A new proof of Gromov's theorem on groups of polynomial growth

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Groups as geometric objects

Let G a group with a finite generating set $S \subset G$. Assume that S is symmetric:

$$S^{-1} = S\,.$$

The **word norm** of an element $g \in G$ is

$$\|g\| \ = \ \min\{ \ k \ \mid \ g = \sigma_1 \ldots \sigma_k\,, \quad \sigma_i \in S\}\,.$$

The **word metric** on G is the distance function

$$d(g_1, g_2) = \|g_2^{-1} g_1\|\,.$$

The associated Cayley graph Γ is the graph with vertex set G, where two vertices $g_1, g_2 \in G$ are adjacent if and only if $g_1 = g_2 s$ for some $s \in S$.

Examples

- $G = \mathbb{Z}^n$, $S = \{\pm e_1, \ldots, \pm e_n\}$.

- $G = F_2$, the free nonabelian group on two generators, $S = \{a^{\pm 1}, b^{\pm 1}\}$, where $\{a, b\} \subset F_2$ is a free basis.

Lemma. If S' is another finite generating set, with associated word metric $\|\cdot\|_{S'}$, then there is a constant C such that

$$\frac{1}{C}\|g\|_S \;\leq\; \|g\|_{S'} \;\leq\; C\,\|g\|_S\,,$$

for all $g \in G$.

Growth functions

The **growth function**

$$N : \mathbb{Z}_+ \to \mathbb{R}$$

of a group G is given by

$$N(r) = |\{g \in G \mid d(g, e) \leq r\}| = |B(e, r)|.$$

For $G = \mathbb{Z}^n$, $N(r) \sim r^n$.

For $G = F_2$, $N(r) \sim 3^r$.

The growth function depends on the generating set, but its asymptotic behavior does not.

Suppose $N : \mathbb{Z}_+ \to \mathbb{R}$ and $N' : \mathbb{Z}_+ \to \mathbb{R}$ are functions.

We say that $N \preceq N'$ if there are constants $A, B, C \in \mathbb{Z}$ such that for all $r \in \mathbb{Z}_+$,

$$N(r) \leq C\ N'(Ar + B)\ .$$

We say that N is **asymptotically equivalent** to N' if $N \preceq N'$ and $N \succeq N'$. This defines an equivalence relation $\asymp$ on such functions.

Suppose $N \asymp N'$. Then:

• If N is bounded by above (or below) by a polynomial function of degree d, then so is N'.

• If N has exponential growth, then so does N'.

Properties of growth functions (Milnor, Svarc)

• If N_S and $N_{S'}$ are two growth functions for a group G, then

$$N_S \asymp N_{S'}.$$

• If G contains a subgroup isomorphic to H, or there is a surjective homomorphism $G \to H$, then

$$N_H \preceq N_G.$$

• If G contains a finite index subgroup isomorphic to H, then

$$N_H \asymp N_G.$$

• Suppose G is the fundamental group of a closed Riemannian manifold M, and X is the universal cover of M. Then the growth function of G is asymptotically equivalent to the function

$$V : \mathbb{R}_+ \to \mathbb{R}$$

defined by

$$V(r) := |B(p, r)|,$$

where $B(p, r)$ is the r-ball in X.

• If M is a complete Riemannian n-manifold of nonnegative Ricci curvature, then every finitely generated subgroup $G \subset \pi_1(M)$ of satisfies

$$N_G \preceq r^n .$$

Theorem. (Wolf) Finitely generated nilpotent groups have polynomial growth.

Recall that for a group G, the descending central series

$$G = G_0 \supset G_1 \supset \dots$$

is defined inductively by $G_k := [G, G_{k-1}]$. The derived series is defined similarly, but with $G_k := [G_{k-1}, G_{k-1}]$.

A group is nilpotent (respectively solvable) if its descending central series (respectively derived series) terminates with the trivial group.

Theorem. (Gromov) Any group of polynomial growth contains a finite index nilpotent subgroup.

Definition. A group G has **weakly polynomial growth** if

$$\liminf_{r \to \infty} \frac{N(r)}{r^d} < \infty$$

for some $d \in \mathbb{R}$.

Theorem. (Wilkie-Van Den Dries) Any group of weakly polynomial growth contains a finite index nilpotent subgroup.

Some applications of Gromov's theorem

• (Wolf, Bass-Guivarch, Pansu) If a group G has weakly polynomial growth, then the ratio

$$\frac{N(r)}{r^d}$$

has a nonzero limit for some $d \in \mathbb{Z}_+$.

• If M is a complete Riemannian manifold with nonnegative Ricci curvature, then any finitely generated subgroup of $\pi_1(M)$ contains a finite index nilpotent subgroup.

• (Gromov, Hirsch, Shub, Franks, Epstein) If $f : M \to M$ is an expanding map, then M has a finite cover $\hat{M} \to M$ where $\hat{M}$ is a nilmanifold.

• (Varopoulos) Random walks on G are recurrent if and only if G contains a finite index sugroup isomorphic to $\mathbb{Z}^k$ for $k \in \{0, 1, 2\}$.

• (Mess, Gabai, Casson-Jungreis) The Seifert fibered space conjecture for 3-manifolds.

• (Gersten, Bass, Dunwoody) Classification of groups quasi-isometric to $\mathbb{Z}^n$, or to nilpotent groups.

• (Papasoglu) Quasi-isometry invariance of JSJ splittings for finitely presented groups.

The proof of Gromov's theorem

Gromov's proof uses the following earlier work:

Theorem. (Tits) Any finitely generated linear group either contains a nonabelian free subgroup, or is virtually solvable.

Theorem. (Wolf, Milnor) A finitely generated solvable group has polynomial growth if it is virtually nilpotent, and exponential growth otherwise.

Corollary. The theorem holds for linear groups and solvable groups.

The main work in Gromov's argument goes into proving the following:

Theorem. (Infinite Representation)
If G is an infinite group of weakly polynomial growth, then G has a finite dimensional linear representation

$$f : G \to GL(n, \mathbb{R})$$

with infinite image.

Assuming the above theorem, the sketch goes as follows.

If G has weakly polynomial growth, define the **degree** of G to be the minimal $d \in \mathbb{Z}_+$ such that

$$\liminf_{r \to \infty} \frac{N(r)}{r^d} < \infty.$$

The theorem holds trivially for groups of degree 0, since they are finite. Pick $k \in \mathbb{N}$, and assume inductively that the theorem holds for groups of degree $< k$.

Suppose G has degree k, and that

$$f : G \to GL(n, \mathbb{R})$$

is a homomorphism with infinite image H.

By theorems of Tits, Wolf, and Milnor, H has a finite index nilpotent subroup. This implies that after passing to a finite index subgroup, there will be a surjective homomorphism

$$\alpha : G \to \mathbb{Z}.$$

One then shows that the kernel K of α is finitely generated and has degree $< k$; therefore it is virtually nilpotent by the induction hypothesis.

Let K' be a finite index nilpotent subgroup of K which is normal in G, and let $L \subset G$ be a copy of $\mathbb{Z}$ which maps isomorphically under α onto $\mathbb{Z}$.

Then $K'L$ is a finite index solvable subgroup of G. Invoking Wolf/Milnor again, one concludes that G has a finite index nilpotent subgroup.

Gromov's construction of an infinite representation

Let G be an infinite group of polynomial growth equipped with a word metric d.

Gromov showed that for certain sequences $\{\lambda_k\} \subset (0, \infty)$ with $\lambda_k \to 0$, the sequence of metric spaces

$$\{ \, (G, \lambda_k \, d) \, \}$$

Gromov-Hausdorff converges to a metric space X, which is

- Geodesic.
- Proper (i.e. closed balls are compact).
- Finite dimensional.

Example.

$$\{ \, (\mathbb{Z}^n, \frac{1}{k} \, d) \, \} \longrightarrow (\mathbb{R}^n, \ell^1) \quad \text{as} \quad k \longrightarrow \infty.$$

The transitive action of G on itself by left translation converges to a transitive isometric action on the limit space X.

The isometry group $\text{Isom}(X)$ equipped with the compact open topology is a locally compact topological group, by the Arzela-Ascoli theorem and the fact that X is proper.

From work of Montgomery-Zippin and Yamabe, it follows that $\text{Isom}(X)$ is a Lie group with finitely many connected components.

Finally, by a clever scaling argument, Gromov shows that either

a) A finite index subgroup of G is abelian, or

b) A finite index subgroup of G has a homomorphism to $\text{Isom}(X)$ with infinite image.

The new proof

The proof begins with the same reduction: it suffices to show that an infinite group of polynomial growth has a finite dimensional linear representation with infinite image.

The argument for this is based on harmonic maps, and avoids Gromov-Hausdorff limits, as well as the theory of locally compact groups.

Harmonic functions on graphs

Let Γ be a locally finite graph, metrized so that each edge has length 1.

Definition. If $u : \Gamma \to \mathbb{R}$ is a piecewise smooth function and $\Gamma' \subset \Gamma$ is a finite subgraph, the **energy** of u on Γ' is

$$E_{\Gamma'}(u) := \int_{\Gamma'} |Du|^2 \, ds.$$

The function u is **harmonic** if it minimizes energy on compact subsets, i.e. if $v : \Gamma \to \mathbb{R}$ is any function which agrees with u outside some finite subgraph $\Gamma' \subset \Gamma$, then

$$E_{\Gamma'}(v) \geq E_{\Gamma'}(u).$$

Lemma. A function $u : \Gamma \to \mathbb{R}$ is harmonic if and only if

• The restriction of u to each edge of Γ has constant derivative.

and

• For every vertex $v \in \Gamma$, the average of u over the vertices adjacent to v is $u(v)$.

Remarks. One may also interpret everything discretely, working instead on the vertex set $V \subset \Gamma$.

Similar definitions apply to maps $u : \Gamma \to \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space. Moreover, a map into Hilbert space is harmonic if and only if the composition

$$\Gamma \xrightarrow{f} \mathcal{H} \xrightarrow{\phi} \mathbb{R}$$

is harmonic for every $\phi \in \mathcal{H}^*$.

Outline of the proof of the infinite representation theorem

Let G be an infinite group of weakly polynomial growth with $S \subset G$ as before, and let Γ be the Cayley graph of (G, S).

Step 1. There exists a fixed point free isometric action $G \curvearrowright \mathcal{H}$ on a Hilbert space, and a nonconstant G-equivariant harmonic map $f : \Gamma \to \mathcal{H}$.

Step 2. The map f takes values in a finite dimensional subspace of $\mathcal{H}$.

Step 3. The group G has a finite dimensional representation with infinite image.

Step 1

Since G does not have exponential growth, it follows that there is a sequence of radii $\{r_k\}$ such that

$$\frac{|S(e, r_k)|}{|B(e, r_k)|} \longrightarrow 0 \,.$$

Define $u_k : G \to \mathbb{R}$ by

$$u_k := \frac{\chi_{B(e,r_k)}}{|B(e, r_k)|^{\frac{1}{2}}} \;.$$

The sequence $\{u_k\} \subset \ell^2(G)$ is a sequence of unit vectors which are **almost fixed** by the left regular representation of G, in the sense that

$$\max_{s \in S} \|s \cdot u_k - u_k\|_{\ell^2} \longrightarrow 0 \,.$$

Let (G, S) be a group with a symmetric finite generating set.

If $G \curvearrowright X$ is an action on a metric space X, the **energy function** $E : X \to \mathbb{R}$ is defined by

$$E(x) := \sum_{s \in S} d^2(sx, x) .$$

Theorem. (Mok, Korevaar-Schoen)
Let (G, S) be as above. Then one of the following holds.

A. There is a constant $D \in (0, \infty)$ such that if $G \curvearrowright \mathcal{H}$ is any isometric action of G on a Hilbert space, and $x \in \mathcal{H}$, then there is a fixed point of G in the ball

$$B(x, D\sqrt{E(x)}) .$$

B. There is an isometric action $G \curvearrowright \mathcal{H}$ which has no fixed points, and a G-equivariant harmonic map

$$f : \Gamma \to \mathcal{H} .$$

Now suppose G is an infinite group with weakly polynomial growth, and let $\{u_k\} \subset G$ be the sequence defined above. Then

$$E(u_k) \to 0 \ .$$

On the other hand, G is infinite, so the only fixed point in $\ell^2(G)$ is 0.

This means that alternative A in Mok/Korevaar-Schoen fails, and therefore alternative B must hold.

Remark. Alternative A in Mok/Korevaar-Schoen is equivalent to having Property (T).

Remark. More generally, if G does not have Property (T), then there is a fixed point free isometric action $G \curvearrowright \mathcal{H}$ and a G-equivariant harmonic map $f : \Gamma \to \mathcal{H}$.

Sketch of the proof of Mok/Korevaar-Schoen.

Lemma. Suppose $G \curvearrowright \mathcal{H}$ is an isometric action of G on a Hilbert space. Pick $x \in \mathcal{H}$, and let

$$f : \Gamma \longrightarrow \mathcal{H}$$

be the G-equivariant map such that

$$f(g) = gx \quad \text{for all} \quad g \in G \subset \Gamma,$$

and f is has constant derivative on each edge of Γ. Then G is harmonic if and only if x is the minimum of the energy function $E : \mathcal{H} \to \mathbb{R}$.

Therefore, it suffices to show that if condition A fails, there is an isometric action $G \curvearrowright \mathcal{H}$ on some Hilbert space, such that the energy function attains a nonzero minimum.

This follows from a rescaling argument.

Let G be a group of weakly polynomial growth, and $G \curvearrowright \mathcal{H}$, $f : \Gamma \to \mathcal{H}$ be as above.

We now concentrate on Step 2, that f takes values in a finite dimensional subspace of $\mathcal{H}$.

Lemma. The harmonic map f is Lipschitz.

Theorem. (Colding-Minicozzi) Pick $\delta \in \mathbb{R}$. The space of harmonic functions on Γ with polynomial growth at most δ is finite dimensional.

$$\sup_{x \in G} \frac{|u(x)|}{(1 + d(x,e))^{\delta}} < \infty .$$

Corollary. The map f takes values in a finite dimensional subspace of $\mathcal{H}$.

Proof. Define

$$\Phi : \mathcal{H}^* \to \mathrm{Lip}(\Gamma)$$

by

$$\Phi(\alpha) = \alpha \circ f.$$

The range is finite dimensional, so the kernel $K \subset \mathcal{H}^*$ has finite codimension in $\mathcal{H}^*$.

The image of f lies in the finite dimensional space

$$K^{\perp} \subset \mathcal{H}.$$

□

The proof of the finite dimension theorem given by Colding-Minicozzi relies on Gromov's polynomial growth theorem in an essential way.

We give a new proof of Colding-Minicozzi finite dimensionality which is independent of Gromov's theorem.

Colding-Minicozzi

The finite dimension theorem holds for polynomial growth harmonic functions on a graph Γ (or Riemannian manifold) satisfying two conditions:

• Γ is doubling, i.e. every ball can be covered by a controlled number of balls of half the radius.

• Γ satisfies a uniform scale-invariant Poincare inequality:

$$\int_{B(R)} |u - u_R|^2 \leq C\, R^2 \int_{B(CR)} |\nabla u|^2,$$

where u_R denotes the average of u over $B(R)$.

The proof that these conditions hold for groups of polynomial growth depends on Gromov's theorem.

The new proof of the finite dimension theorem

The first ingredient is a new Poincare inequality:

Theorem. Suppose (G, S) is an arbitrary group with symmetric finite generating set S, and let Γ be the associated Cayley graph. Then

$$\int_{B(R)} |u - u_R|^2$$

$$\leq 8\,|S|^2\,R^2\,\frac{|B(2R)|}{|B(R)|}\int_{B(3R)} |\nabla u|^2,$$

where $R \in \mathbb{Z}_+$, $B(R)$ and $B(CR)$ denote concentric balls of radius R and CR centered at a group element $g \in \Gamma$, and u is a piecewise smooth function on $B(CR)$.

Pick $d \in \mathbb{R}$, and use $e \in \Gamma$ as the basepoint for balls.

Let $\mathcal{V}$ be a finite dimensional vector space of harmonic functions on Γ with polynomial growth at most d.

For each $R \in \mathbb{Z}_+$, let Q_R be the quadratic form on $\mathcal{V}$ given by

$$Q_R(u,u) := \int_{B(R)} u^2,$$

where $B(R) = B(e,R)$.

Sketch of the rest of the proof.

One selects a scale R such that the ratios

$$\frac{\det Q_{LR}}{\det Q_R} \quad \text{and} \quad \frac{|B(LR)|}{|B(R)|}$$

are controlled, for a suitable $L \in [1, \infty)$.

This scale selection argument is a hybrid of arguments in Gromov and Colding-Minicozzi. The idea is that polynomial growth implies doubling behavior at many scales.

The key point is that one has the doubling condition at this scale, and the Poincare inequality has controlled constant at this scale (because of the doubling condition).

Now one can use the strategy of Colding-Minicozzi to bound the dimension of $\mathcal{V}$.

The main estimate

We find a pair of radii $R_1 < R_2$, a cover

$$\mathcal{B} = \{B_j\}_{j \in J}$$

of $B(R_2)$ by balls of radius R_1, and a subspace $\mathcal{W} \subset \mathcal{V}$ of dimension at least $\frac{1}{2} \dim \mathcal{V}$, such that:

- The cardinality of $\mathcal{B}$ is controlled.
- The ratio $\frac{|B(2R_1)|}{|B(R_1)|}$ is controlled.
- $Q_{16R_2}(w, w) \leq C\, Q_{R_2}(w, w)$ for all $w \in \mathcal{W}$.

Applying the Poincare inequality, one concludes that the map $W \to \mathbb{R}^J$

$$w \mapsto \left\{ \int_{B_j} w \right\}_{j \in J}$$

is injective, and this implies that $\dim \mathcal{W} \leq |J|$.