REAL ALGEBRAIC THREEFOLDS III. CONIC BUNDLES

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1. Introduction

This paper continues the study of the topology of real algebraic threefolds begun in [Kollár97b, Kollár97c], but the current work is completely independent of the previous ones in its methodology.

The present aim is to understand the topology of a special class of real algebraic threefolds. In many respects, the simplest 3–folds are of the form $\mathbb{P}^1 \times S$ where S is a surface. If one adopts the birational point of view, it is more natural, and considerably more general, to investigate those 3-folds which admit a map onto a surface $f : X \to S$ whose general fibers are smooth rational curves. This class of threefolds also appears as one of the 4 possible outcomes of the minimal model program (cf. [Kollár-Mori98]). From the birational point of view the most interesting 3–folds in this class are those which map onto a rational surface. This is also the natural assumption to make in connection with the Nash conjecture [Nash52, p. 421]. Our main theorem gives a nearly complete description of the possible topological types of the set of real points of such a threefold. (See (1.11) for terminology and notation.)

Theorem 1.1. *Let X be a smooth projective real algebraic threefold such that the set of real points $X(\mathbb{R})$ is orientable. Let $f : X \to S$ be a morphism onto a real algebraic surface S whose general fibers are rational curves. Let $M \subset X(\mathbb{R})$ be any connected component. Then*

$$M \sim N \ \# \ a\mathbb{RP}^3 \ \# \ b(S^1 \times S^2) \quad \textit{for some } a, b \geq 0,$$

where one of the following holds:

1. *N is Seifert fibered over a topological surface (1.6, 3.4),*
2. *N is the connected sum of lens spaces (3.1).*

Assume in addition that S is rational over $\mathbb{C}$. Then one of three stronger restrictions holds:

3. *N is Seifert fibered over a nonorientable surface with at most 6 multiple fibers,*
4. *N is Seifert fibered over S^2 or $S^1 \times S^1$ with at most 6 multiple fibers,*
5. *N is the connected sum of at most 6 lens spaces.*

Remark 1.2. I believe that (1–2) are sharp.

The conclusions of (3–5) are, however, probably not optimal. The computation of several examples suggests that the following stronger assertions may hold:

1. In all 3 cases 6 can be replaced by 4.
2. More precisely, if m_i are the multiplicities of the multiple fibers (or the lens spaces are $S^3/\mathbb{Z}_{m_i}$) then $\sum_i (1 - \frac{1}{m_i}) \leq 2$.
3. If N is Seifert fibered over $S^1 \times S^1$ then there are no multiple fibers.

Weaker results hold if $X(\mathbb{R})$ is not orientable (1.10).

The map $f : X \to S$ may have finitely many 2-dimensional fibers. The effect of these fibers can be investigated with the methods of [Kollár97c]. In effect, [Kollár97c] shows that (1.1) can be reduced to the case when we also assume that every fiber of f has dimension 1, at the price of allowing certain isolated singularities on X. Thus for the rest of this paper I consider morphisms $f : X \to S$ where every fiber of f has dimension 1. Two of the resulting classes of morphisms is formalized in the next definition:

Definition 1.3. Let X, Y be normal varieties and $f : X \to Y$ a proper morphism. f is called a *rational curve fibration* if every fiber of f has dimension 1 and the general fiber is a smooth rational curve. (Special fibers may have several components. We do not assume that the irreducible components of the special fibers are rational, though this is a consequence of the other assumptions [Kollár96, I.3.17 and II.2.2].)

f is called a *conic bundle* if Y is smooth and there is a $\mathbb{P}^2$-bundle $g : P \to Y$ and an embedding $j : X \hookrightarrow P$ such that $f = g \circ j$ and every fiber of f becomes a conic in the corresponding $\mathbb{P}^2$. The conic can be smooth, a pair of lines or a double line. (This definition is slightly more general than the one given in [Beauville77, Chap.I] since we allow X to be singular.)

Note that $P = \operatorname{Proj}_Y f_* \mathcal{O}_X(-K_{X/Y})$, thus P is uniquely determined by X.

Assume now that X, Y are real algebraic varieties and f is defined over $\mathbb{R}$ (see (1.11) for the definitions). We can look at the set of real points $X(\mathbb{R})$ (resp. $Y(\mathbb{R})$) and obtain a map of topological spaces $f : X(\mathbb{R}) \to Y(\mathbb{R})$. The preimage of a general point is the set of real points of a smooth rational curve, hence it is either empty or homeomorphic to the circle S^1. Thus we conclude that $X(\mathbb{R})$ contains a dense open set which is an S^1-bundle. Unfortunately, every manifold has this property, and we need to understand the behaviour of f at every fiber in order to get useful topological information about $X(\mathbb{R})$.

In Section 2 we prove the first general result in this direction:

Theorem 1.4. *Let X be a projective 3–fold over $\mathbb{R}$ with isolated singularities and $f : X \to S$ a rational curve fibration. Assume that $X(\mathbb{R})$ is an orientable PL 3–manifold.*

Then there is a PL map $\tilde{f} : X(\mathbb{R}) \to F$ onto a surface with boundary such that $\tilde{f}^{-1}(P)$ is a circle for every $P \in \operatorname{Int} F$ and $\tilde{f}^{-1}(P)$ is a point for every $P \in \partial F$. ($\tilde{f}$ can be obtained as a small perturbation of $f|_{X(\mathbb{R})}$ but it usually can not be chosen algebraic in a natural way.)

Remark 1.5. More generally, the above result also holds if we only assume that $\overline{X(\mathbb{R})}$ is a PL 3–manifold. ($\overline{X(\mathbb{R})}$ denotes the topological normalization of $X(\mathbb{R})$. In our case, $X(\mathbb{R})$ is a manifold except at finitely many points where it is locally like the cone over a possibly disconnected surface. The topological normalization separates these local components.) This is crucial for our applications.

The topological 3–manifolds described in (1.4) can be understood in terms of the usual classification scheme of topological 3–manifolds. This is done in section 3. $\tilde{f}$ is an S^1-bundle over all but finitely many points of $\operatorname{Int} F$ and it is exactly these finitely many points which assert the greatest influence on the topology of $X(\mathbb{R})$. In order to study these special fibers, we need a definition.

Definition 1.6. Let $g : M \to F$ be a proper PL map of a 3–manifold to a surface such that every fiber is a circle. Pick $P \in F$ and let $P \in D_P \subset F$ be a small disc around P. Then $g^{-1}(D_P)$ retracts to $g^{-1}(P)$. Let $P' \in D_P$ be a general point. The retraction gives a map

$$r(P', P) : S^1 \sim g^{-1}(P') \to g^{-1}(P) \sim S^1.$$

The absolute value of the degree of $r(P', P)$ is independent of the choices of P', the retraction and the orientations. It is called the *multiplicity* of the fiber $g^{-1}(P)$ and it is denoted by $m_P(g)$. (A detailed study of the local structure of such maps is given in section 3.)

If M is orientable, then $m_P(g) = 1$ for all but finitely many points $P \in F$. We say that M is *Seifert fibered* over F if $m_P(g) \geq 1$ for every $P \in F$.

The multiple fibers with $m_P(g) > 1$ are in close analogy with multiple fibers of elliptic surfaces. (Historically, Seifert fibers came first.) The $m_P(g) = 0$ case does not have an analog in complex algebraic geometry since it leads to fibers which are homologous to zero.

1.7. The proof of (1.4) given in sections 2 and 3 does not establish any link between the topological multiple fibers and the local algebraic nature of f. It turns out, however, that a very close relationship can be established if we pose additional restrictions on X, for instance if we assume that X is smooth. For many applications this is, however, too restrictive.

One way to obtain rational curve fibrations in practice is through a minimal model program. [Kollár97c, 1.9–11] suggests that we should therefore concentrate on rational curve fibration $f : X \to S$ which satisfy the following properties. (These conditions may seem rather technical, but they are very natural from the point of view of the minimal model program. See, for instance, [Kollár87, Kollár-Mori98]. Also, we only use these conditions to prove (1.8), after which one can forget about what terminal means.)

- **(1.7.1):** X is a real projective 3-fold with *terminal singularities* (4.1) such that K_X is Cartier along $X(\mathbb{R})$ and $\overline{X(\mathbb{R})}$ is a PL 3–manifold.
- **(1.7.2):** $-K_X$ is f-ample.

For varieties over $\mathbb{C}$, the analogous rational curve fibrations have been investigated by many authors. A systematic study of the special fibers was begun in [Prokhorov95, Prokhorov96, Prokhorov97]. In these papers a huge number of examples is described and a partial classification is given. A full description of the general case seems still far away. Usually, adding a real structure only complicates matters, but the assumption that K_X be Cartier along $X(\mathbb{R})$ turns out to be extremely strong. (A similar situation happens for extremal contractions, as observed in [Kollár97c].)

Section 4 uses the methods of [Mori88] and Prokhorov to obtain the following consequence:

Proposition 1.8. *Assume that $f : X \to S$ satisfies the conditions (1.7.1–2). Then there is a finite set $T \subset S$ such that*

1. $f : X \setminus f^{-1}(T) \to S \setminus T$ *is a conic bundle,*
2. $\operatorname{red} f^{-1}(s)$ *is an irreducible, smooth and rational curve for every real point $s \in T$. Moreover, X has only isolated hypersurface singularities along* $\operatorname{red} f^{-1}(s)$ *with the exception of a pair of conjugate points $P, \bar{P} \in f^{-1}(s)$. At these points X is complex analytically isomorphic to the quotient of a hypersurface singularity by a cyclic group $\mathbb{Z}_m$.*

In section 5 we begin the study of the local structure of f near the special points T of (1.8). First we establish that in a neighborhood of $s \in T$, f can be described as the quotient of a conic bundle by a cyclic group (5.6). This result is further developed to a complete description in (6.5). This way we obtain a complete topological description of $f : X(\mathbb{R}) \to S(\mathbb{R})$ near the special points T.

In order to understand $f : X(\mathbb{R}) \to S(\mathbb{R})$, we still need to analyze the topology of conic bundles over $\mathbb{R}$. A complete listing of the cases seems rather difficult, but in section 7 we see that one never obtains *multiple* Seifert fibers. This gives the following precise relationship between the algebraic and topological multiple fibers of f:

Theorem 1.9. *Let X be a real algebraic threefold satisfying (1.7.1) and $f : X \to S$ a rational curve fibration satisfying (1.7.2). Let $\tilde{f} : \overline{X(\mathbb{R})} \to F$ be the PL map constructed in (1.4). Then there is a one–to–one correspondence between the sets:*

1. *Multiple Seifert fibers of $\tilde{f}$ of multiplicty $m(\tilde{f}) \geq 2$, and*
2. *Multiple fibers of f which are real analytically isomorphic to the normal form*

$$\left(\mathbb{P}^2_{x:y:z} \times \mathbb{A}^2_{s,t}\right)/\mathbb{Z}_m \supset (x^2 + y^2 - z^2 = 0)/\mathbb{Z}_m \to \mathbb{A}^2_{s,t}/\mathbb{Z}_m,$$

where $\mathbb{Z}_m$ acts by rotation with angle $2b\pi/m$ on (s,t) for some $(b,m) = 1$, it fixes z and acts by rotation with angle $2\pi/m$ on (x,y).

The most interesting piece of information truns out to be the description of the singularity occurring on S. The quotient $\mathbb{A}^2_{s,t}/\mathbb{Z}_m$ is real analytically isomorphic to the singularity denoted by A^+_{m+1} given by equation $(u^2+v^2-w^m=0)$ (cf. (9.3)). It turns out that it is quite hard for a real surface to contain such a singularity. In section 9 we prove that if S is a real surface which is rational over $\mathbb{C}$ then a connected component of $\overline{S(\mathbb{R})}$ contains at most 6 singularities of type A^+. (By contrast, a rational surface can contain arbitrary many singularities of type $(u^2-v^2-w^m=0)$.)

All these results are assembled in section 8 to obtain the proof of the main theorems.

Finally, section 10 contains some examples of 3–manifolds which can be realized by rational curve fibrations over $\mathbb{R}$.

1.10 (The nonorientable case)**.**

The situation becomes more complicated if we do not assume that $X(\mathbb{R})$ is orientable. Because of [Kollár97c], in this case it is more natural to consider the case when X satisfies (1.7.1) and $f: X \to S$ is a rational curve fibration.

If S is a (not necessarily rational) real algebraic surface then $\overline{X(\mathbb{R})}$ is a connected sum of lens spaces with a 3–manifold where all the pieces of the Jaco–Johannson–Shalen decomposition (cf. [Scott83, p.483]) are Seifert fibered. There are further restrictions but they are somewhat complicated. Also, I do not know how sharp these results are. See (8.3) for details.

If $S_{\mathbb{C}}$ is rational then we should get very few cases, but again I do not have a reasonably sharp answer.

Definition 1.11. By a *real algebraic variety* I mean a variety given by real equations, as defined in most algebraic geometry books (see, for instance, [Shafarevich72, Hartshorne77]). This is consistent with the usage of [Silhol89] but is different from the definition of [BCR87] which essentially considers only the germ of X along its real points. In many cases the two variants can be used interchangeably, but in this paper it is crucial to use the first one.

If X is a real algebraic variety then $X(\mathbb{R})$ denotes the set of real points of X as a topological space and $X(\mathbb{C})$ denotes the set of complex points as a complex space. $X_{\mathbb{C}}$ denotes the corresponding complex variety (same equations as for X but we pretend to be over $\mathbb{C}$).

For all practical purposes we can identify X with the pair ($X(\mathbb{C})$, complex conjugation) (cf. [Silhol89, Sec.I.1]).

A property of X always refers to the variety X. Thus, for instance, X is smooth iff it is smooth at all complex points, not just at its real points. I use the adjective *"geometrically"* to denote properties of the complex variety $X_{\mathbb{C}}$.

1.12 (Piecewise linear 3–manifolds).

In this paper I usually work with piecewise linear manifolds (see [Rourke-Sanderson82] for an introduction). Everey real algebraic variety carries a natural PL structure (cf. [BCR87, Sec.9.2]).

In dimension 3 every compact topological 3–manifold carries a unique PL–manifold structure (cf. [Moise77, Sec. 36]) and a PL–structure behaves very much like a differentiable structure. For instance, let M^3 be a PL 3–manifold, N a compact PL–manifold of dimension 1 or 2 and $g : N \hookrightarrow M$ a PL–embedding. Then a suitable open neighborhood of $g(N)$ is PL–homeomorphic to a real vector bundle over N (cf. [Moise77, Secs. 24 and 26]). (The technical definition of these is given by the notion of *regular neighborhoods*, see [Rourke-Sanderson82, Chap.3].) If $f : M \to N$ is a PL–map and $X \subset N$ a compact subcomplex then there is a regular neigborhood $X \subset U \subset N$ such that $f^{-1}(U)$ is a regular neigborhood of $f^{-1}(X) \subset M$ (cf. [Rourke-Sanderson82, 2.14]).

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2. The topology of rational curve fibrations over $\mathbb{R}$

The aim of this section is to provide an argument describing the topology of rational curve fibrations over $\mathbb{R}$ under rather general assumptions. A weakness of the result is that it does not provide any connection between the algebraic structure of the singular fibers and the topology over $\mathbb{R}$. The precise relationship is worked out in subsequent sections.

Theorem 2.1. *Let X be a projective 3–fold over $\mathbb{R}$ with isolated singularities only and $f : X \to S$ a rational curve fibration. Assume that $\overline{X(\mathbb{R})}$ is an orientable 3–manifold.*

Let $M \subset \overline{X(\mathbb{R})}$ be any connected component. Then there are disjoint solid tori $S^1 \times D^2 \sim T_i \subset M$ such that $M \setminus \cup_i \operatorname{Int} T_i$ is an S^1-bundle over a compact surface with boundary.

We need the description of rational curve fibrations over curves. These are discussed in [Silhol89, Secs. II.6, VI.3] and [Kollár97a, 1.8].

Lemma 2.2. *Let Y be a smooth projective surface over $\mathbb{R}$ and $f : Y \to A$ a rational curve fibration. Every point $0 \in A(\mathbb{R})$ has a neighborhood $0 \in U \subset A(\mathbb{C})$ such that $f^{-1}(U) \to U$ is real analytically equivalent to one of the following normal forms: (In all 4 cases f is the second projection and D is the unit disc in $\mathbb{A}^1_s$.)*

1. *($\mathbb{P}^1$-bundle)* $(x^2+y^2-z^2) \subset \mathbb{P}^2_{(x:y:z)} \times D$,
2. *(empty fibers)* $(x^2+y^2+z^2) \subset \mathbb{P}^2_{(x:y:z)} \times D$,
3. *(collapsed end)* $(x^2+y^2+sz^2) \subset \mathbb{P}^2_{(x:y:z)} \times D$,
4. *(blown up cases) obtained from one of the above by repeatedly blowing up real points and conjugate pairs of complex points.* □

2.3 (Proof of (2.1)).

We may assume that X and S are normal.

Let $A \subset S$ be a general hyperplane section and set $Y = f^{-1}(A)$. $Y(\mathbb{R})$ is an orientable 2–manifold. Since $Y(\mathbb{R})$ is orientable, we can only blow up conjugate pairs of complex points in (2.2.4) (cf. [Silhol89, Sec.II.6], [Kollár97a, 2.2]). Thus $Y(\mathbb{R}) \to A(\mathbb{R})$ is locally (on $A(\mathbb{R})$) homeomorphic to one of three normal forms:

1. S^1-bundle
2. empty fibers
3. $(x^2+y^2+sz^2) \subset \mathbb{RP}^2_{(x:y:z)} \times D$, with projecion to $\mathbb{A}^1_s$.

Let $B \subset S$ be the locus of singular fibers and set $U = f(X(\mathbb{R}))$. $U \subset S(\mathbb{R})$ is a semi algebraic subset. Aside from a finite set $T \subset S(\mathbb{R})$, we have the following description:

1. $\partial U = B(\mathbb{R})$,
2. $f : X(\mathbb{R}) \to U$ is an S^1-bundle over $\operatorname{Int} U$,
3. each fiber over ∂U is a single point.

We still have to determine the local structure of $X(\mathbb{R}) \to U$ near the points in T. Let $P \in T$ be a point. $f^{-1}(P)(\mathbb{R})$ is a real algebraic curve, hence a union of some copies of S^1. The normalization $\overline{X(\mathbb{R})} \to X(\mathbb{R})$ can break apart some of the circles, thus the preimage of $f^{-1}(P)(\mathbb{R})$ in $\overline{X(\mathbb{R})}$ is a 1–complex; let $D_i(P)$ be its connected components.

The boundary of a regular neighborhood of $D_i(P)$ in $\overline{X(\mathbb{R})}$ is connected, hence the map $\overline{X(\mathbb{R})} \to U$ factors through the normalization $\bar{U} \to U$. We denote it by $\bar{f} : \overline{X(\mathbb{R})} \to \bar{U}$. We are in the following situation:

1. $\overline{X(\mathbb{R})}$ is a PL 3-manifold and $\bar{U}$ is a PL 2–manifold with boundary.
2. There is a finite set $T \subset U$ such that
   (a) $f^{-1}(P)$ is 1–complex for $P \in T$,
   (b) f is an S^1-bundle over $\operatorname{Int} U \setminus T$, and
   (c) $f^{-1}(\partial U \setminus T) \to (\partial U \setminus T)$ is a PL homeomorphism.

Pick $P \in T$. If $h^1(D_i(P), \mathbb{Q}) = g_i$, then the boundary of a regular neighborhood of $D_i(P)$ in $\overline{X(\mathbb{R})}$ is an orientable surface of genus g_i (cf. [Hempel76, 2.4]). On the other hand, this boundary is an S^1-bundle over S^1 if $P \in \operatorname{Int} U$ and S^2 if $P \in \partial U$ by (1.12). Thus $\bar{f}^{-1}(P)$ is collapsible to an S^1 in the first case and to a point in the second case. We can do these collapsings in $\overline{X(\mathbb{R})}$. This creates a 3–manifold $\overline{X(\mathbb{R})}^*$ which is PL homeomorphic to $\overline{X(\mathbb{R})}$ (cf. [Rourke-Sanderson82, 3.27]). We replace $\overline{X(\mathbb{R})}$ with a connected component M of $\overline{X(\mathbb{R})}^*$ to obtain $g : M \to V$, satisfying the following conditions:

1. M is a compact PL 3-manifold and V a PL 2–manifold with boundary.
2. There is a finite set $T \subset \operatorname{Int} V$ such that
   (a) $f^{-1}(P) \sim S^1$ for $P \in T$,
   (b) f is an S^1-bundle over $\operatorname{Int} U \setminus T$, and
   (c) $f^{-1}(\partial V) \to \partial V$ is a PL homeomorphism.

Let $F^0 \subset V$ be obtained by removing small open discs U_P around each $P \in T$ and a small annulus U_C along each boundary component $C \subset \partial V$. Then $M^0 := f^{-1}(F^0)$ is an S^1-bundle over F^0. Moreover, each $f^{-1}(\bar{U}_P)$ is a regular neighborhood of $f^{-1}(P) \sim S^1$ and each $f^{-1}(\bar{U}_C)$ is a regular neighborhood of $f^{-1}(C) \sim S^1$. By the orientability of M, the $f^{-1}(\bar{U}_P)$ and the $f^{-1}(\bar{U}_C)$ are solid tori. □

3. Surgery on circle bundles

The aim of this section is to give a more detailed description of the 3–manifolds that appear in the study of real conic bundles. For the purposes of (2.1) we need only the orientable case, but later on we encounter nonorientable examples as well.

Definition 3.1 (Lens spaces). For relatively prime $0 < q < p$ consider the action $(x, y) \mapsto (e^{2\pi i/p}x, e^{2\pi iq/p}y)$ on the unit sphere $S^3 \sim (|x^2| + |y^2| = 1) \subset \mathbb{C}^2$. The quotient is a 3–manifold, called the *lens space* $L_{p,q}$.

Another way to obtain lens spaces is to glue two solid tori together. The result is a lens space, S^3 or $S^1 \times S^2$. Sometimes one writes $L_{1,0} = S^3$ and $L_{0,1} = S^1 \times S^2$. See, for instance, [Hempel76, p.20].

3.2. [Torus and solid tous]

As a general reference, see [Rolfsen76, Chap.2].

Let S^1 be the unit circle $(|u| = 1) \subset \mathbb{C}_u$ and D^2 the unit disc $(|z| \le 1) \subset \mathbb{C}_z$. $S^1 \times D^2$ is called the *solid torus.* Its boundary

$$\partial(S^1 \times D^2) = (|u| = 1) \times (|z| = 1) \sim S^1 \times S^1$$

is a *torus.* Up to isotopy, the homeomorphisms of a torus are given by

$$(u, z) \mapsto (u^a z^b, u^c z^d), \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z}).$$

Up to isotopy, the homeomorphisms of a solid torus are given by

$$(u,z)\mapsto(u^{\pm1},u^m z)\quad\text{or}\quad(u,z)\mapsto(u^{\pm1},u^m\bar z),\quad\text{where}\quad m\in\mathbb{Z}.$$

Let $g: S^1\times S^1\to S^1$ be an S^1-bundle. Up to isotopy, it can be written as

$$g_{c,d}:(u,z)\mapsto u^c z^d,\quad\text{where}\quad (c,d)=1.$$

Given a solid torus $S^1\times D^2$, let $g_{c,d}:\partial(S^1\times D^2)\to S^1$ be an S^1-bundle. This can be extended to a map

$$G_{c,d}: S^1\times D^2\to D^2\quad\text{by}\quad (u,z)\mapsto\begin{cases} u^c z^d & \text{if}\ \ d>0,\\ u^c\bar z^{(-d)} & \text{if}\ \ d<0,\\ u^c|z| & \text{if}\ \ d=0.\end{cases}$$

The fiber $G_{c,d}^{-1}(1)$ is parametrized by

$$\phi: S^1\to S^1\times D^2\quad\text{given as}\quad t\mapsto(t^d,t^{-c}).$$

(ϕ is injective since $(d,c)=1$.) Composing ϕ with the first projection $S^1\times D^2\to S^1$ we obtain a map of degree m from S^1 to S^1. Thus the fiber of $G_{c,d}$ over the origin has multiplicity $|d|$ (1.6).

Applying a homeomorphism of the solid torus, we can get every $G_{c,d}$ to one of the following normal forms:

1. For a pair of integers c,d satisfying $0\le c<d$ and $(c,d)=1$, define

$$f_{c,d}: S^1\times D^2\to D^2\quad\text{by}\quad f_{c,d}(u,z)=u^c z^d.$$

   $f_{c,d}$ restricts to a fiber bundle $S^1\times(D^2\setminus\{0\})\to D^2\setminus\{0\}$. The fiber of $f_{c,d}$ over the origin is still S^1, but $f_{c,d}^{-1}(0)$ has multiplicity d.
2. $f_{1,0}(u,z)=u\cdot|z|$. In this case the fibers $f_{1,0}^{-1}(t)$ for $t\ne0$ are null homotopic in the solid torus.

   Instead of $f_{1,0}$, I prefer to use the map

$$f_0:(u,z)\mapsto u\frac{1+|z|}{2}\in\{t|1/2\le|t|\le1\}\subset\mathbb{C}_t.$$

   The image is a closed annulus $A_{1/2,1}$ with boundary circles $C_{1/2}$ and C_1. f_0 is an S^1-bundle over the annulus $A_{1/2,1}\setminus C_{1/2}$ and $f_0^{-1}(C_{1/2})\to C_{1/2}$ is a homeomorphism.

   $f_{1,0}$ is obtained by composing f_0 with $t\mapsto 2t(1-\frac{1}{2|t|})$ which contracts the circle $C_{1/2}$ to a point.

3.3. [Klein bottle and solid Klein bottle]

On the solid torus above, consider the map $(u,z)\mapsto(u^{-1},\bar z)$. This is a fixed point free and orientation reversing ivolution. The quotient space is called a *solid Klein bottle*. Its boundary is a *Klein bottle*.

Among the homeomorphisms of the torus, only the maps $(u,z) \mapsto (u^{\pm 1}, z^{\pm 1})$ descend to the Klein bottle, and these are all the homeomorphisms of the Klein bottle modulo isotopy. All of these extend to the solid Klein bottle as

$$(u,z) \mapsto (u^{\pm 1}, z) \quad \text{or} \quad (u,z) \mapsto (u^{\pm 1}, \bar{z}).$$

Thus there is a unique way to glue a solid Klein bottle to a manifold along a boundary component which is a Klein bottle.

Given an S^1-bundle structure on the boundary Klein bottle, the second normal form in (3.2) gives the extension

$$f_0 : (u,z) \mapsto u^2 \frac{1+|z|}{2} \in \{t \mid 1/2 \le |t| \le 1\} \subset \mathbb{C}_t.$$

Definition 3.4 (Seifert fiber spaces).

A *Seifert fibered* 3–manifold is a proper morphism of a 3–manifold to a surface $f : M \to S$ such that every point $s \in S$ has a neighborhood $s \in D_s \subset S$ such that the pair $f^{-1}(\bar{D}_s) \to \bar{D}_s$ is fiber preserving homeomorphic to one of the normal forms $f_{c,d}$ defined in (3.2.1) for some c,d satisfying $0 \le c < d$ and $(c,d) = 1$.

There are other ways of describing the local models. Consider the map

$$h_{c,d} : [0,1] \times D^2 \to S^1 \times D^2 \quad \text{given by} \quad (t,z) \mapsto (e^{2\pi i t}, z e^{-2\pi i c t/d}).$$

Under this map we can view $S^1 \times D^2$ as $[0,1] \times D^2$ with the two ends $\{0\} \times D^2$ and $\{1\} \times D^2$ identified by clockwise rotation by $2\pi c/d$. Moreover, each $[0,1] \times \{z\}$ maps to a single fiber of $f_{c,d}$. This shows that the above definition is equivalent with the usual one (cf. [Scott83]).

Finally, we can consider the action of $\mathbb{Z}_d$ on $S^1 \times D^2$ which is rotation by $2\pi c/d$ on S^1 and by $2\pi/d$ on D^2. The quotient $(S^1 \times D^2)/\mathbb{Z}_d \to D^2/\mathbb{Z}_d \sim D^2$ is Seifert fibered with multiplicity d.

(In several papers, one nonorientable local model is also allowed. For $p = 2$ we can act on D^2 by reflection. Then $D^2/\mathbb{Z}_2$ is a surface with boundary and f is not a fiber bundle near the boundary. This case does not come up for us. An algebraic model of this situation is given by

$$X := (z^2 + s(x^2+y^2) = 0) \subset \mathbb{P}^2_{x,y,z} \times \mathbb{A}^2_{s,t},$$

where f is the second projection. Notice that X is singular along the curves $(z = s = x \pm \sqrt{-1}y = 0)$. This is a good example to show how complex singularities influence the behaviour of the real points.)

Theorem 3.5. *Let F^0 be a compact PL surface with boundary and $F \supset F^0$ the surface obtained from F^0 by attaching discs to all boundary* *components. Let $M^0 \to F^0$ be an S^1-bundle (so its boundary components are tori and Klein bottles). Let M be any 3–manifold obtained from M^0 by attaching solid tori or Klein bottles to all boundary components. Then one of the following holds:*

1. *M is Seifert fibered over F, or*
2. *M is the connected sum of lens spaces, $S^1 \times S^2$ and $S^1 \tilde{\times} S^2$.*

Remark 3.6. In the orientable case, (3.5) is equivalent to the following statement.

If M is obtained from an S^1-bundle by surgery along fibers then either M is Seifert fibered or M is the connected sum of lens spaces and $S^1 \times S^2$.

Complement 3.7. *Notation as in (3.5). The gluing data determine the structure of M as follows.*

Let $A_1, \dots, A_r$ be the boundary components of M^0 homeomorphic to a torus and $T_1, \dots, T_r$ the corresponding solid tori attached by $\phi_i : \partial T_i \sim A_i$. On each T_i choose coordiantes (u_i, z_i) as in (3.2). The S^1-bundle structure of M^0 induces S^1-bundles $p_i : A_i \to S^1$ for every i. We may assume that $p_i \circ \phi_i = u_i^{c_i} z_i^{d_i}$, where c_i, d_i satisfy the usual conditions $0 \leq c_i < d_i$ and $(c_i, d_i) = 1$ or $c_i = 1, d_i = 0$.

Let $B_1, \dots, B_s$ be the Klein bottle boundary components.

1. *If $s = 0$ and $d_i > 0$ for every i then M is Seifert fibered over F with mulitple fibers of multiplicity $d_1, \dots, d_r$.*
2. *If $s > 0$ or $d_i = 0$ for some i, then*
   $M \sim L_{d_1,c_1} \# \cdots \# L_{d_r,c_r} \# a(S^1 \times S^2) \# b(S^1 \tilde{\times} S^2)$
   for some $a, b \geq 0$.

Proof. The local models of attaching a solid torus or solid Klein bottle are described in (3.2) and (3.3). We obtain that there are

1. a surface $F^0 \subset F^1 \subset F$, obtained from F^0 by attaching discs or annuli, and
2. a proper map $g : M \to F^1$ such that $g^{-1}(\operatorname{Int} F^1) \to \operatorname{Int} F^1$ is Seifert fibered and $g^{-1}(\partial F^1) \to \partial F^1$ is a homeomorphism.

If $\partial F^1 = \emptyset$ then we are done.

Assume next that $\partial F^1 \neq \emptyset$ and let $L \subset F^1$ be a simple path starting and ending in ∂V. Then $g^{-1}(L) \sim S^2$, hence cutting along $g^{-1}(L)$ corresponds to connected sum decomposition (if L separates F^1) or to removing a 1–handle (if L does not separate F^1), cf. [Hempel76, 3.8]. After repeated cuts we are reduced to the situation when each of the pieces $\cup V_i = F^1$ is a disc containing at most 1 multiple fiber and M is the connected sum of the corresponding manifolds M_i (possibly with further 1–handles attached). $g_i : M_i \to V_i$ has a unique multiple fiber in

$\operatorname{Int} V_i$ at a point P_i and $g^{-1}(\partial V_i) \sim \partial V_i \sim S^1$. A regular neighborhood of $g^{-1}(P_i)$ is a solid torus and so is a regular neighborhood of $g^{-1}(\partial V_i)$. Thus M_i is obtained by gluing two solid tori together, hence M_i is a lens space, S^3 or $S^1 \times S^2$, depending on d_i, c_i (3.1). □

4. General results on rational curve fibrations over $\mathbb{C}$

Let X be a 3–fold with terminal singularities and $f : X \to S$ a rational curve fibration such that $-K_X$ is f-ample. A considerable effort has been spent on trying to understand the local structure of f (see [Prokhorov95, Prokhorov96, Prokhorov97]), but so far we do not have a complete description. The examples of Prokhorov suggest that there are many cases. The aim of this section is to prove a technical result on rational curve fibrations (4.2). The proof is quite easy but it uses some of the machinery of extremal neighborhoods developed in [Mori88]. This result is used only through (4.3), so readers more interested in the topological aspects can skip the rest of the secion.

4.1 (Terminal singularities)**.**

For a precise definition of terminal singularities the reader is referred to [Reid85, Kollár-Mori98]. We need only the following consequence of their classification:

Every 3–dimensional terminal singularity is either

1. an isolated hypersurface singularity $(0 \in X) \subset \mathbb{C}^4$, (these are the cases when K_X is Cartier), or
2. a quotient of an isolated hypersurface singularity by a $\mathbb{Z}_m$-action with an isolated fixed point.

The value of m is called the *index* of the singularity.

The classification of 3–dimensional terminal singularities over $\mathbb{R}$ is given in [Kollár97b]. We do not need it but in some cases I use their local topological description developed in [Kollár97b, Secs. 4–5].

Proposition 4.2. *Let X be a 3–fold with terminal singularities over $\mathbb{C}$ and $f : X \to S$ a rational curve fibration such that $-K_X$ is f-ample. Let $0 \in S$ be a closed point. Then one of the following alternatives holds:*

1. *K_X is Cartier along $f^{-1}(0)$, or*
2. *K_X is not Cartier at all singular points of* $\operatorname{red} f^{-1}(0)$.

We are mainly interested in the following consequence for rational curve fibrations over $\mathbb{R}$:

Corollary 4.3. *Let X be a real projective 3-fold with terminal singularities such that K_X is Cartier along $X(\mathbb{R})$. Let $f : X \to S$ be a* *rational curve fibration over $\mathbb{R}$ such that $-K_X$ is f-ample. For every point $s \in S(\mathbb{R})$, one of the following holds:*

1. *S is smooth at s and f is a conic bundle over a neighborhood of s, or*
2. $\operatorname{red} f^{-1}(s)$ *is an irreducible, smooth and rational curve and K_X is not Cartier precisely at a conjugate pair of complex points of* $\operatorname{red} f^{-1}(s)$.

Proof. If K_X is Cartier along $f^{-1}(s)$ then S is smooth at 0 and f is a conic bundle over a neighborhood of s by [Cutkosky88, Thm7]. This gives (4.3.1).

If $\operatorname{red} f^{-1}(s)$ is an irreducible, smooth and rational curve then X can have at most 3 singular points along $\operatorname{red} f^{-1}(s)$ by [Mori88, 0.4.13.1] and [Prokhorov96, 2.7]. Thus either K_X is Cartier along $\operatorname{red} f^{-1}(s)$ or there is a unique conjugate pair of complex points of $\operatorname{red} f^{-1}(s)$ where K_X is not Cartier. Thus we have (4.3.2).

In all other cases, $\operatorname{red} f^{-1}(s)_{\mathbb{C}}$ is a tree of smooth rational curves by (4.4) and K_X is not Cartier at all singular points of $\operatorname{red} f^{-1}(s)_{\mathbb{C}}$ by (4.2). $\operatorname{red} f^{-1}(s)_{\mathbb{C}}$ can not have a real singular point by assumption, hence it contains a real irreducible component $D \subset \operatorname{red} f^{-1}(s)_{\mathbb{C}}$. As in (4.4), D can be contracted in an extremal contraction. By [Kollár97c, 7.2] this is only possible if K_X is Cartier along D, a contradiction. □

4.4. [Rational curve fibrations and extremal neighborhoods]

Let X be a 3–fold with terminal singularities and $g : X \to Y$ a birational morphism. Assume that $-K_X$ is g-ample and $g^{-1}(y)$ is a curve for some $y \in Y$. The germ of X along $g^{-1}(y)$ is called an *extremal curve neighborhood* (cf. [Kollár-Mori92, p.549]). Extremal curve neighborhoods have been studied in detail in [Mori88, Kollár-Mori92]. A key point in their study are the vanishings $R^1 g_* \mathcal{O}_X = 0$ and $R^1 g_* \omega_X = 0$. The first of these still holds for rational curve fibrations by the general Kodaira vanishing theorem (cf. [Kollár-Mori98, Sec.2.5]) but not the second.

By an easy argument (cf. [Mori88, 1.2.1]), this implies the following: If $\mathcal{O}_X \twoheadrightarrow Q$ is any quotient supported on $f^{-1}(0)$ then $H^1(X, Q) = 0$. In particular, $H^1(\mathcal{O}_{\operatorname{red} f^{-1}(0)}) = 0$. Any such curve is a tree of smooth rational curves (cf. [Mori88, 1.3]).

For many results in [Mori88], these are the key vanishings needed. In practice this means that many of the results for extremal curve neighborhoods still hold for rational curve fibrations, maybe in slightly modified form. This approach has been carried out by [Prokhorov96].

In some cases one can directly apply the results on extremal curve neighborhoods to rational curve fibrations through the following trick:

Pick $0 \in S$ and let $\cup_i C_i = C = \operatorname{red} f^{-1}(0)$ be the irreducible components. If we replace S by a small analytic neighborhood of 0 then there are divisors $H_i \subset X$ such that H_i intersects C only at a point of C_i. Let $D \subsetneq C$ be any closed subcurve and H_D the sum of those H_i which are disjoint from D. By the relative base point free theorem (cf. [Kollár-Mori98, Sec.3.6]), a mutiple of H gives a proper birational morphism $p_D : X \to Y$. p_D contracts D to a point (p_D may also contract other curves in other fibers of f). $-K_X$ is p_D-ample, thus D determines an extremal curve neighborhood in X. In particular, all the results of [Kollár-Mori92] describing extremal neighborhoods apply to D. This of course does not work if $D = C$.

4.5. [Adjunction formula and $i_P(1)$]

If Y is a smooth 3–fold and $C \subset Y$ a smooth curve with ideal sheaf I_C, then the adjunction formula says that

$$2g(C) - 2 = (C \cdot K_Y) - \deg_C(I_C/I_C^2).$$

If Y is singular but C smooth, a similar formula was developed in [Mori88], which also involves a correction term depending on the singularities of X along C. The two corrections coming from $(C \cdot K_Y)$ and from $\deg_C(I_C/I_C^2)$ are mixed together into a single number $i_{P,C}(1)$ (denoted by $i_P(1)$ in [Mori88] and by i_P in [Prokhorov96]). The definition is not crucial for us, and we need only two properties:

1. $i_{P,C}(1)$ is a nonnegative integer which is zero iff P is a smooth point of Y [Mori88, 2.15],
2. If $C \cong \mathbb{P}^1$ gives an extremal curve neighborhood then $\deg_C I_C/I_C^{(2)} = 1 - \sum_{P \in C} i_{P,C}(1)$ [Mori88, 2.3.2].

4.6 (Proof of (4.2))**.**

Write $f^{-1}(0) = \cup_i C_i$. $\cup_i C_i$ is a tree of smooth rational curves, hence all the singular points of $\operatorname{red} f^{-1}(0)$ are at intersection points of the irreducible components. Let $P \in C_i \cap C_j$ be such a point and set $D = C_i \cup C_j$.

If $D \neq \operatorname{red} f^{-1}(0)$ then D determines an extremal neighborhood, so K_X is not Cartier at P by [Mori88, 1.15]. Thus (4.2.2) holds if $f^{-1}(0)$ has at least 3 irreducible components. (4.2.2) holds vacuously if $f^{-1}(0)$ has only one irreducible component. We are left with the case when $f^{-1}(0) = C_1 \cup C_2$ has two irreducible components. They have a unique intersection point Q. We are done if K_X is not Cartier at Q.

Assume next that K_X is Cartier along C_1. Let $p : X \to Y$ be the contraction of C_1. By [Cutkosky88, Thm.4], p is a divisorial contraction, Y again has terminal singularities and $K_X = p^*K_Y + E$ where E is a p-exceptional Cartier divisor. Let $g : Y \to S$ be the induced rational curve fibration. A general fiber of g is algebraically equivalent to a multiple $n\cdot p(C_2)$ of $p(C_2)$. If $n=1$ then g is a $\mathbb{P}^1$-bundle by [Kollár96, II.2.28], thus K_X is Cartier along C_2 as well and we are in case (4.2.1). Otherwise $(K_Y\cdot p(C_2))\geq -2/n\geq -1$ and so

$$(K_X\cdot C_2)\geq (K_Y\cdot p(C_2))+(E\cdot C_2)\geq -1+1=0,$$

a contradiction.

We are left with the case when K_X is Cartier at Q but not Cartier along some point on either C_i. Let $I_i\subset \mathcal{O}_X$ be the ideal sheaf of C_i. Consider the exact sequence

$$\begin{array}{rlll} 0 & \to\ H^0(\mathcal{O}_X/(I_1^{(2)}\cap I_2^{(2)})) & \to & H^0(\mathcal{O}_X/I_1^{(2)})+H^0(\mathcal{O}_X/I_2^{(2)}) \\ & \to\ H^0(\mathcal{O}_X/(I_1^{(2)}+I_2^{(2)})) & \to & H^1(\mathcal{O}_X/(I_1^{(2)}\cap I_2^{(2)})). \end{array}$$

$H^1(\mathcal{O}_X/(I_1^{(2)}\cap I_2^{(2)}))=0=H^1(\mathcal{O}_X/I_i^{(2)})$ as we noted in (4.4). By (4.5.2) and Riemann–Roch on C_i,

$$h^0(\mathcal{O}_X/I_i^{(2)})=\chi(\mathcal{O}_X/I_i^{(2)})=3+\deg_{C_i}(I_i/I_i^{(2)})=4-\sum_{P\in C_i} i_{P,C_i}(1).$$

$\mathcal{O}_X/(I_1^{(2)}+I_2^{(2)})$ is supported at Q. In order to compute it, we consider two cases:

1. (X is smooth at Q) In suitable local coordinates $I_1=(x,y)$ and $I_2=(y,z)$. Thus $\mathcal{O}_X/(I_1^{(2)}+I_2^{(2)})$ is spanned by $1,x,y,z,xz$ and $h^0(\mathcal{O}_X/(I_1^{(2)}+I_2^{(2)}))=5$.
2. (X is singular at Q) In suitable local coordinates $I_1=(x,y,z)$ and $I_2=(y,z,t)$. Thus $\mathcal{O}_X/(I_1^{(2)}+I_2^{(2)})$ is spanned by $1,x,y,z,t,xt$ and if the equation of X involves xt, then xt is zero in the quotient. Thus $h^0(\mathcal{O}_X/(I_1^{(2)}+I_2^{(2)}))\geq 5$.

Comparing these with the above exact sequence we conclude that

$$8-\sum_{P\in C_1} i_{P,C_i}(1)-\sum_{P\in C_2} i_{P,C_i}(1)\geq 1+5.$$

If X is not smooth at $P\in C_i$ then $i_{P,C_i}(1)\geq 1$ by (4.5.1). Since K_X is not Cartier along either C_i, we conclude that there is a unique point $P_i\in C_i$ such that $i_{P_i,C_i}(1)=1$.

Each C_i defines an extremal curve neighborhood in X, and we are in cases (2.2.1) or (2.2.1') in the list [Kollár-Mori92, 2.2]. (In the cases (2.2.2) or (2.2.2') the singular point has $i_{P,C}(1)=2$ by [Mori88, 6.5] and [Kollár-Mori92, p.549]. In the cases (2.2.3), (2.2.3') and (2.2.4) there are at least 2 singular points.) From this we conclude that $-K_X$ has a good member along each C_i. That is, there are divisors $D_i\subset X$ such that D_i intersects C_1+C_2 only at $P_i\in C_i$ and $D_i\cdot C_i=-K_X\cdot C_i$. Thus D_1+D_2 and $-K_X$ are numerically equivalent in a neighborhood of $C_1 + C_2$. In particular, each D_i intersects the general fiber of f in a single point and $f : D_i \to S$ is an isomorphism near 0.

If S is smooth at 0 then D_i is smooth at P_i. We obtain a contradiction since then X is smooth at P_i. (If $0 \in X$ is a 3–fold terminal singularity and $0 \in H$ is a smooth member of $|-K_X|$ then X is smooth. This is quite easy to prove but the only reference I know is the much too general [Kollár-Mori92, 3.5.1].) Otherwise let $\tilde{S} \to S$ be the universal cover of $S \setminus \{0\}$ and $\pi : \tilde{X} \to X$ the corresponding cover. π ramifies only over P_1 and P_2, thus $\pi^{-1}(C_i)$ is a union of $\deg \pi$ curves intersecting at a single point over P_i. Then $\pi^{-1}(C_1 \cup C_2)$ is not a tree, a contradiction again. $\square$

5. Irreducible fibers

As (4.3) shows, we should study the topology of conic bundles and of rational curve fibrations with geometrically irreducible central fibers. In this section we prove that the geometrically irreducible cases are locally quotients of conic bundles. This result is developed into a complete classification in the next section.

The proof relies on computing the algebraic fundamental group of $X \setminus \operatorname{Sing} X$ in the neighborhood of a fiber. First I recall the definition and basic properties of the algebraic fundamental group of a scheme over $\mathbb{R}$.

5.1 (The algebraic fundamental group of a real variety)**.**

As a general reference, see [SGA1].

For a complex variety $Y_{\mathbb{C}}$, the algebraic fundamental group $\pi_1^{alg}(Y_{\mathbb{C}})$ is the profinite completion of $\pi_1(Y(\mathbb{C}))$ (see, for instance, [SGA1, XII.5.2]). If $Y_{\mathbb{R}}$ is a real variety, then $\pi_1^{alg}(Y_{\mathbb{R}})$ is related to $\pi_1(Y(\mathbb{C}))$ as follows. $\operatorname{Spec}_{\mathbb{R}} \mathbb{C} \to \operatorname{Spec}_{\mathbb{R}} \mathbb{R}$ is a degree 2 étale morphism, and correspondingly we obtain a degree 2 étale cover $Y_{\mathbb{R}} \times_{\mathbb{R}} \operatorname{Spec}_{\mathbb{R}} \mathbb{C} \to \operatorname{Spec} Y_{\mathbb{R}}$. The left hand side is essentially $Y_{\mathbb{C}}$, but viewed as a real variety. It has 2 geometrically connected components (assuming that Y itself is geometrically connected) and the two components are interchanged by complex conjugation. In particular, it has no real points. Any further étale cover is obtained as an étale cover of one of the components, out of which we create the conjugate cover over the other component.

Hence we get an exact sequence (cf. [SGA1, IX.6.1])

$$1 \to \hat{\pi}_1(Y(\mathbb{C})) \to \pi_1^{alg}(Y_{\mathbb{R}}) \to \mathbb{Z}_2 \to 0.$$

We see that if an étale cover $\bar{Y}_{\mathbb{C}} \to Y_{\mathbb{C}}$ corresponds to a subgroup $H \subset \hat{\pi}_1(Y(\mathbb{C}))$, then the choice of a real structure on $\bar{Y}_{\mathbb{C}}$ is equivalent to a splitting of the above sequence modulo H. Such splittings need not exist and they need not be unique.

We need the computation of the real π_1^{alg} in one case:

5.2. [Étale covers of $\mathbb{R}[x,y]/(x^2+y^2-1)$]
Adjoining $\sqrt{-1}$ we obtain $\mathbb{C}[x,y]/(x^2+y^2-1)$, which is isomorphic to $\mathbb{C}^*$. $\pi^{alg}(\mathbb{C}^*)=\hat{\mathbb{Z}}$, thus the algebraic fundamental group of $\mathbb{R}[x,y]/(x^2+y^2-1)$ is an extension of $\hat{\mathbb{Z}}$ by $\mathbb{Z}_2$. I claim that this extension is trivial, that is

$$\pi_1^{alg}(\mathbb{R}[x,y]/(x^2+y^2-1))\cong\hat{\mathbb{Z}}+\mathbb{Z}_2.$$

To see this, we construct the tower of Galois extensions corresponding to the finite quotients of $\hat{\mathbb{Z}}$.

Rotating by $2\pi/n$ acts on $\mathbb{R}[s_n,t_n]/(s_n^2+t_n^2-1)$ and the ring of invariants is generated by x_n and y_n as in (5.3) with a single relation $x_n^2+y_n^2=1$. Hence

$$\mathbb{R}[x,y]/(x^2+y^2-1)\cong\mathbb{R}[x_n,y_n]/(x_n^2+y_n^2-1)\subset\mathbb{R}[s_n,t_n]/(s_n^2+t_n^2-1)$$

is a degree n Galois extension with Galois group $\mathbb{Z}_n$. The corresponding map between the set of real points is the n-sheeted cover $S^1\to S^1$.

If n is odd then $\hat{\mathbb{Z}}\twoheadrightarrow\mathbb{Z}_n$ and $\mathbb{Z}_2\to\{1\}$ give the unique quotient of order n of $\hat{\mathbb{Z}}+\mathbb{Z}_2$. For $n=2m$ even, there are two other quotients. One corresponds to $\hat{\mathbb{Z}}\twoheadrightarrow\mathbb{Z}_m$ and $\mathbb{Z}_2\to\mathbb{Z}_2$. This gives the extension

$$\mathbb{R}[x_m,y_m]/(x_m^2+y_m^2-1)\subset\mathbb{C}[s_m,t_m]/(s_m^2+t_m^2-1).$$

Finally there is also a quotient $\hat{\mathbb{Z}}\twoheadrightarrow\mathbb{Z}_{2m}$ and $\mathbb{Z}_2\to\mathbb{Z}_2\subset\mathbb{Z}_{2m}$. This corresponds to the Galois extension

$$\mathbb{R}[x_n,y_n]/(x_n^2+y_n^2-1)\subset\mathbb{R}[s_n,t_n]/(-s_n^2-t_n^2-1).$$

Notice that $\operatorname{Spec}\mathbb{R}[s_n,t_n]/(-s_n^2-t_n^2-1)$ has no real points.

By contrast one should observe that the algebraic fundamental group of $\mathbb{R}[x,x^{-1}]$ is the completion of the infinite dihedral group. This is shown by the fact that the natural degree n extension $\mathbb{R}[t^n,t^{-n}]\subset\mathbb{R}[t,t^{-1}]$ is not Galois.

Lemma 5.3. *Consider the $\mathbb{Z}_n$-action on $\mathbb{R}^2_{s,t}$ which is rotation by $2\pi/n$. The ring of polynomial invariants is generated by 3 elements*

$$x_n:=\sum(-1)^j\binom{n}{2j}t^{2j}s^{n-2j},\quad y_n:=\sum(-1)^j\binom{n}{2j+1}t^{2j+1}s^{n-2j-1}$$

and $z:=s^2+t^2$, subject to the single relation

$$x_n^2+y_n^2-z^n=0.$$

Proof. The action is

$$s \mapsto \cos(2\pi/n)s + \sin(2\pi/n)t, \quad t \mapsto -\sin(2\pi/n)s + \cos(2\pi/n)t.$$

In order to understand the ring of invariants, note that under this action

$$s + \sqrt{-1}t \mapsto e^{2\pi\sqrt{-1}/n}(s + \sqrt{-1}t).$$

Thus the invariants are $z := s^2 + t^2$ and the real and imaginary parts of $(s + \sqrt{-1}t)^n$. The latter are x_n and y_n. Then $x_n^2 + y_n^2 = (s^2 + t^2)^n = z^n$. □

5.4 (Trouble with terminology)**.**

In describing the local structure of rational curve fibrations over $\mathbb{C}$, it is natural to use the analytic topology on the base. Thus in effect we deal with triplets $f : X \to S$ where S is a germ of a complex analytic space, X is a germ of a complex analytic space along a compact set $f^{-1}(0)$ and f is a projective morphism. It is natural to do the same in the real analytic case. The only problem is that the resulting objects do not seem to have a standard name. (One could use the terminology "complex analytic space with a real structure".)

Let $Y_{\mathbb{R}}$ be a smooth real algebraic variety. Then $Y(\mathbb{C})^{an}$ is a complex analytic space with a real structure on it. This is not the same as a real analytic space $Y(\mathbb{R})^{an}$. The real analytic space $Y(\mathbb{R})^{an}$ corresponds only to the germ of $Y(\mathbb{C})^{an}$ along its real points. It does not contain enough information to describe $Y(\mathbb{C})^{an}$ completely.

For us it is crucial to keep a neighborhood of the whole fiber, not just a neighborhood of the real points of a fiber. Thus if $f : X \to S$ is a rational curve fibration over $\mathbb{R}$ and $0 \in S(\mathbb{R})$ a point then first replace S with a small neighborhood of $0 \in S^0 \subset S(\mathbb{C})^{an}$ and then replace X by $X^0 := f^{-1}(S^0) \subset X(\mathbb{C})^{an}$.

One can think of these objects as complex analytic spaces with an antiholomorphic involution on them. Thus the only problem is that they do not have a name.

Notation 5.5. Let X be a real algebraic 3–fold with terminal singularities such that K_X is Cartier along $X(\mathbb{R})$. Let $f : X \to S$ be a rational curve fibration and $0 \in S(\mathbb{R})$ a point such that $f^{-1}(0)$ is geometrically irreducible.

By (4.3) we know that either K_X is Cartier along $f^{-1}(0)$ or there is a unique conjugate pair of points of index $m > 1$ along $f^{-1}(0)$ (4.1).

In the proof of the next theorem it is essential to distinguish several structures on the central fiber $f^{-1}(0)$. These are the following:

1. $f^{-1}(0)$, the central fiber as a real algebraic variety.

2. $(f^{-1}(0))(\mathbb{R})$, the topological space of real points of the central fiber.
3. $f^{-1}(0)_{\mathbb{C}}$, the central fiber as a complex algebraic variety.
4. $(f^{-1}(0))(\mathbb{C})$, the complex analytic space of complex points of the central fiber.

There is not much problem in mixing up the last 2 objects, but it is cruicial to keep the distinction between $f^{-1}(0)$ and $f^{-1}(0)_{\mathbb{C}}$ in mind.

Theorem 5.6. *Notation and assumptions as in (5.5). There is a real conic bundle $\tilde{f} : \tilde{X} \to \tilde{S}$ and an $\tilde{f}$-equivariant $\mathbb{Z}_m$-action on $\tilde{X}$ and $\tilde{S}$ such that*

$$(f : X \to S) \quad \textit{is real analytically isomorphic to} \quad (\tilde{f} : \tilde{X} \to \tilde{S})/\mathbb{Z}_m.$$

Moreover, we can choose $p_X : \tilde{X} \to X$ such that the following conditions are satisfied:

1. *The induced morphism*
$$S^1 \sim \left(\tilde{f}^{-1}(0)\right)(\mathbb{R}) \to \left(f^{-1}(0)\right)(\mathbb{R}) \sim S^1$$
is a degree m cover.
2. *If $s \in S(\mathbb{R}) \setminus \{0\}$, then $p^{-1}(f^{-1}(s))$ is either isomorphic to m copies of $f^{-1}(s)$ or $(p^{-1}(f^{-1}(s)))(\mathbb{R}) = \emptyset$ (the latter can happen only for m even).*

Proof. Replace S with a small analytic neighborhood of 0. Let $P, \bar{P} \in (f^{-1}(0))(\mathbb{C})$ be the two points of index m.

$$\left(f^{-1}(0)\right)(\mathbb{C}) \setminus \{P, \bar{P}\} \cong \mathbb{C}^* \quad \text{thus} \quad \pi_1(\left(f^{-1}(0)\right)(\mathbb{C}) \setminus \{P, \bar{P}\}) \cong \mathbb{Z}.$$

By [Mori88, 0.4.13.3], the natural map

$$\pi_1(\left(f^{-1}(0)\right)(\mathbb{C}) \setminus \{P, \bar{P}\}) \to \pi_1(X(\mathbb{C}) \setminus \{P, \bar{P}\})$$

is surjective and its kernel is $m\mathbb{Z}$. Thus $\pi_1(X(\mathbb{C}) \setminus \{P, \bar{P}\}) \cong \mathbb{Z}_m$. Let $\tilde{X}_{\mathbb{C}} \setminus \{Q, \bar{Q}\} \to X_{\mathbb{C}}$ denote the universal cover of $X_{\mathbb{C}} \setminus \{P, \bar{P}\}$; this can be extended uniquely to a normal scheme $\tilde{X}_{\mathbb{C}}$ which admits a finite morphism $\tilde{X}_{\mathbb{C}} \to X_{\mathbb{C}}$. $\tilde{f}_{\mathbb{C}}$ and $\tilde{S}_{\mathbb{C}}$ are obtained from the Stein factorization of $\tilde{X}_{\mathbb{C}} \to S_{\mathbb{C}}$. $\tilde{X}_{\mathbb{C}}$ has only index 1 terminal singularities and so $\tilde{f}_{\mathbb{C}}$ is a conic bundle by (4.3). All that remains to do is to find a real algebraic variety $\tilde{X}$ such that $\tilde{X}_{\mathbb{C}}$ is the complexification of $\tilde{X}$.

By (5.1), we have the following exact sequences, the second obtained by taking quotient of the first by $m\hat{\mathbb{Z}}$:

3. $0 \to \hat{\mathbb{Z}} \to \pi_1^{alg}(f^{-1}(0) \setminus \{P, \bar{P}\}) \to \mathbb{Z}_2 \to 0.$
4. $0 \to \mathbb{Z}_m \to \pi_1^{alg}(X \setminus \{P, \bar{P}\}) \to \mathbb{Z}_2 \to 0.$

By (5.2) the above sequence (3) splits, and so the same holds for (4). Moreover, in both cases we have a distinguished splitting which induces a connected m-sheeted covering of $(f^{-1}(0))\,(\mathbb{R})$. The corresponding cover is denoted by $p_X : \tilde{X} \to X$. This satisfies condition (5.6.1) by construction.

Pick $s \in S(\mathbb{R}) \setminus \{0\}$. Then $(f^{-1}(s))_{\mathbb{C}}$ is isomorphic to either $\mathbb{CP}^1$ or two copies of $\mathbb{CP}^1$ meeting to a point. Thus $\pi_1^{alg}(f^{-1}(s)) = \mathbb{Z}_2$. $p^{-1}(f^{-1}(s)) \to f^{-1}(s)$ is étale and Galois. Thus $p^{-1}(f^{-1}(s))$ is either m copies of $f^{-1}(s)$ or $m/2$ copies of the nontrivial degree 2 cover of $f^{-1}(s)$. The latter has no real points by (5.1). $\square$

6. $\mathbb{Z}_m$-actions on real conic bundles

In this section we complete the analysis of rational curve fibrations with an irreducible central fiber. (5.6) reduced this question to the study of $\mathbb{Z}_m$-actions on conic bundles. We start by recalling the representation theory of $\mathbb{Z}_m$ over $\mathbb{R}$, mostly to fix notation.

Notation 6.1. The following are the irreducible real representations of the cyclic group $\mathbb{Z}_m$.

1. the 1–dimensional trivial representation $\mathbf{1}$ (also denoted by $\mathbf{1}^+$),
2. if m is even, we have the 1–dimensional sign representation $\mathbf{1}^-$,
3. for any $a \in \mathbb{Z}$ we have a 2–dimensional representation $R_{a,m}$ (rotation by $a\frac{2\pi}{m}$), given as

$$u \mapsto \cos(a\tfrac{2\pi}{m})u + \sin(a\tfrac{2\pi}{m})v, \quad v \mapsto -\sin(a\tfrac{2\pi}{m})u + \cos(a\tfrac{2\pi}{m})v.$$

   $R_{a,m} \cong R_{b,m}$ iff $a \equiv \pm b \mod m$. $R_{a,m}$ is irreducible iff $a \not\equiv 0, m/2 \mod m$.

$\mathbf{1}^{\pm}(z)$ denotes the vectorspace $\mathbb{R}z$ with $\mathbb{Z}_m$-action $\mathbf{1}^{\pm}$. $R_{a,m}(x,y)$ denotes that we have the above described rotation action on the vector space $\mathbb{R}x + \mathbb{R}y$ in the basis x, y.

6.2. We will need the 3–dimensional representations of $\mathbb{Z}_m$ modulo tensoring by $\mathbf{1}^{\pm}$. The 3–dimensional representations are

1. $\mathbf{1}^{\pm} \oplus \mathbf{1}^{\pm} \oplus \mathbf{1}^{\pm}$, and
2. $\mathbf{1}^{\pm} \oplus R_{a,m}$ for $a \not\equiv 0, m/2 \mod m$.

By tensoring with $\mathbf{1}^-$ if necessary, these can be brought to the form $\mathbf{1}(z) \oplus R_{a,m}(x,y)$ where $0 \leq a < m$ is arbitrary and z, x, y is a choice of a basis. Under this action the vector space of quadratic forms decomposes as follows:

$$\mathbf{1}(z^2) \oplus \mathbf{1}(x^2+y^2) \oplus R_{a,m}(zx, zy) \oplus R_{2a,m}(x^2-y^2, 2xy).$$

Let now $q(x,y,z)$ be a $\mathbb{Z}_m$-equivariant real quadratic form in the above variables which is not the product of two linear forms over $\mathbb{R}$. Assume that every nonidentity element of $\mathbb{Z}_m$ has only isolated fixed points on the conic $(q=0)$. Then q is one of the following:

3. If $m \geq 3$ then $q = \alpha z^2 + \beta(x^2+y^2)$ and $(a,m)=1$.
4. If $m=2$ then $q = \alpha z^2 + q'(x,y)$ and $a=1$.

We need the following lemma whose proof is left to the reader.

Lemma 6.3. *Notation as above. Then*

$$\begin{aligned}
&R_{b,m}(s,t)\otimes S^2(\mathbf{1}(z)+R_{a,m}(x,y)) \cong\\
&\qquad R_{b,m}(sz^2,tz^2)+R_{b,m}(s(x^2+y^2),t(x^2+y^2))+\\
&\qquad R_{a+b,m}(z(sx-ty),z(sy+tx))+R_{a-b,m}(z(sx+ty),z(sy-tx))+\\
&\qquad R_{2a+b,m}(s(x^2-y^2)-2txy,t(x^2-y^2)+2sxy)+\\
&\qquad R_{2a-b,m}(s(x^2-y^2)+2txy,t(x^2-y^2)-2sxy). \quad \square
\end{aligned}$$

Notation 6.4. Let $f: Y \to S$ be a real conic bundle. Assume that $\mathbb{Z}_m$ acts f-equivariantly on Y and S. Let $0 \in S(\mathbb{R})$ be a fixed point. $f_*\mathcal{O}_Y(-K_{Y/S})$ is a rank 3 real vector bundle with a $\mathbb{Z}_m$-action. In a neighborhood of $0 \in S$ we can choose a real analytic linearization of the actions, and we have the following standard form:

S is the germ of a 2–dimensional representation (s,t) of $\mathbb{Z}_m$. V is a 3–dimensional representation (x,y,z) of $\mathbb{Z}_m$ and $f_*\mathcal{O}_Y(-K_{Y/S}) \cong V\otimes_{\mathbb{R}}\mathcal{O}_S$. Thus Y has a $\mathbb{Z}_m$-equivariant embedding $Y \subset \mathbb{P}^2_{x,y,z}\times S$ and g is the second projection.

Theorem 6.5. *Notation as above. Assume that $m \geq 2$, $(f^{-1}(0))(\mathbb{R}) \neq \emptyset$ and every $1 \neq k \in \mathbb{Z}_m$ acts with only isolated and nonreal fixed points. Then $f: Y \to S$ is real analytically equivalent to one of the following normal forms:*

1. *$m \geq 2$ and $Y = (z^2 - x^2 - y^2 + h_1(x,y,z,s,t) = 0)$, or*
2. *m is odd, $Y = (z^2 + sx^2 + 2txy - sy^2 + h_2(x,y,z,s,t) = 0)$ and $b = 2$.*

In both cases $V = \mathbf{1}(z)\oplus R_{1,m}(x,y)$, $S = R_{b,m}(x,y)$ for some $(b,m)=1$ and $h_i \in (s,t)^i$.

Remark 6.6. It is not difficult to see that in both of the above cases, after a further coordinate change we can achieve that $h_i = 0$. Without the $\mathbb{Z}_m$-action this is a standard result on conic bundles (cf. [Beauville77, Chap.I]).

Proof. We are interested in the projectivization of the $\mathbb{Z}_m$-action on $f_*\mathcal{O}_Y(-K_{Y/S}) \cong V\otimes_{\mathbb{R}}\mathcal{O}_S$, thus V can be replaced by $V\otimes \mathbf{1}^-$ if necessary. Thus we may assume that $V = \mathbf{1}(z)\oplus R_{a,m}(x,y)$ for some

$(a, m) = 1$ by (6.2). By choosing the generator of $\mathbb{Z}_m$ suitably, we can assume that $a = 1$.

Let $s \in S(\mathbb{C})$ be a fixed point of $k \in \mathbb{Z}_m$. Then k has a fixed point on $f^{-1}(s)_{\mathbb{C}}$. Hence the $\mathbb{Z}_m$ action on S has an isolated fixed point at 0. Thus $0 \in S$ is real analytically isomorphic to the germ of $R_{b,m}(s,t)$ for some $(b, m) = 1$.

Let $q(x, y, z) = 0$ be the equation of the central fiber. Then $q = \alpha z^2 + \beta(x^2 + y^2)$ if $m \geq 3$ and $q = \alpha z^2 + q'(x, y)$ if $m = 2$ by (6.2).

Consider first the case when $(q = 0)$ is a smooth conic with real points. We can bring q to the normal form $q = z^2 - (x^2 + y^2)$ if $m \geq 3$ and $q = z^2 - (x^2 \pm y^2)$ if $m = 2$. If $q = z^2 - (x^2 - y^2)$, then $\mathbb{Z}_2$ has a real fixed point at $(1 : 1 : 0)$, which is not our situation. This gives the case (6.5.1).

If $(q = 0)$ is a pair of intersecting lines then their real intersection point is fixed by $\mathbb{Z}_m$, so this is impossible.

Finally consider the case when $(q = 0)$ is a double line. Then $q = z^2$, hence the equation of Y can be written as

$$z^2 + sq_s(x, y, z) + tq_t(x, y, z) + (\text{higher order terms in } s, t),$$

where $sq_s(x, y, z) + tq_t(x, y, z)$ is $\mathbb{Z}_m$-invariant. Y has ony isolated singularities along the central fiber, thus $z = q_s(x, y, z) = q_t(x, y, z) = 0$ has only finitely many solutions.

From (6.3) we see that an expression of the form $sq_s(x, y, z) + tq_t(x, y, z)$ can be $\mathbb{Z}_m$-invariant iff $b \equiv \pm a, \pm 2a \mod m$. In the former case q_s and q_t are both divisible by z (these are the $R_{a\pm b,m}$ summands in (6.3)). Thus $b \equiv \pm 2a \mod m$ and so m is odd since $(b, m) = 1$. So $sq_s(x, y, z) + tq_t(x, y, z)$ is an element of one of the summands $R_{2a\pm b,m}$ in (6.3). These can always be brought to the form $sx^2 + 2txy - sy^2$ by a suitable change of the s, t-coordinates. □

Lemma 6.7. *Notation as in (6.5). Then* $(Y/\mathbb{Z}_m)(\mathbb{R}) = (Y(\mathbb{R}))/\mathbb{Z}_m$.

This is of interest only if m is even. By (5.3), the quotient $S/\mathbb{Z}_m$ is locally isomorphic to the singularity $(u^2 + v^2 - w^m = 0)$, which has 2 real components if m is even. The main assertion of the lemma is that our conic fibration has real points over only one of these components.

Proof. $\mathbb{Z}_m$ acts freely on $Y(\mathbb{R})$, so $Y/\mathbb{Z}_m$ is smooth along $(Y(\mathbb{R}))/\mathbb{Z}_m$. Thus $(Y(\mathbb{R}))/\mathbb{Z}_m$ is a connected component of $(Y/\mathbb{Z}_m)(\mathbb{R})$. We know that $(f^{-1}(0))(\mathbb{R}) \sim S^1$ is connected, so $(Y/\mathbb{Z}_m)(\mathbb{R})$ is connected. □

Corollary 6.8. *Let X be a real projective 3-fold with terminal singularities such that K_X is Cartier along $X(\mathbb{R})$. Let $f : X \to S$ be a rational curve fibration over $\mathbb{R}$ such that $-K_X$ is f-ample. Let $N \subset X(\mathbb{R})$*

be a connected component. Then $f(N)$ intersects only one of the connected components of $S(\mathbb{R}) \setminus \operatorname{Sing} S$.

Proof. $f(N)$ is connected, thus if it intersects two connected components of $S(\mathbb{R}) \setminus \operatorname{Sing} S$ then there is a point $P \in \operatorname{Sing} S$ such that P locally disconnects $S(\mathbb{R})$ and $f(N)$ intersects two of the local components of $S(\mathbb{R})$. By (4.3), all real singular points of S arise as in (6.5). At each of these, even $f(X(\mathbb{R}))$ intersects only one of the local components of $S(\mathbb{R}) \setminus \operatorname{Sing} S$ by (6.7), a contradiction. □

7. Local topological description of real conic bundles

In this section we study the topology of real conic bundles. Aside from finitely many points the usual methods of conic bundle theory (cf. [Beauville77, Chap.I]) give the following local analytic description:

Lemma 7.1. *Let S be a quasiprojective real algebraic surface and $f : X \to S$ a real conic bundle over S. There is a finite set of points $T \subset S(\mathbb{R})$ such that every $s \in S(\mathbb{R}) \setminus T$ has a Euclidean neighbourhood $s \in W_s \subset S$ such that $f : f^{-1}(W_s) \to W_s$ is fiber preserving real analytically equivalent to one of the following normal forms. (In all 4 cases f is the second projection and $B^2_{s,t}$ is the unit ball in $\mathbb{A}^2_{s,t}$.)*

1. *(S^1-bundle) $(x^2 + y^2 - z^2) \subset \mathbb{P}^2_{(x:y:z)} \times B^2_{s,t}$,*
2. *(empty fibers) $(x^2 + y^2 + z^2) \subset \mathbb{P}^2_{(x:y:z)} \times B^2_{s,t}$,*
3. *(collapsed end) $(x^2 + y^2 + tz^2) \subset \mathbb{P}^2_{(x:y:z)} \times B^2_{s,t}$,*
4. *(blown up S^1-bundle) $(x^2 - y^2 + tz^2) \subset \mathbb{P}^2_{(x:y:z)} \times B^2_{s,t}$.* □

In order to understand $X(\mathbb{R})$, we have to describe the local structure of f over the exceptional points T. This is quite difficult to do up to real analytic equivalence. It turns out, however, that if $X(\mathbb{R})$ is a manifold, then a suitable PL perturbation of f becomes very simple everywhere.

Theorem 7.2. *Let S be a quasiprojective real algebraic surface and $f : X \to S$ a real conic bundle over S. Let $T \subset S(\mathbb{R})$ be as in (7.1). Let $\bar{n} : \overline{X(\mathbb{R})} \to X(\mathbb{R})$ denote the topological normalization. Let $M \subset \overline{X(\mathbb{R})}$ be a connected component which is a PL 3-manifold and set $U := f(M)$. Let $T \cap U \subset W_T \subset U$ be any open neighborhood.*

Then there is a 2-manifold with boundary $F \supset (U \setminus W_T)$ and a map $g : M \to F$ such that

1. *$g|(M \setminus f^{-1}(W_T)) = f|(M \setminus f^{-1}(W_T))$, and*
2. *every $s \in F$ has a neighbourhood $s \in W_s \subset F$ such that $g : g^{-1}(W_s) \to W_s$ is fiber preserving PL homeomorphic to one of the normal forms (7.1.1–4).*

Moreover, if $\bar n(M)$ is not a connected component of $X(\mathbb{R})$, then $g : M \to F$ contains collapsed ends.

Proof. Let $n : \bar U \to U$ denote the normalization. As we noted during the proof of (2.1), f lifts to a morphism between the normalizations $\bar f : M \to \bar U$. The locus of blown up fibers gives a curve $B \subset \bar U$. We know that there is a finite set $\bar T := n^{-1}(T \cap U) \subset \bar U$ such that over $\bar U \setminus \bar T$ the map $\bar f$ is described by one of the normal forms (7.1.1–4).

We study the various possibilities for the points $P \in \bar T$.

First consider the case when $f^{-1}(n(P))$ is reducible. Then in suitable local coordinates X can be written as $(x^2 - y^2 + h(s,t)z^2 = 0)$ for some real power series $h(s,t)$. Along the fiber, X has a unique singular point Q at $x = y = 0$ with local equation $(x^2 - y^2 + h(s,t) = 0)$.

By [Kollár97b, 4.4], the normalization of $X(\mathbb{R})$ has a manifold component near Q iff one of the following conditions holds:

3. $(0,0)$ is an isolated point of $(h = 0)$. Possibly after interchanging x and y, we may assume that $h(s,t) > 0$ near $(0,0)$. The projection
$$\overline{X(\mathbb{R})} \to \mathbb{RP}^1_{(x:z)} \times \mathbb{R}^2_{s,t}$$
is a 2-sheeted unramified cover. Thus $\overline{X(\mathbb{R})} \to \mathbb{R}^2_{s,t}$ is an S^1-bundle.
4. $(h = 0)$ is a PL 1–manifold near $(0,0)$. Then, up to a PL coordinate change, we can assume that $h(s,t)$ is one of the local coordinates. Thus, locally we have the standard equation $(x^2 - y^2 + tz^2 = 0)$ for X.

In all other cases, $(f^{-1}(n(P)))(\mathbb{R})$ is S^1 or a point. In particular, in all other cases, $\bar f^{-1}(P)$ is either S^1 or is collapsible to a point. Let $P \in W_P \subset \bar U$ be a regular neighborhood of P such that $N_P := \bar f^{-1}(W_P)$ is a regular neighborhood of $\bar f^{-1}(P)$. Note that W_P is a disc if $P \in \operatorname{Int} \bar U$ and a half disc (with P a boundary point) if $P \in \partial \bar U$.

If $P \in \operatorname{Int} \bar U$ then ∂N_P is an S^1-bundle over a circle with a point blown up for each intersection point $B \cap \partial W_P$. If $P \in \partial \bar U$ then ∂N is S^2 with a point blown up for each intersection point $B \cap \partial W_P$.

We consider separately the various topological types for N_P.

If $\bar f^{-1}(P)$ is collapsible to a point, then N_P is a ball, hence orientable. Therefore, $P \notin B$ and $P \in \partial \bar U$. After collapsing $\bar f^{-1}(P)$ to a point we have a collapsed end as in (7.1.3).

Assume next that $\bar f^{-1}(P) \sim S^1$ and N_P is a solid Klein bottle. If $P \in \operatorname{Int} \bar U$ then $P \notin B$. As explained in (3.2.2), we can modify $\bar f$ to get a map to an annulus with a collapsed end along the inner circle. If $P \in \partial \bar U$ then ∂N_P is S^2 blown up at 2 points. So $B \cap W_P$ is an interval and we can move B away from P. Thus we get disjoint curves of collapsed ends and blown up S^1-bundles.

Finally consider the case when $\bar{f}^{-1}(P) \sim S^1$ and N_P is a solid torus. Then $P \notin \partial \bar{U}$ (for such a point ∂N_P would be S^2 or S^2 blown up in a few points) and $P \notin B$ (for such a point ∂N_P would be a torus or Klein bottle blown up in at least one point).

Here we have to look at the algebraic structure of f. If $f^{-1}(n(P))$ is a smooth conic, then $X(\mathbb{R}) \to S(\mathbb{R})$ is an S^1-bundle near $n(P)$ and $M \to X(\mathbb{R})$ is a homeomorphism near $f^{-1}(n(P))$.

We are left with the case when $f^{-1}(n(P))$ is a double line. Set $C = \operatorname{red} f^{-1}(n(P))$. Then $C \cong \mathbb{P}^1$ and $(C \cdot K_X) = -1$. $S^1 \sim \bar{f}^{-1}(P) \to C(\mathbb{R})$ is a homeomorphism, thus $\overline{X(\mathbb{R})}$ is not orientable along $f^{-1}(P)$ by (7.3), a contradiction.

The last assertion of the theorem is equivalent to the following local statement which we checked in each case:

Let $n(P) \in D_P \subset S(\mathbb{R})$ be a small neighborhood. If $(f \circ \bar{n})^{-1}(D_P)$ is disconnected then M contains a collapsed end. □

Lemma 7.3. *Let X be a normal real algebraic variety and $C \subset X$ an irreducible real algebraic curve which is not contained in* $\operatorname{Sing} X$. *Assume that K_X is Cartier along C and $(C \cdot K_X)$ is odd. Let $n : \overline{X(\mathbb{R})} \to X(\mathbb{R})$ denote the normalization. Let $\gamma \subset \overline{X(\mathbb{R})}$ be a simple closed loop such that $n : \gamma \to C(\mathbb{R})$ is a homeomorphism.*

Then $\overline{X(\mathbb{R})}$ is not orientable along small perturbations of γ.

Proof. We can throw away the closed subset of X along which K_X is not Cartier. Let ω be a meromorphic real section of K_X which is nonzero and defined at all points of $C(\mathbb{R}) \cap \operatorname{Sing} X$. Since complex zeros and poles of $\omega|C$ come in conjugate pairs, we see that the number of zeros of $\omega|C(\mathbb{R})$ minus the number of poles of $\omega|C(\mathbb{R})$ is odd. $n^*(\omega|X(\mathbb{R}))$ restricts to the orientation bundle on the set of smooth points (cf. [Kollár97a, 2.7]). Let γ' be a small perturbation of γ which is contained in the smooth locus. Then $n^*(\omega|X(\mathbb{R}))$ has an odd number of sign changes along γ', thus $\overline{X(\mathbb{R})}$ is not orientable along γ'. □

Here are some examples of real conic bundles where the central fiber is a double line.

Example 7.4. All of the examples sit in $\mathbb{P}^2 \times \mathbb{A}^2$, f is the second projection and $U := f(X(\mathbb{R}))$.

$(z^2 + sx^2 + 2txy - sy^2 = 0)$. X is smooth and $f : X(\mathbb{R}) \to \mathbb{R}^2$ is an S^1-bundle outside the origin.

$(z^2 + sx^2 + ty^2 = 0)$. $f : X(\mathbb{R}) \to \mathbb{R}^2$ has empty fibers over the positive quadrant and is an S^1-bundle over the other 3 open quadrants.

We have a collapsed end along the positive s and t axes and a blown up S^1-bundle along the negative s and t axes.

$(z^2 + sx^2 + (t^2 - s^2)y^2 = 0)$. $f : X(\mathbb{R}) \to \mathbb{R}^2$ has empty fibers over $0 < s < |t|$ and is an S^1-bundle outside $st(s^2 - t^2) = 0$ otherwise. U has 2 components: $(s \le 0)$ and $(|t| \le s)$. We have a colapsed end everywhere along the $(|t| = s)$ curve. In the $(s \le 0)$ component, there is a blown up S^1-bundle along the curve $B = (s + |t| = 0)$. In the normalization map $\overline{X(\mathbb{R})} \to X(\mathbb{R})$ the preimage of the central fiber is the disjoint union of a circle and of a point.

$(z^2 + s^2x^2 + 2txy + s^2y^2 = 0)$. U has two components, $(t \ge s^2)$ and $(-t \ge s^2)$. No blown up S^1-bundle curve. In the normalization map $\overline{X(\mathbb{R})} \to X(\mathbb{R})$ the central fiber breaks up into 2 intervals.

8. Proof of the main theorems

The next theorem contains all the information about the topology of rational curve fibrations over $\mathbb{R}$ that we have obtained. Later we translate this information to the language of the classification scheme of 3–manifolds.

Theorem 8.1. *Let X be a real projective 3-fold with terminal singularities such that K_X is Cartier along $X(\mathbb{R})$. Let $f : X \to S$ be a rational curve fibration over $\mathbb{R}$ such that $-K_X$ is f-ample. Let $\bar{n} : \overline{X(\mathbb{R})} \to X(\mathbb{R})$ be the topological normalization and $M \subset \overline{X(\mathbb{R})}$ a connected component which is a PL 3-manifold. Then there is a 2-manifold with boundary F and a map $g : M \to F$ such that every $s \in F$ has a neighbourhood $s \in W_s \subset F$ such that $g : g^{-1}(W_s) \to W_s$ is fiber preserving PL homeomorphic to one of the following normal forms. (In all 4 cases g is the second projection and $B^2_{s,t}$ is the unit ball in $\mathbb{A}^2_{s,t}$.)*

1. *(S^1-bundle) $(x^2 + y^2 - z^2) \subset \mathbb{P}^2_{(x:y:z)} \times B^2_{s,t}$,*
2. *(collapsed end) $(x^2 + y^2 + tz^2) \subset \mathbb{P}^2_{(x:y:z)} \times B^2_{s,t}$,*
3. *(blown up S^1-bundle) $(x^2 - y^2 + tz^2) \subset \mathbb{P}^2_{(x:y:z)} \times B^2_{s,t}$.*
4. *(Seifert fiber) $(x^2 + y^2 - z^2)/\mathbb{Z}_m \subset (\mathbb{P}^2_{(x:y:z)} \times B^2_{s,t})/\mathbb{Z}_m$ where $\mathbb{Z}_m$ acts by rotation with angle $2\pi/m$ on (x, y), fixes z and acts by rotation with angle $2b\pi/m$ on (s, t) for some $(b, m) = 1$.*

Moreover, the following additional conditions are also satisfied:

5. *If $\bar{n}(M)$ is not a connected component of $X(\mathbb{R})$ then $g : M \to F$ contains collapsed ends.*
6. *There is an injection from the set of Seifert fibers of g to the set of singular points of S contained in $f(M)$ with real analytic equation $(u^2 + v^2 - w^m = 0)$. Under this injection the multiplicity*

of the Seifert fiber equals the exponent of w in the corresponding equation.

Proof. Let $P_j \in S(\mathbb{R})$ be the set of points such that f is not a conic bundle over any neighborhood of P_j. This set is finite and (7.2) describes the behaviour of f over $S \setminus \{\cup_j P_j\}$. Over suitable small neighborhoods of each P_j, f is described by (6.5). If (6.5.1) holds then we obtain a Seifert fiber by (3.4). If (6.5.2) holds then by (7.3) and (3.3) we obtain a curve of collapsed ends. □

8.2 (Proof of (1.1))**.**

Let us run the minimal model program for $f : X \to S$ over $\mathbb{R}$ [Kollár97c, Sec.3]. At the end we obtain:

1. A real algebraic threefold X^* with only terminal singularities such that K_X is Cartier along $X^*(\mathbb{R})$ and $\overline{X^*(\mathbb{R})}$ is a 3–manifold (see [Kollár97c, 1.9, 1.11]).
2. A morphism $f^{(*)} : X^* \to S^*$ to a real algebraic surface S^* such that $-K_{X^*}$ is $f^{(*)}$-ample. (There is a birational morphism $S^* \to S$ but they need not be equal.)

Moreover, by [Kollár97c, 1.1], $X(\mathbb{R})$ is obtained from $\overline{X^*(\mathbb{R})}$ by the following operations:

: throwing away all isolated points of $\overline{X^*(\mathbb{R})}$,
: taking connected sums of connected components,
: taking connected sum with $S^1 \times S^2$,
: taking connected sum with $\mathbb{RP}^3$.

Let $M \subset X(\mathbb{R})$ be a connected component and $M^* \subset X^*(\mathbb{R})$ its image. M^* is connected, but its preimage $\overline{M^*} \subset \overline{X^*(\mathbb{R})}$ may have several connected components, say $M_1, \dots, M_s$. M_j minus finitely many points is homeomorphic to an open subset of M, so each M_j is orientable.

We can apply (8.1) to each M_j to obtain maps $g_j : M_j \to F_j$. M_j is orientable, thus we do not have blown up S^1-bundles, so every fiber of g_j is either a circle or a point. By (3.5), each M_j is either Seifert fibered over F_j or is a connected sum of lens spaces. If $s \geq 2$ then each M_j contains collapsed ends by (7.2), thus one of the following holds:

3. $M \sim M_1 \# a\mathbb{RP}^3 \# b(S^1 \times S^2)$ and $g_1 : M_1 \to F_1$ is Seifert fibered, or
4. $M \sim M_1 \# \cdots \# M_s \# a\mathbb{RP}^3 \# b(S^1 \times S^2)$ end each M_i is a connected sum of lens spaces.

By (1.9) there is an injection from the set of the multiple Seifert fibers in case (3) (resp. the lens space summands in case (4)) to the set of singular points of S on $f^{(*)}(M^*)$. By (6.8), $f^{(*)}(M^*)$ intersects only one of the components $\bar{U}$ of $\overline{S^*(\mathbb{R})}$.

Assume first that $f^{(*)}(M^*) \subsetneq \bar{U}$. Then $M \to F$ contains a curve of collapsed ends, so we are in case (4). A singularity of type $(u^2 + v^2 - w^2 = 0)$ on $\bar{U}$ gives rise to a connected summand $L_{1,2} \cong \mathbb{RP}^3$ by (3.7). If $S_{\mathbb{C}}$ is rational then there are at most 6 singular points of type $(u^2 + v^2 - w^m = 0)$ with $m \geq 3$ on $\bar{U}$ by (9.2), thus there are at most 6 lens spaces $L_{p,q}$ with $q \geq 3$.

Next consider the case when $f^{(*)}(M^*) = \bar{U}$. By (6.8) $\bar{U}$ can not map to both local components of a singularity of type $(u^2 + v^2 - w^{2m} = 0)$, thus every such singular point of $\bar{U}$ is separating (9.1). If $S_{\mathbb{C}}$ is rational then there are at most 6 singular points of type $(u^2 + v^2 - w^m = 0)$ with $m \geq 2$ on $\bar{U}$ by (9.2), thus $M \to F$ has at most 6 multiple fibers.

An orientable component of $\overline{S^*(\mathbb{R})}$ is either a sphere or a torus by (9.2). □

In the nonorientable case the topological description is more complicated. The theorem below gives substantial restrictions. The proof yields a more precise result but I am not sure how much further can one go.

Theorem 8.3. *Let X be a real projective 3-fold with terminal singularities such that K_X is Cartier along $X(\mathbb{R})$. Let $f : X \to S$ be a rational curve fibration over $\mathbb{R}$ such that $-K_X$ is f-ample. Let $M \subset \overline{X(\mathbb{R})}$ be a connected component which is a PL 3-manifold. Then*

$$M \sim N \# L_1 \# \cdots \# L_s \# a(S^1 \times S^2) \# b(S^1 \tilde{\times} S^2),$$

where each each L_i is a lens space and all the pieces of the Jaco–Johannson–Shalen decomposition of N (cf. [Scott83, p.483]*) are Seifert fibered.*

Proof. By (8.1) there is a map $g : M \to F$ whose local structure is given by (8.1.1–4). In F we have boundary curves $C_1, \dots, C_m$ and curves corresponding to blown up S^1-bundles $B_1, \dots, B_k$.

We start with the boundary curves. As in the proof of (3.5), we cut F along simple paths starting and ending in $\cup C_j$ which are disjoint from $\cup B_i$. Each time we obtain either a connected sum decomposition or we remove a 1–handle. At the end we may assume that each C_j is parallel to a B_i. Let B_i' be the boundary of a regular neighborhood $U_i \supset B_i$. We assume that the B_i' are disjoint. Let us cut F along all the B_i'. We obtain the pieces U_i and some other pieces V_k. $g^{-1}(V_k) \to V_k$ is Seifert fibered. Each U_i is either an annulus or a Möbius band so can be relaized as an interval bundle over a circle. Correspondingly, we obtain fibrations $g^{-1}(U_i) \to S^1$ whose fibers are annuli with one point blown up. Equivalently, these fibers can be thought of as

$$R := \mathbb{RP}^2 \setminus \{(1:0:0), (0:1:0)\}.$$

Fibrations over S^1 with this fiber are classified by diffeomorphisms of R modulo isotopy. It turns out there there are only 8 such, and they can be realized by the matrices

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$$

acting on the first 2 coordinates on $R \subset \mathbb{RP}^2$. It is easy to see that each of these 8 cases gives a Seifert fibered 3–manifold with $\mathbb{H}^2 \times \mathbb{R}^1$-geometry. It is important to note that this Seifert fibration is not compatible with the Seifert fibration of the pieces $g^{-1}(V_k)$.

The preimages of the B_i' in M are tori or Klein bottles, so this starts to look like the Jaco–Johannson–Shalen decomposition. We have to be careful, since we may have cut too much.

First, we can have a component V_k which is a disc with at most one multiple Seifert fiber in it. Then $g^{-1}(V_k)$ is a solid torus which can be pasted to the neighbouring $g^{-1}(U_j)$. If V_k does not contain any multiple fibers then we get a collapsed end and we can further decompose M as above. If V_k contains a multiple fiber the we get a Seifert fiber space. If there is no such V_k then the cutting tori and Klein bottles are all incompressible (cf. [Scott83, p.432]).

Second, we can have a component V_k which is an annulus with no multiple fibers. If one of the boundary components of V_k is B_j' the other C_i then over V_k we have a solid torus or Klein bottle which can be pasted to $g^{-1}(U_j)$ to obtain a new Seifert fibered 3–manifold. If the two boundary components of V_k are in B_j' and $B_{j'}'$ then $g^{-1}(U_j), g^{-1}(U_{j'})$ and $g^{-1}(V_k)$ paste together to a Seifert fibered 3–manifold with $\mathbb{H}^2 \times \mathbb{R}^1$-geometry.

After all these operations we have Seifert fibered 3–manifolds pasted along incompressible tori and Klein bottles. Furthermore, no two pieces can be glued together to get a Seifert fibered manifold (cf. [Scott83, p.440]). So this is the Jaco–Johannson–Shalen decomposition. □

9. Real surfaces with Du Val singularities

The aim of this section is to study the configurations of Du Val singularities that can simultaneously occur on a real algebraic surface which is rational over $\mathbb{C}$.

Definition 9.1. Let S be a real algebraic surface and $P \in S(\mathbb{R})$ a singular point with equation $(u^2 + v^2 - w^{2m} = 0)$. $\overline{S(\mathbb{R})}$ has two connected components locally near P. We say that P is *separating* if these 2 local components are on different connected components of

$\overline{S(\mathbb{R})}$ and *nonseparating* if the 2 local components are on the same connected component of $\overline{S(\mathbb{R})}$.

Theorem 9.2. *Let S be a projective real algebraic surface such that $S_{\mathbb{C}}$ is rational. Assume that along $S(\mathbb{R})$ we have only singularities of real analytic type $(x^2 + y^2 - z^m = 0)$. Let $M \subset \overline{S(\mathbb{R})}$ be a connected component.*

1. *If M is orientable then $M \sim S^2$ or $M \sim S^1 \times S^1$.*
2. *M can contain arbitrary many points which map to a nonseparating singular point of type $(x^2 + y^2 - z^2 = 0)$ but at most 6 points which map to other singular points of S.*

The minimal model theory of real surfaces has been studied in detail in the papers [Comessatti14, Silhol89, Kollár97a]. It is not difficult to generalize these results to the case when we allow singularities of the form $(x^2 + y^2 - z^m = 0)$. Most of the proofs run very close to the arguments in [Kollár97a], thus I will only explain the points where some differences arise.

For the minimal model theory the natural setting is to allow all Du Val singularities (9.3). (See [Reid85] or [Kollár-Mori98, 4.2] for the relevant background on Du Val singularities over $\mathbb{C}$.) This will show the surprising difference between the $(x^2 + y^2 - z^m = 0)$ and $(x^2 - y^2 - z^m = 0)$ singularities. (For $m = 2$ the two singularities are isomorphic which leads to some complications.)

Definition 9.3. A real Du Val singularity is a surface singularity $(0 \in S) \subset (0 \in \mathbb{A}^3)$ which is real analytically equivalent to one of the following normal forms:

A_n^+**:** $(x^2 + y^2 - z^{n+1} = 0)$ for $n \geq 1$,
A_n^-**:** $(x^2 - y^2 - z^{n+1} = 0)$ for $n \geq 1$,
A_n^{++}**:** $(x^2 + y^2 + z^{n+1} = 0)$ for n odd,
D_n^+**:** $(x^2 + y^2 z + z^{n-1} = 0)$ for $n \geq 4$,
D_n^-**:** $(x^2 + y^2 z - z^{n-1} = 0)$ for $n \geq 4$,
E_6^+**:** $(x^2 + y^3 + z^4 = 0)$,
E_6^-**:** $(x^2 + y^3 - z^4 = 0)$,
E_7**:** $(x^2 + y^3 + yz^3 = 0)$,
E_8**:** $(x^2 + y^3 + z^5 = 0)$.

$A_1^+ \cong A_1^-$ but otherwise the singularities with different name are not isomorphic. It is easy to see that these are all the real forms of the complex Du Val singularities. (My setting and notation are not the same as in [AGV85, I.17.1], but the above list is easy to obtain from the results there.)

Definition 9.4. Let $0 \in S$ be a smooth real point and x, y local coordinates at 0. The $(1,m)$-blow up of (x,y) is the surface $S' \subset S \times \mathbb{P}^1_{u:v}$ given by equation $ux - vy^m = 0$. For $m = 1$ this is the ordinary blow up. The $(1,m)$-blow up has a unique singular point of type A^-_{m-1} at $(0,0,0,1)$.

If $P, \bar{P} \in S$ are smooth and conjugate complex points, then we can choose conjugate coordinate systems to do a $(1,m)$-blow up at both points. The result is again a real algebraic surface with a conjugate pair of A_{m-1}-points (for nonreal points the signs in the equations do not matter).

Lemma 9.5. *Let $p : S' \to S$ be the $(1,m)$-blow up of a smooth real point. If m is even, then a small perturbation of p gives a homeomorphism $\overline{S'(\mathbb{R})} \to S(\mathbb{R})$. If m is odd then $S'(\mathbb{R}) \sim S(\mathbb{R}) \# \mathbb{RP}^2$.*

In both cases a $(1,m)$-blow up creates a nonseparating singular point.

Proof. If m is odd then $(x,y) \mapsto (x, y^m)$ is a homeomorphism, so the $(1,m)$-blow up is homeomorphic to the ordinary blow up.

Assume that $m = 2n$ is even. Then S' has a singular point $(ux - y^{2n} = 0)$. In the normalization this splits into 2 parts, and correspondingly the preimage of the exceptional S^1 becomes an interval. Thus $p : \overline{S'(\mathbb{R})} \to S(\mathbb{R})$ contracts an interval to a point, hence suitable small perturbations of p are homeomorphisms. □

The minimal model program for real surfaces is explained in [Kollár97a], see also [Silhol89, Kollár-Mori98]. The general theory applies equally to surfaces with Du Val singularities. For the applications the key point is the following description of the extremal contractions:

Theorem 9.6. *Let F be a projective surface over $\mathbb{R}$ with Du Val singularities and $R \subset \overline{NE}(F)$ a K_F-negative extremal ray. Then R can be contracted $f : F \to F'$, and we obtain one of the following cases:*

- (B) *(Birational) F' is a projective surface over $\mathbb{R}$ with Du Val singularities and $\rho(F') = \rho(F) - 1$. F is the $(1,m)$-blow up of F' at a smooth point P of F' for some $m \geq 1$. We have two cases:*
  - (a) *$P \in F'(\mathbb{R})$ is a real point, or*
  - (b) *P is a pair of conjugate points.*
- (C) *(Conic bundle) $B := F'$ is a smooth curve, $\rho(F) = 2$ and $F \to B$ is a conic bundle.*
- (D) *(Del Pezzo surface) F' is a point, $\rho(F) = 1$, $-K_F$ is ample and $\rho(F_{\mathbb{C}}) \leq 9$.*

Proof. The proof can be put together from the appropriate pieces in [Cutkosky88, Morrison85, Kollár97a]. I just explain the main point:

Why does the list of birational contractions involve only A^--type singularities?

Let $E \subset F$ be the exceptional curve. Let $G \to F$ be the minimal resolution of the singularities of F along E and $G \to F'$ the composition. The exceptional divisor of $G \to F'$ consists of -2-curves (coming from curves exceptional over F) and (-1)-curves (the irreducible components of the birational transform of E). We can factor $G \to F'$ by repeatedly contracting (-1)-curves. Two (-1)-curves can never intersect since then we would get an exceptional curve with selfintersection zero after contracting one of them. By looking at the various cases we see that each connected component of $\operatorname{Ex}(G \to F')$ contains a unique (-1)-curve. In fact, we have one of the following configuration of curves:

$$\overset{-1}{\circ} - \overset{-2}{\circ} - \cdots - \overset{-2}{\circ}$$

Consider curve germs C_x and C_y intersecting the (-1)-curve on the left (resp. the (-2)-curve on the right) transversally at a smooth point. After we contract everything, C_x and C_y become a pair of transversally intersecting curves C'_x and C'_y on F'. Choose coordinates such that $C'_x = (x = 0)$ and $C'_y = (y = 0)$. We see that $F \to F'$ is the $(1, m)$-blow up where $m - 1$ is the number of (-2)-curves above. □

Since birational contractions can eliminate only nonseparating cA^--type singularities, we obtain:

Corollary 9.7. *Let F be a projective surface over $\mathbb{R}$ with Du Val singularities and $F \to F^*$ the result of the MMP over $\mathbb{R}$. Then there is a one–to–one correspondence between the two sets:*

1. *Real Du Val singular points of F which are either not of type A^- or are separating of type A^-.*
2. *Real Du Val singular points of F^* which are either not of type A^- or are separating of type A^-.* □

Corollary 9.8. *Let F be a projective surface over $\mathbb{R}$ with Du Val singularities. Assume that $F_{\mathbb{C}}$ is rational. Then every connected component of $\overline{F(\mathbb{R})}$ contains at most 6 real Du Val singular points which are either not of type A^- or are separating of type A^-.*

Proof. By (9.7) it is sufficient to consider the cases when F is either a conic bundle or a Del Pezzo surface.

In the latter case it is sufficient to consider degree 1 Del Pezzo surfaces, since every other can be made into degree 1 by blowing up a few points. Every degree 1 Del Pezzo surface is a double cover of a quadric cone $Q \subset \mathbb{P}^3$, ramified along a curve B not pasing through the vertex which is a complete intersection of Q and of a cubic C. B has the maximum number of singular points 6 when C is the union of 3 planes. ([Furushima86] contains a partial list of singular Del Pezzo surfaces.)

If $f : F \to B$ is a conic bundle, then the number of singular points of $\overline{F(\mathbb{R})}$ is not bounded, so we have to analyze the connected components of $\overline{F(\mathbb{R})}$.

The local structure of $f : F \to B$ is described by one of the following equations. Here s is a local coordinate on B and we write F as a hypersurface in $\mathbb{P}^2_{x:y:z} \times \mathbb{A}^1_s$.

1. $(x^2 + y^2 \pm z^2 = 0)$,
2. $(x^2 + y^2 \pm s^m z^2 = 0)$,
3. $(x^2 + s(y^2 + z^2) = 0)$,
4. $(x^2 + syz = 0)$,
5. $(x^2 + sy^2 \pm s^m z^2 = 0)$,

The first one describes an S^1-bundle or the empty set. In case of the second equation, $F(\mathbb{R})$ lies entirely in $(\mp s \geq 0)$ if m is odd and $\overline{F(\mathbb{R})}$ decomposes into two connected components if m is even and the sign is negative. In the third case there are no real singular points and $F(\mathbb{R})$ lies entirely in $(\mp s \geq 0)$. In the fourth case $\overline{F(\mathbb{R})}$ decomposes into two connected components, one in $yz - s \geq 0$ and one in $yz + s \geq 0$. Each of the components passes through the two singular points $(0, 1, 0, 0)$ and $(0, 0, 1, 0)$.

In the last case we have a D-type singular point. If we have the plus sign then $F(\mathbb{R})$ lies in $(s \leq 0)$. In the other case the local structure is easy to work out by projecting to the $(x = 0)$ line bundle. We obtain that $\overline{F(\mathbb{R})}$ decomposes into two connected components, one for $(s \geq 0)$ and one for $(s \leq 0)$.

Thus we see that each connected component of $\overline{F(\mathbb{R})}$ pases through at most 4 singular points. $\square$

9.9 (Proof of (9.2))**.**

We can resolve the nonreal singular points of S without changing the set of real points. Thus we may assume that S has only Du Val singularities. Let $f : S \to S^*$ be the result of the minimal model program.

The statement about the number of singular points now follows from (9.8).

Let $0 \in S(\mathbb{R})$ be a singular point with equation $(x^2 + y^2 - z^m = 0)$ and $p : S' \to S$ the blow up of $0 \in S$. Then $S'(\mathbb{R}) \to S(\mathbb{R})$ is a homeomorphism for $m \geq 3$. Thus the question of orientablility can be reduced to the case when all singular points have equation $(x^2 + y^2 - z^2 = 0)$.

Assume that $M \subset \overline{S(\mathbb{R})}$ is orientable. M gives a connected component of $M^* \subset \overline{S^*(\mathbb{R})}$, and M^* is homeomorphic to M by (9.5).

If S^* is a conic bundle, then we obtain from the proof of (9.8) that every connected component of $\overline{S^*(\mathbb{R})}$ is $S^2, S^1 \times S^1, \mathbb{RP}^2$ or a Klein bottle (these are the only surfaces that map to S^1 such that every fiber is S^1, collapses to a point or is empty).

We are left with the case when S^* is a Del Pezzo surface. For every singularity $(x^2+y^2-z^2=0)$ I choose the deformation $(x^2+y^2-z^2+\epsilon=0)$. If these local deformations are simultaneously realized by a global deformation S^*_ϵ of S^*, then it is easy to see that $\overline{S^*(\mathbb{R})} \sim S^*_\epsilon(\mathbb{R})$. Thus (9.2.1) follows from the corresponding result of Comessatti in the smooth case (cf. [Comessatti14, Silhol89, Kollár97a]).

For the existence of S^*_ϵ it is sufficient to consider the case when S^* has degree 1 or 2. In the degree 2 case S^* is realized as a double cover of $\mathbb{P}^2$ given by an equation $u^2 = f_4(x,y,z)$. We look at deformations of the form $u^2 = f_4(x,y,z) + \epsilon g_4(x,y,z)$. We need to choose g_4 to have prescribed signs at the singular points of S^*. Given at most 6 points in $\mathbb{P}^2$ with no 5 on a line, one can always find a quartic g_4 which has prescribed values at the points. The degree 1 case is similar. □

10. Examples

The aim of this section is to present examples of 3–manifolds which can be realized as the real points of rational curve fibrations over rational surfaces. In many cases our examples are unirational over $\mathbb{R}$, and they are always rationally connected (cf. [Kollár96, IV.3]).

10.1 (Circle bundles).

Let F be a compact topological surface (without boundary). Circle bundles over F are classified as follows (cf. [Scott83, p.434]).

Let $p : M \to F$ be a circle bundle. We can assume that it has structure group $O(2,\mathbb{R})$, since the homeomorphism group of the circle retracts to $O(2,\mathbb{R})$. (This result goes back to Poincaré, see [Wood71, Sec. 4] for a proof.) Using the standard 2–dimensional representation of $O(2,\mathbb{R})$, this induces an $\mathbb{R}^2$-bundle $E \to X(\mathbb{R})$ with structure group $O(2,\mathbb{R})$. That is, in addition to the vector space structure, each fiber carries an inner product, unique up to a positive multiplicative constant. By a partition of unity argument we can choose a continuously varying inner product. Thus M can be thought of as the unit circles of an $\mathbb{R}^2$-bundle $E \to F$ with an inner product. The first Stiefel–Whitney class $w_1(E)$ of E gives the first invariant. The secondary invariant is an element of $H^1(F, R^1 p_* \mathbb{Z})$. The latter group is $\mathbb{Z}_2$, except when $w_1(E)$ is the orientation class of F, in which case it is $\mathbb{Z}$. (If F is orientable and the structure group is $SO(2,\mathbb{R})$, then E is naturally a $\mathbb{C}$-bundle and this invariant coincides with the first Chern class.)

Assume now that $F = S(\mathbb{R})$. We would like to realize every topological circle bundle as a smooth conic bundle, or better, as an algebraic circle bundle. The above arguments suggest that $H^1(S(\mathbb{R}), \mathbb{Z}_2)$ should be "algebraic" for this to be possible. This can indeed be made precise, but for us the main point is the converse:

Proposition 10.2. *Let S be a smooth, projective real algebraic surface such that $H^1(S(\mathbb{R}), \mathbb{Z}_2)$ is generated by the cohomology classes of algebraic curves. Then every topological circle bundle over $S(\mathbb{R})$ is fiber preserving homeomorphic to an algebraic circle bundle.*

Proof. Let $M \to S(\mathbb{R})$ be a circle bundle. We can assume that M is the unit circle bundle of an $\mathbb{R}^2$-bundle E with an inner product. The choice of inner products is equivalent to a section σ_0 of S^2E^*.

By [BCR87, 12.5.3] and our assumption on $H^1(S(\mathbb{R}), \mathbb{Z}_2)$, there is a strongly algebraic vector bundle $F \to S(\mathbb{R})$ which is topologically isomorphic to E. (Strongly algebraic means that it is generated by its global section, cf. [BCR87, 12.1.6–7].) S^2F^* is also strongly algebraic by [BCR87, 12.1.8], hence σ_0 can be approximated by algebraic sections σ_t by [BCR87, 12.3.2]. If t is near 0, then σ_t defines an inner product on each fiber and so we get an algebraic circle bundle. □

If $S_{\mathbb{C}}$ is rational then $H^1(S(\mathbb{R}), \mathbb{Z}_2)$ is generated by the cohomology classes of algebraic curves by [Silhol89, p.67], thus we obtain:

Corollary 10.3. *Let S be a smooth, projective real algebraic surface such that $S_{\mathbb{C}}$ is rational. Then every topological circle bundle over $S(\mathbb{R})$ is fiber preserving homeomorphic to an algebraic circle bundle.* □

10.4 (Manifolds with spherical geometry).

The standard n-sphere is given by equation

$$S^n = (x_1^2 + \cdots + x_{n+1}^2 = 1) \subset \mathbb{R}^{n+1}.$$

Its group of automorphisms is $O(n+1, \mathbb{R})$, which acts by real algebraic automorphisms. Thus every quotient by a finite subgroup is a unirational real algebraic variety.

In dimensions 3 every finite subgroup of $O(4, \mathbb{R})$ acting freely on S^3 is conjugate to a subgroup which leaves the Hopf fibration invariant (cf. [Scott83, 4.10]). The Hopf fibration is easiest to write down as a map to the Riemann sphere

$$(x_1, x_2, x_3, x_4) \mapsto (x_1 + \sqrt{-1}x_2)/(x_3 + \sqrt{-1}x_4).$$

Combined with the inverse of the stereographic projection

$$s+\sqrt{-1}t \mapsto \left(\frac{2s}{1+s^2+t^2}, \frac{2t}{1+s^2+t^2}, \frac{1-s^2-t^2}{1+s^2+t^2}\right)$$

we get a real algebraic map

$$p : (x_0^2+x_1^2+x_2^2+x_3^2=1) \to (u_1^2+u_2^2+u_3^2=1).$$

The coordinate functions of p are easy to work out but they are somewhat messy. The main point is that the fibers of p are conics. Indeed, from the first representation one sees that the fiber over $s+\sqrt{-1}t$ is given by equations

$$x_3^2+x_4^2=(1+s^2+t^2)^{-1}, \quad x_1=sx_3-tx_4, \quad x_2=sx_4+tx_3.$$

From this we see that the indeterminacy locus of p on the complex projective quadric consists of the pair of conjugate lines

$$(x_0=x_2-\sqrt{-1}x_1=x_4-\sqrt{-1}x_3),\ (x_0=x_2+\sqrt{-1}x_1=x_4+\sqrt{-1}x_3).$$

After blowing them up, p becomes a $\mathbb{P}^1$-bundle over the projective quadric $(u_1^2+u_2^2+u_3^2=u_0^2)$.

Any finite subgroup as above acts on this conic bundle, so we see that every 3-manifold with spherical geometry can be realized as a unirational real rational curve fibration.

It would be interesting to study the rationality question of these quotients. This should be of interest even over $\mathbb{C}$.

10.5 (3-manifolds with Euclidean geometry)**.** Consider the following 4 finite cyclic subgroups of $GL(2,\mathbb{Z})$:

$$\left\langle\begin{pmatrix} 0 & -1\\ 1 & 1\end{pmatrix}\right\rangle \cong \mathbb{Z}_6, \quad \left\langle\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\right\rangle \cong \mathbb{Z}_4,$$
$$\left\langle\begin{pmatrix} -1 & -1\\ 1 & 0\end{pmatrix}\right\rangle \cong \mathbb{Z}_3, \quad \left\langle\begin{pmatrix} -1 & 0\\ 0 & -1\end{pmatrix}\right\rangle \cong \mathbb{Z}_2.$$

$GL(2,\mathbb{Z})$ acts on the torus $S^1\times S^1$ as described in (3.2). We can realize $S^1\times S^1$ as the real algebraic group

$$W := \operatorname{Spec}\mathbb{R}[x_1,x_2]/(x_1^2+x_2^2-1)\times \operatorname{Spec}\mathbb{R}[y_1,y_2]/(y_1^2+y_2^2-1),$$

and the action of $GL(2,\mathbb{Z})$ is by algebraic automorphisms. (For instance, the generator of $\mathbb{Z}_4$ acts as $(x_1,x_2,y_1,y_2)\mapsto(y_1,-y_2,x_1,x_2)$.)

Take $W\times \operatorname{Spec}\mathbb{R}[z_1,z_2]/(z_1^2+z_2^2-1)$. Act on the W factor by one of the above groups $\mathbb{Z}_m$ and on the second factor by rotation with angle $2\pi/m$.

The quotient $X_m := (W\times \operatorname{Spec}\mathbb{R}[z_1,z_2]/(z_1^2+z_2^2-1))/\mathbb{Z}_m$ is a unirational real variety. The resulting quotient $M_m := (S^1\times S^1\times S^1)/\mathbb{Z}_m$ is one of the components of $X_m(\mathbb{R})$. M_m is Seifert fibered over

S^2. In each of the above cases we have 3 or 4 multiple fibers with the following multiplicities:

$$\mathbb{Z}_6 : (6,3,2), \quad \mathbb{Z}_4 : (4,4,2), \quad \mathbb{Z}_3 : (3,3,3), \quad \mathbb{Z}_2 : (2,2,2,2).$$

In the $\mathbb{Z}_2$-case we can take a further quotient. Let another copy of $\mathbb{Z}_2$ act on W by $(x_1,x_2,y_1,y_2) \mapsto (-x_1,-x_2,y_1,-y_2)$. This is orientation reversing on the set of real points. The quotient $W(\mathbb{R})/\mathbb{Z}_2 \times \mathbb{Z}_2$ is homeomorphic to $\mathbb{RP}^2$. On $\operatorname{Spec}\mathbb{R}[z_1,z_2]/(z_1^2+z_2^2-1)$ one can act either as the identity or as $(z_1,z_2) \mapsto (z_1,-z_2)$. In both cases the quotient $(S^1 \times S^1 \times S^1)/\mathbb{Z}_2 \times \mathbb{Z}_2$ is Seifert fibered over $\mathbb{RP}^2$ with 2 fibers of multiplicity 2. The quotient is nonorientable in the fist case and orientable in the second case.

These 6 examples, together with the S^1-bundles found in (10.1) exhaust all classical Seifert fiber spaces with Euclidean geometry (cf. [Scott83, p.446]).

I have not tried to decide if these varieties are rational or not.

10.6 (Manifolds with Euclidean geometry)**.**

More generally, let M be any compact manifold with Euclidean geometry. That is, M is the quotient of Euclidean n-space by a group Γ of isometries. By the theorem of Bieberbach (cf. [Raghunathan72, 8.26]) Γ contains n linearly independent translations, so M is a quotient of the flat n-torus by a finite subgroup of $GL(n,\mathbb{Z})$. If we represent the n-torus as the real algebraic variety

$$T_n := \prod_{i=1}^{n} \operatorname{Spec}\mathbb{R}[x_i,y_i]/(x_i^2+y_i^2-1),$$

then $GL(n,\mathbb{Z})$ acts by algebraic automorphisms. So every quotient by a finite group is a unirational real algebraic variety.

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