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Rationally Connected Varieties and Fundamental Groups

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The Lefschetz hyperplane theorem says that if $X \subset \mathbb{P}^N$ is a smooth projective variety over $\mathbb{C}$ and $C \subset X$ is a smooth curve which is a complete intersection of hyperplanes with X then

$$\pi_1^{\rm top}(C) \to \pi_1^{\rm top}(X) \quad \text{is surjective,}$$

where $\pi_1^{\rm top}$ is the topological fundamental group. Later a quasiprojective version of this result was also established (see, for instance, [13] for a discussion and further generalizations). This says that if $X^0 \subset X$ is open and $C \subset X$ as above is *sufficiently general,* then

$$\pi_1^{\rm top}(C \cap X^0) \to \pi_1^{\rm top}(X^0) \quad \text{is surjective.}$$

On a rationally connected variety one would like to use rational curves to obtain a similar result. Complete intersection curves are essentially never rational. (For instance, if $X \subset \mathbb{P}^n$ is a hypersurface then a general complete intersection with hyperplanes is a rational curve iff X is a hyperplane or a quadric.) Therefore we have to proceed in a quite different way.

Let X be a smooth, projective, rationally connected variety over $\mathbb{C}$ and $X^0 \subset X$ an open set whose complement is a normal crossing divisor $\sum D_i$. Let $C \subset X$ be a smooth rational curve which intersects $\sum D_i$ transversally everywhere. It is then easy to see that the *normal* subgroup of $\pi_1^{\rm top}(X^0)$ generated by the image of $\pi_1^{\rm top}(C \cap X^0)$ is $\pi_1^{\rm top}(X^0)$ itself. One can also achieve that the image of $\pi_1^{\rm top}(C \cap X^0)$ has finite index in $\pi_1^{\rm top}(X^0)$. The map $\pi_1^{\rm top}(C \cap X^0) \to \pi_1^{\rm top}(X^0)$ is, however, not always onto. Some examples are given in [19, 4.5].

The group $\pi_1^{\rm top}(X^0)$ is typically an infinite group and in many questions the difference between it and its finite index subgroups is minor. However, in the arithmetic applications, for instance in (34) and in [22] it is crucial to prove surjectivity.

In order to work in arbitrary characteristic, we should use the algebraic fundamental group, which we denote by π_1. (A very good introduction to algebraic fundamental groups is [24]. A more thorough treatment is given in [27] and the ultimate reference is [14].) All the crucial points of the arguments in this paper can be seen by concentrating on the case of complex varieties and the topological fundamental group.

We care only about finite quotients of the algebraic fundamental group. Classically, quotients of the fundamental group correspond to covering spaces, and similarly, finite étale covers correspond to finite index open subgroups of the algebraic fundamental group. (The relevant topology plays no essential role in what follows.) A consequence of this is that a surjectivity between fundamental groups can be checked on finite quotients. We adopt this as a definition:

Definition-Theorem 1 (cf. [27, p. 94])**.** Let k be a field and $p : Y \to X$ a morphism of normal, geometrically connected k-schemes. Then $\pi_1(Y) \to \pi_1(X)$ is *surjective* if for every finite étale cover $X' \to X$ with X' connected, the pull back $Y \times_X X'$ is also connected.

Similarly, the image of $\pi_1(Y) \to \pi_1(X)$ is said to have index $\leq m$ if for every finite étale cover $X' \to X$ with X' connected, the pull back $Y \times_X X'$ has $\leq m$ connected components.

There are infinite groups without finite quotients. Thus there are compact manifolds which are not simply connected but have no finite sheeted covering spaces. It is not known if this also occurs for smooth projective varieties over $\mathbb{C}$. In any case, the algebraic fundamental group detects only the finite quotients of the topological fundamental group in characteristic zero.

Remark 2 (Base points)**.** As in topology, the fundamental group can be defined only with a fixed base point. Fundamental groups of the same scheme with different base points are isomorphic, but the isomorphism is well defined only up to inner automorphisms. For some questions, for instance for surjectivity of maps, this does not matter. In such cases we occasionally omit the base point from the notation.

It is, however, important to keep in mind that ignoring the base point can easily lead to wrong conclusions.

In algebraic geometry the base point is a *geometric point* of X. That is, a morphism $\bar{x} \to X$ where $\bar{x}$ is the spectrum of an algebraically closed field.

If $x \in X$ is a point and K is any algebraically closed field containing the residue field $k(x)$, then we get a geometric point $\bar{x} := \operatorname{Spec} K$ with a map $\bar{x} \to X$. The resulting fundamental group $\pi_1(X, \bar{x})$ does not depend on the choice of K and we usually denote it by $\pi_1(X, x)$.

As in topology, we have the equivalence

$$\left\{ \begin{array}{c} \text{finite index open} \\ \text{subgroups of } \pi_1(X, x) \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} \text{connected finite étale} \\ \text{covers of } X \text{ plus a} \\ \text{lifting of the base point} \end{array} \right\}$$

1. Statements of the Main Results

Separably rationally connected (or SRC) varieties were introduced in [21]. See [1, Sec. 34] in this volume for the definition and basic properties. (Note in particular that an SRC variety is always assumed to be geometrically connected.) A weaker version of our main Theorem (3) was proved in [19]. The key improvement is that the present proof, besides being simpler, does not need the resolution of singularities, thereby extending the range of applications to positive characteristic. It is also nicer that the choice of the curve C is largely independent of the open set X^0.

We formulate the main result in two versions, with fixed base point and with variable base point. Both are quite useful and it seems very cumbersome to have a unified statement.

Theorem 3. *Let X be a smooth, projective, SRC variety over a field k and $x \in X(k)$. Then there is a dominant family of rational curves through x (defined over k)*

$$F : W \times \mathbb{P}^1 \to X, \quad F\bigl(W \times (0{:}1)\bigr) = \{x\}$$

with the following properties:

1. *W is geometrically irreducible, smooth and open in* $\operatorname{Hom}\bigl(\mathbb{P}^1, X, (0{:}1) \mapsto x\bigr)$ *and $F : W \times (1{:}0) \to X$ is smooth.*
2. *There is a k-compactification $\bar{W} \supset W$ such that $\bar{W}$ has a smooth k-point.*
3. *The pullback $F_w^* T_X$ is ample for every $w \in W$.*

4. *Let $K \supset k$ be an algebraic closure and $X^0_K \subset X_K$ any open K-subvariety containing x. Then for every $w \in W(K)$ the induced map on the geometric fundamental groups*

$$\pi_1\bigl(F^{-1}(X^0_K), \bigl(w, (0{:}1)\bigr)\bigr) \twoheadrightarrow \pi_1(X^0_K, x) \quad \textit{is surjective.}$$

Remark 4. From the proof we see that we have considerable freedom in constructing W and F. Thus we can construct them with additional properties that are useful in some applications.

5. If $\dim X \geq 3$ then we can also assume that F is an embedding on every $\mathbb{P}^1_w$. If $\dim X = 2$ then we can also assume that F is an immersion on every $\mathbb{P}^1_w$.
6. Given an integer d, we can also assume that $F^*_w T_X(-d)$ is ample for every $w \in W$.
7. Fix an embedding $X \subset \mathbb{P}^N$. Then the degrees of the curves $F(\mathbb{P}^1_w)$ can be bounded from above in terms of the degree and dimension of X (and the above integer d).

We point out at the end of the proof (26) how to achieve these additional properties.

These comments also apply to (6).

Based on topology, one would expect that for $w \in W(K)$ general, the induced map

$$\pi_1\bigl(\mathbb{P}^1_w \cap F^{-1}(X^0_K), (0{:}1)\bigr) \to \pi_1(X^0_K, x)$$

is surjective. This is indeed so in characteristic zero, but in characteristic p the étale covers of p-power degree behave unexpectedly (see, for instance, example (15). The following weaker result, however, still holds, and this is enough for most applications.

Corollary 5. *Notation and assumptions as in* (3)*. Let $\pi_1(X^0_K, x) \twoheadrightarrow G$ be a finite quotient (whose kernel is open). Then there is a dense open subset $W^0_G \subset W$ such that for every $w \in W^0_G(K)$ the composed map*

$$\pi_1\bigl(\mathbb{P}^1_w \cap F^{-1}(X^0_K), (0{:}1)\bigr) \to \pi_1(X^0_K, x) \to G \quad \textit{is surjective.}$$

The base point free version of (3) is the following:

Theorem 6. *Let X be a smooth, projective SRC variety over a field k. Then there is a dominant family of rational curves (defined over k)*

$$F : W \times \mathbb{P}^1 \to X$$

with the following properties:

1. *W is geometrically irreducible and smooth.*
2. *F is smooth and $F_w^* T_X$ is ample for every $w \in W$.*
3. *Let $K \supset k$ be an algebraic closure and $X_K^0 \subset X_K$ any open K-subvariety. Then for every $z \in F^{-1} X_K^0(K)$ the induced map on the geometric fundamental groups*

$$\pi_1(F^{-1} X_K^0, z) \twoheadrightarrow \pi_1\big(X_K^0, F(z)\big) \quad \text{is surjective.}$$

Corollary 7. *Notation and assumptions as in* (6). *Let Y_K be an irreducible K-variety and $h : Y_K \to X_K$ a dominant morphism.*

Then there is a dense open set $W_Y^0 \subset W_K$ such that for every $w \in W_Y^0(K)$ the fiber product $Y_K \times_{X_K} \mathbb{P}^1_w$ is irreducible.

2. Fundamental Groups in Families

In this section we gather some fundamental group lemmas. The main result is (17) comparing the fundamental groups of different fibers of a morphism.

Lemma 8. *Let X be a normal variety and $X^0 \subset X$ an open subvariety. Then for every geometric point $x \to X^0$ the induced map $\pi_1(X^0, x) \twoheadrightarrow \pi_1(X, x)$ is surjective.*

Proof. Let $p : Y \to X$ be a finite étale cover with Y connected. Since Y is normal, it is also irreducible. Then $p^{-1}(X^0) \subset Y$ is a dense open subscheme, hence also irreducible and connected. ■

Lemma 9. *Let $p : Y \to X$ be a dominant morphism of normal varieties. Then there is an open subset $X^0 \subset X$ and a factorization*

$$p^{-1}(X^0) \xrightarrow{q} Z^0 \xrightarrow{r} X^0$$

such that q has only geometrically irreducible fibers and r is finite and étale. Z^0 is also normal and if Y is irreducible then so is Z^0.

Proof. Let $\bar{Y} \supset Y$ be a normal variety such that p extends to a proper morphism $\bar{p} : \bar{Y} \to X$. Let $\bar{Y} \to W \to X$ be the Stein factorization of $\bar{p}$. $W \to X$ is finite and dominant, hence W is the normalization of X in the function field $k(W)$. Let $F \subset k(W)$ denote the separable closure of $k(X)$ in $k(W)$. Let Z be the normalization of X in the function field F. Then $k(W)/k(Z)$ is purely inseparable and $k(Z)/k(X)$ is separable. Thus $Z \to X$ is étale over an open set $X^1 \subset X$.

Next we claim that there is an open subset $W^1 \subset W$ such that the fibers of $\bar{Y} \to W$ are geometrically irreducible over W^1. This is a little tricky in positive characteristic. Set $K = k(W)$.

The generic fiber $\bar{Y}_K$ is normal and $H^0(\bar{Y}_K, \mathcal{O}) = K$ by the definition of Stein factorization. Unfortunately this does not imply that the geometric generic fiber $\bar{Y}_{\bar{k}}$ is also normal. Nonetheless, by (10) the geometric generic fiber $\bar{Y}_{\bar{K}}$ is irreducible.

Thus there is an open subset $W^1 \subset W$ such that the fibers of $\bar{Y} \to W$ are geometrically irreducible over W^1 (cf. [8, IV.12]). Let $X^2 \subset X$ be an open subset whose preimage in W is contained in W^1. Taking $X^0 := X^1 \cap X^2$ we are done. ■

Lemma 10. *Let Z_K be a normal proper variety over a field K and assume that $H^0(Z, \mathcal{O}_Z) = K$.*

Then Z is geometrically irreducible, that is, $Z_{\bar{K}}$ is irreducible.

Proof. If L/K is a separable field extension then $Z_L := Z_K \times_K \operatorname{Spec} L$ is also normal (see, for instance [23, 21.E]). If we are in characteristic zero then $\bar{K}/K$ is separable, thus $Z_{\bar{K}}$ is also normal. Cohomologies of sheaves change by tensoring under field extensions, thus $H^0(Z_{\bar{K}}, \mathcal{O}_{Z_{\bar{K}}}) = \bar{K}$, and so $Z_{\bar{K}}$ is connected. A normal and connected scheme is irreducible.

In positive characteristic we have a problem with inseparable extensions of K.

Let L/K be a finite degree purely inseparable extension. Let Z_L^n denote the normalization of Z_L. Then Z_L^n is also the normalization of Z_K in the function field $L(Z)$. $L(Z)/K(Z)$ is purely inseparable, thus there is a p-power q such that the q-power map $F_q : g \mapsto g^q$ maps $L(Z)$ into $K(Z)$. This implies that, up to a power of the Frobenius map, $Z_L^n \to Z_K$ can be inverted. In particular, $Z_L^n \to Z_K$ is a *homeomorphism.*

Every finite extension of K can be written as a separable extension followed by an inseparable one. Putting these two steps together we get that if Y_K is normal and geometrically connected then it is geometrically irreducible. ■

Lemma 11. *Let $p : Y \to X$ be a dominant morphism of geometrically irreducible normal varieties over an algebraically closed field k. Let $\bar{y} \to Y$ be a geometric point of Y and $\bar{x} = p(\bar{y})$. Then* $\operatorname{im}\left[\pi_1(Y, \bar{y}) \to \pi_1(X, \bar{x})\right]$ *is a finite index closed subgroup of $\pi_1(X, \bar{x})$.*

Proof. Let $Y^0 := p^{-1}(X^0) \xrightarrow{q} Z^0 \xrightarrow{r} X^0$ be as in (9). The group $\pi_1(Z^0)$ has finite index closed image in $\pi_1(X^0)$, which surjects onto $\pi_1(X)$. Thus it is enough to prove that $\pi_1(Y^0)$ surjects onto $\pi_1(Z^0)$.

Let $Z^1 \to Z^0$ be a connected étale cover. Then $Y^0 \times_{Z^0} Z^1 \to Z^1$ has geometrically irreducible fibers, hence the fiber product is also connected. Thus $\pi_1(Y^0)$ surjects onto $\pi_1(Z^0)$ according to the definition (1). ∎

The following result was stated by [4] over $\mathbb{C}$ but the proof applies in any characteristic:

Corollary 12. *Let X be a normal, proper rationally connected variety over an algebraically closed field k. Then $\pi_1(X)$ is finite.*

Proof. Let $x \in X$ be any point. Since X is rationally connected, there is a dominant morphism

$$F : W \times \mathbb{P}^1 \to X \quad \text{such that} \quad F\big(W \times (0{:}1)\big) = \{x\}.$$

By (11) the image of $\pi_1(W \times \mathbb{P}^1)$ has finite index in $\pi_1(X)$. On the other hand, $\pi_1(W \times \mathbb{P}^1) = \pi_1\big(W \times (0{:}1)\big)$ and the latter is mapped to the identity. Thus $\pi_1(X)$ is finite. ∎

In the smooth case, we even have simple connectedness. In characteristic zero this was proved by [4], it is also explained in [7]. The positive characteristic case follows from a recent result of [16], a proof is written up in [6, 3.6].

Theorem 13 [20]**.** *Let X be a smooth projective SRC variety over an algebraically closed field k of arbitrary characteristic. Then $\pi_1(X) = \{1\}$.* ∎

14 (Fundamental groups in families)**.** We also need to know how the fundamental group varies in families. In characteristic zero the question is topological and easy. There are some new twists in positive characteristic. Several of these are illustrated by the following example:

Example 15. Let us work over an algebraically closed field of characteristic p. Consider the surface $S := (f = 0) \subset \mathbb{A}^3$ where $f = y + z - xz^p + z^{2p}$. The derivative $\partial f/\partial z = 1$, so the projection onto the $z = 0$ plane $S \to \mathbb{A}^2$ is finite and étale. (This already shows that $\pi_1(\mathbb{A}^2)$ is nontrivial.)

Consider the line $L_c := (y - cx = 0)$. The preimage of L_c in S is isomorphic to the curve $cx + z - xz^p + z^{2p} = 0$. If $c = 0$ then this curve is reducible but if $c \neq 0$ then it is irreducible. This shows that the image of

$$\pi_1(\mathbb{A}^1) \cong \pi_1(L_c) \to \pi_1(\mathbb{A}^2)$$

depends on c, and in fact the map is not surjective for $c = 0$. Since any two lines are equivalent under automorphisms of $\mathbb{A}^2$, we get that for any line $L \subset \mathbb{A}^2$, the natural map

$$\pi_1(\mathbb{A}^1) \cong \pi_1(L) \to \pi_1(\mathbb{A}^2)$$

is not surjective. It is not hard to see, however, that the following slightly weaker version holds:

Let $\pi_1(\mathbb{A}^2) \to G$ be a finite quotient (with open kernel). Then the composite

$$\pi_1(\mathbb{A}^1) \cong \pi_1(L) \to \pi_1(\mathbb{A}^2) \to G$$

is surjective for a general line L, the notion of general depending on G.

The latter observation holds very generally:

Proposition 16. *Let k be an algebraically closed field. Let $f : Y \to X$ be a morphism of irreducible and normal k-varieties. Let $S \subset Y$ be an irreducible sub-variety such that $S \to X$ is dominant. For $s \in S(k)$ let Y_s denote the fiber of f over $f(s)$. Let Z be a variety, $z \in Z(k)$ a point and $\pi_1(Z, z) \to G$ a finite quotient (with open kernel). Let $h : Y \to Z$ be a morphism such that $h(S) = \{z\} \in Z$.*

Then there is a dense open subset $S^0 \subset S$ such that for every $s \in S^0(k)$ the two maps

$$\pi_1(Y_s, s) \to \pi_1(Z, z) \to G \quad \text{and} \quad \pi_1(Y, s) \to \pi_1(Z, z) \to G$$

have the same image.

Proof. We are allowed to change base from X to S and then replace S by an open subset. Thus we can assume that $S \cong X$ is a section.

Let $Z' \to Z$ be the cover corresponding to the image of $\pi_1(Y, s) \to \pi_1(Z, z) \to G$. We need to prove that the number of irreducible components of $Z' \times_Z Y$ is the same as the number of irreducible components of $Z' \times_Z Y_s$ for $s \in S^0$. Let $W \subset Z' \times_Z Y$ be an irreducible component. Then W is an irreducible étale cover of Y which is trivial over S. Let $g : W \to X$ be the composite.

As in (9) there are open sets $X^0 \subset X$, $W^0 \subset W$ and a factorization $W^0 \to V^0 \to X^0$ where $W^0 \to V^0$ has geometrically irreducible fibers and $V^0 \to X^0$ is finite and étale. Since $S \subset W$ is a section of g, we have $X^0 \cong S \cap W^0 \to V^0 \to X^0$, thus $V^0 \cong X^0$. Therefore $W^0 \to X^0$ has geometrically irreducible fibers. ■

As a corollary we obtain a lower semicontinuity statement for fundamental groups of fibers. We do not compare the groups themselves, just their images in finite groups whenever this can be done sensibly.

Corollary 17. *Let k be an algebraically closed field. Let $f : Y \to X$ be a morphism of irreducible and normal k-varieties with a section $S \subset Y$. For $s \in S$ let Y_s denote the fiber of f over $f(s)$. Let Z be a k-variety, $z \in Z(k)$ a point and $h : Y \to Z$ a morphism such that $h(S) = \{z\} \in Z$.*

1. *For any $s, s' \in S$,*
$$\operatorname{im}\left[\pi_1(Y_s, s) \to \pi_1(Z, z)\right] \subset \operatorname{im}\left[\pi_1(Y, s') \to \pi_1(Z, z)\right].$$
2. *Let $S^0 \subset S$ be any dense open subset. Then*
$$\operatorname{im}\left[\pi_1(Y, s') \to \pi_1(Z, z)\right] = \Big\langle \operatorname{im}\left[\pi_1(Y_s, s) \to \pi_1(Z, z)\right] \ : \forall s \in S^0 \Big\rangle.$$

Proof. The maps $Y_s \to Y \to Z$ give the inclusion
$$\operatorname{im}\left[\pi_1(Y_s, s) \to \pi_1(Z, z)\right] \subset \operatorname{im}\left[\pi_1(Y, s) \to \pi_1(Z, z)\right].$$
Since S is irreducible and contracted to z,
$$\operatorname{im}\left[\pi_1(Y, s) \to \pi_1(Z, z)\right] = \operatorname{im}\left[\pi_1(Y, s') \to \pi_1(Z, z)\right]$$
for any $s, s' \in S$, proving (1).

This implies the containment $\supset$ in (2), and we only need the reverse inclusion. By the correspondence between the fundamental group and finite étale covers we need to prove that if $Z' \to Z$ is a finite étale cover such that $Y_s \times_Z Z'$ is reducible for every $s \in S^0$ then $Y \times_Z Z'$ is also reducible. This is so by (16). ■

Remark 18. Several times we will apply (17) to smoothings of combs, as defined in [1, 43]. Let $C = C_0 \cup \cdots \cup C_n$ be a pointed comb with handle $C_0 \ni p$. Let $c_i \in C_i$ be the nodes and $C^* \subset C$ an open subset containing C_0. The injections $C_i \hookrightarrow C$ induce maps $\pi_1(C^* \cap C_i, c_i) \to \pi_1(C^*, c_i)$. (These are in fact injections, but this is not crucial to us.) Thus we obtain maps $\pi_1(C^* \cap C_i, c_i) \to \pi_1(C^*, p)$ which are well defined modulo conjugation by an element of $\pi_1(C_0, p)$.

In particular, if $F : C \to X$ is assembled from the maps $f_i : C_i \to X$ and $X^0 \subset X$ is an open subset containing x then

$$\operatorname{im}\left[\pi_1(F^{-1}(X^0), p) \to \pi_1(X^0, x)\right]$$

contains all the images

$$\operatorname{im}\left[\pi_1(f_i^{-1}(X^0), c_i) \to \pi_1(X^0, x)\right] \quad \text{for } i = 1, \ldots, n.$$

At least when the base field is algebraically closed, the conjugation by elements of $\pi_1(C_0, p)$ does not matter since $F(C_0) = \{x\}$ so $\pi_1(C_0, p)$ is mapped to the identity.

Note that a similar assertion would not hold without the assumption $F(C_0) = x$.

3. Proofs of the Theorems

First we reduce (3) to the a priori much weaker (19). Then in the second part we prove (19). Finally we explain that (3) implies the other statements. We repeatedly use several properties of stable curves discussed in [1, Sec.8].

Proposition 19. *Let X be a smooth, projective, SRC variety over an algebraically closed field k and $x \in X(k)$. Then there is a family of free genus zero stable maps with base point x*

$$(C/S,\ F : C \to X \times S,\ \sigma : S \to C)$$

parametrized by a scheme of finite type S such that for every open subset $X^0 \subset X$ with $x \in X^0$ the images

$$\operatorname{im}\left[\pi_1\big(F_s^{-1}(X^0),\ \sigma_s(s)\big) \to \pi_1(X^0, x)\right] \quad \textit{for all } s \in S$$

(topologically) generate $\pi_1(X^0, x)$.

20 (Proof of (19) $\Rightarrow$ (3))**.** Assume first that k is algebraically closed. We start with the family $(C/S,\ F : C \to X \times S,\ \sigma : S \to C)$ in (19) and using it we repeatedly construct other families until we end up with one as required for (3).

$(C/S,\ F : C \to X \times S,\ \sigma : S \to C)$ gives the moduli morphism $S \to \bar{M}_0(X, \operatorname{Spec} k \to x)$ (cf. [1, 41]); let S^* be its image (as a constructible set).

Pick any $s \in S(k)$ and let

$$\begin{array}{ccc} C_s \subset C_{U_s} & \stackrel{\Phi_{U_s}}{\to} & X \\ \downarrow \qquad \downarrow & & \\ s \ \in \ U_s & & \end{array}$$

be a versal family as in [1, 42.5].

To each U_s there corresponds a natural morphism $U_s \to \bar{M}_0(X, \operatorname{Spec} k \to x)$ with open image. Finitely many of these cover S^*. From now on we work with these finitely many families

$$(C_{U_s}/U_s,\ \Phi_{U_s} : C_{U_s} \to X \times U_s,\ \sigma_{U_s} : U_s \to C_{U_s}).$$

From (17) we see that the conclusion of (19) remains true if we replace $\cup U_s$ by a dense open subset. By [1, 42.3] the points corresponding to irreducible curves are dense in each U_s, thus we may replace each U_s by the open subset of irreducible and free curves.

We now have finitely many (say N) families of stable and free maps

$$\Phi_j : U_j \times \mathbb{P}^1 \to X \quad \text{with} \quad \Phi_j\big(U_j \times (0{:}1)\big) = x,$$

where each U_j is irreducible. We are free to add one more family $\Phi_{N+1} : U_{N+1} \times \mathbb{P}^1 \to X$ consisting of very free curves through x. In order to create a single family, we use a trick introduced in [18].

Pick points $u_j \in U_j$ and, as in [1, 43], assemble a pointed comb out of the maps $\Phi_{j,u_j} : \mathbb{P}^1 \to X$. Since $N + 1 \geq 2$, these pointed combs are stable genus zero curves over X. All these combs are parametrized by an irreducible variety (the product of the U_j times the space of $N + 1$ points on $\mathbb{P}^1$). Thus all these combs are in the same irreducible component $\mathcal{U} \subset \bar{M}_0(X, \operatorname{Spec} k \to x)$.

As before, using (18) we can first replace the families $\Phi_j : U_j \times \mathbb{P}^1 \to X$ with the irreducible component $\mathcal{U}$ of the family of stable curves containing all of the above combs

$$f_C : C = C_0 \cup \cdots \cup C_{N+1} \to X,$$

and then by (17) with the open subfamily $\mathcal{U}^0 \subset \mathcal{U}$ of irreducible and free curves. The moduli map $\mathcal{U}^0 \to \operatorname{Hom}\big(\mathbb{P}^1, X, (0{:}1) \mapsto x\big)$ is dominant onto one of the irreducible components. Let $V \subset \operatorname{Hom}\big(\mathbb{P}^1, X, (0{:}1) \mapsto x\big)$ be an open set contained in the image of $\mathcal{U}^0$.

Thus we obtain a single irreducible family of stable maps

$$\Phi : V \times \mathbb{P}^1 \to X \quad \text{with} \quad \Phi\big(V \times (0{:}1)\big) = x.$$

We claim that this satisfies all the requirements of (3) for k algebraically closed.

Indeed, V is geometrically irreducible, smooth and open in

$$\operatorname{Hom}\big(\mathbb{P}^1, X, (0{:}1) \mapsto x\big)$$

by construction. By [17, II.3.4], the tangent map of $F : \big(V \times (1{:}0)\big) \to X$ at $\big(v, (1{:}0)\big)$ is given by the restriction

$$H^0\big(\mathbb{P}^1, F_v^* T_X\big(-(0{:}1)\big)\big) \to F_v^* T_X|_{(1:0)} \cong T_{X, F(v,(1:0))}$$

which is surjective if $H^1\big(\mathbb{P}^1, F_v^* T_X(-2)\big) = 0$. We prove the latter below, thus $F : \big(V \times (1{:}0)\big) \to X$ is smooth, at least after passing to a suitable open subset of V.

Condition (3.2) is not interesting when k is algebraically closed.

Ampleness of a vector bundle E on $\mathbb{P}^1$ is equivalent to $H^1\big(\mathbb{P}^1, E(-2)\big) = 0$. By upper semicontinuity, $H^1\big(\mathbb{P}^1, F_v^* T_X(-2)\big) = 0$ for general $v \in V$ if $H^1\big(C, f_C^* T_X(-2p)\big) = 0$ for one of the combs $f_C : C \to X$ and for some smooth point $p \in C$. Pick $p \in C_{N+1}$. By our choice of the $(N+1)$st family, $F^* T_X$ restricted to C_{N+1} is ample and all the other restrictions to C_i are semipositive. This easily implies that $H^1\big(C, f_C^* T_X(-2p)\big) = 0$, cf. [1, 18]. This proves (3.3) and also (3.1).

Applying (17) several times, we get that the closed subgroup of $\pi_1(X^0, x)$ generated by the images of $\pi_1\big(\mathbb{P}^1_v \cap F^{-1}(X^0), (0{:}1)\big) : v \in V$ is $\pi_1(X^0, x)$ itself. On the other hand, the image of $\pi_1\big(F^{-1}(X^0_K), \big(v, (0{:}1)\big)\big) \to \pi_1(X^0_K, x)$ is closed by (11), thus $\pi_1\big(F^{-1}(X^0_K), \big(v, (0{:}1)\big)\big) \to \pi_1(X^0_K, x)$ is surjective, giving (3.4).

Assume next that k is not algebraically closed, and let $\bar{k}$ be its algebraic closure. The family constructed above is already defined over a finite Galois extensions k'/k with Galois group G. (This is automatic in characteristic 0. In general, note that free curves give smooth points of $\operatorname{Hom}(\mathbb{P}^1, X)$, and the smooth geometric components of a k-scheme are defined over a separable extension.) We may also assume that V has a distinguished k'-point $v \in V(k')$. By [17, II.3.14.2] we may also assume that the corresponding map $\{v\} \times \mathbb{P}^1 \to X$ is an immersion.

For every $\rho \in G$ let

$$\Phi^\rho : V^\rho \times \mathbb{P}^1 \to X$$

be the conjugate family obtained by applying ρ to all the defining equations.

We now proceed as above. Pick arbitrary points $v_\rho \in V^\rho(\bar{k})$ and assemble a pointed comb out of the maps $\Phi^\rho_{v_\rho} : \mathbb{P}^1 \to X$. All these combs are parametrized by an irreducible variety which is now defined over k. (Over k' this parameter space is the product of the V^ρ times the space of N points on $\mathbb{P}^1$. Over k, it is fibered over the Weil restriction $\mathfrak{R}_{k'/k}V$ (see, for instance [3, 7.6]) with various forms of the space of $|G|$ points on $\mathbb{P}^1$ as fibers.) As before, all these combs are in the same geometrically irreducible component of $\mathcal{W} \subset \bar{M}_0(X, \operatorname{Spec} k \to x)$.

By (17) we can replace $\mathcal{W}$ with the subfamily W of irreducible and free curves.

In order to check the required properties, the only new aspect is (3.2). The comb C assembled from the conjugates of $\{v\} \times \mathbb{P}^1$ gives a smooth k-point of $\bar{\mathcal{W}}$. Indeed, all the conjugates of $\{v\} \times \mathbb{P}^1 \to X$ are immersions, thus C has no automorphisms commuting with $C \to X$ and fixing the base point. Therefore C gives a smooth point of $\bar{\mathcal{W}}$ by [10, 2.iii]. ■

21 (Idea of the proof of (19))**.** Take all possible free, stable, genus 0 curves $g : (c \in C) \to (x \in X)$ and let $H \subset \pi_1(X^0, x)$ be the subgroup (topologically) generated by all the images $\pi_1(C^0, c) \to \pi_1(X^0, x)$ where $C^0 := g^{-1}(X^0)$. We want to prove that $H = \pi_1(X^0, x)$. If not, then there is a finite index closed subgroup $H \subset H' \subsetneq \pi_1(X^0, x)$. The subgroup H' corresponds to a degree d étale cover $Y^0 \to X^0$ together with a geometric point $y \to Y^0$ lying over $x \to X^0$. Extend it to a finite morphism $p : Y \to X$.

Since X is simply connected (13), p is not étale and so it ramifies along a divisor $D \subset Y$ (using the purity of branch loci).

Pick a general point $x' \in p(D)$ and assume that $C \subset X$ is a rational curve passing through x, x' and transversal to $p(D)$ everywhere. We hope that $C \to X$ cannot be lifted to Y. The local picture we should have in mind is

$$Y = \mathbb{C}^2_{y_1,y_2} \xrightarrow{(y_1,y_2) \mapsto (y_1,y_2^m)} \mathbb{C}^2_{x_1,x_2} = X.$$

Here $D = (y_2 = 0)$ and $p(D) = (x_2 = 0)$. Set $y' := (0,0) \in Y$. If C is the line $(x_1 = 0)$ through $x' := (0,0)$ then $p^{-1}(C)$ is the irreducible curve $(y_1 = 0)$ through y'. Thus the inclusion $C \hookrightarrow X$ cannot be lifted to $C \to Y$ in such a way that $x' \in C$ is mapped to $y' \in Y$.

This is, however, only the local picture. The global problem is that Y has many other points over x' besides y', and p ramifies only at some of these. The best we can say is the following.

The fiber product $C_Y := Y \times_X C$ is a smooth curve and $C_Y \to C$ definitely ramifies at $C_Y \cap D$. Thus C_Y is *not* the disjoint union of d copies of C. This means that there is at least one point $y_{i_0} \in p^{-1}(x)$ such that the inclusion $C \hookrightarrow X$ cannot be lifted to $C \to Y$ sending c to y_{i_0}. For other choices $y_i \in p^{-1}(x)$ a lifting may exist. It is very hard to tell which preimage is which.

Choosing a preimage y_i is essentially equivalent to choosing a conjugate of the subgroup H', thus we cannot move freely between the different preimages.

We circumvent this problem as follows.

First we choose a general smooth rational curve $B \subset X$ passing through a general point $z \in X$ and intersecting $p(D)$ transversally everywhere. Let $v_i \in Y$ $i = 1, \dots, d$ be all the preimages of z. As with C, there is at least one v_{i_0} such that $B \hookrightarrow X$ cannot be lifted to $B \to Y$ sending z to v_{i_0}.

Then we pick other rational curves A_k passing through x and z. We furthermore achieve (and this turns out to be easy for a *general* point) that the liftings of the curves A_k connect y with *all* the different points v_i.

This implies that at least one of the reducible curves $A_k \cup B$ cannot be lifted to Y. (See Fig. 1.)

There are only two points that need refinement in this method. First, in positive characteristic the local description of the ramification is much more complicated than over $\mathbb{C}$. Second, in order to make these choices more uniform, we should work with a whole family of curves B. This actually also takes care of the ramification problem. This is done next.

22 (Main construction)**.** Let X be a smooth irreducible variety over an algebraically closed field k. Assume that we are given two families of curves on X

$U \stackrel{u}{\leftarrow} A \stackrel{F}{\rightarrow} X$ and $V \stackrel{v}{\leftarrow} B \stackrel{G}{\rightarrow} X$ with the following properties.

1. The k-varieties U, V are irreducible and u, v are proper, smooth morphisms with irreducible fibers.

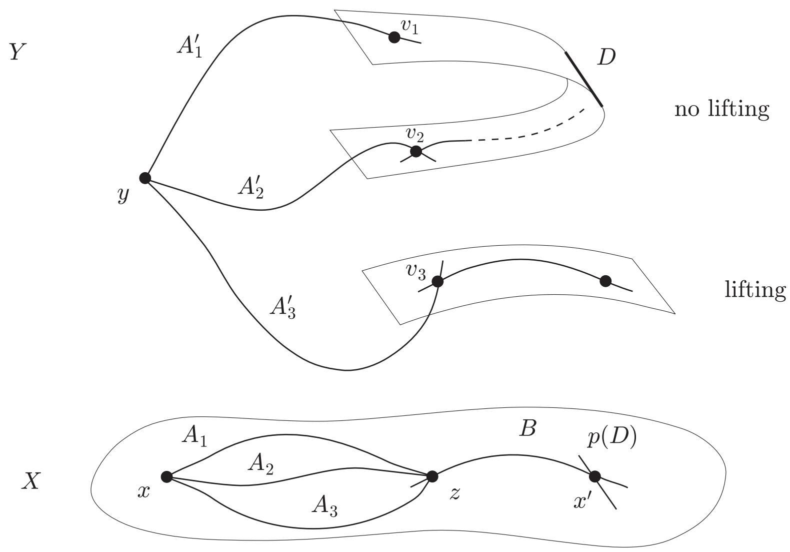

Fig. 1. A non-liftable reducible curve

2. The map F is dominant.
3. The map u has a section $s : U \to A$ such that $F \circ s$ maps U to a single point $x \in X(k)$.
4. The images of the maps $\pi_1\big(A, s(u)\big) \to \pi_1(X, x)$ for $u \in U$ topologically generate $\pi_1(X, x)$.
5. The map G is smooth and for every divisor $D \subset X$ the general $G(B_v)$ intersects D.

Every point of the fiber product $W := A \times_X B$ can be thought of as a quadruplet $(a_u \in A_u, b_v \in B_v)$ such that $F(a_u) = G(b_v)$. To this corresponds a reducible nodal curve $A_u \cup_{a_u \sim b_v} B_v$ obtained from the disjoint union of A_u and B_v by identifying the points a_u and b_v. Thus we have a flat family of reducible but connected curves

$$W \leftarrow C \stackrel{H}{\to} X.$$

The map $s : U \to A$ gives a section $S : W \to C$ such that $H \circ S$ maps W to $\{x\}$. (Here C can be explicitly constructed as

$$A \times_U A \times_X B \cup A \times_X B \times_V B \subset A \times_U A \times_X B \times_V B,$$

where the embedding is with $Id\times_X\Delta_B\cup\Delta_A\times_X Id$ where $\Delta_B : B\to B\times_V B$ and $\Delta_A : A\to A\times_U A$ are the diagonal embeddings. The map $C\to W$ is projection to the middle $A\times_X B$ of the above product. Finally S is given by the fiber product of s with the identity map $A\times_X B = U\times_U A\times_X B\to A\times_U A\times_X B$.)

Proposition 23. *Notation and assumptions as above. Then, for every open set $X^0\subset X$ containing x*

$$\pi_1(X^0,x)=\Big\langle \operatorname{im}\big[\pi_1\big(C_w\cap H^{-1}(X^0),s(w)\big)\to\pi_1(X^0,x)\big] : \forall w\in W(k)\Big\rangle,$$

where $\langle\cdots\rangle$ denotes the closed subgroup generated by the images.

Proof. It is enough to show that if $(Y^0,y)\to(X^0,x)$ is a pointed finite étale cover with Y^0 irreducible and $p : Y\to X$ its extension to a (possibly ramified) finite morphism then

$$H_w : C_w = A_u\cup_{a_u\sim b_v} B_v\to X$$

cannot be lifted to Y for some w.

If $Y\to X$ is unramified then there is a $u\in U(k)$ such that $A_u\to X$ cannot be lifted by assumption 4. Thus we are left to consider the case when $Y\to X$ is ramified and every $A_u\to X$ can be lifted. We claim that in fact $F : A\to X$ lifts to $F_Y : A\to Y$ with $F_Y\circ s : U\to y$. To see this, consider the fiber product $A\times_X Y$. $s : U\to A$ and the constant map $U\to y$ provide $\sigma : U\to A\times_X Y$. Let $A_Y\subset A\times_X Y$ be the unique irreducible component containing $\sigma(U)$. By assumption $A_Y\to A$ is an isomorphism on every fiber of $A\to U$, hence an isomorphism. Thus A_Y is the graph of the required lifting F_Y.

The map F_Y is dominant, so there is an open set $X^1\subset X^0$ such that $F_Y(A)\supset p^{-1}(X^1)$ and $X^1\subset G(B)$.

Let $z\in X^1(k)$ be a point and $p^{-1}(z)=\{v_1,\dots,v_m\}$. By assumption there are $u_i\in U$ and $a_i\in A_{u_i}$ such that $F_Y(a_i)=v_i$. Pick any $v\in V(k)$ and $b\in B_v$ such that $G(b)=z$.

The quadruplets $(a_i\in A_{u_i},\ b\in B_v)$ give connected curves $C_i := A_{u_i}\cup_{a_i\sim b} B_v$. If $H_i : C_i\to X$ lifts to $H_{i,Y} : C_i\to Y$ then $H_{i,Y}(B_v)$ is a lifting of $G : B_v\to X$ which passes through v_i. Thus we obtain:

Claim 24. Under the above assumptions, the fiber product $B_v\times_X Y$ has m irreducible components, each isomorphic to B_v.

We show that this is impossible, which implies that our assumption is incorrect. Thus not all maps $H_i : C_i \to X$ lift to $H_{i,Y} : C_i \to Y$. This will complete the proof of (23).

It should be emphasized that at this point we only know that $B_v \times_X Y$ has m irreducible components, each isomorphic to B_v. We do not know that $B_v \times_X Y$ is a *disjoint* union of its irreducible components. The key point is precisely to establish this for general $v \in V(k)$.

Set $B_Y := B \times_X Y$. $G : B \to X$ is smooth, hence $B_Y \to Y$ is also smooth, thus B_Y is normal.

Consider the diagram

$$\begin{array}{ccccc} B_Y & \stackrel{p_B}{\to} & B & \stackrel{G}{\to} X \\ v_Y \downarrow & & \downarrow v & \\ V & = & V. & \end{array}$$

In characteristic zero, B_Y normal implies that the geometric generic fiber of B_Y/V is also normal, hence it is a disjoint union of its irreducible components.

In positive characteristic, we have only a weaker result (9), which still says that the geometric generic fiber of B_Y/V is a disjoint union of its irreducible components. In particular, the generic fiber of B_Y/V is smooth over the generic fiber of B/V. Thus, after possibly shrinking V, we may assume that p_B is smooth. G is smooth by assumption, hence the composite $B_Y \to X$ is also smooth.

One can, however, factor this as $B_Y \to Y \to X$. By the purity of branch loci, $Y \to X$ ramifies over a whole divisor $D \subset X$, and so $B_Y \to X$ ramifies over every point of D. This is only possible if the image of $B_Y \to B \to X$ does not intersect D. This contradicts the assumption 5. ■

25 (Proof of (19))**.** All that is left is the construction of the families $U \leftarrow A \to X$ and $V \leftarrow B \to X$ as in (22). The rest is already taken care of by (23).

For $U \leftarrow A \to X$ we take a family of rational curves through x that covers X. Property (22.4) holds by (13).

In order to get the family $B \to V$, it is enough to get a very free curve $\mathbb{P}^1 \to X$ which has positive intersection number with every divisor. A free curve has nonnegative intersection number with every divisor. Start with a family $H : U \times \mathbb{P}^1 \to X$ that shows that X is SRC. The only problem is that its image may not intersect finitely many divisors $D_1, \dots, D_s \subset X$. By [1, 29.5] there is a free morphism $g : \mathbb{P}^1 \to X$ whose image intersects every D_i. Now take $g(\mathbb{P}^1)$ and a curve $H(\mathbb{P}^1_u)$ intersecting it. By [17, II.7] their union can be smoothed to a free curve which has positive intersection number with every divisor. The resulting morphism $G : B \to X$ is smooth by [17, II.3.5.3]. ■

26. The additional properties listed in (4) are also easy to get.

First, by [17, II.3.14], F is automatically an embedding for general $w \in W$ if $\dim X \geq 3$, and we can replace W with an open subset at any time.

Assume now that we want more ampleness from $F_w^* T_X$. Take a family of morphisms $h : U \times \mathbb{P}^1 \to X$ such that $h_u^* T_X(-d)$ is ample and h is smooth. Let us now look at all maps of one tooth combs $g : C_0 \cup C_1 \to X$ where $g|_{C_0}$ is from the family F and $g|_{C_1}$ from the family h. Smoothing these as in (20), we get a new family $F_d : W_d \times \mathbb{P}^1 \to X$ that also satisfies (4.6)

Fix next an embedding $X \subset \mathbb{P}^N$. All smooth varieties of given degree and dimension form a bounded family (the Chow variety). Being SRC is an open property (cf. [17, IV.3.11]), hence all SRC varieties of given degree and dimension form a bounded family. This implies that we can choose bounded degree curves for the families A, B in (22). Thus we get families of bounded degree curves in (19).

There is only one problem left, namely that during the proof in (20) we have to take all conjugates of a family of curves, and their number does depend on the field. The families we have are open subsets of an irreducible component of the Hom scheme $\mathrm{Hom}(\mathbb{P}^1, X)$. We are dealing with maps of bounded degree to a bounded family of varieties. The relative Hom scheme of bounded degree maps is quasiprojective (cf. [17, I.1.10]), hence there is a bound for the number of irreducible families of bounded degree maps. The number of conjugates cannot exceed this bound. ■

27 (Proof of (5))**.** From (3) we know that $\pi_1\big(F^{-1}(X^0)\big) \to G$ is surjective. Applying (16) to $F^{-1}(X^0) \to W$ we conclude that $\pi_1\big(\mathbb{P}^1_w \cap F^{-1}(X^0)\big) \to G$ is surjective for all w in some open subset $W^G \subset W$. ■

28 (Proof of (6))**.** Let us start with a variety X and let $k(X)$ denote its field of rational functions. The generic point $x_g \in X$ is a $k(X)$-point of

X, thus we can apply (3) to $X_{k(X)}$ and x_g. Thus we get a geometrically irreducible $k(X)$-variety V_g and a morphism

$$F_g : V_g \times \mathbb{P}^1 \to X_{k(X)}.$$

V_g can be thought of as the generic fiber of a map $V \to X$ where V is a geometrically irreducible k-variety. F_g extends to a map which can be composed with the second projection to obtain

$$F_V : V \times \mathbb{P}^1 \dashrightarrow X \times X \to X.$$

F_V need not be everywhere defined, but it becomes a morphism after restriction to a suitable open set of the form $V' \times \mathbb{P}^1$. By choosing V' small enough we can also assume that F_V is free on $\{v\} \times \mathbb{P}^1$ for every $v \in V'$.

Let us now look at the relative Hom–scheme (cf. [17, I.1.19]) $H := \mathrm{Hom}_{V'}(V' \times \mathbb{P}^1, V' \times X)$. The restriction of F_V to V' determines a section $s : V' \to H$ and the universal morphism $H \times \mathbb{P}^1 \to V' \times X$ is smooth over this section by [17, II.3.5.3]. This also imples that $H \to V'$ is smooth along $s(V')$, hence H has a unique irreducible component $W' \subset H$ which contains $s(V')$ and W' is geometrically irreducible. Furthermore, there is an open subset $W \subset W'$, containing $s(V')$ such that the universal morphism $F : W \times \mathbb{P}^1 \to V' \times X \to X$ is smooth. This gives the required morphism. ∎

29 (Proof of (7))**.** Choose a partial compactification $Y \subset \bar{Y}$ such that h extends to a proper morphism $\bar{h} : \bar{Y} \to X$. By the upper semicontinuity of fiber dimensions, there is a closed subset $T \subset X$ of codimension 2 such that every fiber of h over $X \setminus T$ has the same dimension.

This implies that if B is any irreducible curve and $B \to X \setminus T$ a morphism then every irreducible component of $B \times_X Y$ dominates B. Indeed, the fiber product $B \times_X Y$ is the preimage of the diagonal under the morphism $B \times Y \to X \times X$. Therefore every irreducible component of $B \times_X Y$ has dimension at least $\dim Y - \dim X + 1$. If B maps to $X \setminus T$ then every fiber of $B \times_X Y \to B$ has dimension $\dim Y - \dim X$, hence the claim.

Choose $F : W \times \mathbb{P}^1 \to X$ as in (6). The map F is smooth, so by an easy dimension count, there is an open subset $W^1 \subset W$ such that $F(W^1 \times \mathbb{P}^1) \subset X \setminus T$ (cf. [1, 9]).

Let

$$Y^0 := p^{-1}(X^0) \xrightarrow{q} Z^0 \xrightarrow{r} X^0$$

be as in (9), where Z^0 is normal and irreducible. The map $q_w : Y^0 \times_X \mathbb{P}^1_w \to Z^0 \times_X \mathbb{P}^1_w$ has geometrically irreducible fibers, thus it is enough to prove that $Z^0 \times_X \mathbb{P}^1_w$ is irreducible if $w \in W^0$.

The product $W^1 \times \mathbb{P}^1 \times_X Z^0$ is irreducible by (3) and also smooth, and it has a section over W^1, for instance $w \mapsto \bigl(w, 0, F(w,0)\bigr)$. So the generic fiber of $W^1 \times \mathbb{P}^1 \times_X Z^0 \to W^1$ is irreducible over $k(W^1)$, smooth and it also has a $k(W^1)$-point. Therefore it is geometrically irreducible. Thus there is an open subset $W^0 \subset W^1$ such that the fibers of $W^1 \times \mathbb{P}^1 \times_X Z^0 \to W^1$ are geometrically irreducible over W^0 (cf. [8, IV.12]). ∎

4. Applications to Non-closed Fields

In this section we derive some consequences of the previous results to fields which are not algebraically closed.

Remark 30 (Fundamental groups over non-closed fields)**.** If k is a field then the connected étale covers of $\operatorname{Spec} k$ are exactly the maps $\operatorname{Spec} K \to \operatorname{Spec} k$ where K/k is a finite separable field extension. This implies that

$$\pi_1\bigl(\operatorname{Spec}(k)\bigr) = \operatorname{Gal}(k^{sep}/k)$$

where k^{sep} is the separable closure of k. In general, the fundamental group of a k-scheme X_k is made up of the fundamental group of $X_{\bar{k}}$ and the Galois group $\operatorname{Gal}(k^{sep}/k)$. To be precise, if X_k is geometrically connected, then there is an exact sequence

$$1 \to \pi_1(X_{\bar{k}}, \bar{x}) \to \pi_1(X_k, \bar{x}) \to \operatorname{Gal}(k^{sep}/k) \to 1$$

and every point of $X(k)$ defines a splitting of the sequence. See [14, IX.6.1] for details.

This has an important consequence which indicates that surjectivity assertions between fundamental groups are essentially geometric in nature:

Proposition 31. *Let k be a field and $f : Y \to X$ a morphism of geometrically connected k-schemes. Let $\bar{y} \to Y$ be a geometric point. Then the map of algebraic fundamental groups*

$$f_* : \pi_1(Y, \bar{y}) \to \pi_1\bigl(X, f(\bar{y})\bigr)$$

is surjective (resp. its image has finite index) iff the map of geometric fundamental groups

$$f_* : \pi_1(Y_{\bar{k}}, \bar{y}) \to \pi_1\big(X_{\bar{k}}, f(\bar{y})\big)$$

is surjective (resp. its image has finite index). ∎

For arithmetic applications of these results it is useful to find maps $\mathbb{P}^1 \to X$ defined over the ground field. The current results work very well for certain fields:

32 (Large fields)**.** We are interested in fields K which have the property that on any variety with one smooth K-point the K-points are Zariski dense. Such fields are called *large fields* in [28]. The following are some interesting classes of such fields:

1. Fields complete with respect to a discrete valuation. This in particular includes the finite extensions of the p-adic fields $\mathbb{Q}_p$ and the power series fields $\mathbb{F}_q\big((t)\big)$ over finite fields $\mathbb{F}_q$.
2. More generally, quotient fields of local Henselian domains.
3. $\mathbb{R}$ and all real closed fields.
4. Infinite algebraic extensions of finite fields and, more generally, pseudo algebraically closed fields, cf. [9, Chap. 10]

For large fields, the existence of a smooth k point in $\bar{W}$ in (3) implies that $W(k)$ is dense in W. Using (3) and (5) we obtain the following:

Theorem 33. *Let X be a smooth, projective, SRC variety over a large field k and $x \in X(k)$. Then there is a dominant family of rational curves through x (defined over k)*

$$F : W \times \mathbb{P}^1 \to X, \quad F\big(W \times (0{:}1)\big) = \{x\}$$

with the following properties:

1. *The pullback $F_w^* T_X$ is ample for every $w \in W$.*
2. *Let $X^0 \subset X$ be any open k-subvariety containing x and $\pi_1(X^0, x) \twoheadrightarrow G$ a finite quotient (whose kernel is open). Then there is a dense subset of k-points $W^0_G(k) \subset W$ such that for every $w \in W^0_G(k)$ the composed map of algebraic fundamental groups*

$$\pi_1\big(\mathbb{P}^1_w \cap F^{-1}(X^0), (0{:}1)\big) \to \pi_1(X^0, x) \to G \quad \textit{is surjective.}$$ ∎

An arithmetic application of these results is a simple proof of the following theorem, proved in increasing generality in [15, 5, 19]. Moret-Bailly [25, 26] proved analogous results for finite and smooth group schemes in positive characteristic.

Theorem 34. *Let K be a characteristic zero large field, G a linear algebraic group scheme over K and A a principal homogeneous G-space. Then there is an open set $V \subset \mathbb{A}^1_K$ containing 0 and a geometrically irreducible G-torsor $g : V_G \to V$ such that $g^{-1}(0) \cong A$ (as a G-space).*

Proof. Assume that G acts on A from the left and choose an embedding $G \subset GL(n)$ over K. The group $A \times GL(n)$ admits a diagonal left action by G and a right action by $GL(n)$ acting only on $GL(n)$.

The right G-action makes the morphism

$$h : Y^0 := G\backslash\big(A \times GL(n)\big) \to G\backslash\big(A \times GL(n)\big)/G =: X^0$$

into a G-torsor. Let $G^0 \subset G$ be the connected component of the identity. Then

$$Z^0 := G\backslash\big(A \times GL(n)\big)/G^0 \to X^0$$

is a finite étale cover, hence it corresponds to a finite quotient $\pi_1(X^0, x) \to H$ where $x \in X(K)$ is the image of $G\backslash(A \times G)$. Let $X \supset X^0$ be a smooth compactification of X^0. The variety $Z^0_{\bar{K}}$ is isomorphic to $GL(n)$, hence X is unirational and so SRC. The fiber of h over x is isomorphic to A.

Apply (33) to X with the point x to get $f : (0 \in \mathbb{P}^1) \to (x \in X)$ such that $\mathbb{P}^1 \times_X Z^0$ is geometrically irreducible. The fibers of $Y^0 \to Z^0$ are torsors over G^0, hence irreducible. Thus $\mathbb{P}^1 \times_X Y^0$ is also geometrically irreducible. Set $V = f^{-1}(X^0)$ and $V_G = V \times_X Y^0$. ■

Remark 35. We have used the characteristic zero assumption in two places. The first is the existence of a smooth compactification X. Conjecturally this is always satisfied. A second point is that in general we can conclude that X is SRC only if $h : Y^0 \to X^0$ is separable, that is when G is smooth. The computations of [11, 12] suggest that in general this may be a quite subtle point.

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