The starting point for the work that I want to discuss is a quantitative unique continuation theorem that arose 5 years ago as a key step in the work of Bourgain-K., in which we proved Anderson localization at the bottom of the spectrum for the continuous Bernoulli model in higher dimensions. Briefly, we considered random Schrödinger operators on $\mathbb{R}^n$, of the form $H_\varepsilon = -\Delta + V_\varepsilon$, where $V_\varepsilon = \sum_{j \in \mathbb{Z}^n} \varepsilon_j \varphi(x-j)$, where $\varphi \in C_0^\infty(B(0,1/10))$, $0 \leq \varphi \leq 1$, $\varepsilon_j \in \{0,1\}$ are independent. Our result is that for energies E, with $0 < E < \delta$, H_ε has, a.s., pure point spectrum, with exponentially decaying eigenfunctions for $n \geq 1$.

To establish this result, we were lead to the following deterministic quantitative unique continuation theorem: Suppose that $\Delta u + Vu = 0$ in $\mathbb{R}^n$, $|V| \leq 1$, $|u| \leq C_0$, $u(0) = 1$. For R large, define

$$M(R) = \inf_{|x_0| = R} \sup_{B(x_0,1)} |u(x)|.$$

Note that by u.c., $\sup_{B(x_0,1)} |u(x)| > 0$. How small can $M(R)$ be?

<u>Theorem</u>: (Bourgain-K 2004)

$$M(R) \geq C \exp(-CR^{4/3} \log R)$$

It turns out that this was a quantitative version of a conjecture of E.M. Landis, who conjectured (late 60's) that if $\Delta u + Vu = 0$ in $\mathbb{R}^n$, $|V| \leq 1$, $|u| \leq C_0$, $|u(x)| \leq C \exp(-C|x|^{1+}) \Rightarrow u \equiv 0$. The conjecture was disproved by Meshkov (1992), who constructed $V, u \not\equiv 0$, $|u(x)| \leq C \exp(-C|x|^{4/3})$. However, u, V are complex valued and Landis onjecture is still open for real valued u, V. Our proof uses a rescaling procedure, combined with Carleman estimates.

Let us next turn our attention to parabolic equations. Thus, consider solutions to

$$\partial_t u - \Delta u + W(x,t)\cdot\nabla u + V(x,t)\,u = 0 \text{ in } \mathbb{R}^n \times (0,1],$$

$|W| \leq N$, $|V| \leq M$. Then, the following back-ward uniqueness result holds: if $|u(x,t)| \leq C_0$, $u(x,1) \equiv 0$ then $u \equiv 0$. This was extended by Escauriaza-Seregin-Sverak (2002), who showed that it is enough to assume that u is a solution in $\{(x', x_n) = X, x_n > 0\} \times (0,1]$. This was a crucial ingredient in their proof that weak solutions of the Navier-Stokes system in $\mathbb{R}^3 \times [0,1]$, which have uniformly bounded L^3_x norm are regular and unique. In 1974, in parallel to Landis elliptic conjecture, Landis-Oleinik asked: assume that u is as in the backward uniqueness situation, but instead of $u(x,1) \equiv 0$, we have $|u(x,1)| \leq C\exp(-C|x|^{2+\varepsilon})$, $\varepsilon > 0$. Is $u \equiv 0$? (Note 2 is sharp here).

<u>Answer</u>: Yes (Escauriaza-K-Ponce-Vega 2005) and with quantitative bounds. The extension to variable coefficient top order terms and to half-spaces is in the dissertation (2007) of my former student Tu Nguyen.

We now turn to dispersive equations, the proper subject of this lecture. We will consider here linear Schrödinger equations of the form $\partial_t u = i(\Delta u + V(x,t)u)$, but applications to non-linear problems follow easily.

Uniqueness results which are the proper analog here of "backward unique continuation" were obtained

in works of K-Ponce-Vega (2002) and Ionescu-K (2004,05)
To explain the nature of these results and move forward, it is important to go backward in time, to recall the classical Hardy uncertainty principle: let f be a function on $\mathbb{R}^n$, $\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx$.
Then, if $f(x) = O(e^{-|x|^2/\beta^2})$ and $\hat{f}(\xi) = O(e^{-4|\xi|^2/\alpha^2})$ and $\alpha\beta < 4$, then $f \equiv 0$. Also, if $\alpha\beta = 4$, then $f \equiv c\, e^{-|x|^2/\beta^2}$. The only known proof of this result (and its variants) (as far as I know) uses complex analysis. There has been considerable interest in a better understanding of this result and on extensions of it to other settings (Hörmander, Beurling, etc) (Bonami-Demange (2006), Bonami-Demange-Jaming (2003), Cowling-Price (1983), Sitaram-Sundari-Thangavelu (1995), etc). The last authors formulat

in L^2 version, namely:

If $e^{|x|^2/\beta^2} f$ and $e^{4|\xi|^2/\alpha^2} \hat{f}$ are in $L^2(\mathbb{R}^n)$ and $\alpha\beta \le 4$, then $f \equiv 0$. 4 is sharp ($f = e^{-|x|^2/\beta^2}$).

These results can be re-written in terms of the free solution of the Schrödinger equation in $\mathbb{R}^n \times (0,\infty)$, $\partial_t u + i\Delta u = 0$, with initial data f: (say for the L^2 result)

$$u(x,t) = (4\pi i t)^{-n/2} \int_{\mathbb{R}^n} e^{i|x-y|^2/4t} f(y)\,dy =$$

$$= (2\pi i t)^{-n/2} e^{i|x|^2/4t} \widehat{e^{i|\cdot|^2/4t} f}\left(\frac{x}{2t}\right), \text{ as}$$

If $e^{|x|^2/\beta^2} u(x,0)$ and $e^{|\xi|^2/\alpha^2} u(x,T)$ are in $L^2(\mathbb{R}^n)$ and $\alpha\beta \le 4T$, then $u \equiv 0$. 4 is sharp.

We now turn to the main subject matter of this lecture. We say that $V(x,t)$ is admissible ($V \in A$) if

i) $\|V\|_\infty \le M$,

ii) Either $V(x,t) = V(x)$, V real valued or

$V(x,t)$ is complex valued and $\lim_{R\to\infty} \|V\|_{L^1_{[0,T]} L^\infty_x(|x|>R)} = 0$

Our main result is:

Theorem: (Escauriaza - K - Ponce - Vega 09). Assume that V is admissible and u solves $\partial_t u = i(\Delta u + V(x,t)u)$ in $\mathbb{R}^n \times [0,T]$. If $e^{|x|^2/\beta^2} u(x,0)$ and $e^{|\xi|^2/\alpha^2} u(x,T)$ are in $L^2(\mathbb{R}^n)$ and $\alpha\beta < 4T$, then $u \equiv 0$.

Remarks: The result is sharp up to the end-point. In fact, for admissible potentials, the end-point $\alpha\beta = 4T$ fails, as we have shown for a suitable complex valued potential $V(x,t)$, verifying $|V(x,t)| \lesssim (1+|x|^2)^{-1}$. The proof does not use any complex analysis, it only uses calculus! This provides the first proof (up to the end-point) of Hardy's uncertainty principle for the Fourier transform, without the use of coplex andysis, and gives a wide generalization. Previous works of E-K-P-V (06, 08) established the corresponding result when $\alpha\beta < C_n T$ and then when $\alpha\beta < 2T$. I will now turn to a sketch of the ideas in the proof. To minimize technicalities I will do it in the case $V \equiv 0$, i.e. the free Schrödinger evolution. The first step is to reduce to the case when $\alpha^2 = \beta^2 = \frac{1}{\mu}$. This is accoplished through the use of the conformal or Appel transform. This is the following fact:

if $\partial_s u = i \Delta u(y,s)$ in $\mathbb{R}^n \times [0,1]$, and

$$\tilde{u}(x,t) = \left(\frac{\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t}\right)^{n/2} u\left(\frac{\sqrt{\alpha\beta}\, x}{\alpha(1-t)+\beta t}, \frac{\beta t}{\alpha(1-t)+\beta t}\right) \cdot e^{(\alpha-\beta)|x|^2/4i(\alpha(1-t)+\beta t)},$$

then

$\partial_t \tilde{u} = i \Delta \tilde{u}$ in $\mathbb{R}^n \times [0,1]$ and