Interface dynamics for incompressible flows in 2D

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Equation

$$\begin{aligned} &\rho_t + u \cdot \nabla \rho = 0, \\ &\nabla \cdot u = 0, \end{aligned} \qquad \rho(x_1, x_2, t) = \begin{cases} \rho^1, & x \in \Omega^1(t) \\ \rho^2, & x \in \Omega^2(t), \end{cases}$$

with $\rho(x,t)$ an active scalar, $(x,t) \in \mathbb{R}^2 \times \mathbb{R}^+$, $\rho^1 \neq \rho^2$ are constants, and $\Omega^2(t) = R^2 \setminus \Omega^1(t)$.

1. SQG sharp front: ρ temperature,

$$u = (-R_2\rho, R_1\rho), \quad \widehat{R}_j = i\xi_j/|\xi|.$$

2. Muskat: ρ density, μ viscosity

$$\frac{\mu}{\kappa} u = -\nabla p - (0, \mathrm{g}\rho), \quad \text{Darcy's law.}$$

3. Water wave: ρ density,

$$\rho(u_t + (u \cdot \nabla)u) = -\nabla p - (0, \mathrm{g}\rho), \quad \text{Euler.}$$

The SQG equation

Equation

$$\theta_t + u \cdot \nabla\theta = 0,$$

$$u = \nabla^{\perp}\psi, \quad \theta = -(-\Delta)^{1/2}\psi,$$

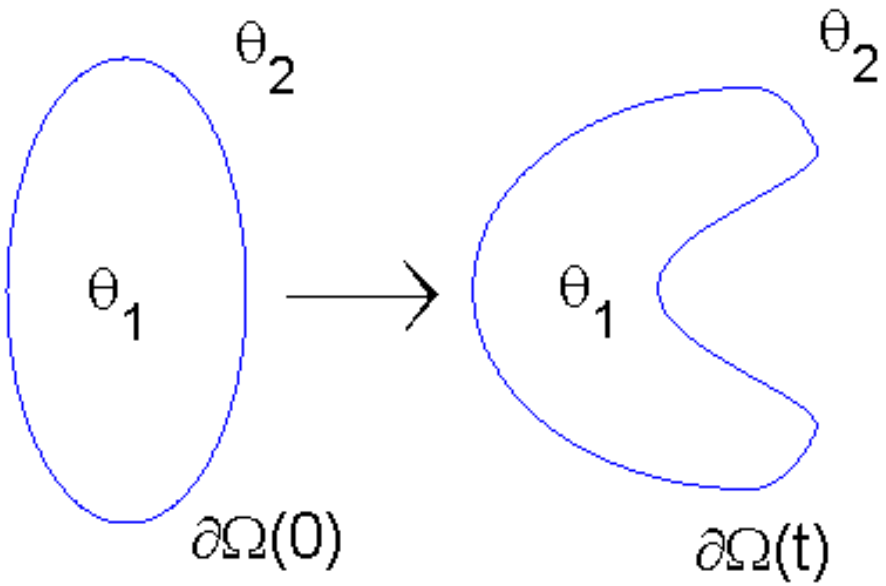
with $\theta(x,t)$ *the temperature,* $(x,t) \in \mathbb{R}^2 \times \mathbb{R}^+$.

- Constantin, Majda, and Tabak (1994)

Sharp fronts

We consider weak solutions given by

$$\theta(x_1, x_2, t) = \begin{cases} \theta_1, & \Omega(t) \\ \theta_2, & \mathbb{R}^2 \setminus \Omega(t). \end{cases}$$



The 2-D vortex patch problem

Contour equation

$$w_t + u \cdot \nabla w = 0,$$

$$u = \nabla^{\perp}\psi, \quad w = \Delta\psi,$$

where the vorticity is given by

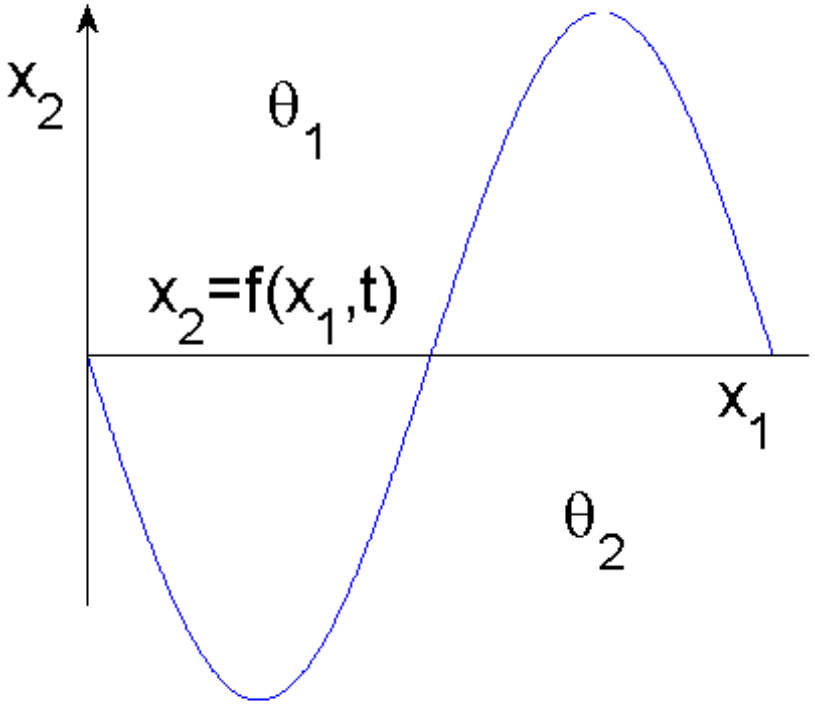
$$w(x_1, x_2, t) = \begin{cases} w_0, & \Omega(t) \\ 0, & \mathbb{R}^2 \setminus \Omega(t). \end{cases}$$

- Chemin (1993)
- Bertozzi and Constantin (1993)

The 2-D patch problem for SQG

- Rodrigo (2005), for a periodic C^∞ front, i.e.

$$\theta(x_1, x_2, t) = \begin{cases} \theta_1, & \{f(x_1, t) > x_2\} \\ \theta_2, & \{f(x_1, t) \leq x_2\}. \end{cases}$$



The α-patch model

- Córdoba, Fontelos, Mancho and Rodrigo (2005)

Contour equation

$$\theta_t + u \cdot \nabla\theta = 0,$$

$$u = \nabla^{\perp}\psi, \quad \theta = -(-\Delta)^{1-\alpha/2}\psi, \quad 0 < \alpha \leq 1,$$

where the active scalar $\theta(x,t)$ *satisfies*

$$\theta(x_1, x_2, t) = \begin{cases} \theta_1, & \Omega(t) \\ \theta_2, & \mathbb{R}^2 \setminus \Omega(t). \end{cases}$$

The contour equation

$$\partial\Omega(t) = \{x(\gamma, t) = (x_1(\gamma, t), x_2(\gamma, t)) \,:\, \gamma \in [-\pi, \pi] = \mathbb{T}\},$$

where $x(\gamma, t)$ is one to one. We have

$$\nabla^{\perp}\theta = (\theta_1 - \theta_2)\, \partial_\gamma x(\gamma, t)\, \delta(x - x(\gamma, t)),$$

$$u = -(-\Delta)^{\alpha/2-1}\nabla^{\perp}\theta,$$

and it gives

$$u(x, t) = -\frac{\Theta_\alpha}{2\pi} \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma - \eta, t)}{|x - x(\gamma - \eta, t)|^\alpha} d\eta.$$

The normal velocity of the systems reads

$$u(x(\gamma,t),t)\cdot\partial_\gamma^\perp x(\gamma,t) = -\frac{\Theta_\alpha}{2\pi}\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma-\eta,t)\cdot\partial_\gamma^\perp x(\gamma,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|^\alpha}d\eta.$$

The tangential velocity does not change the shape of the boundary, so that we fix the contour α-patch equations as follows:

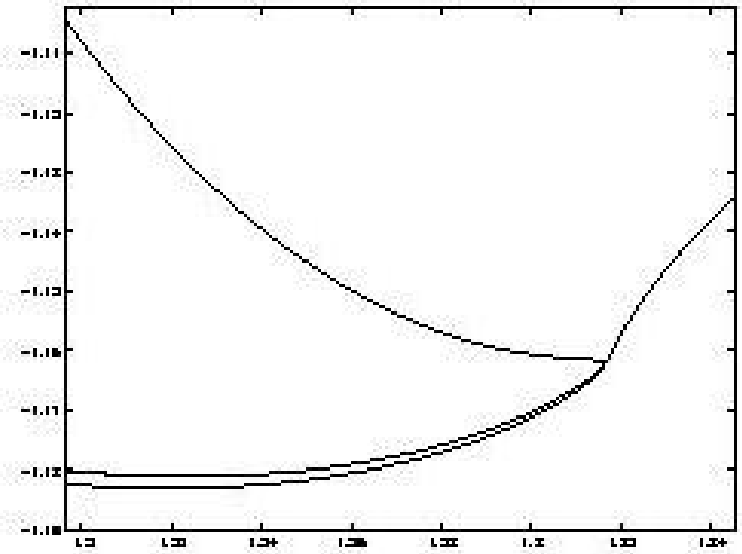
Contour equation

$$x_t(\gamma,t) = \frac{\Theta_\alpha}{2\pi}\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|^\alpha}d\eta, \quad 0<\alpha\leq 1,$$

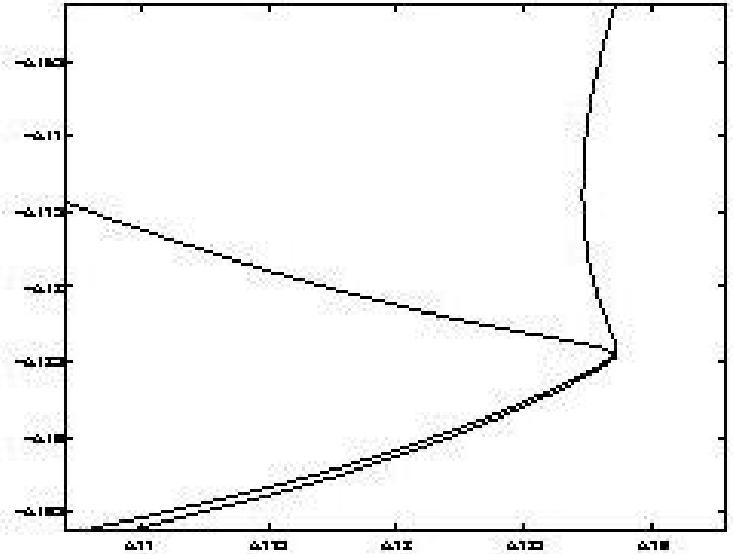
$$x(\gamma,0) = x_0(\gamma).$$

Numerical simulations

Close caption of the corner region at t=16.515 for $\alpha = 0.5$ (a) and t= 4.464 for $\alpha = 1$ (b). Observe that the singularity is point-like in both cases.



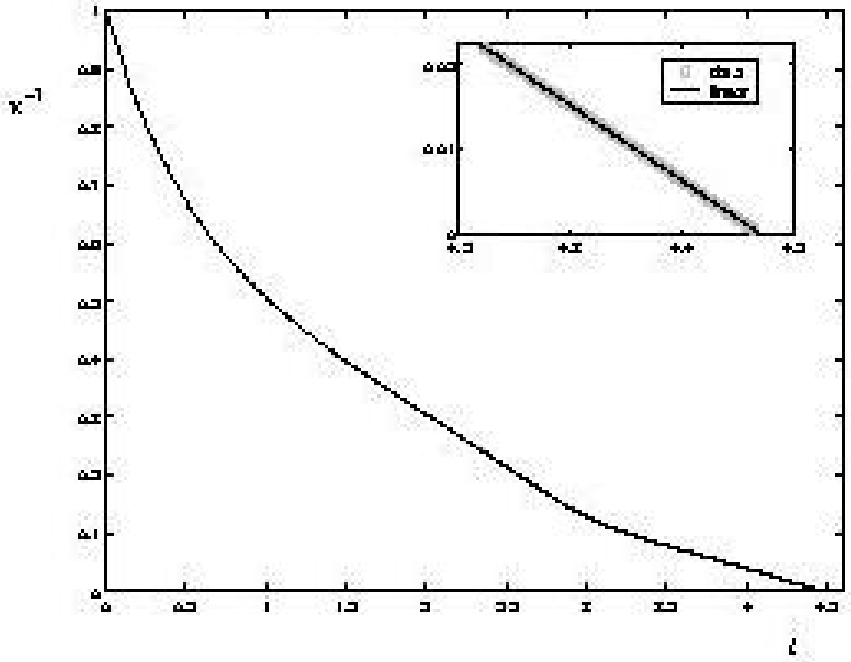
a)



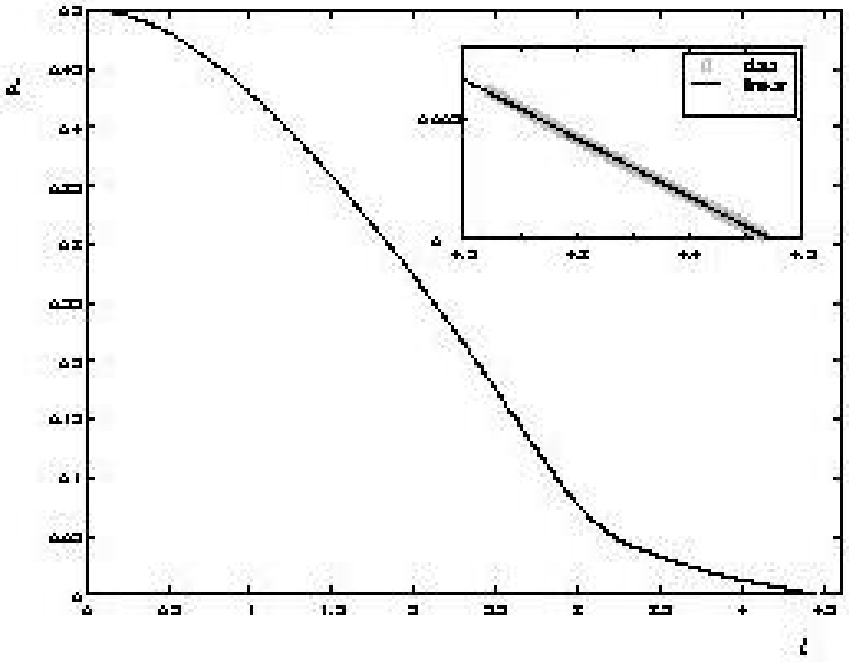
b)

Numerical simulations

Evidence of finite time singularity. (a) Evolution of the inverse of the maximum curvature with time. (b) Evolution of the minimum distance between patches with time. (Insets) Here, we represent the latest stage of the evolution together with its linear interpolation.



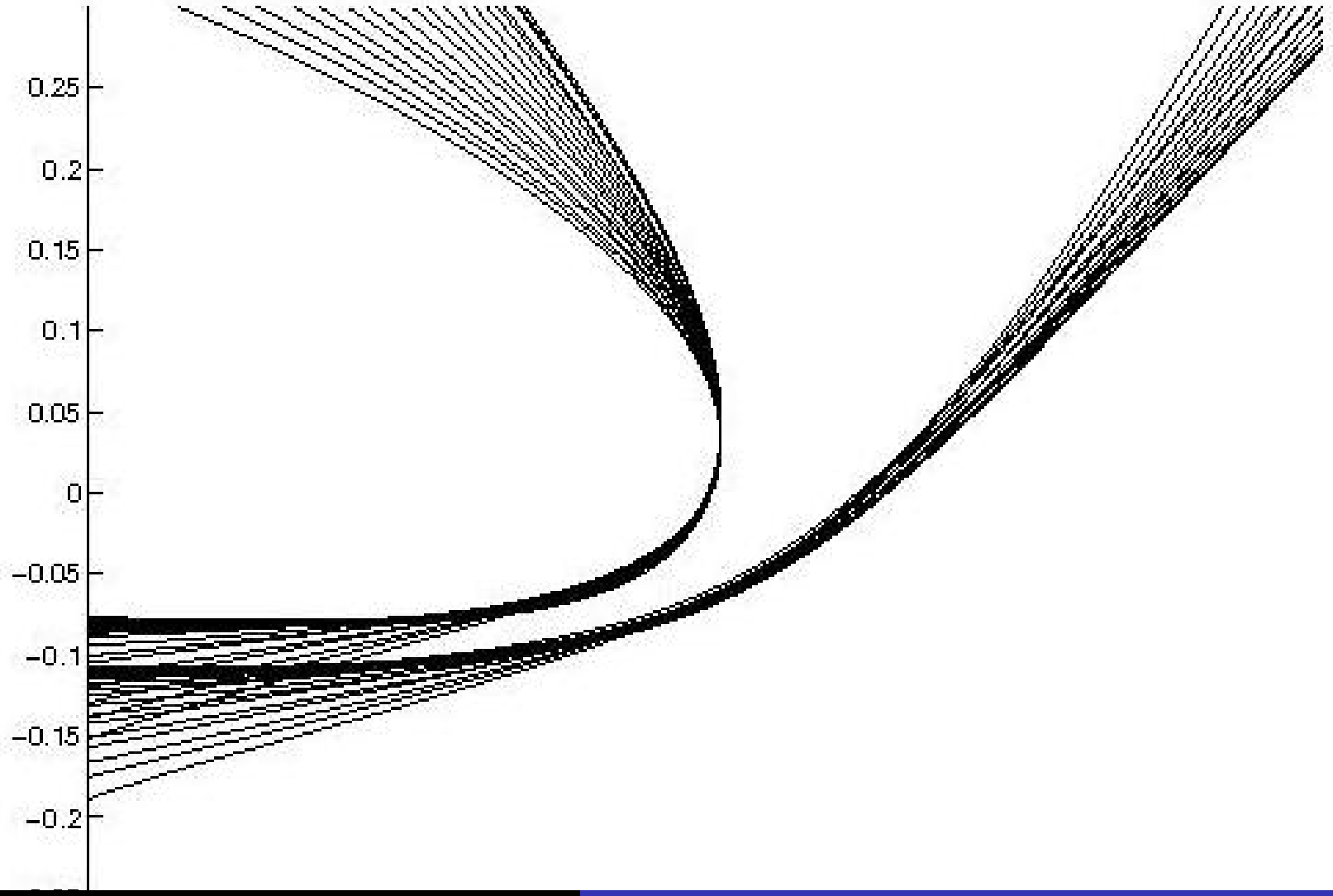
a)



b)

Numerical simulations

Rescaled profiles at 20 different times in the interval (3.46,4.46) for the case $\alpha = 1$.



Local well-posedness in H^s for $0 < \alpha \leq 1$

We define

$$F(x)(\gamma, \eta, t) = \frac{|\eta|}{|x(\gamma, t) - x(\gamma - \eta, t)|} \quad \forall \gamma, \eta \in [-\pi, \pi],$$

with

$$F(x)(\gamma, 0, t) = |\partial_\gamma x(\gamma, t)|^{-1}.$$

- Gancedo (2008)

Theorem

Let $x_0(\gamma) \in H^k(\mathbb{T})$ for $k \geq 3$ with $F(x_0)(\gamma, \eta) < \infty$. Then there exists a time $T > 0$ so that there is a unique solution to the α-patch model for $0 < \alpha \leq 1$ in $C^1([0, T]; H^k(\mathbb{T}))$, with $x(\gamma, 0) = x_0(\gamma)$.

Existence for $\alpha = 1$; the SQG sharp front

We modify the equation as follows:

Contour equation

$$x_t(\gamma,t) = \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t) - \partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t) - x(\gamma-\eta,t)|} d\eta + \lambda(\gamma,t)\partial_\gamma x(\gamma,t), \quad (1)$$

with

$$\lambda(\gamma,t) = \frac{\gamma+\pi}{2\pi}\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t)}{|\partial_\gamma x(\gamma,t)|^2}\cdot\partial_\gamma\Big(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|} d\eta\Big) d\gamma$$
$$-\int_{-\pi}^{\gamma} \frac{\partial_\gamma x(\eta,t)}{|\partial_\gamma x(\eta,t)|^2}\cdot\partial_\eta\Big(\int_{\mathbb{T}} \frac{\partial_\gamma x(\eta,t)-\partial_\gamma x(\eta-\xi,t)}{|x(\eta,t)-x(\eta-\xi,t)|} d\xi\Big) d\eta. \quad (2)$$

We get

$$|\partial_\gamma x(\gamma, t)|^2 = A(t).$$

- Hou, Lowengrub and Shelley (1997)

Extra cancellations:

$$\partial_\gamma x(\gamma, t) \cdot \partial_\gamma^2 x(\gamma, t) = 0,$$

and

$$\partial_\gamma x(\gamma, t) \cdot \partial_\gamma^3 x(\gamma, t) = -|\partial_\gamma^2 x(\gamma, t)|^2.$$

Darcy's law

Law

$$\frac{\mu}{\kappa} v = -\nabla p - (0, 0, \mathrm{g}\,\rho),$$

v velocity, p pressure, μ viscosity, κ permeability, ρ density, and g acceleration due to gravity.

- Muskat (1937)
- Saffman and Taylor (1958)

Equation (Hele–Shaw)

$$\frac{12\mu}{b^2} v = -\nabla p - (0, \mathrm{g}\,\rho),$$

b distance between the plates.

Smooth initial data with $\mu = const.$

Equation

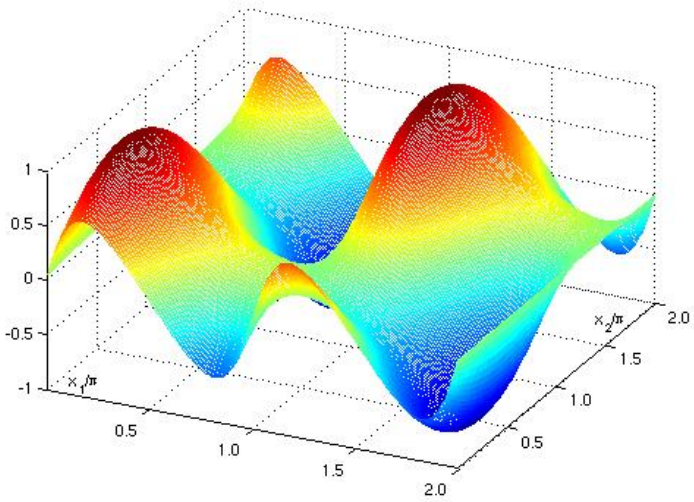
$$\left.\begin{array}{l} \rho_t + v \cdot \nabla \rho = 0 \\ v = -\nabla p - (0, \rho) \\ \operatorname{div} v = 0 \end{array}\right\}$$

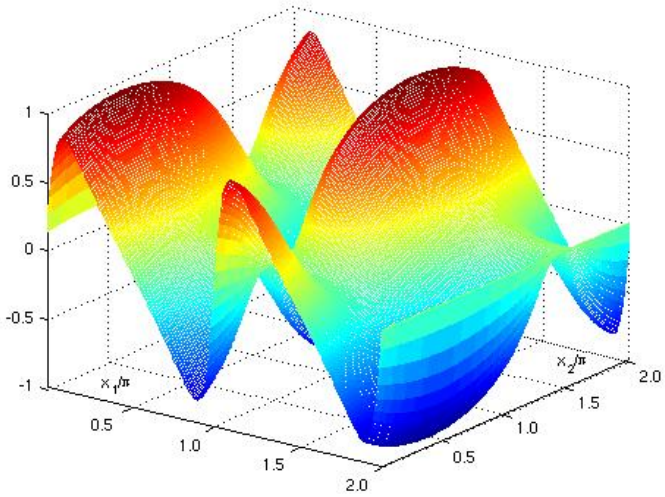
Two-dimensional mass balance equation in porous media (2DPM)

$$v(x) = \frac{1}{2\pi} PV \int_{\mathbb{R}^2} \left(-2\frac{y_1 y_2}{|y|^4}, \frac{y_1^2 - y_2^2}{|y|^4}\right) \rho(x-y) dy \; - \frac{1}{2}\left(0, \rho(x)\right),$$

$$(\partial_t + v \cdot \nabla)\nabla^\perp \rho = (\nabla v)\nabla^\perp \rho.$$

- Local existence.
- Singularities with infinite energy, with a stream function given by
$$\psi(x_1, x_2, t) = x_2 f(x_1, t) + g(x_1, t).$$
- Blow-up $\Leftrightarrow \quad \int_0^T \|\nabla\rho\|_{BMO}(t)\, dt = \infty.$
- Geometric constraints: $\eta = \dfrac{\nabla^\perp \rho}{|\nabla^\perp \rho|}.$
- Numerical simulations: $\|\nabla\rho\|_{L^\infty}(t) \sim e^t.$





- Cordoba-Gancedo (2007)

We consider the case where the fluid has different densities, that is ρ is represented by

$$\rho(x_1, x_2, t) = \begin{cases} \rho_1, & \{x_2 > f(x_1, t)\} \\ \rho_2, & \{x_2 < f(x_1, t)\} \end{cases}$$

being f the interface. Then we have

Equation

$$\frac{df}{dt}(x,t) = \frac{\rho_2 - \rho_1}{2\pi} PV \int_{\mathbb{R}} \frac{(\partial_x f(x,t) - \partial_x f(x-\alpha,t))\alpha}{\alpha^2 + (f(x,t) - f(x-\alpha,t))^2} d\alpha$$

$$f(x,0) = f_0(x); \quad x \in \mathbb{R}.$$

$$\left.\begin{array}{l} \rho_t + v \cdot \nabla \rho = 0 \\ v = -\nabla p - (0, \rho) \\ \operatorname{div} v = 0 \end{array}\right\} \quad \Leftrightarrow \quad \text{2DPM contour equation}$$

The linearized equation

If we neglect the terms of order greater than one

$$f_t = \frac{\rho_1 - \rho_2}{2}(H\partial_x f) = \frac{\rho_1 - \rho_2}{2}\Lambda f,$$
$$f(x,0) = f_0(x).$$

Applying the Fourier transform we get

$$\hat{f}(\xi) = \hat{f}_0(\xi)e^{\frac{\rho_1-\rho_2}{2}|\xi|t}.$$

Problem

- $\rho_1 < \rho_2$ *stable case,*
- $\rho_1 > \rho_2$ *unstable case.*

Local well-posedness for the stable case ($\rho_2 > \rho_1$)

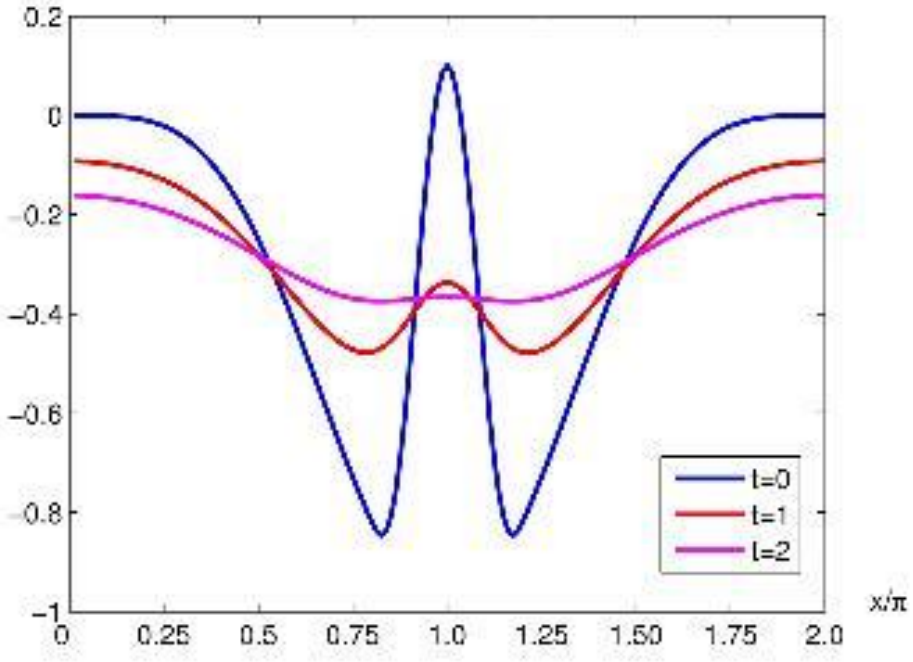
The following theorem follows

Theorem

Let $f_0(x) \in H^k$ for $k \geq 3$ and $\rho_2 > \rho_1$. Then there exists a time $T > 0$ so that there is a unique solution to 2DPM contour equation in $C^1([0,T]; H^k)$ with $f(x,0) = f_0(x)$.

Global existence?

- L^∞ decays.
- $\int f(x,t)\,dx = \int f_0(x)\,dx.$
- Global existence for small initial data: $\sum |\xi||\hat{f}(\xi)| << 1.$



Ill-posedness for the unstable case ($\rho_1 > \rho_2$)

We define

$$\|g\|_b = \sum |\hat{g}(\xi)| e^{b|\xi|} \quad b \geq 0.$$

If $\|g\|_b < \infty$, g is analytic in $|\Im z| < b$.

We get global solutions for the 2-D stable case $f(x,t)$, $x \in \mathbb{R}$, with

$$\|\partial_x f\|_a(t) \leq C(\varepsilon) \exp((2\sigma a - |\rho_2 - \rho_1| t)/4),$$

$$\|\Lambda^{1+\gamma} f_0\|_0 < C,$$

and

$$\|\Lambda^{1+\gamma+\zeta} f_0\|_0 = \infty,$$

for $\gamma, \zeta > 0$.

We take

$$f_\lambda(x_1,t) = \lambda^{-1} f(\lambda x_1, -\lambda t + \lambda^{1/2}),$$

$\{f_\lambda\}_{\lambda>0}$ a family of solutions to the unstable case. Then

$$\|f_\lambda\|_{H^s}(0) = |\lambda|^{s-\frac{3}{2}} \|f\|_{H^s}(\lambda^{\frac{1}{2}}) \leq C|\lambda|^{s-\frac{3}{2}} \|f\|_1(\lambda^{\frac{1}{2}}) \leq C|\lambda|^{s-\frac{3}{2}} e^{-\lambda^{1/2}},$$

and

$$\|f_\lambda\|_{H^s}(\lambda^{-1/2}) = |\lambda|^{s-\frac{3}{2}} \|f\|_{H^s}(0) \geq |\lambda|^{s-\frac{3}{2}} C \|\Lambda^{1+\gamma+\zeta} f_0\|_0 = \infty,$$

for $s > 3/2$ and γ, ζ small enough.

Theorem

Let $s > 3/2$, then for any $\varepsilon > 0$ there exists a solution f of 2DPM contour equation with $\rho_1 > \rho_2$ and $0 < \delta < \varepsilon$ such that $\|f\|_{H^s}(0) \leq \varepsilon$ and $\|f\|_{H^s}(\delta) = \infty$.

The Muskat problem

Equation

$$\begin{aligned} &\rho_t + u \cdot \nabla \rho = 0, \\ &\nabla \cdot u = 0, \\ &\frac{\mu}{\kappa} u = -\nabla p - (0, \mathrm{g}\rho), \end{aligned} \qquad (\mu, \rho)(x,t) = \begin{cases} (\mu^1, \rho^1), & x \in \Omega^1(t) \\ (\mu^2, \rho^2), & x \in \Omega^2(t), \end{cases}$$

- Siegel, Caflish & Howison (2004)
- Ambrose (2004)

We shall consider $z(\alpha + 2\pi k, t) = z(\alpha, t) + (2\pi k, 0)$.
Darcy's law yields

$$\omega(x, t) = \varpi(\alpha, t)\delta(x - z(\alpha, t)),$$

Biot-Savart

$$u(x, t) = \frac{1}{2\pi} PV \int \frac{(x - z(\beta, t)^{\perp}}{|x - z(\beta, t)|^2} \varpi(\beta, t) d\beta,$$

and taking limits

$$u^j(z(\alpha, t), t) = BR(z, \varpi)(\alpha, t) \mp \frac{\varpi(\alpha, t)}{2|\partial_\alpha z(\alpha, t)|^2} \partial_\alpha z(\alpha, t) \quad j = 1, 2,$$

where

$$BR(z, \varpi)(\alpha, t) = \frac{1}{2\pi} PV \int \frac{(z(\alpha, t) - z(\beta, t)^{\perp}}{|z(\alpha, t) - z(\beta, t)|^2} \varpi(\beta, t) d\beta.$$

Closing the equation: In search of a relationship between z and ϖ Beginning with Darcy's law:

$$\frac{\mu_j}{\kappa} u^j = -\nabla p_j - (0, \mathrm{g}\, \rho_j)$$

First we substitute u^j by its formula involving Birkhoff-Rott. Next, for each j, we write its scalar product with $\partial_\alpha z(\alpha)$, and finally we subtract both expressions to get

$$\frac{\mu^2 + \mu^1}{2\kappa} \varpi(\alpha, t) + \frac{\mu^2 - \mu^1}{\kappa} BR(z, \varpi)(\alpha, t) \cdot \partial_\alpha z(\alpha, t)$$

$$= -(\nabla p^2(z(\alpha, t), t) - \nabla p^1(z^1(\alpha, t), t)) \cdot \partial_\alpha z(\alpha, t) - \mathrm{g}(\rho^2 - \rho^1)\, \partial_\alpha z_2(\alpha, t)$$

$$= -\partial_\alpha(p^2(z(\alpha, t), t) - p^1(z(\alpha, t), t)) - \mathrm{g}(\rho^2 - \rho^1)\, \partial_\alpha z_2(\alpha, t)$$

Since

$$\partial_\alpha(p^2(z(\alpha, t), t) - p^1(z(\alpha, t), t)) = 0$$

Darcy's law implies

$$\Delta p(x,t) = -\mathrm{div}\,(\frac{\mu(x,t)}{\kappa}u(x,t)) - \mathrm{g}\,\partial_{x_2}\rho(x,t),$$

therefore

$$\Delta p(x,t) = \Pi(\alpha,t)\delta(x - z(\alpha,t)),$$

where $\Pi(\alpha,t)$ is given by

$$\Pi(\alpha,t) = (\frac{\mu^2 - \mu^1}{\kappa}u(z(\alpha,t),t)\cdot\partial_\alpha^\perp z(\alpha,t) + \mathrm{g}(\rho^2 - \rho^1)\partial_\alpha z_1(\alpha,t)).$$

It follows that:

$$p(x,t) = -\frac{1}{2\pi}\int_{\mathbb{T}} \ln\big(\cosh(x_2 - z_2(\alpha,t)) - \cos(x_1 - z_1(\alpha,t))\big)\Pi(\alpha,t)d\alpha,$$

for $x \neq z(\alpha,t)$, implying the important identity

$$p^2(z(\alpha,t),t) = p^1(z(\alpha,t),t), \tag{3}$$

which is just a mathematical consequence of Darcy's law, making unnecessary to impose it as a physical assumption.

Then

$$\varpi(\alpha,t) + A_\mu T(\varpi)(\alpha,t) = -2g\kappa C(\rho,\mu)\partial_\alpha z_2(\alpha,t)$$

where

$$T(\varpi) = 2BR(z,\varpi)\cdot\partial_\alpha z, \quad A_\mu = \frac{\mu^2-\mu^1}{\mu^2+\mu^1}, \quad C(\rho,\mu) = \frac{\rho^2-\rho^1}{\mu^2+\mu^1}.$$

- Baker, Meiron & Orszag (1982) $|\lambda| < 1$ for $(I + A_\mu T)$.

Contour equation

$$\begin{aligned} z_t(\alpha,t) &= BR(z,\varpi)(\alpha,t) + c(\alpha,t)\partial_\alpha z(\alpha,t), \\ \varpi(\alpha,t) &= (I + A_\mu T)^{-1}(-2g\kappa C(\rho,\mu)\partial_\alpha z_2)(\alpha,t). \end{aligned}$$

- Cordoba, Cordoba & Gancedo (2008).

Theorem

Let $z_0(\gamma) \in H^k(\mathbb{T})$ for $k \geq 3$, $F(z_0)(\alpha, \beta) < \infty$ and

$$\sigma_0(\alpha) = -(\nabla p^2(z_0(\alpha), 0) - \nabla p^1(z_0(\alpha), 0)) \cdot \partial_\alpha^\perp z_0(\alpha) > 0.$$

Then there exists a time $T > 0$ so that there is a solution to the contour equation in $C^1([0, T]; H^k(\mathbb{T}))$ with $z(\alpha, 0) = z_0(\alpha)$.

where

$$\begin{aligned}\sigma(\alpha) &= -(\nabla p^2(z(\alpha)) - \nabla p^1(z(\alpha))) \cdot \partial_\alpha^\perp z(\alpha) \\ &= \tfrac{\mu^2 - \mu^1}{\kappa} BR(z, \varpi)(\alpha) \cdot \partial_\alpha^\perp z(\alpha) + g(\rho^2 - \rho^1)\partial_\alpha z_1(\alpha).\end{aligned}$$

Energy estimates:

$$|||z|||^2 = \|z\|_{H^3}^2 + \|F(z)\|_{L^\infty}^2, \quad \|(I + A_\mu T)^{-1}\|_{H^{\frac{1}{2}} \to H^{\frac{1}{2}}} \leq e^{C|||z|||^2},$$

We can estimate as follows:

$$\|\varpi\|_{H^2} \leq e^{C|||z|||^2}.$$

By energy estimates we obtain

$$\frac{d}{dt}|||z|||^2(t) \leq -K\int_{\mathbb{T}}\sigma(\alpha,t)\partial_\alpha^3 z(\alpha,t)\cdot\Lambda(\partial_\alpha^3 z)(\alpha,t)d\alpha + e^{C(|||z|||^2)}.$$

Pointwise inequality:

$$f\Lambda(f) \geq \frac{1}{2}\Lambda(f^2).$$

$$-\int_{\mathbb{T}}\sigma\partial_\alpha^3 z\cdot\Lambda(\partial_\alpha^3 z)d\alpha \leq -\frac{1}{2}\int_{\mathbb{T}}\sigma\Lambda(|\partial_\alpha^3 z|^2)d\alpha = -\frac{1}{2}\int_{\mathbb{T}}\Lambda(\sigma)|\partial_\alpha^3 z|^2 d\alpha$$

as long as

$$\sigma(\alpha,t) > 0.$$

Sketch of the proof

- Estimates on the arc-chord condition

Lemma

$$\frac{d}{dt}\|\mathcal{F}(z)\|^2_{L^\infty}(t) \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|z\|^2_{H^3}(t))$$

Sketch of the proof

- Estimate for the evolution of the minimum of the difference of the gradients of the pressure in the normal direction. This quantity is given by

$$\sigma(\alpha) = \frac{\mu^2 - \mu^1}{k} BR(z, \varpi)(\alpha) \cdot \partial_\alpha^\perp z(\alpha) + \mathrm{g}(\rho^2 - \rho^1)\partial_\alpha z_1(\alpha).$$

Lemma Let $z(\alpha)$ be a solution of the system with $z(\alpha, t) \in C^1([0, T]; H^3)$, and

$$m(t) = \min_{\alpha \in \mathbb{T}} \sigma(\alpha, t).$$

Then

$$m(t) \geq m(0) - \int_0^t \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2(s) + \|z\|_{H^3}^2(s))ds.$$

Sketch of the proof

- Regularizing the system

Let $z^{\varepsilon}(\alpha,t)$ be a solution of the following system:

$$
\begin{aligned}
z_t^{\varepsilon,\delta}(\alpha,t) &= BR^{\delta}(z^{\varepsilon,\delta},\varpi^{\varepsilon,\delta})(\alpha,t) + c^{\varepsilon,\delta}(\alpha,t)\partial_\alpha z^{\varepsilon,\delta}(\alpha,t),\\
z^{\varepsilon,\delta}(\alpha,0) &= z_0(\alpha),
\end{aligned}
$$

where

$$
\begin{aligned}
BR^{\delta}(z,\varpi)(\alpha,t) = \Big(&-\frac{1}{4\pi}\int_{\mathbb{T}} \varpi(\beta,t)\frac{\tanh(\frac{z_2(\alpha,t)-z_2(\beta,t)}{2})(1+\tan^2(\frac{z_1(\alpha,t)-z_1(\beta,t)}{2}))}{\tan^2(\frac{z_1(\alpha,t)-z_1(\beta,t)}{2})+\tanh^2(\frac{z_2(\alpha,t)-z_2(\beta,t)}{2})+\delta}d\beta,\\
&\frac{1}{4\pi}\int_{\mathbb{T}} \varpi(\beta,t)\frac{\tan(\frac{z_1(\alpha,t)-z_1(\beta,t)}{2})(1-\tanh^2(\frac{z_2(\alpha,t)-z_2(\beta,t)}{2}))}{\tan^2(\frac{z_1(\alpha,t)-z_1(\beta,t)}{2})+\tanh^2(\frac{z_2(\alpha,t)-z_2(\beta,t)}{2})+\delta}d\beta\Big),
\end{aligned}
$$

$$\varpi^{\varepsilon,\delta}(\alpha,t) = -\frac{\mu^2-\mu^1}{\mu^2+\mu^1}\phi_\varepsilon * \phi_\varepsilon * (2BR(z^{\varepsilon,\delta},\varpi^{\varepsilon,\delta})\cdot\partial_\alpha z^{\varepsilon,\delta})(\alpha)$$
$$-2k\mathrm{g}\frac{\rho^2-\rho^1}{\mu^2+\mu^1}\phi_\varepsilon * \phi_\varepsilon * (\partial_\alpha z_2^{\varepsilon,\delta})(\alpha),$$

$$c^{\varepsilon,\delta}(\alpha,t) = \frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}} \frac{\partial_\alpha z^{\varepsilon,\delta}(\alpha,t)}{|\partial_\alpha z^{\varepsilon,\delta}(\alpha,t)|^2}\cdot\partial_\alpha BR^\delta(z^{\varepsilon,\delta},\varpi^{\varepsilon,\delta})(\alpha,t)d\alpha$$
$$-\int_{-\pi}^{\alpha}\frac{\partial_\alpha z^{\varepsilon,\delta}(\beta,t)}{|\partial_\alpha z^{\varepsilon,\delta}(\alpha,t)|^2}\cdot\partial_\beta BR^\delta(z^{\varepsilon,\delta},\varpi^{\varepsilon,\delta})(\beta,t)d\beta,$$

$$\phi\in C_c^\infty(\mathbb{R}),\quad \phi(\alpha)\geq 0,\quad \phi(-\alpha)=\phi(\alpha),\quad \int_{\mathbb{R}}\phi(\alpha)d\alpha=1,\quad \phi_\varepsilon(\alpha)=\phi(\alpha/\varepsilon)/\varepsilon,$$

for $\varepsilon>0$ and $\delta>0$.

Sketch of the proof

- The next step is to integrate the system during a time T with T independent of ε. If $z_0(\alpha) \in H^n$, we have the solution $z^\varepsilon \in C^1([0, T^\varepsilon]; H^n)$ and that

$$m^\varepsilon(t) \geq m(0) - \int_0^t \exp C(\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(s) + \|z^\varepsilon\|^2_{H^3}(s))ds,$$

for

$$m^\varepsilon(t) = \min_{\alpha \in \mathbb{T}} \sigma^\varepsilon(\alpha, t),$$

and $t \leq T^\varepsilon$.

If initially $\sigma(\alpha, 0) > 0$, there is a time depending on ε, denoted by T^ε again, in which $\sigma^\varepsilon(\alpha, t) > 0$.

Sketch of the proof

For $t \leq T^\varepsilon$

$$\frac{d}{dt}\|z^\varepsilon\|^2_{H^n}(t) \leq I + \exp C(\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(t) + \|z^\varepsilon\|^2_{H^n}(t)),$$

for

$$\begin{aligned} I &= -\frac{k}{2\pi(\mu_1+\mu_2)}\int_{\mathbb{T}} \frac{\sigma^\varepsilon(\alpha,t)}{|\partial_\alpha z^\varepsilon(\alpha,t)|^2}\phi_\varepsilon * (\partial^n_\alpha z^\varepsilon)(\alpha,t)\cdot\Lambda(\phi_\varepsilon * (\partial^n_\alpha z^\varepsilon))(\alpha,t)d\alpha \\ &\leq -\frac{k}{2\pi(\mu_1+\mu_2)A^\varepsilon(t)}\int_{\mathbb{T}} \sigma^\varepsilon(\alpha,t)\frac{1}{2}\Lambda(|\phi_\varepsilon * (\partial^n_\alpha z^\varepsilon)|^2)(\alpha,t)d\alpha. \\ &= -\frac{k}{2\pi(\mu_1+\mu_2)A^\varepsilon(t)}\int_{\mathbb{T}} \Lambda(\sigma^\varepsilon)(\alpha,t)\frac{1}{2}|\phi_\varepsilon * (\partial^n_\alpha z^\varepsilon)|^2(\alpha,t)d\alpha, \end{aligned}$$

we obtain

$$I \leq C\|\mathcal{F}(z^\varepsilon)\|_{L^\infty}\|\Lambda(\sigma^\varepsilon)\|_{L^\infty}\|\partial^n_\alpha z^\varepsilon\|^2_{L^2} \leq \exp C(\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty} + \|z^\varepsilon\|^2_{H^3}).$$

Sketch of the proof

Finally, for $t \leq T^\varepsilon$ we have

$$\frac{d}{dt}\|z^\varepsilon\|^2_{H^n}(t) \leq C \exp C(\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(t) + \|z^\varepsilon\|^2_{H^n}(t)).$$

and

$$\frac{d}{dt}\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(t) \leq C \exp C(\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(t) + \|z^\varepsilon\|^2_{H^3}(t)),$$

Therefore

$$\frac{d}{dt}(\|z^\varepsilon\|^2_{H^n}(t) + \|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(t)) \leq C \exp C(\|z^\varepsilon\|^2_{H^n}(t) + \|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(t))$$

for $t \leq T^\varepsilon$.

Integrating together with

$$m^\varepsilon(t) \geq m(0) - \int_0^t \exp C(\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(s) + \|z^\varepsilon\|^2_{H^3}(s))ds,$$

q.e.d.

The fluid interface problem

Equation

$$\begin{aligned} &\rho_t + u \cdot \nabla \rho = 0, \\ &\nabla \cdot u = 0, \\ &\rho(u_t + (u \cdot \nabla)u) = -\nabla p - (0, \mathrm{g}\rho), \end{aligned} \qquad \rho(x,t) = \begin{cases} \rho^1, & x \in \Omega^1(t) \\ \rho^2, & x \in \Omega^2(t), \end{cases}$$

Previous works

- Rayleigh (1879)
- Taylor (1950)
- Craig (1985)
- Ebin (1987)
- Beale, Hou & Lowengrub (1993)
- Wu (1997)
- Christodoulou & Lindblad (2000)
- Lindblad (2005)
- Ambrose & Masmoudi (2005)
- Coutand & Shkoller (2007)
- Shatah & Zeng (2008)
- Zhang & Zhang (2008)

Contour equation

We consider $\partial\Omega^j(t) = \{z(\alpha,t) = (z_1(\alpha,t), z_2(\alpha,t)) \,:\, \alpha \in \mathbb{R}\}$:

- $z(\alpha + 2\pi k, t) = z(\alpha,t) + (2\pi k, 0)$,
- $z(\alpha + 2\pi k, t) = z(\alpha,t)$.

We study the irrotational case

$$\omega(x,t) = \varpi(\alpha,t)\delta(x - z(\alpha,t)).$$

Biot-Savart yields

$$u(x,t) = \frac{1}{2\pi} PV \int \frac{(x - z(\beta,t))^{\perp}}{|x - z(\beta,t)|^2} \varpi(\beta,t) d\beta,$$

for $x \neq z(\alpha,t)$.

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for $x \neq z(\alpha, t)$.

Then (Bernoulli's law)

Contour equation

$$\varpi_t = -2A_\rho \partial_t BR(z,\varpi)\cdot\partial_\alpha z - A_\rho \partial_\alpha\left(\frac{|\varpi|^2}{4|\partial_\alpha z|^2}\right) + \partial_\alpha(c\,\varpi) + 2A_\rho c\,\partial_\alpha BR(z,\varpi)\cdot\partial_\alpha z(\alpha,t) - 2A_\rho g\partial_\alpha z_2, \tag{4}$$

for $A_\rho = (\rho^2 - \rho^1)/(\rho^2 + \rho^1)$ and c parametrization freedom.

Contour equation

$$z_t = BR(z,\varpi) + c\,\partial_\alpha z. \tag{5}$$

Equations (4) and (5) give weak solutions of Euler.

We can choose c to get $|\partial_\alpha z(\alpha,t)|^2 = A(t)$.

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Conditions

- Arc-chord:

$$\mathcal{F}(z)(\alpha,\beta,t) = \frac{|\beta|}{|z(\alpha,t) - z(\alpha-\beta,t)|} \qquad \forall\, \alpha,\beta \in (-\pi,\pi),$$

and

$$\mathcal{F}(z)(\alpha,0,t) = \frac{1}{|\partial_\alpha z(\alpha,t)|}.$$

- Rayleigh-Taylor:

$$\sigma(\alpha,t) = -(\nabla p^2(z(\alpha,t),t) - \nabla p^1(z(\alpha,t),t)) \cdot \partial_\alpha^\perp z(\alpha,t) > 0.$$

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Local-existence

- Cordoba, Cordoba & Gancedo (2008)

Theorem

Let $z_0(\alpha) \in H^k$ *and* $\varpi_0(\alpha) \in H^{k-1}$ *for* $k \geq 4$,

$$\mathcal{F}(z_0)(\alpha, \beta) < \infty, \quad \text{and} \quad \sigma(\alpha, 0) > 0.$$

Then there exists a time $\mathrm{T} > 0$ *so that we have a solution to* (4) *and* (5) *in the case* $\rho^1 = 0$, *where* $z(\alpha, t) \in C^1([0, \mathrm{T}]; H^k)$ *and* $\varpi(\alpha, t) \in C^1([0, \mathrm{T}]; H^{k-1})$ *with* $z(\alpha, 0) = z_0(\alpha)$ *and* $\varpi(\alpha, 0) = \varpi_0(\alpha)$.

The linearized equation

$$f_{tt}(\alpha,t) = -\sigma\Lambda(f)(\alpha,t), \qquad \begin{cases} \sigma < 0 \Rightarrow & e^{|\sigma\xi|^{\frac{1}{2}}t}, \quad e^{-|\sigma\xi|^{\frac{1}{2}}t} \\ \sigma > 0 \Rightarrow & \cos(|\sigma\xi|^{\frac{1}{2}}t), \quad \sin(|\sigma\xi|^{\frac{1}{2}}t) \end{cases}$$

It can be written as follows:

$$f_t(\alpha,t) = H(g)(\alpha,t),$$
$$g_t(\alpha,t) = -\sigma\partial_\alpha f(\alpha,t),$$

and therefore

$$E_L(t) = \int (\sigma|\partial_\alpha f|^2 + |\Lambda^{\frac{1}{2}} g|^2) d\alpha.$$

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Energy of the system

Case $z \in H^4$:

$$E(t) = \|z\|^2_{H^3}(t) + \int_{-\pi}^{\pi} \frac{\sigma(\alpha,t)}{\rho^2|\partial_\alpha z(\alpha,t)|^2}|\partial_\alpha^4 z(\alpha,t)|^2 d\alpha$$
$$+ \|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|\varpi\|^2_{H^2}(t) + \|\varphi\|^2_{H^{4-\frac{1}{2}}}(t),$$

for $\sigma(\alpha,t) > 0$ and $\varphi(\alpha,t)$ given by

$$\varphi(\alpha,t) = \frac{\varpi(\alpha,t)}{2|\partial_\alpha z(\alpha,t)|} - c(\alpha,t)|\partial_\alpha z(\alpha,t)|.$$

- Beale, Hou & Lowengrub (1993)
- Ambrose & Masmoudi (2005)

Equation in terms of $\varphi(\alpha, t)$

- Case $A_\rho = 0$ (vortex sheet problem):

$$\varpi_t(\alpha, t) = \partial_\alpha(c\,\varpi)(\alpha, t).$$

Kelvin-Helmholtz instability arises.

The linear formulation is given by

$$f_{tt}(\alpha, t) = -\partial_\alpha^2 f(\alpha, t) \quad \Rightarrow \quad e^{|\xi| t}, \quad e^{-|\xi| t}$$

- Case $A_\rho = 1$ (water wave):

$$\varphi_t = -\frac{\partial_\alpha(\varphi^2)}{2|\partial_\alpha z|} - B(t)\,\varphi - \partial_t BR(z, \varpi) \cdot \frac{\partial_\alpha z}{|\partial_\alpha z|} - g\frac{\partial_\alpha z_2}{|\partial_\alpha z|} - \partial_t(c|\partial_\alpha z|).$$

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Estimates on the inverse operator $(I+T)^{-1}$

Let T be defined by $\quad T(u)(\alpha,t) = 2BR(z,u)(\alpha)\cdot\partial_\alpha z(\alpha),$

then $T : L^2 \to H^1$ and $\quad \|T\|_{L^2\to H^1} \leq \|\mathcal{F}(z)\|^4_{L^\infty}\|z\|^4_{C^{2,\delta}}.$

$$\varpi_t(\alpha,t) = (I+T)^{-1}(R(z,\varpi))(\alpha,t).$$

- Baker, Meiron & Orszag (1982).
- Córdoba, Córdoba & Gancedo (2008):

$$\|(I+T)^{-1}\|_{L^2\to L^2} \leq e^{C(\|z\|^2_{H^3}+\|F(z)\|^2_{L^\infty})},$$

(conformal mappings, Hopf lemma and Dahlberg-Harnack).

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A priori energy estimates

$$E(t) = \|z\|^2_{H^3}(t) + \int_{-\pi}^{\pi} \frac{\sigma(\alpha,t)}{\rho^2|\partial_\alpha z(\alpha,t)|^2}|\partial_\alpha^4 z(\alpha,t)|^2 d\alpha$$
$$+ \|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|\varpi\|^2_{H^2}(t) + \|\varphi\|^2_{H^{4-\frac{1}{2}}}(t).$$

Lemma

Let $z(\alpha,t)$ and $\varpi(\alpha,t)$ be a solution of (4) and (5). Then, the following a priori estimate holds:

$$\frac{d}{dt}E(t) \leq \frac{C}{m^q(t)}\exp(CE(t)),$$

for $m(t) = \min_{\alpha\in[-\pi,\pi]} \sigma(\alpha,t) = \sigma(\alpha_t,t) > 0$ and C and q some universal constants.

The Rayleigh-Taylor condition

The true energy: $E_{RT}(t) = E(t) + \frac{1}{m^\varepsilon(t)}$.

One gets $\sigma^\varepsilon(\alpha, t) \in C^1([0, T^\varepsilon] \times [-\pi, \pi])$,
and therefore $m^\varepsilon(t)$ is Lipschitz.

$$\frac{d}{dt}(m^\varepsilon)(t) = \sigma_t^\varepsilon(\alpha_t, t), \qquad \frac{d}{dt}(\frac{1}{m^\varepsilon})(t) = \frac{-\sigma_t^\varepsilon(\alpha_t, t)}{(m^\varepsilon(t))^2},$$

for almost every t.It yields

$$\frac{d}{dt}E_{RT}(t) \leq C\exp(CE_{RT}(t)),$$

almost everywhere and therefore

$$E_{RT}(t) \leq -\frac{1}{C}\ln(\exp(-CE_{RT}(0)) - C^2 t).$$

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Vortex Sheets

Equation

$$u_t + (u \cdot \nabla)u = -\nabla p,$$

$$\nabla \cdot u = 0,$$

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Contour equation

$$\begin{aligned} z_t &= BR(z,\varpi) + c\partial_\alpha z, \\ \varpi_t &= \partial_\alpha(c\,\varpi) \end{aligned} \tag{6}$$

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The Parametrization

Linearizing

$$\varpi(\alpha, t) = 1 \Rightarrow \quad e^{|\xi|t}, \quad e^{-|\xi|t}.$$

We study the case: $\int \varpi(\alpha, 0) d\alpha = 0.$

We take a parametrization given by $c = \frac{1}{2} H(\varpi)$.Let us point out that for $z(\alpha, t) = (\alpha, 0)$ it follows:

$$BR(z, \varpi)(\alpha, t) = \frac{1}{2} H(\varpi)(\alpha, t)(0, 1).$$

For the amplitude one finds

$$\varpi_t - \frac{1}{2} \partial_\alpha (\varpi H \varpi) = 0.$$

- Castro, Córdoba & Gancedo (2009)

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Ill-posedness for the amplitude equation

If we take

$$\Omega(\alpha, t) = H\varpi(\alpha, t) - i\varpi(\alpha, t),$$

by using the Green's function, Ω can be extended analytically on the unit ball and satisfies

$$\Omega_t(u, t) - \frac{1}{2} iu\Omega(u, t)\Omega_u(u, t) = 0, \quad \text{with} \quad u = re^{i\alpha}.$$

Complex trajectories:

$$X_t(u, t) = -\frac{1}{2} iX(u, t)\Omega(X(u, t), t), \quad \text{with} \quad X(u, 0) = u.$$

Therefore $\Omega(X(u, t), t) = \Omega(u, 0)$ and $X(u, t) = ue^{-\frac{1}{2} i\Omega(u,0)t}$.

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