Fourth order Paneitz operator and $Q$ curvature equation

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1 What is $Q$ curvature?

Given $(M^n, g)$, $n \geq 3$,

\[
Q = -\frac{1}{2(n-1)} \Delta R - \frac{2}{(n-2)^2} |Rc|^2 + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} R^2. 
\]

To simplify the formulas, in conformal geometry people use

\[
J = \frac{R}{2(n-1)}, \quad A = \frac{1}{n-2} (Rc - Jg). 
\]

Then

\[
Q = -\Delta J - 2 |A|^2 + \frac{n}{2} f^2. 
\]

The Paneitz operator [Paneitz 1983]

\[
P\varphi = \Delta^2 \varphi + \text{div} \left( 4A (\nabla \varphi, e_i) e_i - (n-2) J \nabla \varphi \right) + \frac{n-4}{2} Q \varphi. 
\]

Here $e_1, \cdots, e_n$ is a local orthonormal frame.
For $n \neq 4$, 

$$P_{\frac{4}{\rho^{n-4}g}} \varphi = \rho^{-\frac{n+4}{n-4}} P_g(\rho \varphi).$$

In particular

$$Q_{\frac{4}{\rho^{n-4}g}} = \frac{2}{n-4} \rho^{-\frac{n+4}{n-4}} P_g \rho.$$  

This can be compared to

$$L = -\frac{4(n-1)}{n-2} \Delta + R,$$

which satisfies

$$L_{\frac{4}{\rho^{n-2}g}} \varphi = \rho^{-\frac{n+2}{n-2}} L_g(\rho \varphi)$$

and

$$R_{\frac{4}{\rho^{n-2}g}} = \rho^{-\frac{n+2}{n-2}} L_g \rho.$$  

For $n = 4$, 

$$P_{e^{2w}g} \varphi = e^{-4w} P_g \varphi.$$
and

\[ Q_{e^{2w}g} = e^{-4w} (P_g w + Q_g) . \]

Moreover

\[ \int_M Q d\mu + \frac{1}{4} \int_M |W|^2 \, d\mu = 8\pi^2 \chi (M) . \]

This can be compared to Laplacian on surface

\[ \Delta_{e^{2w}g} \varphi = e^{-2w} \Delta_g \varphi \]

and

\[ K_{e^{2w}g} = e^{-2w} (K_g - \Delta_g w) . \]

There are higher order analogue called GJMS operator with \((-\Delta)^m\) as leading term. There are even fractional order GJMS operators by scattering theory (Branson, Fefferman-Graham, Graham-Zworski, Juhl …).
2 \( Q \) curvature equation in dimension 4

Problems in spectral geometry motivate the analytical study of Paneitz operator in dimension 4 [Branson-Chang-Yang, 1992], [Chang-Yang 1995]. Paneitz operator appears in the log determinant of conformal covariant operators. One typical breakthrough made along the way is

**Theorem 1 (Chang-Gursky-Yang 2002)**  If \((M^4, g)\) is compact with \(Y(g) > 0\) and \(\int_M Qd\mu > 0\), then there exists a \(\tilde{g} \in [g]\) such that \(\tilde{Rc} > 0\).

Both \(Y(g)\) and \(\int_M Qd\mu\) are conformal invariants. The study also reveals the usefulness of \(\sigma_2(A)\) equation.

[Djadli-Hebey-Ledoux 2000] started the question of finding constant \(Q\) curvature in a fixed conformal class (or
closely related prescribing \( Q \) curvature problem) in dimension \( n \geq 5 \). [Xu-Yang 2002] started similar problem for dimension 3.

**Theorem 2 (Ge-Lin-Wang, Catino-Djadli 2010)** Assume \((M^3, g)\) is compact with \( R > 0, Q > 0 \), then there exists a \( \tilde{g} \in [g] \) such that \( \tilde{R}c > 0 \).
Let \((M^n, g)\) be smooth and compact, \(K : M \times M \to \mathbb{R}\), \(\varphi : M \to \mathbb{R}\),

\[(T_K \varphi)(p) = \int_M K(p, q) \varphi(q) \, d\mu(q).\]

Given another \(K' : M \times M \to \mathbb{R}\),

\[(K * K')(p, q) = \int_M K(p, s) K'(s, q) \, d\mu(s).\]

This makes

\[T_K T_{K'} = T_{K * K'}.\]

Denote

\[Y(g) = \inf_{\tilde{g} \in [g]} \tilde{\mu}(M)^{-\frac{n-2}{n}} \int_M \tilde{R} \, d\tilde{\mu}.\]
Assume $Y (g) > 0$, $G_L$ is the Green's function of $L$, $G_{L,p} (q) = G_L (p, q)$. Denote

$$H (p, q) = \frac{2^{n-6}}{n^{n-2} (n - 2)(n - 4)} G_L (p, q) \frac{n-4}{2} \frac{2^{n-2} (n - 1)^{n-2}}{n^{n-2} (n - 2) (n - 4) \omega_n^{n-2}} G_L (p, q)^{n-4}.$$ 

and

$$\Gamma_1 (p, q) = \frac{2^{n-6} (n - 1)^{n-2}}{n^{n-2} (n - 2)^3 \omega_n^{n-2}} G_L (p, q) \frac{n-4}{2} \frac{2^{n-2} (n - 1)^{n-2}}{n^{n-2} (n - 2) (n - 4) \omega_n^{n-2}} G_L (p, q)^{n-4} \left| \text{Rc} \frac{4}{G_{L,p}^n} g \right| \left( g \right).$$

$\omega_n$ is the volume of unit ball in $\mathbb{R}^n$. Note $G_{L,p}^n g$ is the stereographic projection at $p$. Then

$$\Gamma_1 (p, q) = O \left( \frac{pq^{4-n}}{} \right).$$

If $\tilde{g} = \rho G_{L, p}^n g$, then

$$T_{\Gamma_1} (\varphi) = \rho^{-1} T_{\Gamma_1} (\rho \varphi).$$

They have the same spectrum and spectral radius i.e.

$$\sigma \left( T_{\Gamma_1} \right) = \sigma \left( T_{\Gamma_1} \right) \quad \text{and} \quad r \sigma \left( T_{\Gamma_1} \right) = r \sigma \left( T_{\Gamma_1} \right).$$
Theorem 3 (Hang-Yang 2015) Assume $Y (g) > 0$, then the following statements are equivalent

1. there exists a $\tilde{g} \in [g]$ with $\tilde{Q} > 0$.

2. ker $P = 0$ and the Green’s function $G_P (p, q) > 0$ for $p \neq q$.

3. ker $P = 0$ and there exists $p \in M$ such that $G_P (p, q) > 0$ for $q \neq p$.

4. $r_\sigma (T_{\Gamma_1}) < 1$.

Moreover if $r_\sigma (T_{\Gamma_1}) < 1$, then

$$G_P = H + \sum_{k=1}^{\infty} \Gamma_k \ast H,$$

here $\Gamma_k = \Gamma_1 \ast \ldots \ast \Gamma_1$ ($k$ times). In particular, $G_P \geq H$, moreover if $G_P (p, q) = H (p, q)$ for some $p \neq q$, then $(M, g)$ is conformal equivalent to the standard $S^n$. 
If \( \ker P = 0 \), \( \tilde{g} = \rho^{\frac{n-4}{4}} g \), then
\[
 G_{	ilde{P}}(p, q) = \rho(p)^{-1} \rho(q)^{-1} G_P(p, q). 
\]

The above statement can be compared to
\[
 \exists \tilde{g} \in [g] \text{ with } \tilde{R} > 0 \iff \lambda_1(L_g) > 0 \iff Y(g) > 0,
\]
an observation of [Kazdan-Warner 1975]; and [Aubin 1974] about Green’s function of the Laplacian. For relations between \( L \) and \( P \), [Hijazi-Raulot 2007] shows if \( Y(g) > 0 \), then
\[
 \lambda_1(L_g)^2 \geq \frac{16n(n-1)^2}{(n+2)(n-2)(n-4)} \lambda_1(P_g),
\]
equality holds iff \( g \) is Einstein metric.

**Theorem 4 ([Gursky-Malchiodi, Hang-Yang 2015])**
If \( Y(g) > 0 \), \( r_\sigma(T_{\tilde{1}}) < 1 \), then there exists a \( \tilde{g} \in [g] \) with \( \tilde{Q} = 1 \). Such \( \tilde{g} \) can be found as extremal metric of some functionals. If in addition we know \( R > 0 \) and \( Q > 0 \), then \( \tilde{R} > 0 \).
To find metrics with both positive scalar and $Q$ curvature,

**Theorem 5** (Gursky-Hang-Lin 2015) *For* $n \geq 6$,

$\exists \tilde{g} \in [g]$ *with* $\tilde{R} > 0$ *and* $\tilde{Q} > 0 \iff L_g > 0$ *and* $P_g > 0$.

For some reason the approach does not work for $n = 5$. But it seems the statement should valid for $n = 5$ too.

Recall

$Y (g) = \inf_{\substack{u \in C^\infty (M) \\ u > 0}} \frac{\int_M \left( \frac{4(n-1)}{n-2} |\nabla u|^2 + Ru^2 \right) d\mu}{\| u \|^2_{L^{2n} / L^{n-2}}}$

$= \inf_{u \in H^1 (M) \setminus \{0\}} \frac{\int_M \left( \frac{4(n-1)}{n-2} |\nabla u|^2 + Ru^2 \right) d\mu}{\| u \|^2_{L^{2n} / L^{n-2}}}$,

by the fact $u \in H^1 (M) \Rightarrow |u| \in H^1 (M)$ (this simple fact is the reason why the first eigenfunction of $L_g$ is strictly positive or negative, it is also a basic block for
DeGiorgi-Nash-Moser theory for second order scalar elliptic equations). In particular if the minimizer exists, it must be strictly positive or negative.

[Aubin 1976] shows $Y (g) \leq Y (S^n)$; if $Y (g) < Y (S^n)$, then $Y (g)$ is achieved; for $n \geq 6$, $g$ not locally conformally flat, then $Y (g) < Y (S^n)$. [Schoen 1984] solves all the remain case by applying positive mass theorem on stereographic projection.

To solve

$$Pu = \text{const} \cdot u^{\frac{n+4}{n-4}}, \quad u > 0;$$

let

$$Y_4 (g) = \inf_{u \in H^2 (M) \backslash \{0\}} \frac{\int_M P u \cdot u d\mu}{\| u \|_{L^n}^{2n}} ,$$

then $Y_4 (g) \leq Y_4 (S^n)$; if $Y_4 (g) < Y_4 (S^n)$, then $Y_4 (g)$ is achieved; for $n \geq 8$, $g$ not locally conformally flat, then $Y_4 (g) < Y_4 (S^n)$. Unfortunately $u \in H^2 (M) \not\ni |u| \in H^2 (M)$, how can we know the minimizer is strictly
positive or negative? It needs positivity to qualify for being conformal factors.

[Robert 2009] made an observation in an unpublished lecture notes: if $P > 0$, $G_P > 0$ and $Y_4 (g)$ is achieved, then the minimizer must be strictly positive or negative.

In fact $u$ is a minimizer with $\|u\| \frac{2n}{L^{n-4}} = 1$ and $u^+ \neq 0$, then

$$Pu = Y_4 (g) |u|^{\frac{8}{n-4}} u.$$ 

Let $Pv = |Pu|$, then $v > 0$ and $|u| \leq v$. 

$$Y_4 (g) \leq \frac{\int_M Pv \cdot vd\mu}{\|v\|^2 \frac{2n}{L^{n-4}}} = Y_4 (g) \frac{\int_M |u|^{\frac{n+4}{n-4}} vd\mu}{\|v\|^2 \frac{2n}{L^{n-4}}}$$

$$\leq Y_4 (g) \|v\|^{-\frac{1}{L^{n-4}}} \|u\|^{\frac{2n}{L^{n-4}}} \leq Y_4 (g).$$

Hence $\|v\| \frac{2n}{L^{n-4}} = 1 = \|u\| \frac{2n}{L^{n-4}}$ and $u = v$.

[Humbert-Raulot 2009] If $(M, g)$ is locally conformally flat or $n = 5, 6, 7$, ker $P = 0$, then under the conformal
normal coordinate at \( p \),

\[
2n (n - 2) (n - 4) \omega_n G_{p,p} = r^{4-n} + A + O(4) (r).
\]

If \( Y (g) > 0 \), \( G_{p,p} > 0 \), then

\[
A = c(n) \int_M G_{p,p} G_{L,p}^{\frac{n-4}{n-2}} \left. Rc \frac{4}{G_{L,p}^{\frac{n-2}{4}}} g \right|_g^2 \, d\mu \geq 0.
\]

\( A = 0 \) iff \((M, g)\) is conformal equivalent to the standard \( S^n \). Originally Humbert-Raulot only considered locally conformally flat manifolds, Gursky-Malchiodi pointed out their argument works in dimension 5, 6, 7.

Indeed on \( M \setminus \{p\} \),

\[
P_g G_{p,p} = 0.
\]

Use conformal covariant property,

\[
P \frac{4}{G_{L,p}^{\frac{n-2}{4}}} g \left( \frac{-n-4}{n-2} G_{L,p} G_{p,p} \right) = 0.
\]

Integrate the equation on \( M \setminus B_{g,\varepsilon} (p) \) with respect to 
\[
d\mu \frac{4}{G_{L,p}^{\frac{n-2}{4}}} g
\]
and let \( \varepsilon \to 0^+ \) we get the needed identity.
How can we know $G_P > 0$? Green’s function for fourth order elliptic operator is not as simple as second order ones.

**Example 6** On $S^1$ consider the operator $u \mapsto u^{(4)} + \lambda u$, $\lambda > 0$, we have

$$G(x, y) = \frac{1}{2\sqrt{\lambda}} \text{Im} \left( \frac{\cosh \alpha (\pi - xy)}{\alpha \sinh \pi \alpha} \right)$$

$$= \frac{\mu}{2\lambda (\cosh 2\mu \pi - \cos 2\mu \pi)} [\cosh \mu xy \sin \mu (2\pi - xy) + \cosh \mu (2\pi - xy) \sin \mu xy + \sinh \mu xy \cos \mu (2\pi - xy) + \sinh \mu (2\pi - xy) \cos \mu xy].$$

Here $\mu = \sqrt[4]{\lambda/4}$, $\alpha = \mu + \mu i$. Note

$$G(1, -1) = \frac{\mu (\cosh \mu \pi \sin \mu \pi + \sinh \mu \pi \cos \mu \pi)}{\lambda (\cosh 2\mu \pi - \cos 2\mu \pi)}.$$

If $\mu = 2k + 1$, $k \in \mathbb{Z}_+$, then $G(1, -1) < 0$. Indeed careful study shows if $\lambda \geq 4$, $G$ is negative somewhere.
Example 7  If $Rc = (n - 1) g$, then

\[ Q = \frac{n(n - 2)(n + 2)}{8}, \]
\[ P = \left( -\Delta + \frac{n(n - 2)}{4} \right) \left( -\Delta + \frac{(n + 2)(n - 4)}{4} \right). \]

Hence $P > 0$ and

\[ G_P = G_{-\Delta + \frac{(n+2)(n-4)}{4}} * G_{-\Delta + \frac{n(n-2)}{4}} > 0. \]

This is basically applying maximum principle twice. On $S^n$, let $N$ be the north pole, $x = \pi_N$ as the coordinate, then

\[ G_{P,N} = \frac{(|x|^2 + 1)^{\frac{n-4}{2}}}{n(n - 2)(n - 4) 2^{n-3} \omega_n}. \]

Example 8  If

\[ \Delta^2 u = \frac{n+4}{u^{n-4}}, \quad u > 0, \]

then by [Lin 1998] using method of moving plane and
applying maximum principle twice, we have

\[ u = c_n \left( \frac{\lambda}{|x - x_0|^2 + \lambda^2} \right)^{\frac{n-4}{2}} \text{ for some } \lambda > 0. \]

[Chen-Li-Ou 2006] achieve this without using maximum principle, namely if for \( 0 < \alpha < n, \)

\[ u(x) = \int_{\mathbb{R}^n} \frac{u(y)^{\frac{n+\alpha}{n-\alpha}}}{|x - y|^{n-\alpha}} dy, \quad u > 0, \]

then

\[ u = c_{n,\alpha} \left( \frac{\lambda}{|x - x_0|^2 + \lambda^2} \right)^{\frac{n-\alpha}{2}} \text{ for some } \lambda > 0. \]

They develop the integral form of the method of moving planes based on the property of the kernel. This approach, together with [Schoen-Yau 1988]’s result on locally conformally flat manifolds and Kleinian groups, enable [Qing-Raske 2006] to solve the \( Q \) curvature equation for locally conformally flat manifolds with \( Y(g) > 0, Q > 0. \) These are all based on explicit formulas of the Green’s function.
[Gursky-Malchiodi 2014] makes a breakthrough: If \( R > 0, Q > 0 \), then \( P > 0, G_P > 0 \); moreover if \( \tilde{g} \in [g] \) satisfies \( \tilde{Q} > 0 \), then \( \tilde{R} > 0 \).

Method: try to show

\[
Pu \geq 0 \Rightarrow u \geq 0.
\]

For \( \lambda > 0 \), let \( u_\lambda = u + \lambda, \ g_\lambda = u_\lambda^{\frac{4}{n-4}} g \).

\[
Q_\lambda = \frac{2}{n-4} u_\lambda^{\frac{n+4}{n-4}} Pu_\lambda > 0;
\]

i.e.

\[
-\Delta_\lambda J + \frac{n}{2} J_\lambda^2 \geq Q_\lambda > 0.
\]

By method of continuity \( J_\lambda > 0 \). Hence \( u_\lambda \) is super-harmonic. By strong maximum principle, \( u_\lambda > 0 \) for all \( \lambda > 0 \). Hence \( u \geq 0 \).

[Hang-Yang 2014] Recall how we deal with second order operators. Given \( Su = -\Delta u + cu, \ c > 0 \). We need

\[
Su \geq 0 \Rightarrow u \geq 0.
\]
If not, let \( \min_M u = -\lambda, \lambda > 0 \), then
\( S(u + \lambda) > 0, \ u + \lambda \geq 0 \) and touches zero somewhere.

Hence by strong maximum principle \( u + \lambda \equiv 0 \), a contradiction. It follows that \( G_S \geq 0 \). By strong maximum principle we have \( G_S' > 0 \).

Assume \( Y(g) > 0, Q > 0 \), we need
\[
Pu \geq 0 \implies u \geq 0.
\]
If not let \( u(p) = \min_M u = -\lambda, \lambda > 0 \), then
\[
P(u + \lambda) > 0, \ u + \lambda \geq 0 \) and \( u(p) + \lambda = 0 \).

How to rule this out? The crucial equality
\[
P_q H(p, q) = \delta_p(q) - \Gamma_1(p, q).
\]
This equality is closely related to [Humbert-Raulot 2009].
Recall
\[
H(p, q) = \frac{2^{n-6} (n - 1)^{n-4}}{n^{n-2} (n - 2)(n - 4) \omega_n^{n-2}} G_L(p, q)^{n-4}.
\]
and
\[
\Gamma_1 (p, q) = \frac{2^{n-6} (n - 1)^{n-2}}{n^{n-2} (n - 2)^3 \omega_n^{n-2}} G_L (p, q)^{n-4} \bigg| \frac{Rc}{GL, p} \bigg|_g^{n-2} (q).
\]

Hence
\[
\int_M H (p, q) P (u + \lambda) (q) d\mu (q) = - \int_M \Gamma_1 (p, q) (u + \lambda) (q) d\mu (q).
\]

A contradiction.

How to get formulas for \( G_P \)? For any \( \varphi \in C^\infty (M) \),
\[
T_H (P\varphi) = \varphi - T\Gamma_1 \varphi.
\]

If \( r_\sigma (T\Gamma_1) < 1 \), then
\[
\varphi = (I - T\Gamma_1)^{-1} T_H (P\varphi).
\]

Then use geometric series expansion.

How to get positive mass for \( G_P \)? For locally conformally flat manifold or \( n = 5, 6, 7 \),
\[
G_{P, p} - H_p = A + o (1).
\]
Since
\[ P \left( G_{P,p} - H_p \right) = \Gamma_{1,p}, \]
we see
\[ A = \int_M G_P (p, q) \Gamma_1 (p, q) \, d\mu (q). \]
This is exactly the formula in [Humbert-Raulot 2009].

**Remark 9** There is a difference between \( R > 0, Q > 0 \) and \( Y (g) > 0, Q > 0 \).

Assume \( Y (g) > 0, Q > 0 \), then \( \ker P = 0 \) and \( G = G_P > 0 \). To solve
\[ Pu = u^{n+4}, \quad u > 0. \]
We write it as
\[ u = T_G u^{n+4}. \]
Let \( f = u^{n+4} \), it becomes
\[ T_G f = f^{\frac{n-4}{n+4}}, \quad f > 0. \]
This can be solved by

\[ \Theta_4(g) = \sup_{f \in L^{2n+4}(M) \setminus \{0\}} \frac{\int_M T_G f \cdot f \, d\mu}{\|f\|^2 L^{2n} \frac{2n}{2n+4}}. \]

It has similar structure as the solution to Yamabe problem. \( \Theta_4(g) \geq \Theta_4(S^n) \), with equality iff \((M, g)\) is conformal equivalent to \(S^n\). Hence \(\Theta_4(g)\) is always achieved.

Note

\[ \Theta_4(g) = \frac{2}{n-4} \sup_{\tilde{g} \in [g]} \frac{\int_M \tilde{Q} d\tilde{\mu}}{\|\tilde{Q}\|^2 L^{2n} \frac{2n}{2n+4}(\tilde{\mu})}. \]
To begin, note

\[
\sigma_2(A) = \frac{1}{2} \left( J^2 - |A|^2 \right);
\]

\[
Q = -\Delta J + \frac{n - 4}{2} J^2 + 4\sigma_2(A),
\]

Hence

\[
\int_M Q d\mu = \frac{n - 4}{2} \int_M J^2 d\mu + 4 \int_M \sigma_2(A) d\mu.
\]

For \(\lambda \geq 1\) consider the functional

\[
\frac{n - 4}{2} \lambda \int_M J^2 d\mu_g + 4 \int_M \sigma_2(A_g) d\mu_g.
\]

A critical metric of this functional restricted to the space of conformal metrics of unit volume satisfies

\[
\lambda \left( -\Delta J + \frac{n - 4}{2} J^2 \right) + 4\sigma_2(A) = \text{const}.
\]

Fix \(\lambda_0 \gg 1\) such that there exists \(g_0 = u_0^{\frac{n - 4}{4}}\) satisfying

\[
\lambda_0 \left( -\Delta_0 J_0 + \frac{n - 4}{2} J_0^2 \right) + 4\sigma_2(A_0) > 0
\]
and $J_0 > 0$. Define $f$ as

$$
\lambda_0 \left(-\Delta_0 J_0 + \frac{n - 4}{2} J_0^2\right) + 4\sigma_2 \left(A_0\right) = f u_0^{-\frac{n+4}{n-4}}.
$$

Then for $1 \leq \lambda \leq \lambda_0$ try to solve

$$
\lambda \left(-\widetilde{\Delta} J + \frac{n - 4}{2} \widetilde{J}^2\right) + 4\sigma_2 \left(\widetilde{A}\right) = f u^{-\frac{n+4}{n-4}}, \quad \tilde{g} = u^{\frac{4}{n-4}} g
$$

by method of continuity. This approach has similar spirit as [Chang-Gursky-Yang 2002].
4 \( Q \) curvature equation in dimension 3

On \((M^3, g)\),

\[
Q = -\frac{1}{4} \Delta R - 2 |Rc|^2 + \frac{23}{32} R^2;
\]

and

\[
P\phi = \Delta^2 \phi + 4 \text{div} (Rc (\nabla \phi, e_i) e_i) - \frac{5}{4} \text{div} (R \nabla \phi) - \frac{1}{2} Q \phi.
\]

The transformation law is

\[
P_{\rho^{-4} g} \phi = \rho^7 P_g (\rho \phi).
\]

Hence seeking \( \tilde{g} \in [g] \) with \( \tilde{Q} = \text{const} \) is the same as

\[
P_g u = \text{const} \cdot u^{-7}, \quad u > 0.
\]

Example 10 On \( S^3 \),

\[
Q = \frac{15}{8},
\]

\[
P = \Delta^2 + \frac{1}{2} \Delta - \frac{15}{16} = \left( -\Delta + \frac{3}{4} \right) \left( -\Delta - \frac{5}{4} \right).
\]
\[ \lambda_1(P) = -\frac{15}{16}, \quad \lambda_2(P) = \frac{105}{16} > 0. \]

Let \( N \) be north pole, \( x = \pi_1N \), then

\[ G_N = -\frac{1}{4\pi} \frac{1}{\sqrt{1 + |x|^2}}. \]

Recall on \( \mathbb{R}^3 \), the fundamental solution of \( \Delta^2 \) is \(-\frac{r}{8\pi}\).

**Example 11** On \( S^2 \times S^1 \), \( Q = -\frac{9}{8}, \quad P > 0 \).

**Example 12** If \( \ker P = 0 \), then under the conformal normal coordinate at \( p \),

\[ G_p = A - \frac{r}{8\pi} + \sum_{i=1}^{3} a_i x_i + O \left( r^2 \right). \]

Note \( PG_p = \delta_p \in H^{-2} \), hence \( G_p \in H^2(M) \).

Assume \( Y(g) > 0 \),

\[ H(p, q) = -\frac{G_L(p, q)^{-1}}{256\pi^2}, \]

\[ \Gamma_1(p, q) = \frac{G_L(p, q)^{-1}}{256\pi^2} \left\| R_{G_L, p^g} \right\|^2_g(q). \]
Then

\[ \Gamma_1 (p, q) = O \left( \frac{1}{pq} \right) \]

and

\[ P_q H (p, q) = \delta_p (q) - \Gamma_1 (p, q). \]

[Hang-Yang 2015] Assume \( Y (g) > 0 \), then the following statements are equivalent

1. there exists a \( \tilde{g} \in [g] \) with \( \tilde{Q} > 0 \).

2. \( \ker P = 0 \) and the Green’s function \( G_P (p, q) < 0 \) for \( p \neq q \).

3. \( \ker P = 0 \) and there exists \( p \in M \) such that \( G_P (p, q) < 0 \) for \( q \neq p \).

4. \( r_\sigma \left( T_{\Gamma_1} \right) < 1 \).
Moreover if \( r_\sigma \left( T_{\Gamma_1} \right) < 1 \), then

\[
G_P = H + \sum_{k=1}^{\infty} \Gamma_k * H,
\]

In particular, \( G_P \leq -\frac{G_L^{-1}}{256\pi^2} \); if equality holds somewhere, then \((M, g)\) is conformal equivalent to the standard \( S^3 \).

**Example 13** If \( Y(g) > 0 \), \( r_\sigma \left( T_{\Gamma_1} \right) < 1 \) and \((M, g)\) is not conformal equivalent to \( S^3 \), then the set \( \{ \tilde{g} \in [g] : \tilde{Q} = 1 \} \) is nonempty and compact.

Indeed, \( G = G_P < 0 \). Let \( \tilde{g} = u^{-4}g \), then

\[
Pu = -\frac{1}{2}u^{-7}, \quad u > 0.
\]

In another way,

\[
u(p) = \int_M \left( -\frac{G(p, q)}{2} \right) u(q)^{-7} d\mu(q) \\
\sim \int_M u^{-7} d\mu,
\]
hence $0 < c_1 \leq u \leq c_2$. This gives compactness. Degree theory gives existence.

**Problem 14** Find a conformal invariant condition which is equivalent to the existence of a $\tilde{g} \in [g]$ with $\tilde{R} > 0$ and $\tilde{Q} > 0$. 