Example 7 *If $Rc = (n-1)\,g$, then*

$$\begin{aligned} Q &= \frac{n\,(n-2)\,(n+2)}{8}, \\ P &= \left(-\Delta + \frac{n\,(n-2)}{4}\right)\left(-\Delta + \frac{(n+2)\,(n-4)}{4}\right). \end{aligned}$$

Hence $P > 0$ and

$$G_P = G_{-\Delta + \frac{(n+2)(n-4)}{4}} * G_{-\Delta + \frac{n(n-2)}{4}} > 0.$$

This is basically applying maximum principle twice. On S^n, let N be the north pole, $x = \pi_N$ as the coordinate, then

$$G_{P,N} = \frac{\left(|x|^2 + 1\right)^{\frac{n-4}{2}}}{n\,(n-2)\,(n-4)\,2^{n-3}\omega_n}.$$

Example 8 *If*

$$\Delta^2 u = u^{\frac{n+4}{n-4}}, \quad u > 0,$$

then by [Lin 1998] using method of moving plane and *applying maximum principle twice, we have*

$$u = c_n \left(\frac{\lambda}{|x - x_0|^2 + \lambda^2} \right)^{\frac{n-4}{2}} \quad \textit{for some } \lambda > 0.$$

[Chen-Li-Ou 2006] achieve this without using maximum principle, namely if for $0 < \alpha < n$,

$$u(x) = \int_{\mathbb{R}^n} \frac{u(y)^{\frac{n+\alpha}{n-\alpha}}}{|x - y|^{n-\alpha}} dy, \quad u > 0,$$

then

$$u = c_{n,\alpha} \left(\frac{\lambda}{|x - x_0|^2 + \lambda^2} \right)^{\frac{n-\alpha}{2}} \quad \textit{for some } \lambda > 0.$$

They develop the integral form of the method of moving planes based on the property of the kernel. This approach, together with [Schoen-Yau 1988] 's result on locally conformally flat manifolds and Kleinian groups, enable [Qing-Raske 2006] to solve the Q *curvature equation for locally conformally flat manifolds with* $Y(g) > 0, Q > 0$*. These are all based on explicit formulas of the Green's function.*

[Gursky-Malchiodi 2014] makes a breakthrough: If $R > 0, Q > 0$, then $P > 0, G_P > 0$; moreover if $\tilde{g} \in [g]$ satisfies $\tilde{Q} > 0$, then $\tilde{R} > 0$.

Method: try to show

$$Pu \geq 0 \Rightarrow u \geq 0.$$

For $\lambda > 0$, let $u_\lambda = u + \lambda$, $g_\lambda = u_\lambda^{\frac{4}{n-4}} g$.

$$Q_\lambda = \frac{2}{n-4} u_\lambda^{-\frac{n+4}{n-4}} P u_\lambda > 0;$$

i.e.

$$-\Delta_\lambda J + \frac{n}{2} J_\lambda^2 \geq Q_\lambda > 0.$$

By method of continuity $J_\lambda > 0$. Hence u_λ is superharmonic. By strong maximum principle, $u_\lambda > 0$ for all $\lambda > 0$. Hence $u \geq 0$.

[Hang-Yang 2014] Recall how we deal with second order operators. Given $Su = -\Delta u + cu$, $c > 0$. We need

$$Su \geq 0 \Rightarrow u \geq 0.$$

If not, let $\min_M u = -\lambda$, $\lambda > 0$, then

$S(u+\lambda) > 0$, $u+\lambda \geq 0$ and touches zero somewhere.

Hence by strong maximum principle $u + \lambda \equiv 0$, a contradiction. It follows that $G_S \geq 0$. By strong maximum principle we have $G_S > 0$.

Assume $Y(g) > 0, Q > 0$, we need

$$Pu \geq 0 \Longrightarrow u \geq 0.$$

If not let $u(p) = \min_M u = -\lambda$, $\lambda > 0$, then

$$P(u+\lambda) > 0,\ u+\lambda \geq 0 \text{ and } u(p)+\lambda = 0.$$

How to rule this out? The crucial equality

$$P_q H(p,q) = \delta_p(q) - \Gamma_1(p,q).$$

This equality is closely related to [Humbert-Raulot 2009]. Recall

$$H(p,q) = \frac{2^{\frac{n-6}{n-2}}(n-1)^{\frac{n-4}{n-2}}}{n^{\frac{2}{n-2}}(n-2)(n-4)\,\omega_n^{\frac{2}{n-2}}} G_L(p,q)^{\frac{n-4}{n-2}}.$$

and

$$\Gamma_1(p,q) = \frac{2^{\frac{n-6}{n-2}}(n-1)^{\frac{n-4}{n-2}}}{n^{\frac{2}{n-2}}(n-2)^3\,\omega_n^{\frac{2}{n-2}}} G_L(p,q)^{\frac{n-4}{n-2}} \left| Rc_{G_{L,p}^{\frac{4}{n-2}}g} \right|_g^2 (q).$$

Hence

$$\begin{aligned} &\int_M H(p,q)\,P(u+\lambda)(q)\,d\mu(q) \\ = \ & -\int_M \Gamma_1(p,q)(u+\lambda)(q)\,d\mu(q). \end{aligned}$$

A contradiction.

How to get formulas for G_P? For any $\varphi \in C^\infty(M)$,

$$T_H(P\varphi) = \varphi - T_{\Gamma_1}\varphi.$$

If $r_\sigma\left(T_{\Gamma_1}\right) < 1$, then

$$\varphi = \left(I - T_{\Gamma_1}\right)^{-1} T_H(P\varphi).$$

Then use geometric series expansion.

How to get positive mass for G_P? For locally conformally flat manifold or $n = 5, 6, 7$,

$$G_{P,p} - H_p = A + o(1).$$

Since

$$P\left(G_{P,p}-H_p\right)=\Gamma_{1,p},$$

we see

$$A=\int_M G_P\left(p,q\right)\Gamma_1\left(p,q\right)d\mu\left(q\right).$$

This is exactly the formula in [Humbert-Raulot 2009].

Remark 9 *There is a difference between $R>0, Q>0$ and $Y\left(g\right)>0, Q>0$.*

Assume $Y\left(g\right)>0, Q>0$, then $\ker P=0$ and $G=G_P>0$. To solve

$$Pu=u^{\frac{n+4}{n-4}},\quad u>0.$$

We write it as

$$u=T_G u^{\frac{n+4}{n-4}}.$$

Let $f=u^{\frac{n+4}{n-4}}$, it becomes

$$T_G f=f^{\frac{n-4}{n+4}},\quad f>0.$$

This can be solved by

$$\Theta_4(g) = \sup_{f \in L^{\frac{2n}{n+4}}(M)\setminus\{0\}} \frac{\int_M T_G f \cdot f d\mu}{\|f\|^2_{L^{\frac{2n}{n+4}}}}.$$

It has similar structure as the solution to Yamabe problem. $\Theta_4(g) \geq \Theta_4(S^n)$, with equality iff (M, g) is conformal equivalent to S^n. Hence $\Theta_4(g)$ is always achieved.

Note

$$\Theta_4(g) = \frac{2}{n-4} \sup_{\widetilde{g} \in [g]} \frac{\int_M \widetilde{Q} d\widetilde{\mu}}{\left\|\widetilde{Q}\right\|^2_{L^{\frac{2n}{n+4}}(\widetilde{\mu})}}.$$

[Gursky-Hang-Lin 2015]

$n \geq 6, L_g > 0, P_g > 0 \Rightarrow \exists \tilde{g} \in [g]$ with $\widetilde{R} > 0, \widetilde{Q} > 0$.

To begin, note

$$\begin{aligned} \sigma_2(A) &= \frac{1}{2}\left(J^2 - |A|^2\right); \\ Q &= -\Delta J + \frac{n-4}{2} J^2 + 4\sigma_2(A), \end{aligned}$$

Hence

$$\int_M Q d\mu = \frac{n-4}{2} \int_M J^2 d\mu + 4 \int_M \sigma_2(A) d\mu.$$

For $\lambda \geq 1$ consider the functional

$$\frac{n-4}{2} \lambda \int_M J_g^2 d\mu_g + 4 \int_M \sigma_2(A_g) d\mu_g.$$

A critical metric of this functional restricted to the space of conformal metrics of unit volume satisfies

$$\lambda\left(-\Delta J + \frac{n-4}{2} J^2\right) + 4\sigma_2(A) = const.$$

Fix $\lambda_0 \gg 1$ such that there exists $g_0 = u_0^{\frac{4}{n-4}}$ satisfying

$$\lambda_0\left(-\Delta_0 J_0 + \frac{n-4}{2} J_0^2\right) + 4\sigma_2(A_0) > 0$$

and $J_0 > 0$. Define f as

$$\lambda_0 \left(-\Delta_0 J_0 + \frac{n-4}{2} J_0^2 \right) + 4\sigma_2 \left(A_0 \right) = f u_0^{-\frac{n+4}{n-4}}.$$

Then for $1 \leq \lambda \leq \lambda_0$ try to solve

$$\lambda \left(-\widetilde{\Delta} J + \frac{n-4}{2} \widetilde{J}^2 \right) + 4\sigma_2 \left(\widetilde{A} \right) = f u^{-\frac{n+4}{n-4}}, \ \widetilde{g} = u^{\frac{4}{n-4}} g$$

by method of continuity. This approach has similar spirit as [Chang-Gursky-Yang 2002].

4 Q curvature equation in dimension 3

On $\left(M^3, g\right)$,

$$Q = -\frac{1}{4}\Delta R - 2\left|Rc\right|^2 + \frac{23}{32}R^2;$$

and

$$P\varphi = \Delta^2\varphi + 4\operatorname{div}\left(Rc\left(\nabla\varphi, e_i\right)e_i\right) - \frac{5}{4}\operatorname{div}\left(R\nabla\varphi\right) - \frac{1}{2}Q\varphi.$$

The transformation law is

$$P_{\rho^{-4}g}\varphi = \rho^7 P_g\left(\rho\varphi\right).$$

Hence seeking $\widetilde{g} \in [g]$ with $\widetilde{Q} = const$ is the same as

$$P_g u = const \cdot u^{-7}, \quad u > 0.$$

Example 10 *On S^3,*

$$\begin{aligned} Q &= \frac{15}{8}, \\ P &= \Delta^2 + \frac{1}{2}\Delta - \frac{15}{16} = \left(-\Delta + \frac{3}{4}\right)\left(-\Delta - \frac{5}{4}\right). \end{aligned}$$

$\lambda_1(P) = -\frac{15}{16}$, $\lambda_2(P) = \frac{105}{16} > 0$. *Let* N *be north pole,* $x = \pi_N$*, then*

$$G_N = -\frac{1}{4\pi}\frac{1}{\sqrt{1+|x|^2}}.$$

Recall on $\mathbb{R}^3$*, the fundamental solution of* Δ^2 *is* $-\frac{r}{8\pi}$.

Example 11 *On* $S^2 \times S^1$, $Q = -\frac{9}{8}$, $P > 0$.

Example 12 *If* $\ker P = 0$*, then under the conformal normal coordinate at* p,

$$G_p = A - \frac{r}{8\pi} + \sum_{i=1}^{3} a_i x_i + O\left(r^2\right).$$

Note $PG_p = \delta_p \in H^{-2}$*, hence* $G_p \in H^2(M)$.

Assume $Y(g) > 0$,

$$\begin{aligned} H(p,q) &= -\frac{G_L(p,q)^{-1}}{256\pi^2}, \\ \Gamma_1(p,q) &= \frac{G_L(p,q)^{-1}}{256\pi^2}\left|Rc_{G_{L,p}^4 g}\right|_g^2(q). \end{aligned}$$

Then

$$\Gamma_1 (p, q) = O \left(\overline{pq}^{-1} \right)$$

and

$$P_q H (p, q) = \delta_p (q) - \Gamma_1 (p, q) .$$

[Hang-Yang 2015] Assume $Y (g) > 0$, then the following statements are equivalent

1. there exists a $\widetilde{g} \in [g]$ with $\widetilde{Q} > 0$.

2. $\ker P = 0$ and the Green's function $G_P (p, q) < 0$ for $p \neq q$.

3. $\ker P = 0$ and there exists $p \in M$ such that $G_P (p, q) < 0$ for $q \neq p$.

4. $r_\sigma \left(T_{\Gamma_1} \right) < 1$.

Moreover if $r_\sigma\left(T_{\Gamma_1}\right) < 1$, then

$$G_P = H + \sum_{k=1}^{\infty} \Gamma_k * H,$$

In particular, $G_P \le -\frac{G_L^{-1}}{256\pi^2}$; if equality holds somewhere, then (M, g) is conformal equivalent to the standard S^3.

Example 13 *If $Y(g) > 0$, $r_\sigma\left(T_{\Gamma_1}\right) < 1$ and (M, g) is not conformal equivalent to S^3, then the set $\left\{\widetilde{g} \in [g] : \widetilde{Q} = 1\right\}$ is nonempty and compact.*

Indeed, $G = G_P < 0$. Let $\widetilde{g} = u^{-4} g$, then

$$Pu = -\frac{1}{2} u^{-7}, \quad u > 0.$$

In another way,

$$\begin{aligned} u(p) &= \int_M \left(-\frac{G(p,q)}{2}\right) u(q)^{-7} \, d\mu(q) \\ &\sim \int_M u^{-7} d\mu, \end{aligned}$$

hence $0 < c_1 \leq u \leq c_2$. *This gives compactness. Degree theory gives existence.*

Problem 14 *Find a conformal invariant condition which is equivalent to the existence of a* $\widetilde{g} \in [g]$ *with* $\widetilde{R} > 0$ *and* $\widetilde{Q} > 0$.