REAL ALGEBRAIC THREEFOLDS IV. DEL PEZZO FIBRATIONS

JÁNOS KOLLÁR

Contents

1. Introduction

This paper continues the study of the topology of real algebraic threefolds begun in [Kollár97b, Kollár97c, Kollár98a], but the current work is independent of the previous ones in its methodology.

The present aim is to understand the topology of the set of real points of threefolds which admit a morphism to a curve whose general fiber is a rational surface. This class of threefolds also appears as one of the 4 possible outcomes of the minimal model program (cf. [Kollár-Mori98, Sec. 3.7]). Our main theorem gives a nearly complete description of the possible topological types of the set of real points of such a threefold in the orientable case.

Theorem 1.1. *Let X be a smooth projective real algebraic threefold such that the set of real points $X(\mathbb{R})$ is orientable. Assume that there is a morphism $f : X \to C$ onto a real algebraic curve C whose general fibers are rational surfaces. Let $M \subset X(\mathbb{R})$ be any connected component. Then*

$$M \sim N \# a\mathbb{RP}^3 \# b(S^1 \times S^2) \quad \textit{for some } a, b \geq 0,$$

where one of the following holds.

1. *N is a connected sum of lens spaces (cf. (1.8)).*

2. *N is Seifert fibered over a topological surface (cf. (1.9)).*
3. *N is either an $S^1 \times S^1$-bundle over S^1 or is doubly covered by such a bundle.*

1.2 (First reduction step)**.**

As with many results for 3–folds, the proof starts with a suitable minimal model program. The minimal model program over $\mathbb{R}$ was studied in greater generality in [Kollár97c] so I just outline the main conclusions. The end result is that (1.1) follows from (1.3) and for the rest of the paper we do not have to know anything about the 3–dimensional minimal model program.

Let us run the relative minimal model program for $X \to C$ over $\mathbb{R}$ (cf. [Kollár-Mori98, Sec. 3.7] or [Kollár97c, Sec. 3]). The change in the topology of real points in the course of the program is described in [Kollár97c, 1.2]. There are two possibilities for the final step of the program.

First, we may get a conic bundle over a surface. These are described in [Kollár98a] and we obtain cases 1 or 2 of (1.1).

Otherwise the program ends with a morphism $f^m : X^m \to C$ which has the following properties:

1. X^m has only isolated singularities which are $\mathbb{Q}$-factorial over $\mathbb{R}$ (that is, if D is a Weil divisor defined over $\mathbb{R}$ then mD is Cartier for some $m > 0$).
2. $-K_{X^m}$ is f^m-ample.
3. K_{X^m} is Cartier at all real points.
4. Every fiber of f is irreducible (over $\mathbb{R}$).
5. The topological normalization $\overline{X^m(\mathbb{R})}$ of $X^m(\mathbb{R})$ is a 3–manifold.

In view of [Kollár97c, 1.2] and [Kollár98a, 1.1], (1.1) is reduced to proving the following:

Theorem 1.3. *Let X be a projective real algebraic threefold and $f : X \to C$ a morphism onto a real algebraic curve C. Assume that X satisfies the conditions (1.2.1–5) and that $\overline{X(\mathbb{R})}$ is orientable. Let $M \subset \overline{X(\mathbb{R})}$ be any connected component. Then*

$$M \sim N \ \# \ a\mathbb{RP}^3 \ \# \ b(S^1 \times S^2) \quad \textit{for some } a, b \geq 0,$$

where one of the following holds.

1. *N is a connected sum of lens spaces.*
2. *N is either an $S^1 \times S^1$-bundle over S^1 or is doubly covered by such a bundle.*

Remark 1.4. The conclusion of the theorem can probably be considererably improved.

In case (1.3.1) I do not have any examples where N is not a single lens space. A more detailed analysis of the present methods may prove that this is always the case.

$(S^1 \times S^1)$-bundles over S^1 are classified by the monodromy map on $H_1(S^1 \times S^1, \mathbb{Z}) \cong \mathbb{Z}^2$. If this map is hyperbolic then N has a geometry modelled on *Sol* (cf. [Scott83, p.470]). I believe that N can never be of this type but the methods of my proof do not say anything about the monodromy.

1.5 (Second reduction step)**.**

In this step we reduce the proof of (1.3) to the study of the singular fibers of f.

Let $A \sim S^1$ be a connected component of $C(\mathbb{R})$ and $p_1, \dots, p_s \in A$ the points (in cyclic order) over which f is not smooth. For each i pick a point $q_i \in (p_i, p_{i+1})$. Then $X_{q_i} := f^{-1}(q_i)$ is a smooth Del Pezzo surface and $X_{q_i}(\mathbb{R})$ is orientable. Thus $X_{q_i}(\mathbb{R})$ is either $S^1 \times S^1$ or a disjoint union of copies of S^2 by a result of [Comessatti14]. (See also [Silhol89, V.3.4,VI.4.6 and VI.6.3].) Gluing 3-manifolds along such surfaces is a relatively simple operation, thus one can expect to get a good description of $X(\mathbb{R})$ by describing the pieces $Z_i := (f^{-1}[q_{i-1}, q_i])(\mathbb{R})$ for every i; see (6.3).

$f : Z_i \to [q_{i-1}, q_i]$ is a function whose only critical value is p_i. Thus Z_i can be viewed as a regular neighborhood of the critical level set $(f^{-1}(p_i))(\mathbb{R})$. In fact, once we have a topological description of $(f^{-1}(p_i))(\mathbb{R})$, it is easy to figure out what Z_i is, at least up to finite ambiguity. This is done in (6.1).

The complex projective surface $S_i := f^{-1}(p_i)$ is a "singular Del Pezzo" surface which appears as a degeneration of smooth Del Pezzo surfaces. Quite a lot is known about such surfaces. A structural theory of normal Del Pezzo surfaces is developed in [Keel-M^{c}Kernan98]. Degenerations of $\mathbb{P}^2$ and some other Del Pezzo surfaces are studied in [Manetti91, Manetti93]. Unfortunately, the conclusion of these works is that a complete classification of such singular Del Pezzo surfaces is not feasible because of the combinatorial complexity of the problem. On the other hand, the methods developed by these and other authors can be used to answer many questions about singular Del Pezzo surfaces.

The main part of this work is devoted to understanding the topology of the set of real points of certain singular Del Pezzo surfaces. It should be emphasized that, as opposed to almost all previous studies, we have to consider nonnormal surfaces as well. In fact, as far as the topology of the real points is concerned, normal surfaces present no difficulties.

Nonnormal Del Pezzo surfaces whose canonical divisor is Cartier have been enumerated in [Reid94]. Many of the irreducible ones appear as singular fibers in our case. For all such examples one can perform a small perturbation of $f : Z_i \to [q_{i-1}, q_i]$ such that we obtain a Morse function whose fibers stay *real algebraic* Del Pezzo surfaces. These are again very easy to understand topologically.

I have been unable to find any other nonnormal irreducible examples, thus it is possible that the work of sections 2–4 is entirely superfluous. On the other hand, the methods of these sections lead to a description of the singular fibers in many cases and it may be possible to develop them further to obtain a complete list of the irreducible fibers occurring in (1.3).

Some interesting examples are given in section 7.

1.6 (The fundamental group of a real Del Pezzo surface)**.**

Let $S := f^{-1}(p_i)$ be one of the fibers and $Z := (f^{-1}[q_{i-1}, q_i])(\mathbb{R})$ the corresponding piece of the above decomposition of $X(\mathbb{R})$.

Here I would like to outline a simple argument giving some information about the fundamental group of Z. Since $S(\mathbb{R})$ is a retract of Z, we see that $\pi_1(Z) \cong \pi_1(S(\mathbb{R}))$.

Let $B \subset S$ be the singular locus and U_j the connected components of $S(\mathbb{R}) \setminus B(\mathbb{R})$. $\pi_1(B(\mathbb{R}))$ is a free group and each U_j gives one relation. Hence we see that the number of relations of $\pi_1(S(\mathbb{R}))$ is bounded by the number of connected components of $S(\mathbb{R}) \setminus B(\mathbb{R})$.

Since S is a limit of smooth Del Pezzo surfaces, $|-K_S|$ has a member; call it D. (This is a bit nebulous since S may not even be normal; see (3.3) for its precise meaning.) Let $h : S' \to S$ be the minimal resolution of S. It is easy to see that $|-K_{S'}|$ has a unique member D' such that $h_*(D') = B + D$. Let $f : S' \to S''$ be a minimal model and set $D'' := f_*D'$.

f is a composite of blow ups and one checks that

$$|\pi_0(S''(\mathbb{R}) \setminus D''(\mathbb{R}))| \geq |\pi_0(S'(\mathbb{R}) \setminus D'(\mathbb{R}))|.$$

Using [Kollár-Mori98, 3.39] one easily checks that every fiber of h is either contained in $\operatorname{Supp} D'$ or is disjoint from it. This gives that

$$|\pi_0(S'(\mathbb{R}) \setminus D'(\mathbb{R}))| \geq |\pi_0(S(\mathbb{R}) \setminus (B + D)(\mathbb{R}))|.$$

Thus we are led to counting the number of connected components of $S''(\mathbb{R}) \setminus D''(\mathbb{R})$. A typical case is when $S'' = \mathbb{P}^2$ and D'' is a cubic curve. We see that there are at most 4 connected components, and 4 is achieved when D'' consists of 3 lines. If we take into account that we want to use only $S(\mathbb{R}) \setminus B(\mathbb{R})$ and not $S(\mathbb{R}) \setminus (B + D)(\mathbb{R})$, we obtain that $S(\mathbb{R}) \setminus B(\mathbb{R})$ has at most 2 connected components in this case.

The other possible minimal models S'' can be similarly treated. (Ad hoc arguments are needed to exclude the case when S'' is a minimal ruled surface with negative section E and $D'' = 2E +$ (many fibers).) At the end we obtain that

($*$) $\pi_1(Z)$ is the free product of groups with 1 relation.

On the one hand ($*$) is quite strong since most 3–manifold groups do not have this property. On the other hand, ($*$) is not strong enough to exclude all hyperbolic 3–manifolds.

The proof of (1.3) given in sections 2–6 is essentially an elaboration of this approach. The steps of going from S to S' and S'' are studied in more detail. At the end we obtain a rather complete geometric description of $S(\mathbb{R})$.

Remark 1.7 (PL three manifolds)**.** In this paper I usually work with *piecewise linear manifolds* ([Rourke-Sanderson82] is a good introduction). Every real algebraic variety carries a natural PL structure (cf. [BCR87, Sec.9.2]).

In dimension 3 every compact topological 3–manifold carries a unique PL–manifold structure (cf. [Moise77, Sec. 36]) and a PL–structure behaves very much like a differentiable structure. For instance, let M^3 be a PL 3–manifold, N a compact PL–manifold of dimension 1 or 2 and $g : N \hookrightarrow M$ a PL–embedding. Then a suitable open neighborhood of $g(N)$ is PL–homeomorphic to a real vector bundle over N (cf. [Moise77, Secs. 24 and 26]). (The technical definition of such neighborhoods is given by the notion of *regular neighborhood*, see [Rourke-Sanderson82, Chap.3]). If $f : M \to N$ is a PL–map and $X \subset N$ a compact subcomplex then there is a regular neigborhood $X \subset U \subset N$ such that $f^{-1}(U)$ is a regular neigborhood $f^{-1}(X) \subset M$ (cf. [Rourke-Sanderson82, 2.14]).

Definition 1.8 (Lens spaces)**.** For relatively prime $0 < q < p$ consider the action $(x, y) \mapsto (e^{2\pi i/p}x, e^{2\pi iq/p}y)$ on the unit sphere $S^3 \sim (|x^2| + |y^2| = 1) \subset \mathbb{C}^2$. The quotient is a 3–manifold called the *lens space* $L_{p,q}$.

Another way to obtain lens spaces is to glue two solid tori together. The result is a lens space, S^3 or $S^1 \times S^2$. Sometimes one writes $L_{1,0} = S^3$ and $L_{0,1} = S^1 \times S^2$, though these are usually not considered lens spaces. (See, for instance, [Hempel76, p.20].)

Definition 1.9 (Seifert fiber spaces)**.** A 3–manifold M is called *Seifert fibered* if there is a morphism $f : M \to F$ to a topological surface such that every $P \in F$ has a neighborhood $P \in U \subset F$ such that $f : f^{-1}(U) \to U$ is fiber preserving homeomorphic to one of the normal forms $f_{c,d}$ defined below.

Let $S^1 \subset \mathbb{C}$ be the unit circle with coordinate u and $D^2 \subset \mathbb{C}$ the closed unit disc with coordinate z. For a pair of integers c, d satisfying $0 \leq c < d$ and $(c, d) = 1$, define

$$f_{c,d} : S^1 \times D^2 \to D^2 \quad \text{by} \quad f_{c,d}(u, z) = u^c z^d.$$

$f_{c,d}$ restricts to a fiber bundle $S^1 \times (D^2 \setminus \{0\}) \to D^2 \setminus \{0\}$. The fiber of $f_{c,d}$ over the origin is still S^1, but $f_{c,d}^{-1}(0)$ has multiplicity d.

(This is the classical definition of Seifert fibered spaces, which is slightly more restrictive than the one in [Scott83].)

Acknowledgments . I thank M. Kapovich and S. Kahrlamov for answering my numerous questions about 3-manifold topology and real algebraic geometry. Partial financial support was provided by the NSF under grant number DMS-9622394.

2. Real Surfaces with Du Val Singularities

The minimal model theory of real surfaces has been studied in detail in the papers [Comessatti14, Silhol89, Kollár97a]. It is not difficult to generalize these results to the case when we allow Du Val singularities (2.1). These results were explained in [Kollár98a].

Definition 2.1. Let $(0 \in S)$ be a normal surface singularity over $\mathbb{R}$ with minimal resolution $g : S' \to S$. $(0 \in S)$ is called a *Du Val* singularity (or rational double point) iff $K_{S'} \cong g^* K_S$. Equivalently, $(0 \in S)$ is Du Val iff every g-exceptional curve is a smooth rational curve with selfintersection -2. (See [Reid85] or [Kollár-Mori98, 4.2] for the relevant background on Du Val singularities over $\mathbb{C}$.)

It is not hard to see that every real Du Val singularity is real analytically equivalent to one of the following normal forms (cf. [AGV85, I.17.1] or [Kollár-Mori98, Sec. 4.2]).

A_n^+**:** $(x^2 + y^2 - z^{n+1} = 0)$ for $n \geq 2$,
A_n^-**:** $(x^2 - y^2 - z^{n+1} = 0)$ for $n \geq 0$,
A_n^{++}**:** $(x^2 + y^2 + z^{n+1} = 0)$ for n odd,
D_n^+**:** $(x^2 + y^2 z + z^{n-1} = 0)$ for $n \geq 4$,
D_n^-**:** $(x^2 + y^2 z - z^{n-1} = 0)$ for $n \geq 4$,
E_6^+**:** $(x^2 + y^3 + z^4 = 0)$,
E_6^-**:** $(x^2 + y^3 - z^4 = 0)$,
E_7**:** $(x^2 + y^3 + yz^3 = 0)$,
E_8**:** $(x^2 + y^3 + z^5 = 0)$.

Definition 2.2. Let S be a surface with Du Val singularities. A curve $C \subset S$ is called a (-1)-*curve* if its birational transform on the minimal resolution is a smooth rational curve with selfintersection -1. Thus $(C \cdot K_S) = -1$.

Definition 2.3. Let $P \in S(\mathbb{R})$ be a smooth real point and x, y local coordinates at P. The surface $S' \subset S \times \mathbb{P}^1_{(u:v)}$ given by equation $ux - vy^m = 0$ is called a $(1,m)$-*blow up* of P on S. For $m = 1$ this is the ordinary blow up. A $(1,m)$-blow up has a unique singular point of type A^-_{m-1} at $(0,0,0,1)$.

It should be noted that for $m \geq 2$ the $(1,m)$-blow up does depend on the choice of the local coordinates.

It is frequently better to think of a $(1,m)$-blow up as follows. First blow up $0 \in S$ to get $S_1 \to S$. Then blow up a point on the exceptional divisor of $S_1 \to S$ to obtain $S_2 \to S_1$. Then blow up a point on the exceptional divisor of $S_2 \to S_1$ which is not on (the birational transform of) any previous exceptional divisor to obtain $S_3 \to S_2$. After m-times we have m exceptional curves in the following configuration:

$$\overset{-1}{\circ} - \overset{-2}{\circ} - \cdots - \overset{-2}{\circ}.$$

We can now contract all the (-2)-curves to get a $(1,m)$-blow up. This shows that the exceptional curve of a $(1,m)$-blow up is a (-1)-curve.

If $P, \bar{P} \in S(\mathbb{C})$ are smooth and conjugate complex points, then we can choose conjugate coordinate systems to do a $(1,m)$-blow up at both points. The result is again a real algebraic surface with a conjugate pair of A_{m-1}-points (for nonreal points the signs in the equations do not matter).

Definition 2.4. Let F_1, F_2 be real algebraic surfaces with Du Val singularities and assume that $g : F_1 \to F_2$ is a composite of $(1,m)$-blow ups. A $(1,m)$-blow up of a conjugate point pair is an isomorphism in the neighborhood of the real points, so if a $(1,m)$-blow up of a conjugate point pair is followed by a $(1,m)$-blow up of a real point then their order can be reversed. Repeating if necessary, g can be factored uniquely as

$$g : F_1 \xrightarrow{g^c} F^r \xrightarrow{g^r} F_2$$

where g^c is a composite of $(1,m)$-blow ups of conjugate point pairs and g^r is a composite of $(1,m)$-blow ups of real points.

Definition 2.5. Let F be a normal projective surface such that K_F is $\mathbb{Q}$-Cartier. F is called a (singular) *Del Pezzo* surface if $-K_F$ is ample. F is called a *weak Del Pezzo* surface if $-K_F$ is nef and big. A morphism $g : F \to C$ is called a *conic bundle* if every fiber is isomorphic to a plane conic (which can be smooth, a pair of intersecting lines or a double line). Every conic bundle can be embedded into a $\mathbb{P}^2$-budle over C such that the fibers become conics.

Combining the results of [Kollár98a, Sec. 9] with (2.4) we obtain the following.

Theorem 2.6. *Let F be a projective surface over $\mathbb{R}$ with Du Val singularities. Then there are surfaces and morphisms*

$$g : F \xrightarrow{g^c} F^r \xrightarrow{g^r} F^*$$

with the following properties.

1. *F^r and F^* are projective surfaces over $\mathbb{R}$ with Du Val singularities.*
2. *g^c is a composite of $(1,m)$-blow ups of conjugate point pairs. In particular $F(\mathbb{R}) \cong F^r(\mathbb{R})$.*
3. *g^r is a composite of $(1,m)$-blow ups of real points.*
4. *F^* falls in one of the following 3 cases:*
   (C) *(Conic bundle) $\rho(F^*) = 2$ and F^* is a conic bundle over a smooth curve A.*
   (D) *(Del Pezzo surface) $\rho(F^*) = 1$ and $-K_{F^*}$ is ample.*
   (N) *(Nef canonical class) K_{F^*} is nef.* □

Next we collect some auxiliary results that are needed elsewhere.

Lemma 2.7. *Let K be a field and F a Del Pezzo surface over K with Du Val singularities. Assume that $\rho(F) = 1$ and that there is an effective Cartier divisor $0 \neq B \subset F$ such that $-(K + B)$ is ample.*
Then F is isomorphic to $\mathbb{P}^2$ or to a quadric hypersurface in $\mathbb{P}^3$.

Proof. Let H be a generator of $\operatorname{Pic}(F)/(\text{torsion})$ and write $B \equiv bH$, $K_F \equiv -aH$. This implies that $(K_F^2) = a^2(H^2)$. By assumption $a > b > 0$, so $a \geq 2$. Since $(K_F^2) \leq 9$, thare three possibilities:

1. $a = 3$ and $(H^2) = 1$. Then $F_{\bar{K}} \cong \mathbb{P}^2$ and F contains a line defined over K, thus $F \cong \mathbb{P}^2$.
2. $a = 2$ and $(H^2) = 2$. Then $F_{\bar{K}}$ is a quadric and $\mathcal{O}(1)$ is defined over K. Thus F itself is a quadric.
3. $a = 2$ and $(H^2) = 1$. Then $2p_a(B) - 2 = B\cdot(B + K_F) = H\cdot(-H) = -1$, which is impossible. □

Lemma 2.8. *Let F be a real surface with Du Val singularities and $\rho(F) = 2$. Let $F \to A$ be a conic bundle with a section H. Then H intersects every singular fiber at a singular point of F.*

Proof. A section can never intersect a multiple fiber at a smooth point of F. If $f^{-1}(a)$ is a pair of conjugate lines then their intersection point P is the only real point, hence H passes through P. If P is a smooth point of F then the intersection number of H and $f^{-1}(a)$ is at least 2, which is impossible. □

3. The Basic Set-up

Notation 3.1. Let X be a real algebraic threefold and $f : X \to C$ a proper morphism to a smooth real algebraic curve. As a generalization of the conditions of (1.3) we assume the following:

1. X has isolated $\mathbb{Q}$-factorial singularities,
2. $-K_X$ is $\mathbb{Q}$-Cartier and f-ample, and
3. every fiber of f is irreducible (over $\mathbb{R}$).

Let $0 \in C(\mathbb{R})$ be a point and $X_0 := f^{-1}(0)$ the fiber over 0.

Lemma 3.2. *Notation as above. There are 3 possibilities for X_0.*

1. *X_0 is reduced and geometrically irreducible.*
2. *$X_0 = mZ_0$ for some $m \geq 2$ where Z_0 is reduced and geometrically irreducible.*
3. *$X_0 = m(Z_0 + \bar{Z}_0)$ for some $m \geq 1$ where Z_0 and $\bar{Z}_0$ are conjugate, reduced and irreducible.*

Proof. Write $(X_0)_{\mathbb{C}} = \sum a_i Z_i$ as the sum of its irreducible and reduced components. For any Z_i let $Z_i' := Z_i$ if Z_i is defined over $\mathbb{R}$ and $Z_i' := Z_i + \bar{Z}_i$ otherwise. Z_i' is defined over $\mathbb{R}$ hence $(X_0)_{\mathbb{C}} = mZ_i'$ for some $m \geq 1$ since X_0 is irreducible over $\mathbb{R}$. □

3.3. Let X be a real algebraic threefold and $f : X \to C$ a proper morphism to a smooth real algebraic curve, $0 \in C(\mathbb{R})$ a point and $Z := f^{-1}(0)$ the fiber over 0. Assume that X has isolated singularities, $-K_X$ is $\mathbb{Q}$-Cartier and f-ample, and Z is reduced and geometrically irreducible. We would like to explain what it means that Z is a (possibly nonnormal) Del Pezzo surface.

Let Σ be the finite set of points of X where K_X is not Cartier. Set $X^0 := X \setminus \Sigma$ and $Z^0 := Z \setminus \Sigma$. (To be completely precise, I should let Σ be the set of points where X is not Gorenstein. At the end, however, this does not make any difference.) Z^0 is a Cartier divisor on the variety X^0, thus

$$\omega_{Z^0} \cong \omega_{X^0}(Z^0) \otimes \mathcal{O}_{Z^0} \cong \omega_{X^0} \otimes \mathcal{O}_{Z^0}.$$

By assumption $\omega_{X^0}^{-m}$ is f-very ample for $m \gg 1$, so we obtain that $\omega_{Z^0}^{-m}$ is very ample for $m \gg 1$.

Let $p : \bar{Z} \to Z$ be the normalization and set $\bar{Z}^0 := \pi^{-1}(Z^0)$. There is an adjunction map $\omega_{\bar{Z}^0} \to \pi^* \omega_{Z^0}$ (cf. [Hartshorne77, Ex.III.7.2.a]). (The adjunction map is defined even over $\bar{Z}$ but $\pi^*\omega_Z$ may be messy since ω_Z need not be locally free.) $\bar{Z}^0$ is a normal surface, so there is an effective divisor $B^0 \subset \bar{Z}^0$ such that

$$\mathcal{O}_{\bar{Z}^0}(K_{\bar{Z}^0} + B^0) \cong \omega_{\bar{Z}^0}(B^0) \cong \pi^* \omega_{Z^0}.$$

(Further information about B^0 is given in (3.7).) Here both sides are locally free (since the right hand side is), so we can raise them to any power to get isomorphisms

$$\mathcal{O}_{\bar Z^0}(m(K_{\bar Z^0}+B^0)) \cong \pi^*\omega_{Z^0}^m.$$

Now choose $m>0$ such that mK_X is Cartier. The reflexive sheaves

$$\mathcal{O}_{\bar Z}(-m(K_{\bar Z}+B)) \quad \text{and} \quad \pi^*\mathcal{O}_X(-mK_X)$$

are isomorphic on $\bar Z^0$, thus outside finitely many points. Hence they are isomorphic. Since $\mathcal{O}_X(-mK_X)$ is locally free we conclude that

1. $K_{\bar Z}+B$ is $\mathbb{Q}$-Cartier and $-(K_{\bar Z}+B)$ is ample, and
2. $(K_{\bar Z}+B)^2 = K_{X_t}^2$ where X_t is any smooth fiber of X/C.

We can also get some information about the global sections. One has to be a little careful since K_X may not be Cartier everywhere. As usual, let $\omega_X^{[r]}$ denote the double dual of $\omega_X^{\otimes r}$. $\omega_X^{[-n]}$ is reflexive, so $\omega_X^{[-n]}\otimes\mathcal{O}_Z$ has no embedded points. Thus

$$\begin{aligned} H^0(Z^0,\omega_{Z^0}^{-n}) &= H^0(Z^0,\omega_X^{[-n]}\otimes\mathcal{O}_{Z^0}) \supset H^0(Z,\omega_X^{[-n]}\otimes\mathcal{O}_Z)\\ &\geq H^0(X_t,\omega_{X_t}^{[-n]}) = \tbinom{n+1}{2}(K_{X_t}^2)+1, \end{aligned}$$

where the inequlity holds by semicontinuity of H^0 and the last equality is a fact about smooth Del Pezzo surfaces. We can pull back these sections to $\bar Z^0$ and extend them across the finitely many points. Thus we conclude that

3. $H^0(\bar Z,\mathcal{O}_{\bar Z}(-(K_{\bar Z}+B))) \geq (K_{X_t}^2)+1 \geq 2$.

Choose a pencil in $|-(K_{\bar Z}+B)|$ and write it as $D+M$ where D is the fixed part and M the moving part. If K_X is Cartier at all points of $X(\mathbb{R})$ then $Z(\mathbb{R})\subset Z^0$ and also $\bar Z(\mathbb{R})\subset \bar Z^0$. This shows that $K_{\bar Z}+B$ is Cartier at all real points. Setting $S:=\bar Z$ we obtain a quadruplet (S,B,D,M) and we have proved the following.

Proposition 3.4. *Let $f: X\to C$ be a family of Del Pezzo surfaces satisfying the conditions (3.1.1–3). Let $Z:=f^{-1}(0)$ be a reduced and geometrically irreducible fiber. Then the quadruplet (S,B,D,M) constructed above satisfies the conditions (3.5).* □

Condition 3.5. For a quadruplet (S,B,D,M) consider the following properties.

1. S is a normal projective surface over $\mathbb{R}$,
2. B and D are Weil divisors on S,
3. M is a pencil on S without fixed components,
4. $K_S+B+D+M\sim 0$.
5. K_S+B is Cartier at all real points of S, and

6. $-(K_S+B)$ is $\mathbb{Q}$-Cartier and ample.

Remark 3.6. (1) Most quadruplets satisfying the above conditions do not arise as special fibers of families of Del Pezzo surfaces. An obvious numerical conditions is that $(K_S+B)^2$ be an integer beween 1 and 9. Even if this holds, there is no reason to assume that the surface Z is smoothable.

(2) Even if we know (S,B,D,M), it is not always easy to reconstruct the fiber Z. First of all, we need to know the map $B \to$ (singular curve of Z); a set theoretic information. If B is reduced then this determines the scheme structure of Z at least when Z is S_2 (which holds if X has terminal singularities). If, however, B is not reduced, additional scheme theoretic information is needed and this seems rather complicated. See [Reid94] for a closely related case.

3.7. Let X be a smooth 3–fold and $Z\subset X$ a reduced surface which is not normal along a curve $C\subset Z$. Let $\pi:\bar{Z}\to Z$ be the normalization and set $\operatorname{red}(\pi^{-1}(C))=\sum_i \bar{C}_i$. As in (3.3), we see that $\omega_{\bar{Z}}(\sum b_i\bar{C}_i)\cong \pi^*\omega_Z$ for some $b_i\geq 0$. We would like to establish a relationship between the coefficients b_i and the local structure of Z along C.

We can cut everything by a general hyperplane, and our problem is reduced to a curve question which has been classically studied in detail, since $(\sum b_i)/2$ is exactly the contribution of the singular point to the arithmetic genus of a curve. We need the classification of singularities with small b_i:

1. $b_1=0$ iff Z is smooth along C.
2. $b_1=1$ iff Z has 2 branches meeting transversely along C.
3. $b_i\leq 2$ for every i iff either Z has one branch with a cusp along C, or 2 branches which are simply tangent along C or 3 branches meeting pairwise transversely along C or $b_i\leq 1$ for every i.

3.8. In general S is singular and so we try to study it through a suitable resolution. The most natural choice would be to take its minimal resolution, but the following partial resolution turns out to be more convenient. (The main reason for this choice is explained in (5.2.3).)

There is a unique morphism $f:S^m\to S$ such that

1. f is an isomorphism above $P\in S$ if S is smooth at P or if S has a Du Val singularity at P and $P\notin \operatorname{Supp} B$
2. f is the minimal resolution over $P\in S$ otherwise.

(Note that this is not the same as the so called "minimal Du Val resolution".)

f is a birational map between normal surfaces, thus we can pull back any divisor by f if we allow rational coefficients. Hence we can define B^m by the formula $K_{S^m} + B^m \equiv f^*(K_S + B)$. $-B^m \equiv_f K_{S^m}$ has nonnegative intersection number with any exceptional curve, hence B^m is effective (cf. [Kollár-Mori98, 3.39]). Moreover, every exceptional curve appears in B^m with positive coefficient (cf. [Kollár-Mori98, 4.3–5]). (Here we use that we did not resolve Du Val points.) Write $f^*M = D' + M^m$ where D' is the fixed part (which may have rational coefficients) and M^m is a pencil without fixed components. Set $D^m := f^*D + D'$.

(∗) In this paper, the quadruplet (S^m, B^m, D^m, M^m) always denotes the one constructed above staring with (S, B, D, M).

Condition 3.9. For a quadruplet (S^m, B^m, D^m, M^m) consider the following properties.

1. S^m is a projective surface over $\mathbb{R}$ with Du Val singularities,
2. B^m and D^m are effective $\mathbb{Q}$-divisors on S such that $B^m + D^m$ is an integral divisor,
3. M^m is a pencil on S^m without fixed components,
4. $K_{S^m} + B^m + D^m + M^m \sim 0$.
5. If $C \subset S^m$ is a geometrically irreducible real curve then C appears in B^m with integer coefficient,
6. $-(K_{S^m} + B^m)$ is nef and big
7. $-(K_{S^m} + B^m)$ has positive intersection number with every (-1)-curve (2.2),
8. S is smooth along $\operatorname{Supp} B$,
9. $-(K_{S^m} + B^m)$ is nef and has positive intersection number with every curve not in $\operatorname{Supp} B$.

Proposition 3.10. *If (S, B, D, M) satisfies the conditions (3.5) then (S^m, B^m, D^m, M^m) satisfies the conditions (3.9).*

Proof. (1,3,4,6,8) are clear from the construction. $B^m + D^m \sim -K_{S^m} - M^m$ now implies (2). If $C \subset S$ is a geometrically irreducible real curve and $f(C)$ is a curve then C appears in B and in B^m with the same integer coefficient. If $f(C) = P$ is a point then P is real, so $K_S + B$ is Cartier there and every curve above P appears with integral coefficient in B^m.

(9) holds since every f-exceptional curve is in B. Since we took minimal resolutions, there are no f-exceptional (-1)-curves, and this shows (7). □

Lemma 3.11. *Let (F, B, D, M) be a quadruplet. Let $g : F \to F'$ be a birational morphism and set $B' := g_*B$, $D' := g_*D$ and $M' := g_*M$. Then:*

1. *If (F, B, D, M) satisfies one of the conditions (3.9.2-6 or 9) then (F', B', D', M') also satisfies the same condition.*
2. *If g^{-1} is a composite of $(1, m)$-blow ups and (F, B, D, M) satisfies one of the conditions (3.9.1-9) then (F', B', D', M') also satisfies the same condition.*

Proof. This is clear for (3.9.2,3 and 5) and $K_{F'} = g_* K_F$ implies it for (3.9.4).

Let E_i be the connected components of the exceptional set of g. We can write $K_F + B \equiv g^*(K_{F'} + B') + E$ where E is g-exceptional. $-(K_F + B)$ is nef, so E is effective and $E_i \subset \operatorname{Supp} E$ unless $K_F + B$ is numerically trivial along E_i (cf. [Kollár-Mori98, 3.39]). If $C \subset F$ is an irreducible curve which is not g-exceptional then

$$g_*(C) \cdot (K_{F'} + B') = C \cdot g^*(K_{F'} + B') = C \cdot (K_F + B) - C \cdot E \leq C \cdot (K_F + B).$$

This shows that $-(K_{F'} + B')$ is nef if $-(K_F + B)$ is, settling the case (3.9.9). If (3.9.6) holds on F then bigness of $-(K_{F'} + B')$ follows from

$$(K_{F'} + B')^2 = (K_F + B) \cdot g^*(K_{F'} + B') = (K_F + B)^2 - (K_F + B) \cdot E \geq (K_F + B)^2.$$

Assume now that g^{-1} is a single $(1, m)$-blow up. The points $g(E_i)$ are smooth on F' by (2.3), which implies the claim for (3.9.8). Every connected component of the exceptional set is a (-1)-curve by (2.3) and $K_F + B$ has negative intersection number with it. So $\operatorname{Supp} E$ coincides with the exceptional set and if C intersects the exceptional set then

$$g(C) \cdot (K_{F'} + B') \leq -C \cdot E < 0.$$

This shows the claim for (3.9.7). By induction this gives (3.11.2). □

4. The Classification of S^r

Notation 4.1. Throughout this section (S^m, B^m, D^m, M^m) denotes a quadruplet which satisfies the conditions (3.9.1–8). Let $g : S^m \to S^*$ be a minimal model of S^m and set $B^* := g_*(B^m)$. For simplicity we assume S^* is not obtained by $(1, m)$-blow ups from another surface. (This could happen in only a handful of cases. For instance, the blow up of $\mathbb{P}^2$ at a point is also a $\mathbb{P}^1$-bundle so it could be S^* as in (2.6.4.C).)

4.2 (How to determine S^m?).

We use the following method to get information about S^m.

(1) First we determine the possible surfaces S^*. This was in fact done using the MMP for real surfaces with Du Val singularities.

(2) Then we get a list of all possible quadruplets (S^*, B^*, D^*, M^*). This involves finding all possible ways of writing $-K_{S^*} \sim (B^* + D^*) + M^*$, so this is equivalent to classifying all pencils in $|-K_{S^*}|$. This is easy to do if $|-K_{S^*}|$ is small. Unfortunately, $|-K_{S^*}|$ gets arbitrarily large

for minimal ruled surfaces and I do not know of any useful classification of all anticanonical pencils.

(3) Given a quadruplet (S^*, B^*, D^*, M^*) we can try to find all possible ways it came from an (S^m, B^m, D^m, M^m). We factor $S^m \to S^*$ into $(1, m)$-blow ups

$$S^m \to \cdots \to S_2 \to S_1 \to S^*.$$

For any of the intermediate stages $g : S^m \to S_i$ set $B_i := g_* B^m, D_i := g_* D^m, M_i := g_* M^m$. A key point to observe is that by (3.11) all the (S_i, B_i, D_i, M_i) satisfy the conditions (3.9.1–8). Thus we can work our way backwards one blow up at a time starting with S^*.

(4) Assume that we already have (S_i, B_i, D_i, M_i) and that $\pi : S_{i+1} \to S_i$ is the ordinary blow up of a point $P \in S_i$ with exceptional curve $E \subset S_{i+1}$. By (3.9.4)

$$K_{S_{i+1}} + B_{i+1} + D_{i+1} + M_{i+1} = \pi^*(K_{S_i} + B_i + D_i + M_i),$$

so we conclude that the

$$\text{coefficient of } E \text{ in } (B_{i+1} + D_{i+1}) = \operatorname{mult}_P(B_i + D_i) + \operatorname{mult}_P M_i - 1.$$

In particular, we can blow up only points in $\operatorname{Supp}(B_i + D_i)$ and the base points of M_i.

(5) Given (S^*, B^*, D^*, M^*) and the sequence of blow ups leading to S^m, we have determined $S^m, B^m + D^m$ and M^m. Conditions (3.9.5–8) give further restrictions on B^m. In many cases these are impossible to satisfy.

(6) The role of (3.9.5) turns out to be crucial for us. Frequently there are many possibilities for B^m such that (S^m, B^m, D^m, M^m) satisfies all the conditions (3.9) except (3.9.5), but there are no choices of B^m where (3.9.5) also holds.

(3.9.5) is especially useful if there are many geometrically irreducible curves in $\operatorname{Supp}(B^m + D^m)$. This is the reason for studying S^r since in $S^r \to S^*$ all exceptional curves are geometrically irreducible. It turns out that S^r is obtained from S^* by at most 1 blow up and this allows us to understand all possible (S^r, B^r, D^r, M^r) quite well.

(7) In (4.3) we subdivide the possible pairs (S^*, B^*) into 5 cases and then do a separate classification in each case.

Lemma 4.3. *With the above notation the pair (S^*, B^*) satisfies one of the following conditions.*

1. *S^* is a Del Pezzo surface with $\rho(S^*) = 1$ which is neither $\mathbb{P}^2$ nor a quadric in $\mathbb{P}^3$.*
2. *S^* is a weak Del Pezzo surface with $\rho(S^*) = 2$ and $B^* = 0$.*

3. *There is a $\mathbb{P}^1$- bundle structure $S^* \to A$ such that $B^*_{\mathbb{C}}$ has a unique irreducible component which dominates A. This component is a section.*
4. *There is a conic bundle structure $S^* \to A$ such that $B^*_{\mathbb{C}}$ has two irreducible components which dominate A. These components are conjugate sections.*
5. *S^* is $\mathbb{P}^2$ or a quadric in $\mathbb{P}^3$.*

Proof. If S^* is a Del Pezzo surface with $\rho(S^*) = 1$ then we have either (1) or (5). In all other cases there is a conic bundle structure $S^* \to A$. Let $F \subset S^*$ be a general fiber. Then

$$2 = -K_{S^*} \cdot F = (B^* + D^*) \cdot F + M^* \cdot F \geq (B^* + D^*) \cdot F.$$

$B^* + D^*$ is a sum curves with integral coefficients, so it intersects F in at most 2 points. Thus $B^*_{\mathbb{C}}$ has at most 2 horizontal components.

$-(K_{S^*} + B^*) \cdot F > 0$ since $-(K_{S^*} + B^*)$ is big, thus $B^* \cdot F < 2$.

If $B^*_{\mathbb{C}}$ has 1 horizontal component H, then H is defined over $\mathbb{R}$ and hence H appears in B^* with integer coefficient. Together with $B^* \cdot F < 2$ this shows that H is a section. Then S^* is a $\mathbb{P}^1$-bundle by (2.8) and (3.9.8), so we are in case (3).

If $B^*_{\mathbb{C}}$ has 2 horizontal components and they are both defined over $\mathbb{R}$ then arguing as above leads to a contradiction.

If $B^*_{\mathbb{C}}$ has 2 horizontal components which are conjugates $H, \bar{H}$ then they are both sections and we are in case (4).

If $B^* = 0$ then we are in case (2).

Finally it may happen that $B^*_{\mathbb{C}} \neq 0$ has no horizontal components. $\rho(S^*) = 2$ and so $\rho(S^*/A) = 1$. S^* is $\mathbb{Q}$-factorial hence every curve in S^* is $\mathbb{Q}$-Cartier. This implies that any vertical curve is numerically equaivalent to a (rational) multiple of a general fiber, in particular B^* is nef. This implies that $-K_{S^*}$ is ample. By the Cone Theorem (cf. [Kollár-Mori98, 3.7]) S^* has another extremal ray with contraction $S^* \to W$. This is not a birational contraction by assumption (4.1). So $S^* \to W$ is another conic bundle structure and the vertical $B^*_{\mathbb{C}}$ becomes horizontal for $S^* \to W$. These cases were treated already. □

Theorem 4.4. *Assume that S^* is a weak Del Pezzo surface with Du Val singularities, not isomorphic to $\mathbb{P}^2$ or to a quadric in $\mathbb{P}^3$. Assume furthermore that either $\rho(S^*) = 1$ or $B^* = 0$. Then S^r is a weak Del Pezzo surface with Du Val singularities and $B^r = 0$.*

Proof. Assume first that $B^* \neq 0$. Write $B^* + D^* = C + C'$ where $\operatorname{Supp} C = \operatorname{Supp} B^*$. $-K \equiv C + C' + M^*$ and C is a Cartier divisor by (3.9.8) which is not empty if $B^* \neq 0$. On S^* every effective Cartier divisor is ample since $\rho(S^*) = 1$, so $C' + M^*$ is ample. This is impossible by (2.7).

Thus $B^* = 0$ and $S^* \not\cong \mathbb{P}^2$. If $B^r = 0$ then $-(K_{S^r} + B^r) = -K_{S^r}$ is nef and big, so S^r is a weak Del Pezzo surface with Du Val singularities.

We need to exclude the case when $B^r \neq 0$. Let

$$\cdots \to S_2 \to S_1 \to S^*$$

be the series of blow ups leading to S^r and $S_{i+1} \to S_i$ the last blow up with exceptional curve $E \subset S_{i+1}$ such that $B_i = 0$. Then $B_{i+1} = aE$ for some $a > 0$ and a is an integer since E is geometrically irreducible. S_{i+1} is smooth along B_{i+1}, so $S_{i+1} \to S_i$ is an ordinary blow up. This is impossible by (4.5). □

Proposition 4.5. *Let F be a projective surface with Du Val singularities over $\mathbb{C}$ and $p : G \to F$ the blow up of a smooth point with exceptional curve $E \subset G$. Assume that $-(K_G + aE)$ is nef and has positive intersection number with every (-1)-curve for some $a \geq 1$.*

Then $F \cong \mathbb{P}^2$ and $a < 2$.

Proof. Let $F' \to F$ and $G' \to G$ be the minimal resolutions. Then F' is the blow up of G' at a point and $-(K_{G'} + aE')$ is the pull back of $-(K_F + aE)$. Thus it is sufficient to consider the case when F itself is smooth.

If $F \not\cong \mathbb{P}^2$ then there is a morphism $g : F \to \mathbb{P}^1$ whose general fiber is $\mathbb{P}^1$. $-K_F$ is g-nef, so it is easy to see that every fiber of g has dual graph

$$\overset{0}{\circ} \quad \text{or} \quad \overset{-1}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \cdots - \overset{-2}{\circ} - \overset{-1}{\circ}.$$

If P is on a fiber of the second type then E intersects a rational curve with self intersection -2 or -3 and $-(K_G + aE)$ is not nef for $a > 0$. In the first case, the birational transform of the fiber becomes a (-1)-curve F intersecting E. Thus $-(K_G + aE) \cdot F = 1 - a \leq 0$. □

Theorem 4.6. *Assume that $S^* \to A$ is a $\mathbb{P}^1$-bundle and $H \subset B^*$ is a (real) section. Then $S^r = S^*$.*

Proof. If $S^r \neq S^*$ then there is a last contraction $S_1 \to S^*$. The inverse of this is a $(1, m)$-blow up of a real point $P \in S^*$ and we need to show that this can not happen. Let F' be the fiber of $S^* \to A$ containing P.

If $S_1 \to S^*$ is an ordinary blow up then the exceptional curve $E \subset S_1$ and the birational transform F of F' are both (-1)-curves intersecting in a point. By (3.9.7) this implies that $E \cdot (K_{S_1} + B_1) < 0$ and $F \cdot (K_{S_1} + B_1) < 0$. We can write $B_1 = b_e E + b_f F + B'$ where B' does not have E, F as components. H intersects either E or F, so $E \cdot B' \geq 1$ or $F \cdot B' \geq 1$. We get a contradiction by (4.8).

The $(1,m)$-blow ups are excluded by (4.7). □

4.7 (Excluding $(1,m)$ blow ups)**.** Given a pair (S^*, B^*) assume that $S_1 \to S^*$ is a $(1,m)$-blow up for some $m \geq 2$ with exceptional curve E. Then S_1 is singular at a point $P \in E$, so E is not in B_1. Let $\pi : \tilde{S}_1 \to S_1$ be the minimal resolution of P. Write $\pi^*(D_1 + M_1) = \tilde{D}_1 + \tilde{M}_1$ where $\tilde{M}_1$ has no fixed components. Then $(\tilde{S}_1, \pi^* B_1, \tilde{D}_1, \tilde{M}_1)$ satisfies the conditions (3.9.1–8) and $\tilde{S}_1$ is obtained from S^* by m ordinary blow ups (2.3). So once we prove that there can be at most one ordinary blow up, this implies that there are no $(1,m)$-blow ups at all for $m \geq 2$.

Lemma 4.8. *Let S be a surface, $E, F \subset S$ effective curves and B' an effective $\mathbb{Q}$-divisor on S whose support does not contain E and F. Assume that $(E^2) = (F^2) = (E \cdot K_S) = (F \cdot K_S) = -1$ and $(E \cdot F) = 1$. Let b_e, b_f be integers and assume that*

$$E \cdot (K_S + b_e E + b_f F + B') < 0 \quad \textit{and} \quad F \cdot (K_S + b_e E + b_f F + B') < 0.$$

Then $b_e = b_f$, $E \cdot B' < 1$ and $F \cdot B' < 1$. □

Theorem 4.9. *Assume that $S^* \to A$ is a conic bundle and $B^*_{\mathbb{C}}$ contains a pair of conjugate sections. Then A is rational and either $S^r = S^*$ or S^r is obtained from S^* by one ordinary blow up.*

Proof. Let $H, \bar{H} \subset \operatorname{Supp} B^*$ be the conjugate sections. First we determine $B^* + D^*$. We can write $-K_{S^*} \equiv H + \bar{H} + aF$ for some a where F is a general fiber. $B^* + D^*$ is a Weil divisor containing $H + \bar{H}$, so $(B^* + D^* - H - \bar{H}) + M \equiv aF$ is effective and moves in a pencil, hence $a \geq 1$. By the adjunction formula

$$2g(H) - 2 = H \cdot (K_{S^*} + H) = -(H \cdot \bar{H}) - a.$$

Thus $g(H) = 0$ and either $H \cdot \bar{H} = 1$ and $B^* + D^* = H + \bar{H}$ or $H \cdot \bar{H} = 0$ and $B^* + D^* = H + \bar{H} + \epsilon C$ where $\epsilon \in \{0, 1\}$ and C is a fiber. In the first case let C denote the fiber passing through the unique point of $H \cap \bar{H}$. In both cases $M = |F|$ is base point free. The real points of $H + \bar{H} + \epsilon C$ are in $C(\mathbb{R})$. So all real blow ups take place over C by (4.2.4). We distinguish three cases according to the type of C.

(1): C is a double fiber. Any section intersects a double fiber at a singular point of S^*. Since $H \subset B^*$ and B^* is disjoint from the singular points of S^*, we see that $S^* \to A$ does not have any double fibers.

(2): C is a smooth fiber. Consider the case of 2 ordinary real blow ups $S_2 \to S_1 \to S^*$. There are 3 cases and after blow up we get one of the following curve configurations where $\bullet$ denotes the birational transform of C.

$$\overset{-1}{\circ} - \overset{-2}{\bullet} - \overset{-1}{\circ} \quad \text{or} \quad \overset{-1}{\circ} - \overset{-2}{\circ} - \overset{-1}{\bullet} \quad \text{or} \quad \overset{-2}{\circ} - \overset{-1}{\circ} - \overset{-2}{\bullet}.$$

Let E_1, E_2, E_3 be these curves from left to right. $B_2 = e_1E_1 + e_2E_2 + e_3E_3 + H_2 + \bar{H}_2$, the e_i are integers, $-(K_{S_2} + B_2)$ is positive on the (-1)-curves and nonnegative on the (-2)-curves. By solving these inequalities we obtain that $H_2 + \bar{H}_2$ does not intersect the (-2)-curves, $e_1 = e_2 = e_3$ in the first 2 cases and $e_1 = e_2/2 = e_3$ in the third case.

If $C \cap (H + \bar{H})$ is a conjugate point pair, then $H_2 + \bar{H}_2$ intersects the curve $\bullet$ and this leads to a contradiction in the first and third cases. In the second case we use (4.2.4) to conclude that $B^* + D^* = H + \bar{H} + C$ and $e_2 = 0$, giving a contradiction.

If $C \cap (H + \bar{H})$ is a real point P, then C has coefficient zero in $B^* + D^*$ hence we get that $e_1 = e_2 = e_3 = 0$ and the only point that we can blow up on S^* is P. Thus the first case can not happen and $H_2 + \bar{H}_2$ intersects E_2 in the second case and E_1 in the third case. Both are impossible.

$(1, m)$-blows up are excluded by (4.7).

(3): C is a reducible fiber. The only real point of C is its singular point, hence we have to start by blowing it up. There is only 1 case and after 2 blow ups we get the following curve configuration.

$$\overset{-1}{\circ} - \overset{-2}{\circ} - \bullet,$$

where $\bullet$ denotes the birational transform of C (thus it is geometrically reducible). Let E_1, E_2, E_3 be these curves from left to right. $B_2 = e_1E_1 + e_2E_2 + e_3E_3 + H_2 + \bar{H}_2$, e_1, e_2 are integers, $-(K_{S_2} + B_2)$ is positive on the (-1)-curve and nonnegative on the (-2)-curves. By solving these inequalities we obtain that $e_1 = e_2 = e_3$ and $H_2 + \bar{H}_2$ does not intersect $\bullet$. This is impossible since H, a section, does not pass through the singular point of the fiber C.

$(1, m)$-blow ups are excluded as above. $\square$

Notation 4.10. There are 5 normal quadrics over $\mathbb{R}$ up to isomorphism. The smooth ones are $Q^{4,0} := (x^2 + y^2 + z^2 + t^2 = 0)$, $Q^{3,1} := (x^2 + y^2 + z^2 - t^2 = 0)$ and $Q^{2,2} := (x^2 + y^2 - z^2 - t^2 = 0)$. The quadric cones are $Q^{3,0} := (x^2 + y^2 + z^2 = 0)$ and $Q^{2,1} := (x^2 + y^2 - z^2 = 0)$.

$Q^{4,0}(\mathbb{R}) = \emptyset$ and $Q^{3,0}(\mathbb{R})$ is a single point, so they are not very interesting for us.

Theorem 4.11. *Assume that S^* is $\mathbb{P}^2$ or a quadric in $\mathbb{P}^3$. Then one of the following holds.*

1. *S^r is $\mathbb{P}^2$ or a quadric in $\mathbb{P}^3$.*

2. *S^r is weak Del Pezzo surface and $B^r = 0$.*
3. *S^r is obtained from $\mathbb{P}^2$ by one ordinary blow up.*
4. *S^r is $\mathbb{P}^2$ blown up in 2 points and $B^r(\mathbb{R}) = \emptyset$.*
5. *S^r is one of the above cases blown up at conjugate pairs of points.*

(The last case occurs since blowing up the quadric $Q^{3,1}$ at a real point is the same as blowing up $\mathbb{P}^2$ at a conjugate pair of points. A more economical choice of S^* would have eliminated this case.)

Proof. Assume that we blow up 2 (possibly infinitely near) real points in $\mathbb{P}^2$. There is unique line passing through the center of both blow ups. This line and the two exceptional curves form a configuration

$$\overset{-1}{\circ} - \overset{-1}{\circ} - \overset{-1}{\circ} \quad \text{or} \quad \overset{-1}{\circ} - \overset{-1}{\circ} - \overset{-2}{\circ}.$$

Let E_1, E_2, E_3 be these curves from left to right. $B_2 = e_1E_1 + e_2E_2 + e_3E_3 + B_2'$, the e_i are integers, $-(K_{S_2} + B_2)$ is positive on the (-1)-curves and nonnegative on the (-2)-curve. By solving these inequalities we obtain that $e_1 = e_2 = e_3 = 0$ and $E_i \cdot B_2' < 1$ for every i.

If $B_2 = 0$ then we end up in case (2) as in (4.4). If B^* contains a curve with positive integer coefficient then its birational transform intersects $E_1 + E_2 + E_3$ and one of the inequalities $E_i \cdot B_2' < 1$ is violated.

A similar argument excludes the possibility of even one blow up when S^* is a quadric and B^* contains a curve with positive integer coefficient.

We are left with three cases: $S^* = \mathbb{P}^2$, $B^* = c(L + \bar{L})$ for a conjugate pair of lines, or $S^* = Q^{3,1}$, $B^* = c(L + \bar{L})$ for a conjugate pair of intersecting lines or $S^* = Q^{2,2}$, $B^* = c(L + \bar{L})$ for a conjugate pair of nonintersecting lines. In the latter case $S^* \cong \mathbb{P}^1 \times \mathbb{P}^1$ and one of the projections lands us in case (4.9), so this is already treated.

Assume that $S^* = \mathbb{P}^2$ and $B^* = c(L + \bar{L})$ for a conjugate pair of lines intersecting at a (necessarily real) point P. Then $B^* + D^* = L + \bar{L}$ and M is a pencil of lines with a base point Q. By (4.2.4) the only possibilities to blow up are P and Q. If $Q \neq P$ and we blow up Q then projection from Q becomes a $\mathbb{P}^1$-bundle and $L, \bar{L}$ become conjugate sections. This case was treated in (4.9) and we get that blowing up P is the only possible further blow up.

If we blow up P (possibly $P = Q$) and the exceptional curve E appears in B with coefficient ≥ 1 then we are in case (4.6) and no more blow ups are possible. So E appears in B with coefficient 0. Another blow up on E would create a (-2)-curve with coefficient 0 in B_2 intersecting the birational transform of L, a contradiction. Thus we can blow up only $Q \neq P$ and then we are in the already discussed case.

As we showed, we can never blow up infinitely near points, so there are no $(1,m)$-blow ups by (4.7).

If $S^* = Q^{3,1}$ then a one point blow up of S^* is also a blow up of $\mathbb{P}^2$ at a conjugate pair of points. So we are reduced to considering the blow ups of $\mathbb{P}^2$. These cases can also be treated directly. □

We can summarize the above results as follows.

Proposition 4.12. *Assume that the quadruplet (S^m, B^m, D^m, M^m) satisfies the conditions (3.9). Then (S^r, B^r) satisfies one of the following conditions.*

1. *S^r is a weak Del Pezzo surface and $B^r = 0$.*
2. *S^r is a $\mathbb{P}^1$-bundle, B^r consists of a section, at most 2 real fibers and some conjugate pairs of fibers.*
3. *S^r is a conic bundle, B^r consists of a conjugate pair of sections and at most one fiber.*
4. *S^r is a conic bundle with one point blown up, B^r consists of a conjugate pair of sections and possibly one fiber which is the union of a $\mathbb{P}^1$ and of a conjugate pair of rational curves.*
5. *$S^r = \mathbb{P}^2$ and B^r is either a line or a pair of lines or a conic.*
6. *$S^r = Q^{2,2} \cong \mathbb{P}^1 \times \mathbb{P}^1$ and B^r is either a line or a pair of intersecting lines or a conic.*
7. *$S^r = Q^{2,1}$ and B^r is a hyperplane section not through the vertex.*
8. *$S^r(\mathbb{R})$ is a finite set.*

Proof. There are only a few points which have not been settled earlier.

In the $\mathbb{P}^1$-bundle case write $B^r = H + B'$. From the adjunction formula we get that

$$-2 \le 2g(H) - 2 = H \cdot (K + H) = -H \cdot (B' + D^r + M^r) \le -(H \cdot B'),$$

so B^r contains at most 2 fibers.

In the conic bundle case we either have no blow ups and get (3) or let C denote the fiber containing the center of the blow up. There are 2 cases to consider. C can be singular giving case (4). If C is smooth then after one blow up we get 2 intersecting (-1)-curves and both appear with coefficient 0 in B^r. □

The following result could have been proved much earlier.

Lemma 4.13. *Let (S, B, D, M) be a quadruplet satisfying the conditions (3.5). Then one of the following holds.*

1. *S is a cone over an elliptic curve and $B = 0$.*
2. *$S_{\mathbb{C}}$ is a rational surface with rational singularities and every irreducible component of B^m is a smooth rational curve.*

Proof. In the previous classification, a nonrational surface can occur only in (4.6). So S^* is a $\mathbb{P}^1$-bundle and $H \subset B^*$ is a section. From the adjunction formula

$$2g(H) - 2 = H(K + H) = -H(B^* - H + D^* + M^*) \leq 0.$$

So H is elliptic, $B^* = H$ and $H(K + B^*) = 0$. We have proved that there are no real blow ups and a similar computation shows that there are no complex blow ups either. So $S^m = S^*$ and $|-r(K + B^*)|$ contracts H for $r \gg 1$.

Let $h : S' \to S$ be any resolution. The Leray spectral sequence gives an exact sequence

$$H^1(S', \mathcal{O}_{S'}) \to R^1 h_* \mathcal{O}_{S'} \to H^2(S, \mathcal{O}_S).$$

$H^1(S', \mathcal{O}_{S'}) = 0$ if S' is rational. $h^2(S, \mathcal{O}_S) = h^0(S, \mathcal{O}_S(K_S))$ and the latter is zero since $\mathcal{O}_S(-rK_S)$ has sections for $r \gg 1$. Thus $R^1 h_* \mathcal{O}_{S'} = 0$ and so S has rational singularities.

Irreducible components of B^* are smooth and rational by the classification and all other curves in B^m appear as exceptional curves of $(1, m)$-blow ups, so they are smooth and rational. □

Lemma 4.14. *Let S be a real algebraic surface with rational singularities and $h : S' \to S$ a proper birational morphism from a normal surface S'. Then $S'(\mathbb{R}) \to S(\mathbb{R})$ has connected fibers.*

Proof. For any $P \in S(\mathbb{R})$, $H^1(h^{-1}(P), \mathcal{O}) = 0$ since S has rational singularities. Thus $C := \operatorname{red}(h^{-1}(P))$ is a tree of smooth rational curves. If $A, B \in C(\mathbb{R})$ are two points then C has a unique chain of rational curves $C_{A,B}$ connecting A and B. Complex conjugation fixes the two ends of the chain, so it fixes every complex irreducible component and every singular point. Thus $C_{A,B}(\mathbb{R})$ is a connected chain of circles. □

5. The Real Points of Singular Fibers

In (3.8) B^m was defined by the formula $K_{S^m} + B^m \equiv f^*(K_S + B)$. This shows that S is obtained from S^m by contracting all the curves $A \subset S^m$ such that $(A \cdot (K_{S^m} + B^m)) = 0$. By (3.9) all such curves are in $\operatorname{Supp} B^m$. Using (4.14), a version of this also holds for real points.

Proposition 5.1. *From $(S^m(\mathbb{R}), B^m(\mathbb{R})) \cong (S^r(\mathbb{R}), B^r(\mathbb{R}))$ we obtain $(S(\mathbb{R}), B(\mathbb{R}))$ by contracting certain connected subcurves of $B^r(\mathbb{R})$ to points and by adding isolated points.* □

Remark 5.2. (1) A real singular point of S may be isolated in $S(\mathbb{R})$ and so invisible in the real part of the resolution.

(2) In the applications I will be able to compute only a Zariski neighborhood of $S(\mathbb{R})$ in $S(\mathbb{C})$. (5.1) allows us to compute the pair $(S(\mathbb{R}), B(\mathbb{R}))$ up to finite ambiguity.

(3) The somewhat complicated choice of the partial resolution $f : S^m \to S$ is important for (5.1). If we take the minimal resolution then the exceptional curves over Du Val points appear in B with zero coefficient. Thus $S^m \to S$ would also involve exceptional curves which are not controled by B^m. It is probably possible to analyze this but some complications definitely do appear.

We have a list of all possible pairs $(S^r(\mathbb{R}), B^r(\mathbb{R}))$, so using (5.1) we can get a list of all possible pairs $(S(\mathbb{R}), B(\mathbb{R}))$. From these pairs the real points of the fibers are obtained by gluing B to itself. It should be kept in mind that if a point $P \in B$ is glued to its conjugate $\bar{P}$ then we obtain a real point. Thus some points of $X_0(\mathbb{R})$ may not come from a point of $S(\mathbb{R})$.

The gluing process is easy if $B(\mathbb{R})$ is not complicated, say empty or a single circle but it may be subtle in general. Some of the worst cases can be avoided if we assume that $X(\mathbb{R})$ is orientable though this is not essential.

Next we define a certain class of 2–complexes. The reason for this rather unnatural definition is that this is what I can prove about the real points of sigular fibers.

Definition 5.3. Let C be a circle, $(B_i, \partial B_i)$ 2–discs and $\phi_i : \partial B_i \to C$ PL–maps. We use the ϕ_i to glue the discs to C. The resulting 2–complex $C \cup_{\phi_i} B_i$ is called a *circle with discs attached*. A disc B_i is called *inessential* if ϕ_i has degree 0. $\pi_1(C \cup_{\phi_i} B_i) = \mathbb{Z}/m$ where m is the gcd of the degrees of the ϕ_i.

Condition 5.4. We consider compact 2–complexes K which satisfy the following conditions:

1. There is a 2–complex K' and a surjective PL–map $h : K' \to K$.
2. K' is the disjoint union of points, intervals, spheres, real projective planes, tori, Klein bottles and circles with discs attached.
3. There are finite subsets $A \subset K$ and $A' \subset K'$ such that $h : K' \setminus A' \to K \setminus A$ is a homeomorphism. Moreover, A' does not contain any interior point of an interval.
4. If K' contains a torus or a Klein bottle then this is its only 2–dimensional connected component.

Observe that the 2–dimensional part of K' is uniquely determined by K. In the applications the lower dimensional parts will play only a minor role.

It is easy to see that if K satisfies the above conditions and G is a finite group acting on K without fixed points then K/G also satisfies the above conditions.

Theorem 5.5. *Let X be a real algebraic threefold and $f : X \to C$ a morphism to a smooth real algebraic curve. Assume that X has isolated singularities which are $\mathbb{Q}$-factorial over $\mathbb{R}$, $-K_X$ is f-ample and the smooth part of $X(\mathbb{R})$ is orientable. Let $0 \in C(\mathbb{R})$ be a point such that $X_0 := f^{-1}(0)$ is irreducible (over $\mathbb{R}$).*

Then $X_0(\mathbb{R})$ is a 2–complex satisfying the conditions (5.4).

Proof. Assume first that X_0 is geometrically reducible. Then $\operatorname{red} X_0 = Z_0 + \bar{Z}_0$ and all real points of X_0 are in $Z_0 \cap \bar{Z}_0$. Thus $X_0(\mathbb{R})$ is a 1–complex and we are done.

Next consider the case when X_0 is geometrically irreducible and reduced. The list of all $(S^r(\mathbb{R}), B^r(\mathbb{R}))$ can be read off from (4.12) and (5.1) then gives a longer list for the pairs $(S(\mathbb{R}), B(\mathbb{R}))$. The orientability assumption can be used through the following two consequences:

1. $S^r(\mathbb{R}) \setminus B^r(\mathbb{R})$ is orientable, and
2. If $\{C_t\}$ is a base point free pencil on S^r such that $(C_t \cdot (K_{S^r} + B^r))$ is odd then the general C_t intersects at least one irreducible component of B^r which gets contracted in S.

The first of these holds since $S^r(\mathbb{R}) \setminus B^r(\mathbb{R})$ injects into $S(\mathbb{R}) \setminus B(\mathbb{R})$ by (5.1) and $X_0(\mathbb{R})$ is 2–sided in $X(\mathbb{R})$, hence its smooth part is orientable. The second assertion holds since otherwise $\{C_t\}$ would give a base point free family of curves in X such that $(C_t \cdot K_X)$ is odd. This is impossible since $X(\mathbb{R})$ is orientable (cf. [Kollár97a, 2.8]).

A real point of $X_0(\mathbb{R})$ is either in the image of $S(\mathbb{R})$ or is in $\operatorname{Sing}(X_0)$. The latter is a 1–complex, and adding a 1–complex does not change the conditions (5.4).

Let us now go through the list of (4.12).

If S^r is a weak Del Pezzo surface such that the smooth part of $S^r(\mathbb{R})$ is orientable then $S^r(\mathbb{R})$ is obtained from spheres and tori by identifying some points. In the smooth case this was proved by [Comessatti14], see also [Silhol89]. The singular case is discussed in [Kollár98a, 9.9] or it can be read off the list of real plane quartics in [GUT66]. See also [Wall95].

Next let S^r be a conic bundle. Then $S^r(\mathbb{R})$ is a union of spheres with possibly some points identified. $B^r(\mathbb{R})$ is either empty or a circle in a sphere. The first case goes as before. If $B^r(\mathbb{R})$ is contracted, we get one more sphere. Otherwise as we go from S to X_0 we have to glue $B^r(\mathbb{R})$ to itself. Since $B^r(\mathbb{R})$ cuts the sphere into 2 discs, we obtain a circle with two discs attached. If S^r is a conic bundle with one point blown up then we have spheres and one $\mathbb{RP}^2$ and $B^r(\mathbb{R})$ is a line in $\mathbb{RP}^2$. If it is contracted then we get a sphere, otherwise we obtain a circle with one disc attached.

Similar arguments apply every time $B(\mathbb{R})$ is empty or a circle. Some of these cases can not happen, for instance $(\mathbb{RP}^2, \text{smooth conic})$ does not occur since the complement of a smooth conic is not orientable.

In case $(\mathbb{RP}^2, \text{two lines})$ a general line in $\mathbb{P}^2$ has intersection number -1 with $(K_{S^r} + B^r)$. Thus at least one of the lines gets contracted in S. Similarly, if $S^r = \mathbb{P}^1 \times \mathbb{P}^1$ then all of B^r has to be contracted.

We are left with the case when S^r is a $\mathbb{P}^1$-bundle. A general fiber in S^r has intersection number -1 with $(K_{S^r} + B^r)$, and this shows that the section H gets contracted. $S^r(\mathbb{R})$ is a torus or a Klein bottle. If we contract $H(\mathbb{R})$ it becomes a sphere with a pair of points pinched together. If B^r contains at most one real fiber then we are in one of the previously studied cases.

When B^r contains two real fibers F_1^r, F_2^r, further discussions are needed. Let $F_i \subset S$ be the birational transform of F_i^r. Since H is contracted and B^r does not contain any other sections, F_1 and F_2 intersect at a single point $P \in S$.

If F_1 and F_2 are mapped to the same curve in X_0 then we obtain a circle with 2 discs attached. This is the typical case and it occurs when X_0 is a cone over a nodal rational curve.

We are left with the cases when the normalization $\pi : S \to X_0$ maps $F_i \to C_i$ and $C_1 \neq C_2$. Each F_i appears in B with coefficient 1, so $F_i \to C_i$ is a degree 2 map and the two branches of X intersect transversely along C_i by (3.7.2).

There are two different degree 2 maps $\mathbb{RP}^1 \to \mathbb{RP}^1$. The first is $(z : 1) \mapsto (z^2 : 1)$. Under this map the circle $\mathbb{P}^1(\mathbb{R})$ maps to an interval. If $F_1 \to C_1$ is such a map then $\pi(S(\mathbb{R}))$ is the circle $C_2(\mathbb{R})$ with a disc attached and we have a 1–complex $C_2(\mathbb{R}) \setminus \pi(F_1(\mathbb{R}))$ added to it.

Finally it may happen that both $F_i(\mathbb{R}) \to C_i(\mathbb{R})$ are given as $(z : 1) \mapsto (z : z^2 - 1)$. I claim that near $\pi(P)$ this leads to a configuration which can not be realized in $\mathbb{R}^3$.

If (H^2) is odd then locally near P

$$(S(\mathbb{R}), F_1(\mathbb{R}), F_2(\mathbb{R}), P) \sim_s (\mathbb{R}^2, (x = 0), (y = 0), (0, 0)),$$

where $\sim_s$ denotes a PL homeomorphism of stratified spaces. There are 2 more branches of $X_0(\mathbb{R})$ passing through $\pi(P)$. One branch B_1 passes through $F_1(\mathbb{R})$ the other branch B_2 passes through $F_2(\mathbb{R})$) and these two branches do not intersect each other. Moreover, all intersections are generically transverse. Intersect everything with a small sphere S^2_ϵ

around the origin. The upper hemi sphere ($z \geq 0$) is a topological disc and B_1 and B_2 intersect it in 2 curves. One of these curves connects the points $(0,1),(0,-1)$ and the other the points $(1,0),(-1,0)$. Hence $B_1 \cap B_2 \cap S^2_\epsilon \neq \emptyset$, a contradiction.

If (H^2) is even then locally near P

$$\begin{aligned}(S(\mathbb{R}), F_1(\mathbb{R}), F_2(\mathbb{R}), P) \sim_s \\ \sim_s ((xy - z^2 = 0), (x = z = 0), (y = z = 0), (0,0,0)) \subset \mathbb{R}^3.\end{aligned}$$

There are 2 more branches of $X_0(\mathbb{R})$ passing through $\pi(P)$. One passes through $F_1(\mathbb{R})$ the other through $F_2(\mathbb{R})$ and these two branches do not intersect each other. Moreover, all intersections are generically transverse. This is again impossible in $\mathbb{R}^3$.

We are left with the case when X_0 is a multiple fiber. Let $m > 0$ be the smallest integer such that mX_0 is Cartier. Then $\mathcal{O}_X(mX_0)$ is a locally free sheaf isomorphic to $\mathcal{O}_X$ in a neighborhood of X_0. The section $1 \in H^0(X, \mathcal{O}_X) \cong H^0(X, \mathcal{O}_X(mX_0))$ determines the corresponding m-sheeted cyclic cover $Y \to X$ (cf. [Kollár-Mori98, Sec. 2.4]). Since X_0 is Cartier at real points, $Y \to X$ is unramified at real points. Thus $Y(\mathbb{R}) \to X(\mathbb{R})$ is a homeomorphism if m is odd. If m is even then over each connected component we get either a 2–sheeted cover or the empty set. In the latter case we switch to $-1 \in H^0(X, \mathcal{O}_X)$.

We have already proved that $Y_0(\mathbb{R})$ satisfies the properties (5.4), thus the same holds for $X_0(\mathbb{R})$, as remarked at the end of (5.4). □

6. Proof of the Main Theorem

In this section I use (5.5) and a purely topological argument to complete the proof of (1.3). First, following (1.5), we identify the neighborhoods of the singular fibers. Then we show how these pieces are assembled to form $\overline{X(\mathbb{R})}$.

Lemma 6.1. *Let M be an orientable 3–manifold and $K \subset M$ a 2-complex satisfying the conditions (5.4). Let $K \subset U \subset M$ be a regular neighborhood of K and assume that ∂U is a union of spheres and tori. Then every connected component of U is one of the following.*

1. *connected sum of lens spaces and $S^1 \times S^2$ minus balls,*
2. *connected sum of lens spaces, $S^1 \times S^2$ and of a solid torus minus balls,*
3. *interval bundle over a torus or a Klein bottle.*

Proof. As a first step we replace $h : K' \to K$ with another map $\tilde{h} : \tilde{K}' \to \tilde{K}$ such that

4. $\tilde{h} : \tilde{K}' \to \tilde{K}$ satisfies the conditions (5.4),

5. U is a regular neighborhood of $\tilde{K}$,
6. $\tilde{h}$ is injective on the set of 2–dimensional points of $\tilde{K}'$.

To achieve this, note that there are only finitely many points $p \in K$ over which h is not one–to–one. We will get rid of these one at a time. Let $p \in V$ be a regular neighborhood. $K \cap \partial V$ is a union of connected 1–complexes A_j and $K \cap V$ is the cone over $K \cap \partial V$. If all the A_j have dimension 0 then we do not need to do anything. Otherwise, there is a 1–dimensional component (say A_1) and a PL-homeomorphism $\psi : (V, \partial V) \to (B^3, S^2)$ such that $\psi(A_1)$ is in the northern hemisphere and all the other 1–dimensional A_j map to the southern hemisphere. Now cut the 3–ball B^3 along the equator and move the two halves apart a little. The center of the ball sweeps out a small interval. We add this interval to obtain K_1. K_1' is K' union an interval.

Repeating this procedure if necessary, at the end we obtain $\tilde{h} : \tilde{K}' \to \tilde{K}$.

To simplify notation let us assume that $h : K' \to K$ already satisfies the above conditions 4–6. Next we use induction on the number of 2–dimensional components of K'.

If there is an $S^2 \sim L \subset K'$ then $h(L)$ is an embedded S^2. Cut U along $h(L)$ and glue 3–balls to the resulting two spheres to get U_1. K' is replaced by $K_1' := K' \setminus L$. U is obtained from U_1 by attaching a 1–handle which is either taking connected sum of two components or taking connected sum with $S^1 \times S^2$. If there is an $\mathbb{RP}^2 \sim L \subset K'$ then $h(L)$ is an embedded $\mathbb{RP}^2$ and the boundary of its regular neighborhood is S^2 (since M is orientable). Cutting along this S^2 corresponds to connected sum with $\mathbb{RP}^3$ (which is the lens space $L_{2,1}$).

Assume now that there is an $L \subset K'$ which is a circle with discs attached. Let V be a regular neighborhood of $h(C)$ which we may assume to be an embedded circle. V is a solid torus. Consider the case when one of the discs (say B_1) is inessential. Then $\partial V \cap h(N_1)$ bounds a disc $D \subset \partial V$. We may asume that no other B_i intersects this disc. We change $h(B_1)$ by replacing $V \cap B_1$ with D and adding an interval connecting D with $h(C)$ to K. This way we have not changed U, we have removed one inessential disc and we created a new embedded sphere. By repeating this procedure we may assume that there are no inessential discs. Then each $h(B_i)$ intersects ∂V in a simple closed curve which is not null homotopic. Since these curves are all disjoint, they are in the same homotopy class γ. Thus $V \cup_i h(B_i)$ is a solid torus with discs attached along parallel curves in ∂V. The boundary of a regular neighborhood W of $V \cup_i h(B_i)$ is a union of spheres. We can again cut U along these spheres. W can also be obtained as attaching first a solid torus V' to V such that the meridian of V' maps to γ and then removing some balls from V'. Gluing two solid tori gives a lens space (1.8).

We are left to deal with the 1–dimensional part of K. Every connected component can be collapsed to a bouquet of circles. If there are ≥ 2 circles then the boundary of its regular neighborhood has genus ≥ 2, a contradiction. Thus we get either a ball (if ther are no cicrcles) or a solid torus (if there is one circle).

If there is an $L \subset K'$ which is a torus or a Klein bottle then the boundary of a regular neighborhood of $h(L)$ is either 2 or 1 tori. There are no other 2–dimensional components in K' by (5.4.4). If K has a 1–dimensional subcomplex which does not collapse into $h(L)$ then it leads to a genus ≥ 2 component in ∂U, a contradiction. □

Remark 6.2. Using (3.7) we obtain that the only lens spaces that can appear in (6.1) are $\mathbb{RP}^3 = S^3/\mathbb{Z}_2$ and $S^3/\mathbb{Z}_3$. I have no example for the latter.

6.3 (Proof of (1.3))**.**

We have established that M is glued together from the following pieces:

1. $S^1 \times S^2$ minus open balls,
2. lens space minus open balls,
3. solid torus minus open balls,
4. interval bundle over a torus or a Klein bottle.

An orientable interval bundle over a torus is a torus times an interval and the boundary of an orientable interval bundle over a Klein bottle is a torus (cf. [Kollár97c, 1.6]).

If the gluing involves a torus times an interval then this piece can be thrown away, except when the two boundary components are glued to each other. This gives a torus bundle over a circle.

Two copies of an interval bundle over a Klein bottle glued together map to an interval such that the fibers are tori over interior porints and Klein bottles over the two boundary points. This 3–manifold is doubly covered by a torus bundle over a circle.

An interval bundle over a Klein bottle glued to a solid torus is homeomorphic to $\mathbb{RP}^3 \# \mathbb{RP}^3$ (cf. [Kollár97c, 12.7]), so in this case we can change the decomposition to one that does not involve any interval bundles over a Klein bottle.

We are left with the case when M is glued together from lens spaces minus open balls. We do one gluing at a time. If a new lens space is glued in, that is connected sum. If two boundary components of a connected component are glued together then that is the same as taking connected sum with $S^1 \times S^2$. $\square$

7. Examples

Example 7.1. Consider $\mathbb{P}^4(1,1,1,2,2)$ with coordinates $(x:y:z:u:v)$ and the affine line $\mathbb{A}^1$ with coordinate t. Let X be the complete intersection

$$\begin{aligned} &X \subset \mathbb{P}^4(1,1,1,2,2) \times \mathbb{A}^1 \quad \text{given by equations} \\ &u^2+v^2 = f_4(x,y,z) \quad \text{and} \quad tv = q_2(x,y,z). \end{aligned}$$

Let $\pi : X \to \mathbb{A}^1$ be the second projection. For $t \neq 0$ we can eliminate v to obtain a degree 2 Del Pezzo surface

$$X_t \cong (u^2 = f_4(x,y,z) - t^{-2} q_2(x,y,z)^2) \subset \mathbb{P}^3(1,1,1,2).$$

For $t=0$ the equations become $u^2+v^2 = f_4(x,y,z)$ and $q_2(x,y,z)=0$. This has two points of index 2 at $(0:0:0:1:\pm\sqrt{-1})$, both are analytically isomorphic to $\mathbb{C}^2/\mathbb{Z}_4(1,1)$. The projection $(x:y:z:u:v) \mapsto (x:y:z)$ is defined outside these two points and the minimal resolution of X_0 becomes a conic bundle over the conic $(q_2(x,y,z)=0)$. The singular fibers correspond to the solutions of the equations $q_2 = f_4 = 0$. We get various cases depending on how the curves $(q_2=0)$ and $(f_4=0)$ intersect.

Another way to obtain this model is as follows. Consider the family

$$Y \subset \mathbb{P}^3(1,1,1,2) \times \mathbb{A}^1 \quad \text{given by} \quad u^2 = t^2 f_4(x,y,z) - q_2(x,y,z)^2.$$

Outside the origin Y is isomorphic to X via the transformation

$$(x,y,z,u,t) \mapsto (x,y,z,ut,t).$$

The central fiber Y_0 consists of a conjugate pair of planes intersecting along the conic $(q_2=0)$. Y is singular along $(u=t=q_2=0)$. If we blow up the singular curve, the two planes in the central fiber become disjoint. Contracting them gives the 3–fold X.

Example 7.2. Consider $\mathbb{P}^4(1,1,2,3,3)$ with coordinates $(x:y:z:u:v)$ and the affine line $\mathbb{A}^1$ with coordinate t. Let X be the complete intersection

$$\begin{aligned} &X \subset \mathbb{P}^4(1,1,2,3,3) \times \mathbb{A}^1 \quad \text{given by equations} \\ &u^2+v^2 = f_6(x,y,z) \quad \text{and} \quad tv = c_3(x,y,z). \end{aligned}$$

Let $\pi : X \to \mathbb{A}^1$ be the second projection. For $t \neq 0$ we can eliminate v to see that the fiber is the degree 1 Del Pezzo surface

$$X_t \cong (u^2 = f_6(x,y,z) - t^{-2} c_3(x,y,z)^2) \subset \mathbb{P}^3(1,1,2,3).$$

For $t=0$ the equations become $u^2+v^2=f_6(x,y,z)$ and $c_3(x,y,z)=0$. This has two points of index 3 at $(0:0:0:1:\pm\sqrt{-1})$, both are analytically isomorphic to $\mathbb{C}^2/\mathbb{Z}_9(1,2)$. The projection $(x:y:z:u:v)\mapsto(x:y:z)$ is defined outside these two points and the minimal resolution of X_0 becomes a conic bundle over the curve $(c_3(x,y,z)=0)$ blown up in one point. Although c_3 has degree 3, this curve is birational to $\mathbb{P}^1$. Indeed, since z has degree 2, it appears in c_3 only linearly, so z can be rationally expressed in terms of x,y.

One can also obtain this model from the degeneration

$$Y\subset\mathbb{P}^3(1,1,2,3)\times\mathbb{A}^1\quad\text{given by}\quad u^2=t^2f_6(x,y,z)-c_3(x,y,z)^2,$$

but the birational transformation between them is more complicated.

Example 7.3. Start with the trivial family of quadrics

$$\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{A}^1\quad\text{with coordinates}\quad(x_1:x_2),(y_1:y_2),t.$$

Consider the $\mathbb{Z}_2$-action

$$\tau_1:(x_1,x_2,y_1,y_2,t)\mapsto(x_2,-x_1,y_2,-y_1,-t).$$

This has 4 fixed points at $(1,\pm\sqrt{-1},1,\pm\sqrt{-1},0)$. Set $X_1:=(\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{A}^1)/\mathbb{Z}_2(\tau_1)$. X_1 has 4 singularities of analytic type $\mathbb{C}^3/\mathbb{Z}_2(1,1,1)$. The central fiber is double and the reduced central fiber has 4 A_1-type points. It is easy to see that $X_1(\mathbb{R})$ is not orientable and its central fiber is $S^1\times S^1$.

Another $\mathbb{Z}_2$-action is given by

$$\tau_2:(x_1,x_2,y_1,y_2,t)\mapsto(x_2,-x_1,y_1,-y_2,-t).$$

This also has 4 fixed points at $(1,\pm\sqrt{-1},1,0,0)$ and $(1,\pm\sqrt{-1},0,1,0)$. Set $X_2:=(\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{A}^1)/\mathbb{Z}_2(\tau_2)$. X_2 has 4 singularities of analytic type $\mathbb{C}^3/\mathbb{Z}_2(1,1,1)$, the central fiber is double and the reduced central fiber has 4 A_1-type points. One can see that $X_2(\mathbb{R})$ is orientable and its central fiber is a Klein bottle.

Example 7.4. Let $(x:y:z)$ and t be coordinates on $\mathbb{P}^2\times\mathbb{A}^1$ and consider the surface $S:=(x^2+y^2-tz^2=0)$. Let $\mathbb{Z}_n$ act on (x,y,z,t) as a rotation of order n on x,y and identity on z,t. This induces an action on $\mathcal{O}_S+\mathcal{O}_S$. Let $F\subset\mathcal{O}_S+\mathcal{O}_S$ be the locally free rank 2 subsheaf which on the $z\neq0$ affine chart is generated by the sections (x,y) and $(-y,x)$. Outside $(x^2+y^2=0)$ the subsheaf F is the same as $\mathcal{O}_S+\mathcal{O}_S$. Let $X:=\mathbb{P}_SF$ be the corresponding $\mathbb{P}^1$-bundle over S. $\mathbb{Z}_n$ acts on X. This action has two isolated conjugate fixed points over $(0,0,1,0)\in S$ and fixes the conjugate surfaces over the curves $(1,\pm\sqrt{-1},0,t)\subset S$. Set $X_n:=X/\mathbb{Z}_n$. The projection $\pi:X_n\to\mathbb{A}^1$ exhibits X_n as a degeneration of quadrics and $-K_{X_n}$ is π-ample. X_n has only terminal singularities, it has 2 points of index n and it is smooth at all real points. The central fiber is geometrically reducible.

The only slight problem with this example is that $\rho(X_n/\mathbb{A}^1) = 2$ since π can be factored as $X_n \to S/\mathbb{Z}_n \to \mathbb{A}^1$. Probably one can globalize this example to get relative Picard number 1.

Example 7.5. Let $S^* \subset \mathbb{P}^3$ be a smooth quadric and $B^* = 0$. Let $P, \bar{P} \in S^*$ be a pair of conjugate points not on a line. Let $S_1 \to S^*$ be the blow up of $P + \bar{P}$ with exceptional curve $E + \bar{E}$. Set $B_1 := (1/2)(E + \bar{E})$. The pencil of planes through $P + \bar{P}$ gives S_1 a conic bundle structure with $E + \bar{E}$ as conjugate sections. Blow up 3 more pairs of conjugate points on $E+\bar{E}$ to get S^m. The birational transforms of $E, \bar{E}$ have self intersection -4, so they can be contracted $S^m \to S$. This S is among the types described in (7.1).

Example 7.6. Set $Q := \mathbb{P}^1_{(x_0:x_1)} \times \mathbb{P}^1_{(y_0:y_1)}$ and let $\sigma : Q \to Q$ be given by $(x_0 : x_1, y_0 : y_1) \mapsto (y_1 : y_0, x_0 : x_1)$. Then σ has order 4 on Q and on $H_1(Q(\mathbb{R}), \mathbb{Z})$. Take $Q \times \mathbb{P}^1_{(z_0:z_1)}$. Let τ be the action which is σ on Q and $(z_0 : z_1) \mapsto (z_0 + z_1 : -z_0 + z_1)$ on the second factor. Set $X := (Q \times \mathbb{P}^1_{(z_0:z_1)})/(\tau)$. Then $X \to \mathbb{P}^1_{(z_0:z_1)}/(\tau)$ is a quadric bundle (except over a conjugate pair of complex points). Its real part gives an $S^1 \times S^1$-bundle over S^1 with order 4 monodromy.

Example 7.7. Let Q be the degree 6 Del Pezzo surface given in $\mathbb{P}^1_{(x_0:x_1)} \times \mathbb{P}^1_{(y_0:y_1)} \times \mathbb{P}^1_{(z_0:z_1)}$ by the equation

$$x_0y_0z_1 + x_0y_1z_0 + x_1y_0z_0 = x_0y_1z_1 + x_1y_0z_1 + x_1y_1z_0.$$

Let $\sigma : Q \to Q$ be given by

$$(x_0 : x_1, y_0 : y_1, z_0 : z_1) \mapsto (z_1 : z_0, x_1 : x_0, y_1 : y_0).$$

σ has order 6 on Q and on $H_1(Q(\mathbb{R}), \mathbb{Z})$. Take $Q \times \mathbb{P}^1_{(t_0:t_1)}$. Let τ be the action which is σ on Q and

$$(t_0 : t_1) \mapsto (\cos(\pi/6)t_0 + \sin(\pi/6)t_1 : -\sin(\pi/6)t_0 + \cos(\pi/6)t_1)$$

on the second factor. Set $X := (Q \times \mathbb{P}^1_{(t_0:t_1)})/(\tau)$. Then $X \to \mathbb{P}^1_{(t_0:t_1)}/(\tau)$ is a degree 6 Del Pezzo surface bundle (except over a conjugate pair of complex points). Its real part gives an $S^1 \times S^1$-bundle over S^1 with order 6 monodromy.

Example 7.8. In $\mathbb{P}^3_{(x_0:x_1:x_2:x_3)} \times \mathbb{P}^1_{(y_0:y_1)}$ consider the 3–fold

$$X := (y_0(x_0^2 + x_1^2) = y_1(x_2^2 + x_3^2)).$$

For p odd and $(p, q) = 1$ let $\sigma : X \to X$ be the action which is rotation by $2\pi/n$ on (x_0, x_1) and rotation by $2q\pi/n$ on (x_2, x_3). Then $X(\mathbb{R}) \sim S^3$ and $X(\mathbb{R})/(\sigma) \sim L_{p,q}$.

The complex 3–fold $X/(\sigma)$ is smooth at its real points but it has nonterminal singularities at complex points. At least for $q=1$ these are easy to resolve and one obtains a quadric bundle X' such that $X'(\mathbb{R}) \sim L_{p,1}$. The other cases seem more complicated.

References

AGV85. V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko, Singularities of Differentiable Maps I–II, Birkhäuser 1985

BCR87. J. Bochnak, M. Coste and M-F. Roy, Géométrie algébrique réelle, Springer 1987

CKM88. H. Clemens, J. Kollár and S. Mori, Higher Dimensional Complex Geometry, Astérisque vol.166, 1988

Comessatti14. A. Comessatti, Sulla connessione delle superfizie razionali reali, Annali di Math. 23(3) (1914) 215-283

Fulton95. W. Fulton, Algebraic topology, Springer, 1995

Furushima86. M. Furushima, Singular Del Pezzo surfaces, Nagoya Math. J. 104 (1986) 1-28

GUT66. D. A. Gudkov, G. A. Utkin and M. L. Taj, The complete classification of irreducible curves of the 4th order (in Russian), Math. Sbornik 69 (1966) 222-256

Hartshorne77. R. Hartshorne, Algebraic Geometry, Springer, 1977

Hempel76. J. Hempel, 3-manifolds, Princeton Univ. Press, 1976

Keel-M^cKernan98. S. Keel and J. M^cKernan, Rational curves on quasi-projective varieties, Mem. AMS (to appear)

Kharlamov76. V. Kharlamov, The topological type of non-singular surfaces in RP^3 of degree four, Funct. Anal. Appl. 10 (1976) 295-305

Kollár87. J. Kollár, The structure of algebraic threefolds - an introduction to Mori's program, Bull. AMS 17 (1987) 211-273

Kollár96. J. Kollár, Rational Curves on Algebraic Varieties, Springer Verlag, Ergebnisse der Math. vol. 32, 1996

Kollár97a. J. Kollár, Real Algebraic Surfaces, Notes of the 1997 Trento summer school lectures, (preprint)

Kollár97b. J. Kollár, Real Algebraic Threefolds I. Terminal Singularities, Collectanea math. (to appear)

Kollár97c. J. Kollár, Real Algebraic Threefolds II. Minimal Model Program, Jour. AMS (to appear)

Kollár98a. J. Kollár, Real Algebraic Threefolds III. Conic Bundles (preprint)

Kollár98b. J. Kollár, The Nash Conjecture for threefolds, ERA of AMS 4 (1998) 63–73

`http://www.ams.org/era`

Kollár-Mori98. J. Kollár - S. Mori, Birational geometry of algebraic varieties, Cambridge Univ. Press, 1998

Manetti91. M. Manetti, Normal degenerations of the complex projective plane, J. f.r.u.a. Math. 419 (1991) 89-118

Manetti93. M. Manetti, Normal projective surfaces with $\rho = 1, P_{-1} \geq 5$, Rend. Sem. Mat. Univ. Padova 89 (1993) 195-205

Moise77. E. Moise, Geometric topology in dimensions 2 and 3, Springer, 1977

Nash52. J. Nash, Real algebraic manifolds, Ann. Math. 56 (1952) 405-421

Reid85. M. Reid, Young person's guide to canonical singularities, in Algebraic Geometry, Proc. Symp. Pure Math. vol.46, pp. 345-414

Reid94. M. Reid, Nonnormal Del Pezzo surfaces, Publ. RIMS Kyoto Univ. 30 (1994) 695-728

Rolfsen76. D. Rolfsen, Knots and links, Publish or Perish, 1976

Rourke-Sanderson82. C. Rourke and B. Sanderson, Introduction to piecewise linear topology, Springer, 1982

Scott83. P. Scott, The geometries of 3–manifolds, Bull. London Math. Soc. 15 (1983) 401-487

Shafarevich72. R. I. Shafarevich, Basic Algebraic Geometry (in Russian), Nauka, 1972. English translation: Springer, 1977, second expanded edition, 1994

Silhol89. R. Silhol, Real algebraic surfaces, Springer Lecture Notes vol. 1392, 1989

Wall95. C. T. C. Wall, Real rational quartic curves, in: W. L. Marar, ed., Real and complex singularities, Pitman Res. Notes in Math. vol. 333 (1995) 1-32.

University of Utah, Salt Lake City UT 84112

kollar@math.utah.edu