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On the work of Charlie Fefferman

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Five Epic Years: 1969-1974

I. Thesis

II. Disc Multiplier

III. H^1-BMO Duality

IV. Mapping Theorem in $\mathbb{C}^n$ and Bergman Kernel

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I. Thesis

1. Strongly singular integrals

$$T(f) = f * K \qquad \text{on } \mathbb{R}^n.$$

Typical example: K distribution

$$K(x) = \frac{e^{\frac{i}{|x|}}}{|x|^n}, \quad 0 < |x| \le 1$$

$$= 0 \qquad |x| \ge 1$$

Previously known: T bounded on L^p, $1 < p < \infty$

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Q1: is T of weak-type $(1,1)$?

Q2: (The "super" strongly singular case). Suppose:

$T_\lambda = f * K_\lambda$, with $K_\lambda = \dfrac{e^{\frac{i}{|x|}}}{|x|^{n+\lambda}}$.

Is T_λ bounded on L^p when
$|1/p - 1/2| \le \frac{1}{2} - \lambda/n$, for $0 < \lambda \le n/2$?

Reformulation of Q1: For fixed θ, $0 \leq \theta < 1$:

$$\begin{cases} \widehat{K}(\xi) = 0\left(|\xi|^{-\theta n/2}\right), \quad \text{as } |\xi| \to \infty \\ \int\limits_{|x| \gtrsim |y|^{1-\theta}} |K(x-y) - K(x)| dx \leq A \end{cases}$$

(Above: $\theta = 1/2$. Note $\theta = 0$ is standard CZ situation.)

Theorem *If $Tf = f * K$ as above, then T is of weak-type* $(1,1)$.

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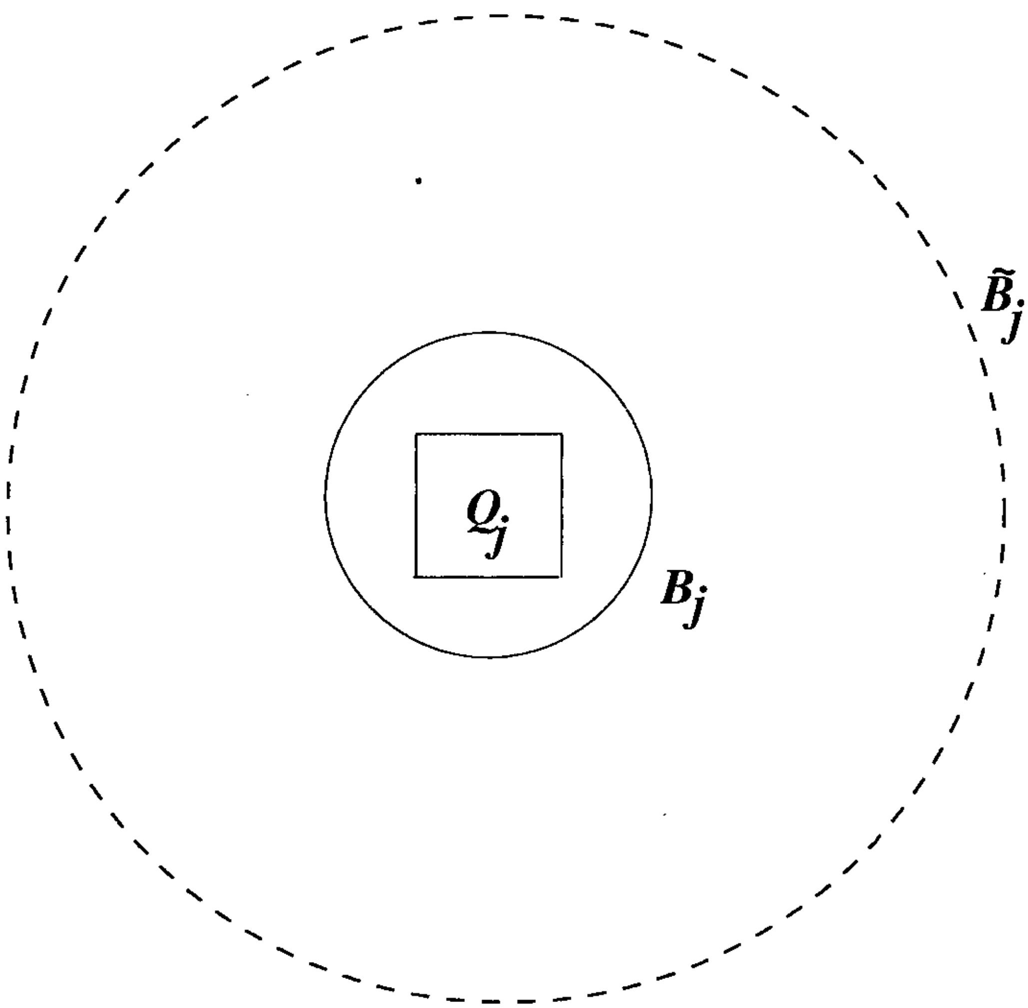
Fix $\alpha > 0$; decompose $f = g + \sum_j b_j$, (CZ).

- Estimate $T(g)$ via Plancherel
- b_j is supported in cube Q_j, and $\frac{1}{|Q_j|} \int_{Q_j} |b_j| \approx \alpha$.

Need to estimate $T(b) = \sum_j T(b_j)$ outside $\bigcup_j B_j$.
(May assume diam $Q_j \leq 1$).

<u>Problem</u> is inside $\widetilde{B}_j$, the ball concentric with B_j but diam $\widetilde{B}_j = (\text{diam } B_j)^{1-\theta}$.



- Idea: replace b_j by $\tilde{b}_j = b_j * \varphi_j$,

$$\varphi_j(x) = \delta_j^{-n} \varphi\left(x/\delta_j\right), \; \delta_j = \left(diam(B_j)\right)^{1/(1-\theta)} .$$

- $\displaystyle\int_{{}^cB_j} |b_j * K - \tilde{b}_j * K| dx \leq c \int_{Q_j} |b_j| dx .$

Suffices then to estimate $T(\tilde{b}) = \sum_j T(\tilde{b_j})$.

Now

$$\| T(\tilde{b}) \|_{L^2} \lesssim \| (1 - \triangle)^{-n\theta/4} \tilde{b} \|_{L^2} .$$

But

- $\| (1 - \triangle)^{-n\theta/4} \tilde{b} \|_{L^2} \lesssim \alpha \| b \|_{L^1}$.

Because

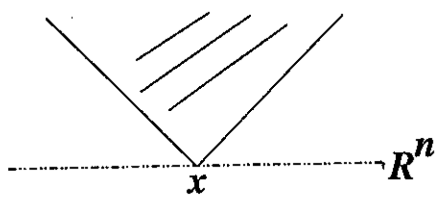
- $\| (1 - \triangle)^{-n\theta/4} \varphi_j \|_{L^2} \leq \frac{1}{m(Q_j)}$.

2. Square Functions

Some familiar square functions:

- $$S^2(f)(x) = \int_{\Gamma(x)} |\nabla u(x-y,t)|^2 \, t^{1-n} \, dy \, dt \,,$$

$u(x,t) = f * P_t =$ Poisson integral of f



, $\Gamma(x)$.

- Littlewood-Paley type: $\left(\sum |\triangle_k(f)|^2\right)^{1/2}$

$$\triangle_k(f)^{\wedge}(\xi) = \widehat{f}(\xi)\, \eta(2^{-k}\xi).$$

These are all bounded on L^p , $1 < p < \infty$.

A more intricate square function: g^*_λ,

$$(g^*_\lambda(f)(x))^2 = \int_{\mathbb{R}^{n+1}_+} |\nabla u(x-y,t)|^2 \left(\frac{t}{|y| + t}\right)^{n\lambda} t^{1-n} \, dy \, dt \,.$$

Majorizes both of the above.

What was known about $f \longrightarrow g^*_\lambda(f)$, $\lambda > 1$:

- maps $L^p \longrightarrow L^p$, if $1 < p < \infty$, and $2/\lambda < p$.

- fails when $p \leq 2/\lambda$.

Question: What can be said about $g^*_\lambda(f)$, when

$$f \in L^p, \quad p = 2/\lambda, \ 1 < p < 2, \ ?$$

Theorem $f \longrightarrow g^*(f)$ *is of weak-type* (p, p) *in this range.*

NOTE: this cannot hold for $p = 1$.

3. Bochner-Riesz

Let $S(f)$ be defined by

$$S(f)^{\wedge} = \widehat{f}(\xi)\,\chi_B(\xi)$$

where $\chi_B =$ characteristic function of unit ball B. Also

$$S^{\delta}(f)^{\wedge} = \widehat{f}(\xi)\,\chi_B(\xi)\,(1-|\xi|^2)^{\delta}.$$

Question (1) Is $S : L^p \longrightarrow L^p$,

when

$$\frac{2n}{n+1} < p < \frac{2n}{n-1}?$$

Question (2) Is $S^{\delta} : L^p \longrightarrow L^p$

if

$$\frac{2n}{n+1+2\delta} < p < \frac{2n}{n-1-2\delta}?$$

(For radial functions, these were known to hold.)

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Restriction Phenomenom:

$$(*)\quad \left(\int_{S^{n-1}} |\widehat{f}(\xi)|^2 \, d\sigma(\xi) \right)^{1/2} \leq A \, \| f \|_{L^p}$$

holds for a range of p's, $1 \leq p < p_0$.

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<u>Theorem</u> *Whenever the (L^p, L^2) restriction $(*)$ holds, then S^δ is bounded on L^p in the optimal range* (i.e. $\frac{2n}{n+1+2\delta} < p \leq 2$, ultimately $\left(1 \leq p \leq \frac{2(n+1)}{n+3}\right)$.

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<u>After-thought:</u> $n = 2$.

The restriction theorem:

$$\left(\int_{S^1} |\widehat{f}|^q \, d\sigma \right)^{1/q} \leq A \, \| f \|_{L^p(\mathbb{R}^2)}$$

holds for

$$1 \leq p < 4/3 \,, \qquad q = (1/3)p' \,.$$

NOTE: as $p \longrightarrow 4/3$, then $q \longrightarrow 4/3$.

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Following this:

Carleson-Sjölin show that when $n = 2$,

$$S^{\delta} : L^p \longrightarrow L^p, \quad 4/3 \leq p \leq 4, \text{ and } \delta > 0.$$

II. Ball Multiplier

Question: Is S bounded on L^p

(e.g. for $4/3 < p < 4$, when $n = 2$)?

Theorem *No! (for $p \neq 2$, $n \geq 2$).*

Background:

Let R_j denote rectangles in the plane. Define

$R_j(f)$, by $R_j(f)^\wedge(\xi) = \widehat{f}(\xi)\,\chi_{R_j}(\xi)$.

Question: Does one have

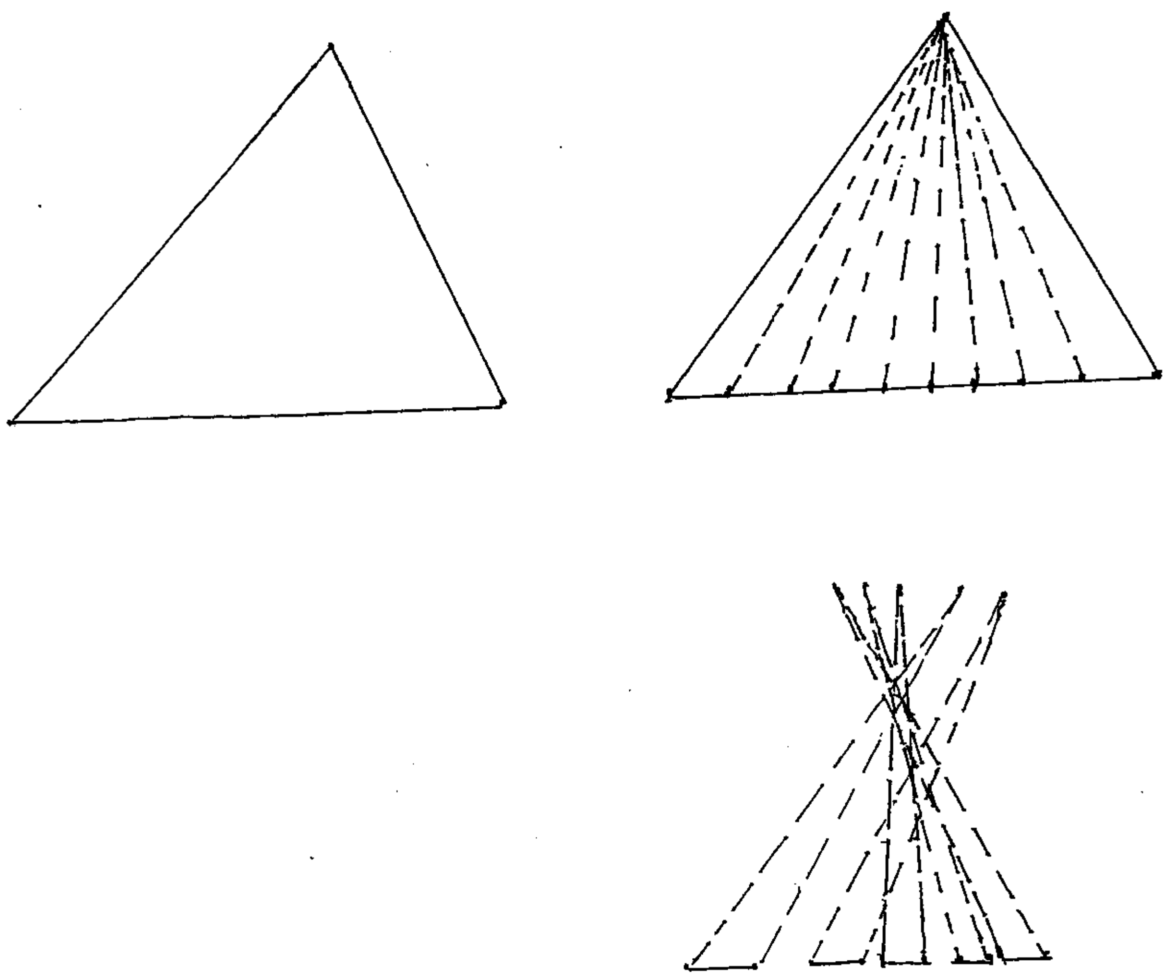
$$(**) \quad \left\| \left(\sum_j |R_j(f_j)|^2 \right)^{1/2} \right\|_{L^p} \lesssim \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{L^p}$$

with R_j having arbitrary orientations?

Y. Meyer: If S is bounded on L^p, then $(**)$ holds for the same p.

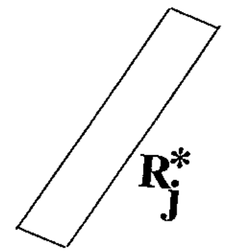
Counter-example sets (in $\mathbb{R}^2$)

- Nikodym Set
- Besicovitch (Kakeya) Set



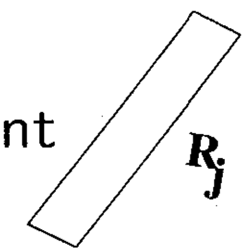
Fefferman's observation:

Given $\epsilon > 0$, there is $N = N_\epsilon$, and rectangles $R_1, R_2, \cdots R_N$ each having side-length $(1, 1/N)$, so that



- $m\left(\bigcup_{j=1}^{N} R_j\right) < \epsilon$, but

- $R_1^*, R_2^*, \cdots, R_N^*$ are all disjoint



Now take $f_j = \chi_{R_j}$, then

$$|R_j(f_j)(x)| \geq 1/10 \text{ for } x \in R_j^* .$$

Hence a contradiction to $(**)$ whenever $p < 2$.

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Further results for Bochner-Riesz and related questions: ... Bourgain, Tao, ...

III. H_1-BMO duality

John-Nirenberg: (1961): $f \in BMO$

If $\sup_Q \frac{1}{m(Q)} \int_Q |f - f_Q| \, dx = \| f \|_{BMO} < \infty$.

Inequality:

$$m\{x \in Q : |f(x) - f_Q| > \alpha\} \leq c_1 e^{-c_2 \alpha} m(Q),$$

all $\alpha > 0$, if $\| f \|_{BMO} \leq 1$.

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Related to work of John, Moser (the latter for Harnack-type inequality leading to DiGorgi-Nash estimates)

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Later observed: BMO good substitute for L^∞ in other settings:

Fact: $Tf = f * K$, and K is a CZ kernel

then: $T : L^\infty \longrightarrow BMO$

(In fact, $T : BMO \longrightarrow BMO$.)

The space H^1.

Classical H^1: Is H^1 of one complex variable (F. & M. Riesz, Hardy).

F analytic in $z = x + iy$, $y > 0$ and

$$\sup_{y>0} \int_{\mathbb{R}} |F(x+iy)|\, dx < \infty .$$

$$H^1 = \left\{ F_0(x) = \lim_{y \to 0} F(x+iy) \right\}, \text{ with}$$

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$F_0 = f + iH(f)$, $H =$ Hilbert transform.

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$$\text{“Real”}\ H^1 = \{ f : f \in L^1 \ \text{ and } H(f) \in L^1 \} .$$

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<u>Next:</u> H^1 in $\mathbb{R}^n$.

$$H^1 = \left\{ f \in L^1 ,\ \text{and } R_j(f) \in L^1 , \quad 1 \le j \le n \right\}$$

H^1 is a good substitute for L^1.

Fact: $Tf = f * K$, K is a CZ kernel

then: $T : H^1 \longrightarrow L^1$,

in fact: $T : H^1 \longrightarrow H^1$.

(Here one used g^*_λ.)

Zygmund's Question:

- What is the Poisson integral characterization of $f \in BMO$?

$$u(x,t) = f * P_t,, \ P_t \text{ the Poisson kernel}.$$

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Note: $\diamond$ For L^p, $f \in L^p \iff \sup_{t>0} \| u(\cdot,t) \|_{L^p} < \infty$

$$1 < p \leq \infty.$$

$\diamond$ For L^p, $f \in L^p \iff S(f) \in L^p$, $1 < p < \infty$.

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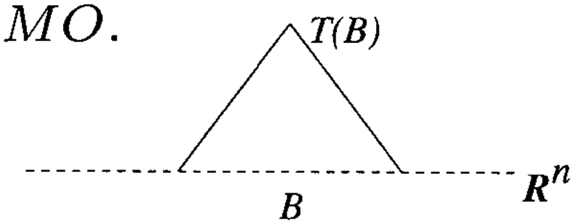
Theorem:

(1) Dual space of H^1 is BMO.



(2) $f \in BMO \iff$

$$\sup_B \frac{1}{m(B)} \int_{T(B)} |\nabla u(x,t)|^2 \, t dt < \infty$$

(3) $\iff f = f_0 + \sum_j R_j(f_j),\ f_0, f_1, \ldots f_n \in L^\infty$.

Condition (2): $d\mu = |\nabla(u(x,t)|^2 \, t \, dx \, dt$

is a "Carleson measure" (on $\mathbb{R}^{n+1}$) i.e.

$$\sup_B \frac{1}{m(B)} \int\limits_{T(B)} d\mu = \| \, d\mu \, \|_{\mathcal{C}} < \infty \, .$$

Fefferman duality:

Let F, G be non-negative functions on $\mathbb{R}^{n+1}_+$.

Then

$$\int\limits_{\mathbb{R}^{n+1}_+} F(x,t) \, G(x,t) \, \frac{dx \, dt}{t} \leq c \, \| \, \widetilde{S}(F) \, \|_{L^1(\mathbb{R}^n)} \| \, G dx \, dt \, \|_{\mathcal{C}}$$

where

$$\widetilde{S}(F)(x) = \left(\int\limits_{\Gamma(x)} (F(y,t))^2 \, \frac{dy \, dt}{t^{1+n}} \right)^{1/2} .$$

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Further Consequences

- Better understanding of H^1, H^p, $p \leq 1$, in particular, atomic decomposition.

- "sharp function":

$$f^{\#}(x) = \sup_{x \in Q} \frac{1}{m(Q)} \int_Q |f(x) - f_Q| \, dx \, .$$

Then

$$f^{\#} \in L^p \implies f \in L^p, \qquad p < \infty.$$

- End-point estimates for $1 < p < 2$ for (super) strong-singular integrals.

IV. Mapping of Domains in $\mathbb{C}^n$:

Question: Suppose Ω_1 and Ω_2 are two bounded smooth domains in $\mathbb{C}^n$. Assume there is a holomorphic bijection $\Phi : \Omega_1 \longrightarrow \Omega_2$. Does Φ extend to a diffeomorphism of $\partial\Omega_1$ to $\partial\Omega_2$?

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Some reasons $n > 1$ is different from $n = 1$.

1. When $n = 1$, the answer is *yes*. In fact, then there always exist "locally" good maps between any pair of smooth arcs.

2. When $n > 1$, and Ω_1 unit ball, Ω_2 is an "ϵ" C^∞ pertubation of Ω_1, then in general such Φ does not exist (even, locally, near a boundary point of Ω_1.)

3. Pseudo-convexity (for $n > 1$).

Theorem: *If Ω_1 and Ω_2 are bounded smooth domains $\Phi : \Omega_1 \longrightarrow \Omega_2$ holomorphic bijection. Then Φ extends smoothly to boundaries if both Ω_1 and Ω_2 are strongly pseudo-convex.*

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(*Later work:* Boutet de Monvel-Sjöstrand, Webster, Bell-Ligocka, Nirenberg, P. Yang, ...)

Fefferman's Approach:

(B) Bergman kernel, K_Ω, of domain Ω.

P_Ω orthogonal projection: $L^2(\Omega) :\longrightarrow L^2(\Omega) \cap (Hol)$

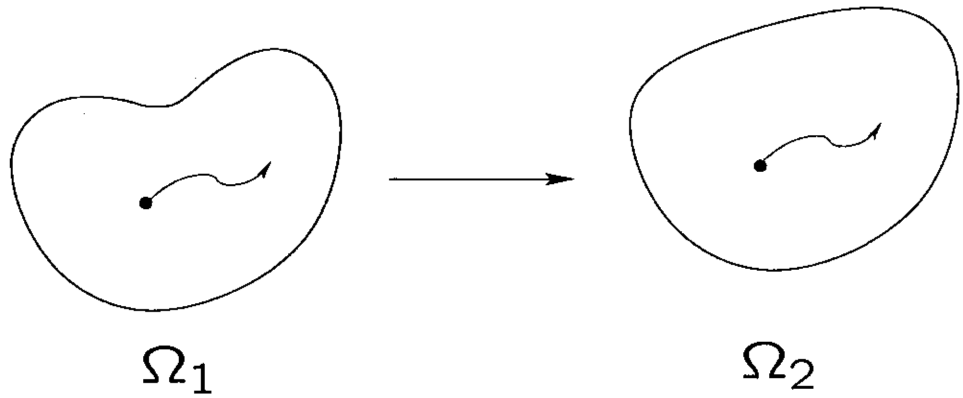
$$P_\Omega(f)(z) = \int_\Omega K_\Omega(z,w)\, f(w)\, dV(w)$$

Bergman metric, $g_{ij}^\Omega = \frac{\partial}{\partial_{z_i} \partial_{\bar{z}_j}} \log K_\Omega(z,z)$.

Fact: $\Phi : (\Omega_1, g^{\Omega_1}) \longrightarrow (\Omega_2, g^{\Omega_2})$ is an isometry.

(G) Follow the geodesics!

Φ:



Main Issues:

(B) What does the Bergman kernel (and Bergman metric) look like near the boundary?

(G) Where do the geodesics lead to? and how?

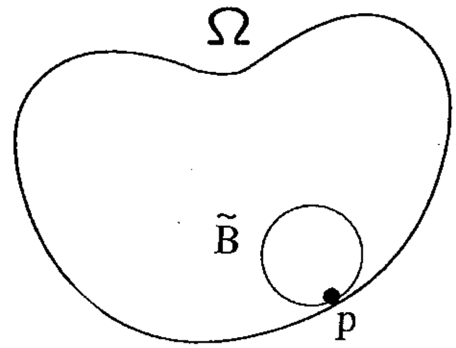
Assume Ω is bounded, smooth, and strongly pseudo-convex. Let $r(z)$ defining function, and $Q(z, w)$ holomorphic part of Taylor expansion of $r(z)$ (up to second-order) at w.

Theorem*:
$K_\Omega(z, w) = \frac{A(z,w)}{Q(z,w)^{n+1}} + B(z, w) \log Q(z, w)$ with $A, B \in C^\infty$.

Note: For unit ball in $\mathbb{C}^n$, $r(z) = 1 - |z|^2$, $K(z, w) = c/(1 - z \cdot \bar{w})^{n+1}$.

Want $I = P_\Omega + Q_\Omega$, $Q_\Omega = P_\Omega^\perp$.

- Find "ball" $\widetilde{B}$ highly tangent (order 4) to Ω at p, but $\widetilde{B} \subset \Omega$



- From explicit identity

$$I = P_{\widetilde{B}} + Q_{\widetilde{B}} \text{ on } \widetilde{B}$$

pass to approximate identity

$$I + E = P^0_\Omega + Q^0_\Omega \qquad \text{(explicit)}$$

($K_{\widetilde{B}}$ extends to $\Omega \times \Omega$ near p. Also one can correct by Kohn's $\bar{\partial}$-Neumann.)

Here $E \approx P^0 \cdot \chi_{\Omega - \widetilde{B}}$

- $P_\Omega = P^0_\Omega (1 + E)^{-1} = P^0_\Omega - P^0_\Omega E + P^0_\Omega E^2 \cdots\cdots$.

(G) Main Lemma

Suppose $X(t) = X(t, z_0, \xi_0)$ *is geodesic starting at* z_0 *in direction* ξ_0*. Assume* $X(t), 0 \leq t < \infty$*, does not lie in a compact set. Then*

(1) $\lim_{t \to \infty} X(t, z_0, \xi_0)$ *converges to a boundary point.*

(2) *The same is true for the geodesic* $X(t, z_0, \xi)$ *for* ξ *near* ξ_0*, and the resulting mapping:* $\xi \to$ *boundary, is a (local) diffeomorphism.*

(3) *All boundary points can be reached this way.*

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Requires several changes of variable:

- New "time" τ, $\frac{d\tau}{dt} = r(X(t))$.

- Further desingularization because of "log term".

V. Several Choices

1. Local solvability of linear p.d.e.

Consider the m^{th} order linear partial differential equation, where p is assumed to be of "principal type."

(*) $p(x, D)u = f$

Theorem: (R. Beals and C. Fefferman) *Suppose p satisfies the condition $\mathcal{P}$ of Nirenberg-Treves. Then (*) is locally solvable.*

Note: $(\mathcal{P})$ means: $\Im p_m$ does not change sign on the null bicharacteristic curves of $\mathcal{R} p_m$.

Theorem was proved by N-T in the real-analytic case.

Proof: requires a refined phase-space decomposition of a transformed problem.

This involves a "stopping-time" argument, in terms of violation of any of three key properties.

2. Convergence of Fourier series

$$f \longrightarrow \sup_{\lambda} \left| \int_{-\pi}^{\pi} \frac{e^{i\lambda y}}{y} f(x-y)\, dy \right| = C(f)\,.$$

Theorem: *A new proof that the Carleson operator C is a (weak-type) mapping $L^2 \longrightarrow L^2$.*

• Consider pairs: (ω, I), where ω and I are dyadic intervals in $\mathbb{R}$ and $[-\pi,\pi]$ respectively, with $|\omega|\,|I| = 1$ (These are later called "tiles".) Endow with ordering $(\omega, I) < (\omega', I')$, if $I \subset I'$, $\omega' \subset \omega$, and study collections of resulting "trees".

• Linearize $C(f)$ as $\displaystyle\int_{-\pi}^{\pi} \frac{e^{iN(x)y}}{y} f(x-y)\, dy$ and decompose according to $\{x \in I\,, N(x) \in \omega\}$, with $\frac{1}{y} = \sum_k \psi_k(y)$, and $|I| = 2^{-k}$, where $\psi_k(y) = 2^k \psi(2^k y)$.

Similar ideas then play important role in "time-frequency" analysis of Lacey and Thiele, for bilinear Hilbert transform, etc.