

Parabolic Invariants: From the Bergman Kernel to the AdS/CFT Correspondence

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Question. Can one express the Taylor expansions of $\varphi \bmod \rho^{n+1}$ and ψ in terms of geometric invariants of $\partial\Omega$?

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In general, each a_j can be expressed as a linear combination of complete contractions

$$\text{contr}(\nabla^{r_1} R \otimes \cdots \otimes \nabla^{r_L} R).$$

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The problem is difficult: the equation is degenerate at the boundary.

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$$u_2 \circ \Phi = |\det \Phi'|^{2/(n+1)} u_1.$$

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Interpretation: the metric with Kähler form $i\partial\bar{\partial} \log u^{-1}$ is a complete Kähler-Einstein metric on Ω .

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In 1982, Lee-Melrose showed that the Cheng-Yau solution has an asymptotic expansion at $\partial\Omega$ of the form

$$u \sim \rho \sum_{k=0}^{\infty} \eta_k (\rho^{n+1} \log \rho)^k, \quad \eta_k \in C^\infty(\bar{\Omega}).$$

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r_1, \dots, r_L must be sufficiently small to avoid the indeterminacy of u at order $n+2$.

Expansion of Bergman Kernel

Theorem (CF, *Parabolic invariant theory in complex analysis*, 1979).

$$K_{\Omega}(z, z) = u^{-n-1} \sum_{j=0}^{n-19} l_j u^j + O(u^{-20}),$$

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The method of reasoning is that since u is invariant, the coefficients in the expansion in powers of u must be CR invariants. Charlie's invariant theory proves that all scalar CR invariants (up to the indicated power) are linear combinations of complete contractions of curvature of \tilde{g} . Thus the result follows.

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The improved invariant theory enabled extension of the characterization of CR invariants and the expansion of the Bergman kernel up to the log term:

$$K_{\Omega}(z, z) = u^{-n-1} \sum_{j=0}^n l_j u^j + O(\log u).$$

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- Construction and characterization of all invariants $I(u)$ depending on $(\partial\Omega, u)$ by complete contractions of a metric \tilde{g} on $\mathbb{C}^* \times \mathbb{C}^n$.
- Full expansion of the log term coefficient: $\psi = \sum_{j=0}^{\infty} I_j(u) u^j$ for any $u \in \mathcal{F}$.

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$$\tilde{g} = \sum_{i=1}^{n+1} (dX^i)^2 - (dX^0)^2 \quad \text{on } \mathbb{R}^{n+2}$$

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The Ricci-flat condition is the conformal analogue of the Monge-Ampère equation $J(u) = 1$.

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One reason: can view CR geometry as a subset of even dimensional conformal geometry. The “Fefferman metric” construction associates a Lorentz signature conformal structure on $\partial\Omega \times S^1$ to a strictly pseudoconvex boundary $\partial\Omega$.

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There are *exceptional* odd invariants (change sign under orientation reversal) if $n \equiv 0 \pmod{4}$. Examples: Pontrajagin invariants.

“Application”: Alexakis’ Theorem

The construction of scalar conformal invariants via the ambient metric plays an essential role in Alexakis’ recent structure theorem for conformally invariant integrals.

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Theorem. Up to a smooth homogeneous diffeomorphism, the inhomogeneous ambient metrics for $(M, [g])$ are parametrized by the choice of an arbitrary trace-free symmetric 2-tensor field A_{ij} on M . This “ambiguity tensor” corresponds to

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Also can construct a finite list of *basic exceptional invariants*.

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These results complete the construction of the ambient metric and the characterization of scalar conformal invariants for n even.

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H^{n+1} also arises as a hyperboloid in Minkowski \mathbb{R}^{n+2} with the induced metric.

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Has led to much interaction and progress which continues now.

- Math \rightarrow Physics Holographic renormalization
- Physics \rightarrow Math Renormalized volume

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Complementary approach via Cartan connections: Cartan, Chern, Tanaka, ...