REAL ALGEBRAIC THREEFOLDS I. TERMINAL SINGULARITIES

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Contents

1. Introduction

In real algebraic geometry, considerable attention has been paid to the study of real algebraic curves (in connection with Hilbert's 16th problem) and also to real algebraic surfaces. See [Viro90], [Riesler93] and the references there.

In higher dimensions one of the main avenues of investigation was initiated by [Nash52], and later developed by many others (see [AK92] for some recent directions). One of these results says that every compact differentiable manifold can be realized as the set of real points of an algebraic variety. [Nash52] posed the problem of obtaining similar results using a restricted class of varieties, for instance rational varieties.

The aim of this series of papers is to develop the theory of minimal models for real algebraic threefolds. This approach gives very strong information about the topology of real algebraic threefolds, and it also answers the above mentioned question of [Nash52].

For algebraic threefolds over $\mathbb{C}$, the minimal model program provides a very powerful tool. The method of the program is the following. (See [Kollár87] or [CKM88] for introductions.)

Starting with a smooth projective variety X, we perform a series of "elementary" birational transformations

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n$$

until we reach a variety X_n whose global structure is "simple". In essence the minimal model program allows us to investigate many questions in two steps: first study the effect of the "elementary" transformations and then consider the "simple" global situation.

In practice both of these steps are frequently rather difficult. For instance, we still do not have a complete list of all possible "elementary" steps, despite repeated attempts to obtain it.

A somewhat unpleasant feature of the theory is that the varieties X_i are not smooth, but have so called terminal singularities. In developing the theory of minimal models for real algebraic threefolds, we again have to understand the terminal singularities that occur.

The aim of this paper is to give a classification of terminal 3-fold singularities over $\mathbb{R}$. Minimal models serve only as a background, the proofs depend entirely on well established methods of singularity theory. I do not even use the definition of terminal singularities!

Subsequent papers of the series will examine the connections between the topology of a real algebraic threefold X and the minimal model program (see [Kollár97b]). Using the classification of this paper as the starting point, we will be able to exclude most terminal singularities from the minimal model of X if the set of real points $X(\mathbb{R})$ is orientable (or satisfies some weaker topological assumptions).

Terminal 3-fold singularities over $\mathbb{C}$ are completely classified. [Reid85] is a very readable introduction and survey. I will take the result of this classification as my definition, since the theory over $\mathbb{R}$ can be most naturally developed in this setting.

The classification is, in some sense, not complete. In a few cases I obtain unique normal forms (4.3), but in most cases this seems nearly impossible (see [Markushevich85] for a special case over $\mathbb{C}$). My aim is to write the singularities in a form that allows one to determine their topology over $\mathbb{R}$. The resulting lists and algorithms are given in sections 4–5.

It turns out that the normal forms of 3-fold terminal singularities are essentially the same over any field of characteristic zero. Thus in sections 2–3 I work with any subfield of $\mathbb{C}$.

As a consequence of the classification over $\mathbb{C}$, we know that 3-fold terminal singularities come in two types. Some are hypersurface singularities, and the others are quotients of these hypersurface singularities by a finite cyclic group. Accordingly, the classification over any field is done in two steps. Section 2 deals with terminal hypersurface singularities. These results are mostly routine generalizations of the theory over $\mathbb{C}$.

Quotient singularities frequently have "twisted" forms over a subfield of $\mathbb{C}$. "Twisted" forms do not appear for 3-fold terminal singularities, and so the classification ends up very similar to the one over $\mathbb{C}$.

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2. Terminal hypersurface singularities

Notation 2.1. For a field K let $K[[x_1,\dots,x_n]]$ denote the ring of formal power series in n variables over K. For $K=\mathbb{R}$ or $K=\mathbb{C}$, let $K\{x_1,\dots,x_n\}$ denote the ring of those formal power series which converge in some neighborhood of the origin.

For any $F\in K\{x_1,\dots,x_n\}$ the set $(F=0)$ is a germ of a real or complex analytic set. I will refer to it as a singularity. If $F\in K[[x_1,\dots,x_n]]$ then by the singularity $(F=0)$ I mean the scheme $\operatorname{Spec}_K K[[x_1,\dots,x_n]]/(F)$.

For a power series F, F_d denotes the degree d homogeneous part. The multiplicity, denoted by $\operatorname{mult}_0 F$, is the smallest d such that $F_d\neq 0$. If we write a power series as $F_{\geq d}$ then it is assumed that its multiplicity is at least d.

We say that two power series $F,G\in K[[x_1,\dots,x_n]]$ are *equivalent* over K if there is an automorphism of $K[[x_1,\dots,x_n]]$ given by $x_i\mapsto \phi_i(x_1,\dots,x_n)\in K[[x_1,\dots,x_n]]$ and an invertible $u(x_1,\dots,x_n)\in K[[x_1,\dots,x_n]]$ such that

$$u(x_1,\dots,x_n)G(x_1,\dots,x_n)=F(\phi_1,\dots,\phi_n).$$

Thus F and G are equivalent iff the corresponding singularities $(F=0)$ and $(G=0)$ are isomorphic (over K).

We have to pay special attention to cases when F and G are not equivalent over K but are equivalent over some larger field. For instance, $F=x_1^2+x_2^2$ and $G=x_1^2-x_2^2$ are not equivalent over $\mathbb{R}$ but are equivalent over $\mathbb{C}$.

If $K=\mathbb{R},\mathbb{C}$ and $F,G\in K\{x_1,\dots,x_n\}$ then I am mainly interested in equivalences where $u,\phi_i\in K\{x_1,\dots,x_n\}$.

If $F,G\in K\{x_1,\dots,x_n\}$ have isolated critical points at the origin, then F and G are equivalent in $K\{x_1,\dots,x_n\}$ iff they are equivalent in $K[[x_1,\dots,x_n]]$ (cf. [AGV85, p.121]), thus we do not have to be careful about this distinction.

Definition 2.2. Let K be a field of characteristic zero with algebraic closure $\bar{K}$. $(F(x,y,z)=0)$ is called a *Du Val* singularity (or a rational double point) iff over $\bar{K}$ it is equivalent to one of the standard forms

A_n: $x^2+y^2+z^{n+1}=0$ for $n\geq 0$;
D_n: $x^2+y^2z+z^{n-1}=0$ for $n\geq 4$;
E_6: $x^2+y^3+z^4=0$;
E_7: $x^2+y^3+yz^3=0$;
E_8: $x^2+y^3+z^5=0$.

Du Val singularities have many interesting intrinsic characterizations, (cf. [Durfee79, Reid85]) but I will not use this.

The following definition introduces our basic objects of study.

Definition 2.3. Let K be a field of characteristic zero with algebraic closure $\bar{K}$. $(F(x,y,z,t)=0)$ is called a *compound Du Val* singularity (or *cDV* for short) iff over $\bar{K}$ it is equivalent to

$$h(x,y,z)+tf(x,y,z,t)=0$$

where $(h=0)$ is a Du Val singularity.

$(F(x,y,z,t)=0)$ is called a cA_n (resp. cD_n or cE_n) singularity if its equation can be written as above with h having type A_n (resp. D_n or E_n), but it does not admit such representation with a smaller value of n. It is called a cA (resp. cD or cE) singularity if the value of n is not specified.

The reason we are interested in cDV singularities is the following:

Theorem 2.4. [Reid80] *A 3-dimensional hypersurface singularity over $\mathbb{C}$ is terminal iff it is an isolated cDV singularity.* □

The aim of this section is to develop "normal forms" for cDV singularities over any field K. This will then give "normal forms" for 3-dimensional terminal hypersurface singularities over K.

The proof is a rather standard application of the methods of [AGV85].

2.5 (How to simplify power series?).

We use 3 methods to bring power series to normal forms:

1. The Weierstrass preparation theorem. This is frequently stated only over $\mathbb{C}$, but it works over any field since the Weierstrass normal form is unique.
2. The elimination of the y^{n-1}-term from the polynomial $a_ny^n+a_{n-1}y^{n-1}+\cdots$ by a coordinate change $y\mapsto y-a_{n-1}/na_n$ when a_n is invertible.
3. Let $M_1,\dots,M_k$ be monomials in the variables $x_1,\dots,x_m$. Assume that $x_0M_1,\dots,x_0M_k$ are multiplicatively independent. Then any

power series of the form $\sum M_i \cdot u_i(x_1, \dots, x_m)$ where $u_i(0) \neq 0$ for all i is equivalent to $\sum M_i \cdot u_i(0)$ by a suitable coordinate change $x_i \mapsto x_i \cdot (\text{unit})$.

These elementary operations are sufficient to deal with the cA and cE cases. In the cD case the following generalization of (2.5.2) is needed.

Construction 2.6. In $K[[x_1, \dots, x_m]]$, assign positive integral weights to the variables $w(x_i) = w_i$. For a monomial set $w(\prod x_i^{c_i}) = \sum c_i w_i$. Write a power series in terms of its weighted homogeneous pieces $F = F_d + F_{d+1} + \cdots$. Our aim is to find a coordinate change such that F is transformed into a power series (or even a polynomial) of the form $\tilde{F}_d + \tilde{F}_{>d}$ where $\tilde{F}_d = F_d$ and $\tilde{F}_{>d}$ contains as few terms as possible.

Choose $g_i \in K[[x_1, \dots, x_m]]$ such that $w(g_i) = w(x_i) + e$ for some $e > 0$. Then

$$F(x_i + g_i) = F(x_i) + \sum g_i \frac{\partial F_d}{\partial x_i} + R_{>(d+e)}(x_i).$$

Repeatedly using this for higher and higher degrees, we see that, for every $N > 0$, F is equivalent to a power series $F^N + R_{>N}$ where F^N is a polynomial of degree N and no linear combination of the monomials in F^N can be written in the form $\sum g_i(\partial F_d / \partial x_i)$ as above.

In the ring of formal power series this can be continued indefinitely, thus at the end we can kill all the degree $> d$ elements of the Jacobian ideal

$$\Delta(F_d) := \left(\frac{\partial F_d}{\partial x_1}, \dots, \frac{\partial F_d}{\partial x_m} \right).$$

If $F \in K\{x_1, \dots, x_m\}$ defines an isolated singularity, then $F^N + R_{>N}$ is equivalent to F^N by an analytic coordinate change for $N \gg 1$ by Tougeron's lemma (cf. [AGV85, p.121]). Thus the final conclusion is the same.

Proposition 2.7. *Any power series $F_{\geq 2}(x_1, \dots, x_n)$ is equivalent to a power series*

$$a_1 x_1^2 + \cdots + a_k x_k^2 + G_{\geq 3}(x_{k+1}, \dots, x_n).$$

Proof. By a linear change of coordinates we can diagonalize F_2, thus we can assume that $F_2 = a_1 x_1^2 + \cdots + a_k x_k^2$. Repeatedly applying (2.5.1) to the variables $x_1, \dots, x_k$ we reach a situation when F is a quadratic polynomial in the variables $x_1, \dots, x_k$. (2.5.2) can then be used to eliminate the linear terms in $x_1, \dots, x_k$. $\square$

Theorem 2.8. *Assume that $F_{\geq 1}(x, y, z, t) \in K[[x, y, z, t]]$ defines a terminal singularity of type cA. Then F is equivalent to one of the following:*

cA_0: x.
cA_1: $ax^2+by^2+cz^2+dt^m$, *where* $abcd \neq 0$.
$cA_{>1}$: $ax^2+by^2+f_{\geq 3}(z,t)$, *where* $ab \neq 0$ *and* $f_{\geq 3}(z,t)$ *has no multiple factors. This has type* cA_n *for* $n = \operatorname{mult}_0 f - 1$.

In all these cases we can multiply through by a^{-1} to get a somewhat simpler form when the coefficient of x^2 is 1.

Proof. If $F_1 \neq 0$ then (2.5.1) gives cA_0. Thus assume that $F_1 = 0$. F has type cA, hence F_2 is a quadric of rank at least 2. If the rank is 2 then (2.7) gives the $cA_{>1}$ cases. $f_{\geq 3}(z,t)$ has no multiple factors since the singularity is isolated.

Assume finally that F_2 has rank 3 or 4. By (2.7) we can write F as $ax^2+by^2+cz^2+g(t)=0$. Using (2.5.3) we obtain $ax^2+by^2+cz^2+dt^m = 0$. □

Theorem 2.9. *Assume that* $F_{\geq 2}(x,y,z,t) \in K[[x,y,z,t]]$ *defines a terminal singularity of type* cD. *Then* F *is equivalent to one of the following:*

cD_4: $x^2+f_{\geq 3}(y,z,t)$, *where* f_3 *is not divisible by the square of a linear form.*
$cD_{>4}$: $x^2+y^2z+ayt^r+h_{\geq s}(z,t)$, *where* $a \in K$, $r \geq 3$, $s \geq 4$ *and* $h_s \neq 0$. *This has type* cD_n *where* $n = \min\{2r, s+1\}$ *if* $a \neq 0$ *and* $n = s+1$ *if* $a = 0$.

Proof. F_2 is a rank one quadric, thus in suitable coordinates the equation becomes $ax^2+f_{\geq 3}(y,z,t)$. Here $f_3 \neq 0$ is not the cube of a linear form since otherwise we would have a type cE singularity. If f_3 is not divisible by the square of a linear form then we have case cD_4.

If f_3 is divisible by the square of a linear form, then $f_3 = l_1^2 l_2$ for two linear forms l_i, and both of them are defined over K. We can change coordinates $l_1 \mapsto y$ and $l_2 \mapsto z$.

At this point our power series is x^2+y^2z+ (higher order terms). Assign weights $w(x)=3, w(y)=w(z)=2, w(t)=6$. The leading term is x^2+y^2z. Using (2.6) we can eliminate all monomials which contain y^2 or yz.

To see the last part, take the hyperplane section $t = \lambda z$. The term $ay\lambda^r z^r$ can be eliminated by a substitution $y \mapsto y + (a/2)\lambda^r z^{r-1}$. This creates a term $-(a/2)^2\lambda^{2r}z^{2r-1}$. The only problem could be that $h(z,\lambda z)$ has multiplicity $2r-1$ and there is cancellation. However, $h_{2r-1}(z,\lambda z) = z^{2r-1}h_{2r-1}(1,\lambda)$ is a polynomial of degree $2r-1$ in λ, thus it does not equal $-(a/2)^2\lambda^{2r}z^{2r-1}$. □

Theorem 2.10. *Assume that $F_{\geq 2}(x,y,z,t) \in K[[x,y,z,t]]$ defines a terminal singularity of type cE. Then F is equivalent to one of the following:*

cE_6: $x^2+y^3+yg_{\geq 3}(z,t)+h_{\geq 4}(z,t)$, *where* $h_4 \neq 0$.
cE_7: $x^2+y^3+yg_{\geq 3}(z,t)+h_{\geq 5}(z,t)$, *where* $g_3 \neq 0$.
cE_8: $x^2+y^3+yg_{\geq 4}(z,t)+h_{\geq 5}(z,t)$, *where* $h_5 \neq 0$.

Proof. F_2 is a rank one quadric by (2.3), thus in suitable coordinates the equation becomes $ax^2+f_{\geq 3}(y,z,t)$. Here $f_3 \neq 0$ and it is the cube of a linear form since otherwise we would have a type cD singularity. (2.5.1–2) gives an equation

$$ax^2+by^3+yg_{\geq 3}(z,t)+h_{\geq 4}(z,t).$$

Multiply the equation by a^3b^2 and then make the substitutions $x \mapsto xa^{-2}b^{-1}$ and $y \mapsto ya^{-1}b^{-1}$ to get the required normal forms. □

3. Higher index terminal singularities

The classification of non-hypersurface terminal 3-fold singularities over $\mathbb{C}$ relies on the following construction:

Let $\mathbb{Z}_n$ denote the cyclic group of order n and ϵ a primitive nth root of unity. Assume that $\mathbb{Z}_n$ acts on $\mathbb{C}^4$ by

$$\sigma : (x,y,z,t) \mapsto (\epsilon^{a_x}x, \epsilon^{a_y}y, \epsilon^{a_z}z, \epsilon^{a_t}t).$$

I will use the shorter notation $\frac{1}{n}(a_x,a_y,a_z,a_t)$ to denote such an action.

If $F(x,y,z,t)$ is equivariant with respect to this action, then $\mathbb{Z}_n$ acts on the hypersurface $(F=0)$ and we can take the quotient, denoted by $(F=0)/\frac{1}{n}(a_x,a_y,a_z,a_t)$.

By [Reid80], every terminal 3-fold singularity X over $\mathbb{C}$ is of the form $(F=0)/\frac{1}{n}(a_x,a_y,a_z,a_t)$, where F defines a terminal hypersurface singularity. The value of n is uniquely determined by X, it is called the *index* of X.

It is not easy to come up with a complete list of terminal 3-fold singularities, but by now the list is well understood; see [Reid85] for a good survey. It turns out that most actions do not produce terminal quotients and we have only a few cases:

Theorem 3.1. [Mori85] *Let $0 \in X$ be a 3-fold terminal nonhypersurface singularity over $\mathbb{C}$. Then $0 \in X$ is isomorphic to a singularity* *described by the following list:*

name	*equation*	*index*	*action*	*condition*
cA/n	$xy+f(z,t)$	n	$(r,-r,1,0)$	$(n,r)=1$
$cAx/2$	$x^2+y^2+f_{\geq 4}(z,t)$	2	$(0,1,1,1)$	
$cAx/4$	$x^2+y^2+f_{\geq 2}(z,t)$	4	$(1,3,1,2)$	$f_2(0,1)=0$
$cD/2$	$x^2+f_{\geq 3}(y,z,t)$	2	$(1,0,1,1)$	
$cD/3$	$x^2+f_{\geq 3}(y,z,t)$	3	$(0,2,1,1)$	$f_3(1,0,0)\neq 0$
$cE/2$	$x^2+y^3+f_{\geq 4}(y,z,t)$	2	$(1,0,1,1)$	

The equations have to satisfy 2 obvious conditions:

1. The equations define a terminal hypersurface singularity.
2. The equations are $\mathbb{Z}_n$-equivariant. (In fact $\mathbb{Z}_n$-invariant, except for $cAx/4$.)

If we work over a field K which does not contain the nth roots of unity, then the action $\frac{1}{n}(a_1,\dots,a_m)$ is not defined over K. There is, however, another way of loking at the quotient which does make sense over any field.

Any action of the cyclic group $\mathbb{Z}_n$ on $\mathbb{C}^m$ defines a $\mathbb{Z}_n$-grading w of $\mathbb{C}[[x_1,\dots,x_m]]$ by

$$w(\prod x_i^{c_i})=a \quad \text{iff} \quad \sigma(\prod x_i^{c_i})=\epsilon^a\cdot\prod x_i^{c_i}.$$

If F is $\mathbb{Z}_n$-equivariant then $(F)\subset\mathbb{C}[[x_1,\dots,x_m]]$ is a homogeneous ideal, hence the grading descends to a grading of $\mathbb{C}[[x_1,\dots,x_m]]/(F)$. The ring of functions on the quotient $(F=0)/\frac{1}{n}(a_1,\dots,a_m)$ can be identified with the ring of grade zero elements of $\mathbb{C}[[x_1,\dots,x_m]]/(F)$.

If K is any field, $n\in\mathbb{N}$ and $a_i\in\mathbb{Z}$, then we obtain a $\mathbb{Z}_n$-grading $w=w(a_1,\dots,a_m)$ of $K[[x_1,\dots,x_m]]$ (or of $\mathbb{R}\{x_1,\dots,x_m\}$) by

$$w(\prod x_i^{c_i})=\sum c_ia_i\in\mathbb{Z}_n.$$

Let $R\subset K[[x_1,\dots,x_m]]$ denote the subring of grade zero elements. Then $\operatorname{Spec}_K R$ gives a singularity over K which is denoted by

$$\mathbb{A}^m/\tfrac{1}{n}(a_1,\dots,a_m).$$

(When $K=\mathbb{R}$, one might be tempted to write $\mathbb{R}^m/\frac{1}{n}(a_1,\dots,a_m)$ instead. However, the set of real points of $\mathbb{A}^m/\frac{1}{n}(a_1,\dots,a_m)$ is not in any sense a quotient of the set $\mathbb{R}^n$ (cf. (5.3)), so this may lead to confusion.)

If $F\in K[[x_1,\dots,x_m]]$ is graded homogeneous, then w gives a grading of $K[[x_1,\dots,x_m]]/(F)$. Let $R/(R\cap(F))\subset K[[x_1,\dots,x_m]]/(F)$ be the subring of grade zero elements. $\operatorname{Spec}_K R/(R\cap(F))$ defines a singularity over K. By construction,

$$\operatorname{Spec}_K R/(R\cap(F))\times_{\operatorname{Spec} K}\operatorname{Spec}\bar{K}\cong(F=0)/\tfrac{1}{n}(a_1,\dots,a_m).$$

Thus $\operatorname{Spec}_K R$ is a terminal singularity over K iff $(F=0)/\frac{1}{n}(a_1,\dots,a_m)$ is a terminal singularity over $\bar{K}$.

Under certain conditions, every K-form of a quotient is obtained this way:

Theorem 3.2. *K be a field of characteristic zero with algebraic closure $\bar{K}$. Let $\mathbb{Z}_n$ denote the cyclic group of order n and ϵ a primitive nth root of unity. Assume that $\mathbb{Z}_n$ acts on $\bar{K}^m$ by $\sigma:(x_i)\mapsto(\epsilon^{w_i}x_i)$. Let $F\in\bar{K}[[x_1,\dots,x_m]]$ be equivariant with respect to this action, and assume that the fixed point set of σ has codimension at least 2 in $(F=0)$. Assume in addition that*

$$w(F)-\sum w_i \quad \text{is relatively prime to} \quad n.$$

Let $0\in X$ be a singularity over K such that

$$X\times_{\operatorname{Spec}K}\operatorname{Spec}\bar{K}\cong(F=0)/\tfrac{1}{n}(w_1,\dots,w_m).$$

Then there is an $F^K\in K[[x_1,\dots,x_m]]$ such that F and F^K are equivalent over $\bar{K}$ and

$$X\cong(F^K=0)/\tfrac{1}{n}(w_1,\dots,w_m).$$

Reid pointed out that the right way to think about this result is the following. If Z_K is a projective variety over a field K such that $Z_{\bar{K}}$ embeds into $\mathbb{P}^m_{\bar{K}}$ then Z_K need not embed into $\mathbb{P}^m_K$. This is, however, true, if the embedding is given by the canonical line bundle. Thus if $\operatorname{Pic}(Z_{\bar{K}})$ is generated by the canonical line bundle, then we do not expect any "unusual" K-forms of Z . As we see in the proof, n and $w(F)-\sum w_i$ are relatively prime iff the canonical line bundle of $X_{\bar{K}}\setminus\{0\}$ generates the Picard group.

The following examples show that the relative prime condition is necessary:

Example 3.3. Consider the quotient singularity $\mathbb{C}[u,v]/\frac{1}{n}(1,-1)$. It is isomorphic to $(xy-z^n=0)$ via the substitutions $x=u^n, y=v^n, z=uv$. Over $\mathbb{C}$ we have a Du Val singularity $A_{n-1}=(x^2+y^2+z^n=0)$.

Over $\mathbb{R}$ we see that $(x^2-y^2-z^n=0)\cong\mathbb{A}^2/\frac{1}{n}(1,-1)$. Another $\mathbb{R}$-form of A_{n-1} is $x^2+y^2-z^n$. This can also be obtained as a quotient, but this time we act on $\mathbb{A}^2$ by rotation with angle $2\pi/n$.

Finally, if n is even, then there is another $\mathbb{R}$-form of A_{n-1} given by $(x^2+y^2+z^n=0)$. The only $\mathbb{R}$-point is the origin, so we do not even have a nonzero map $\mathbb{R}^2\to(x^2+y^2+z^n=0)$.

As another example, take the 4-dimensional terminal singularity $\mathbb{C}^4/\frac{1}{n}(a,-a,b,-b)$ for any $(ab,n)=1$. It has another $\mathbb{R}$-form given as $\mathbb{A}^4/\mathbb{Z}_n$ where we act on the first two coordinates by rotation with angle $2a\pi/n$ and on the last two coordinates by rotation with angle $2b\pi/n$.

In some special cases there are further $\mathbb{R}$-forms. Take for instance $\mathbb{C}^4/\frac{1}{2}(1,1,1,1)$. This can be realized as the cone over $\mathbb{CP}^3$ embedded by the quadrics into $\mathbb{CP}^9$.

Let $C \subset \mathbb{RP}^2$ be a smooth conic. Taking symmetric powers we have $\operatorname{Sym}^3 C \subset \operatorname{Sym}^3 \mathbb{RP}^2$ and $\operatorname{Sym}^3 H^0(\mathbb{RP}^2, \mathcal{O}(1))$ embeds it to $\mathbb{RP}^9$. If C has a real point, then $\operatorname{Sym}^3 C \cong \mathbb{RP}^3$ and we get the Veronese embedding. If C has no real points then the image is a variety over $\mathbb{R}$ without real points. The cone over it is a real form of $\mathbb{A}^4/\frac{1}{2}(1,1,1,1)$ with an isolated real point at the origin.

Proof of (3.2). Let $S = \bar{K}[[x_1, \dots, x_m]]/(F)$ be the ring of functions on $\tilde{X}_{\bar{K}} := (F = 0)$. The $\mathbb{Z}_n$-action defines a $\mathbb{Z}_n$-grading $S = \sum_{i=0}^{n-1} S_i$. S_0, the ring of grade 0 elements, is exactly the ring of functions on $X_{\bar{K}}$. Our aim is to find an algebraic way of reconstructing S from S_0, which then hopefully generalizes to nonclosed fields.

There is another summand which can be easily seen algebraically. Set $d = w(F) - \sum w_i$. Note that

$$\frac{1}{\partial F/\partial x_m} dx_1 \wedge \dots \wedge dx_{m-1}$$

is a local generator of the dualizing sheaf ω_S and it has weight $-d$. Thus

$$\omega_{S_0} \cong S_d \frac{1}{\partial F/\partial x_m} dx_1 \wedge \dots \wedge dx_{m-1}.$$

Once S_d is determined, we obtain S_{jd} as follows. The multiplication maps

$$S_a \otimes_{S_0} S_b \to S_{a+b} \quad \text{(subscripts modulo } n)$$

are isomorphisms over the open set where the $\mathbb{Z}_n$-action is free. We assumed that the complement has codimension at least 2, thus $S_{jd} \cong S_d^{[j]}$, where $S_d^{[j]}$ denotes the double dual of $S_d^{\otimes j}$. If d and n are relatively prime, then we obtain every summand S_i this way. In particular,

$$S = \sum_{i=0}^{n-1} S_i \cong \sum_{j=0}^{n-1} \omega_{S_0}^{[j]}.$$

Over an arbitrary field, we can thus proceed as follows. Let ω_X be the dualizing sheaf of X. This is also the reflexive sheaf $\mathcal{O}_X(K_X)$ where K_X is the canonical class.

Then $\omega_X^{[n]}$ is isomorphic to $\mathcal{O}_X$, where n is the index. (We know this over $\bar{K}$. Tensoring with a field extension is faithfully flat, and a finite module over a local ring is free iff it is free after a faithfully flat extension, cf. [Matsumura80, 4.E].) Fix such an isomorphism $s : \omega_X^{[n]} \to \mathcal{O}_X$.

Consider the $\mathcal{O}_X$-algebra $R(X,s) := \sum_{j=0}^{n-1} \omega_X^{[j]}$, where multiplication for $j+k \geq n$ is given by

$$\omega_X^{[j]} \otimes \omega_X^{[k]} \mapsto \omega_X^{[j+k]} \cong \omega_X^{[n]} \otimes \omega_X^{[j+k-n]} \xrightarrow{s\otimes 1} \omega_X^{[j+k-n]}.$$

This has a $\mathbb{Z}_n$ grading by declaring $\omega_X^{[j]}$ to have grade j.

(Note. Two isomorphisms $s_1, s_2 : \omega_X^{[r]} \to \mathcal{O}_X$ differ by an invertible function $h \in \mathcal{O}_X^*$. If h is an nth power, then the resulting algebras $R(X, s_i)$ are isomorphic, but they need not be isomorphic otherwise. This is connected with the topological aspects observed in (5.3).)

Over $\bar{K}$, $R(X,s)$ is isomorphic to $\mathcal{O}_{\tilde{X}}$. $R(X,s)$ is a K-form of $\mathcal{O}_{\tilde{X}}$, so $R(X,s)$ is an algebra of the form $K[[x_1,\dots,x_m]]/(F^K)$, where F and F^K are equivalent over $\bar{K}$.

The grading lifts to a grading of $K[[x_1,\dots,x_m]]$ such that F^K is graded homogeneous. We can choose x_i to be homogeneous. □

As a corollary, we obtain the following classification of terminal 3-fold nonhypersurface singularities over nonclosed fields:

Theorem 3.4. *Let K be a field of characteristic zero and $0 \in X$ a 3-fold terminal nonhypersurface singularity over K. Then $0 \in X$ is isomorphic over K to a singularity described by the following list:*

name	*equation*	*index*	*weights*	*condition*
$cA/2$	$ax^2+by^2+f(z,t)$	2	$(1,1,1,0)$	
cA/n	$xy+f(z,t)$	$n\geq 3$	$(r,-r,1,0)$	$(n,r)=1$
$cAx/2$	$ax^2+by^2+f_{\geq 4}(z,t)$	2	$(0,1,1,1)$	
$cAx/4$	$ax^2+by^2+f_{\geq 2}(z,t)$	4	$(1,3,1,2)$	$f_2(0,1)=0$
$cD/2$	$x^2+f_{\geq 3}(y,z,t)$	2	$(1,0,1,1)$	
$cD/3$	$x^2+f_{\geq 3}(y,z,t)$	3	$(0,2,1,1)$	$f_3(1,0,0)\neq 0$
$cE/2$	$x^2+y^3+f_{\geq 4}(y,z,t)$	2	$(1,0,1,1)$	

(In the $cA/2$-case, if $-ab$ is a square, the equation can be brought to the form $xy+f(z,t)$, but not otherwise. This is why the $cA/2$ and cA/n cases are treated separately.)

Complement 3.5. *The corresponding quotient singularity is terminal iff the equations satisfy 2 obvious conditions:*

1. *The equations define a terminal hypersurface singularity.*
2. *The equations are graded homogeneous.*

With these assumptions, a terminal singularity corresponds to exactly one case on the above list.

Proof. By looking at the list of (3.1), we see that the assumptions of (3.2) are satisfied. Hence we know that X is of the form $(F^K = 0)/\frac{1}{n}(a_x, a_y, a_z, a_t)$ where $\frac{1}{n}(a_x, a_y, a_z, a_t)$ is on the list of (3.1).

Once we know a $\mathbb{Z}_n$-grading on $K[[x, y, z, t]]$ and a graded homogeneous power series F^K, we can try to bring it to some normal form using the methods (2.5) and (2.6). They are set up in such a way that if F^K is homogeneous in a $\mathbb{Z}_n$-grading then all coordinate changes respect the grading.

The proofs of (2.8, 2.9, 2.10) remain unchanged. The only difference is in (2.7). It is not true that a quadratic form can be diagonalized using a linear transformation which respects the $\mathbb{Z}_n$-grading. The best one can achieve is a sum of forms in disjoint sets of variables $\sum q_i$ where each q_i is either au_i^2 or $u_i v_i$. The latter case is necessary iff the two variables have different $\mathbb{Z}_n$-grading.

In the cD and cE cases the quadric has rank 1, so it can be diagonalized.

In the $cA/2$ and $cAx/2$ cases every grade 0 quadric is diagonalizable.

In the $cAx/4$ case x^2, xz, y^2, z^2 are the only grade 2 quadratic monomials. A quadratic form like this can again be diagonalized.

Finally let us look at the cA/n-case for $n \geq 3$. The only grade 0 degree 2 monomials are xy, t^2 and xz if $r = -1$ or yz if $r = 1$. We need to get a rank ≥ 2 quadric, so xy (or xz if $r = -1$, yz if $r = 1$) must appear. In the $r = \pm 1$ case we may need to perform a linear change of variables to get the normal form $xy + f(z, t)$. $\square$

4. The topology of terminal hypersurface singularities

Let $0 \in X$ be a real singularity. Its real points $X(\mathbb{R})$ form a topological space, which can be triangulated (cf. [BCR87, 9.2]). We may assume that 0 is a vertex of the triangulation. Then locally near 0, $X(\mathbb{R})$ is PL-homeomorphic to the cone over a simplicial complex $L = L(X(\mathbb{R}))$, which is called the *link* of 0 in $X(\mathbb{R})$. The local topology of $X(\mathbb{R})$ at 0 is thus determined by L.

In general one needs to contemplate the dependence of L on various choices made. I am mainly interested in the case when X is a 3-dimensional isolated singularity. In this case L is a compact surface (without boundary) and so L and $X(\mathbb{R})$ determine each other up to homeomorphism.

The aim of this section is to classify terminal singularities over $\mathbb{R}$ according to their local topology. To be precise, we give a classification in the cA cases and provide a procedure in the cD and cE cases which reduces the 3-dimensional problem to some questions about plane curve singularities.

Notation 4.1. $M \sim N$ denotes that M and N are homeomorphic.

$\uplus$ denotes disjoint union. $M \uplus rN$ denotes the disjoint union of M and of r copies of N.

M_g denotes the unique compact, closed and orientable surface of genus g.

We start with a general lemma.

Lemma 4.2. *Let X be a smooth real hypersurface. Then $X(\mathbb{R})$ is orientable.*

Proof. Let $X = (f = 0)$ be a real equation where $f \in \mathbb{R}[x_1, \dots, x_n]$ or $f \in \mathbb{R}\{x_1, \dots, x_n\}$. At each point $p \in X$, X divides a neighborhood of p into two halves. f is positive on one half and negative on the other half. Choosing a sign thus determines an orientation. $\square$

Theorem 4.3. *The following table gives a complete list of 3-dimensional terminal singularities of type cA_1 over $\mathbb{R}$.*

In the table $n \geq 1$. Case 4, $n = 1$ and case 5, $n = 1$ are isomorphic. Aside from this, two singularities are isomorphic iff they correspond to the same case and the same value of n.

case	*equation*	L
$cA_1(1)$	$x^2 + y^2 + z^2 \pm t^{2n+1}$	S^2
$cA_1(2)$	$x^2 + y^2 - z^2 \pm t^{2n+1}$	S^2
$cA_1(3)$	$x^2 + y^2 + z^2 + t^{2n}$	$\emptyset$
$cA_1(4)$	$x^2 + y^2 + z^2 - t^{2n}$	$S^2 \uplus S^2$
$cA_1(5)$	$x^2 + y^2 - z^2 + t^{2n}$	$S^2 \uplus S^2$
$cA_1(6)$	$x^2 + y^2 - z^2 - t^{2n}$	$S^1 \times S^1$

Proof. The equations follow from (2.8), once we note that after multiplying by ± 1 we may assume that the quadratic part has at least 2 positive eigenvalues.

The topology is easy to figure out. Since all the claims are special cases of the next result, I discuss them in more detail there. $\square$

Theorem 4.4. *A 3-dimensional terminal singularity of type $cA_{>1}$ over $\mathbb{R}$ is equivalent to a form*

$$x^2 \pm y^2 \pm h(z,t) \prod_{i=1}^{m} f_i(z,t) = 0,$$

where the f_i are irreducible power series (over $\mathbb{R}$) such that $\mathbb{R}^2 \supset (f_i(z,t) = 0) \neq \{0\}$ and $h(z,t)$ is positive on $\mathbb{R}^2 \setminus \{0\}$. The following table gives a complete list of the possibilities for the topology of

$X(\mathbb{R})$.

case	*equation*	$L(X(\mathbb{R}))$
$cA^+_{>1}(0,+)$	x^2+y^2+h	$\emptyset$
$cA^+_{>1}(0,-)$	x^2+y^2-h	$S^1\times S^1$
$cA^+_{>1}(m)$	$x^2+y^2\pm hf_1\cdots f_m$	$\uplus mS^2$
$cA^-_{>1}(0)$	$x^2-y^2\pm h$	$S^2\uplus S^2$
$cA^-_{>1}(m)$	$x^2-y^2\pm hf_1\cdots f_m$	M_{m-1}

Proof. We already have the form $x^2\pm y^2+f(z,t)$ by (2.8). Write f as a product of irreducible power series over $\mathbb{R}$. Those factors which do not vanish on $\mathbb{R}^2\setminus\{0\}$ are multiplied together to get h. By writing $\pm h$ we may assume that h is positive on $\mathbb{R}^2\setminus\{0\}$. (Since the signs of the other factors are not fixed, the sign of h matters only if there are no other factors.) Let f_i be the remaining factors of f.

Assume now that we are in the cA^+-case: $x^2+y^2\pm h\prod f_i$. Projection to the (z,t)-plane is a proper map whose fibers are as follows:

1. S^1 if $\pm h(z,t)\prod f_i(z,t)<0$,
2. a point if $\pm h(z,t)\prod f_i(z,t)=0$,
3. empty if $\pm h(z,t)\prod f_i(z,t)>0$.

If $m=0$ then $X(\mathbb{R})\setminus\{0\}$ is a circle bundle over either $\mathbb{R}^2\setminus\{0\}$ or over the empty set. The first case gives $L\sim S^1\times S^1$ by (4.2).

If $m>0$, we have to describe the semi-analytic set $U:=(\prod f_i(z,t)\le 0)\subset\mathbb{R}^2$. Semi-analytic sets can be triangulated (cf. [BCR87, 9.2]), thus in a neighborhood of the origin, U is the cone over $U\cap(z^2+t^2=\epsilon)$.

Each $(f_i=0)$ is an irreducible curve germ over $\mathbb{R}$, thus homeomorphic to $\mathbb{R}^1$. So each f_i has 2 roots on the circle $(z^2+t^2=\epsilon)$. Hence $U\cap(z^2+t^2=\epsilon)$ is the disjoint union of m closed arcs. Therefore L has m connected components, each homeomorphic to S^2.

The second possibility is the cA^--case: $x^2-y^2-h\prod f_i$. (The two choices of $\pm h$ are equivalent by interchanging x and y.) Here we project to the (y,z,t)-hyperplane. The fiber over a point (y,z,t) is

1. 2 points if $y^2+h(z,t)\prod f_i(z,t)>0$,
2. 1 point if $y^2+h(z,t)\prod f_i(z,t)=0$,
3. empty if $y^2+h(z,t)\prod f_i(z,t)<0$.

Thus we have to determine the region

$$U:=\{y^2+h(z,t)\prod f_i(z,t)\ge 0\}\subset(y^2+z^2+t^2=\epsilon)\sim S^2,$$

and then take its double cover to get L.

If $m=0$ then $U=S^2$ and so $L=S^2\uplus S^2$. If $m>0$ then $h\prod f_i$ is negative on m disjoint arcs in the circle $(z^2+t^2=\epsilon)$, and $y^2+h(z,t)\prod f_i(z,t)$ is negative in contractible neighborhoods of these intervals. Thus $U = S^2 \setminus (m \text{ discs})$ and so L is a surface of genus $m-1$, orientable by (4.2). □

Example 4.5. It is instructive to consider the following incorrect approach to the topology of cD and cE-type singularities. I illustrate it in the cE_6-case.

Over $\mathbb{R}$, a surface singularity of type E_6 is $x^2 + y^3 \pm z^4$. For either choice of sign, projection to the (x, z)-plane is a homeomorphism. Consider a cE_6-type point X. If $L(X(\mathbb{R}))$ has several connected components, then a suitable hyperplane intersects at least two of them. By a small perturbation we obtain an E_6-singularity as the intersection, thus we conclude that $L(X(\mathbb{R}))$ is connected. This is especially suggestive if we note that instead of a plane we could use a small perturbation of any smooth hypersurface.

Unfortunately the conclusion is false, as we see in (4.16). $L(X(\mathbb{R}))$ can have several components, and some of them are not seen by any hypersurface section. These look like very "thin" cones, as opposed to the main component which is "thick". It would be interesting to give precise meaning to this observation and to see its significance in the study of singularities.

The following approach to the topology of cD and cE-type singularities is taken from [AGV85, Sec.12].

4.6 (Deformation to the weighted tangent cone)**.**

Let $X := (f(x_1, \dots, x_n) = 0)$ be a hypersurface singularity. For simplicity of notation I assume that f converges for $|x_i| < 1+\delta$. Assign integral weights to the variables $w(x_i) = w_i$ and write f as the sum of weighted homogeneous pieces

$$f = f_d + f_{d+1} + f_{d+2} + \cdots,$$

where f_s is weighted homogeneous of degree s. For a parameter $\lambda \neq 0$ set

$$\begin{aligned} f^\lambda(x_1, \dots, x_n) &:= \lambda^{-d} f(\lambda^{w_1} x_1, \dots, \lambda^{w_n} x_n) \\ &= f_d + \lambda f_{d+1} + \lambda^2 f_{d+2} + \cdots \end{aligned}$$

This suggests that if we define $f^0 := f_d$ then

$$X^\lambda := (f^\lambda = 0) \quad \text{for} \quad \lambda \in \mathbb{R}$$

is a "nice" family of hypersurface singularities. For $\lambda \neq 0$ they are all isomorphic to $(f = 0)$ and for $\lambda = 0$ we obtain the weighted tangent cone $(f_d = 0)$.

This can be used to determine the topology of $X(\mathbb{R})$ in 2 steps. First describe X^0 and then try to relate X^λ and X^0 for small values of λ.

Let w be a common multiple of the w_i and set $u_i = w/w_i$.

Proposition 4.7. *Notation as above. Assume that $X = (f = 0)$ is an isolated hypersurface singularity.*

Then there is a $0 < \lambda_0$ such that for every $0 < \lambda \leq \lambda_0$

1. *$L^\lambda := X^\lambda \cap (\sum x_i^{2u_i} = 1)$ is smooth, and*
2. *$X^\lambda \cap (\sum x_i^{2u_i} \leq 1)$ is homeomorphic to the cone over L^λ.*

Proof. The map $\mathbb{R}^m \to \mathbb{R}^+$ given by $(x_1, \dots, x_n) \mapsto \sum x_i^{2u_i}$ is proper. Thus its restriction to X^λ is also proper. The proposition follows once we establish that the resulting map

$$t : X^\lambda \to \mathbb{R}$$

has no critical points with critical value in $(0, 1]$ for $0 < \lambda \leq \lambda_0$.

The critical values of a real algebraic morphism form a semi-algebraic set (cf. [BCR87, 9.5]), thus there is a $0 < \mu_0$ such that $t : X^1 \to \mathbb{R}$ has no critical values in $(0, \mu_0]$. The following diagram is commutative

$$\begin{array}{ccc} X^\lambda & \stackrel{x_i \mapsto \lambda^{-w_i} x_i}{\longrightarrow} & X^1 \\ t \downarrow & & \downarrow t \\ \mathbb{R} & \stackrel{s \mapsto \lambda^{-w} s}{\longrightarrow} & \mathbb{R} \end{array}$$

which shows that (4.7) holds with $\lambda_0 = \mu_0^{1/w}$. □

So far we have not done much, but the advantage of this approach is that we can view L^λ as a deformation of the compact real algebraic variety L^0. If L^0 is smooth then this deformation is locally trivial differentiably. Thus we obtain:

Corollary 4.8. *Assume that $(f_d = 0)$ defines an isolated singularity. Then L^λ is diffeomorphic to L^0.* □

This is sufficient to describe the the topology of "general" members of several families of terminal singularities:

Corollary 4.9. *Let X be a terminal singularity given by one of the following equations:*

cD_4: $x^2 + f_{\geq 3}(y, z, t)$, *where $f_3 = 0$ has no real singular point.*

cE_6: $x^2 + y^3 + yg_{\geq 3}(z, t) + h_{\geq 4}(z, t)$, *where h_4 has no multiple real linear factor.*

cE_8: $x^2 + y^3 + yg_{\geq 4}(z, t) + h_{\geq 5}(z, t)$, *where h_5 has no multiple real linear factor.*

Then:

cD_4: $L(X(\mathbb{R})) \sim S^2$ *if $(f_3 = 0) \subset \mathbb{RP}^2$ has one connected component and* $L(X(\mathbb{R})) \sim S^2 \uplus (S^1 \times S^1)$ *if $(f_3 = 0)$ has two connected components.*

cE_6: $L(X(\mathbb{R})) \sim S^2$.
cE_8: $L(X(\mathbb{R})) \sim S^2$.

Proof. We use deformation to the weighted tangent cone with (suitable integral multiples of the) weights $(1/2, 1/3, 1/3, 1/3)$ in the cD_4-case, $(1/2, 1/3, 1/4, 1/4)$ in the cE_6-case, and $(1/2, 1/3, 1/5, 1/5)$ in the cE_8-case. The equations for X^0 are $x^2 + f_3(y, z, t) = 0$, $x^2 + y^3 + h_4(z, t) = 0$ and $x^2 + y^3 + h_5(z, t) = 0$. Our conditions guarantee that $X^0(\mathbb{R})$ has isolated singularities, thus it is sufficient to determine $L(X^0(\mathbb{R}))$.

In the cE-cases, projection to the (x, z, t) hyperplane is a homeomorphism from $X^0(\mathbb{R})$ to $\mathbb{R}^3$, thus $L(X^0(\mathbb{R})) \sim S^2$

In the cD_4-cases we project to the (y, z, t)-hyperplane. As in the proof of (4.4), we can get $L(X^0(\mathbb{R}))$ once we know the set $U \subset (y^2+z^2+t^2 = 1)$ where f_3 is nonnegative. The boundary ∂U doubly covers the projective curve $(f_3 = 0) \subset \mathbb{RP}^3$. If $(f_3 = 0) \subset \mathbb{RP}^2$ has one connected component then it is a pseudo-line and ∂U is a connected double cover, hence U is a disc. If $(f_3 = 0)$ has two connected components, then one is a pseudo-line, the other an oval. ∂U has 3 connected components, and U is a disc plus an annulus. Thus $L(X^0(\mathbb{R})) \sim S^2 \uplus (S^1 \times S^1)$. □

Remark 4.10. In the cE cases of the above example, projection to the (x, z, t) plane is a homeomorphism from $X^0(\mathbb{R})$ to $\mathbb{R}^3$ even if h_4 or h_5 have multiple factors. In these cases, however, we can not conclude that $X(\mathbb{R})$ is also homeomorphic to $\mathbb{R}^3$. In fact we see in (4.16) that this is not always true.

Similar arguments work in some of the $cD_{>4}$-cases:

Corollary 4.11. *Let X be a terminal singularity given by equation $x^2 + y^2 z + h_{\geq s}(z, t)$, where $z \nmid h_s$ and h_s has no multiple real linear factors. Let s be the number of real linear factors of h_s. There are three cases:*

1. $s = 2r + 1$ *and* $L(X(\mathbb{R})) \sim M_r \uplus rS^2$*;*
2. $s = 2r$, $h(0, 1) < 0$ *and* $L(X(\mathbb{R})) \sim M_r \uplus (r - 1)S^2$*;*
3. $s = 2r$, $h(0, 1) > 0$ *and* $L(X(\mathbb{R})) \sim M_{r-1} \uplus rS^2$*.*

Proof. We use deformation to the weighted tangent cone with weights $(1/2, (s - 1)/2s, 1/s, 1/s)$. Thus we need to figure out the topology of $x^2 + y^2 z + h_{(n-1)}(z, t) = 0$. As before, this reduces to understanding the set where $y^2 z + h_s(z, t) \leq 0$. This can be done by projecting to the (z, t) plane. Details are left to the reader. □

4.12 (Weights for the cD and cE cases)**.**

For many terminal singularities one cannot choose weights so that the weighted tangent cone has an isolated singularity at the origin, but in all cases it is possible to choose weights so that the weighted tangent cone has at worst 1-dimensional singular locus:

name	equation is x^2+	$w(y)$	$w(z)$	$w(t)$
cD_4	$f_{\geq 3}(y,z,t)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$cD_{>4}(1)$	$y^2z + ayt^r + h_{\geq s}(z,t)$	$\frac{s-1}{2s}$	$\frac{1}{s}$	$\frac{1}{s}$
$cD_{>4}(2)$	$y^2z \pm yt^r + h_{\geq s}(z,t)$	$\frac{r-1}{2r-1}-\epsilon$	$\frac{1}{2r-1}+2\epsilon$	$\frac{1}{2r-1}+\frac{\epsilon}{r}$
cE_6	$y^3 + yg_{\geq 3}(z,t) + h_{\geq 4}(z,t)$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{4}$
cE_7	$y^3 + yg_{\geq 3}(z,t) + h_{\geq 5}(z,t)$	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{2}{9}$
cE_8	$y^3 + yg_{\geq 4}(z,t) + h_{\geq 5}(z,t)$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{5}$

In the $cD_{>4}$ case we assume that $h_s \neq 0$ and use the first weight sequence if $a = 0$ or $2r > s+1$ and the second weight sequence if $2r \leq s+1$, where ϵ is a small positive number. (We could use $\epsilon = 0$ except when $2r = s+1$.) Let (w_1, w_2, w_3, w_4) be integral multiples of these weights. The weighted tangent cone, and its singularities are the following:

name	weighted tangent cone	singularities
cD_4	$x^2 + f_3(y,z,t)$	at singular pts of $(f_3 = 0)$
$cD_{>4}(1)$	$x^2 + y^2z + h_s(z,t)$	at multiple real factors of zh_s
$cD_{>4}(2)$	$x^2 + y^2z \pm yt^r$	at the z-axis
cE_6	$x^2 + y^3 + h_4(z,t)$	at multiple real factors of h_4
cE_7	$x^2 + y^3 + yg_3(z,t)$	at real factors of g_3
cE_8	$x^2 + y^3 + h_5(z,t)$	at multiple real factors of h_5

These equations have the form $x^2 + F(y,z,t)$ and the deformation to the weighted tangent cone leaves this form invariant:

$$x^2 + F^\lambda(y,z,t) = x^2 + \lambda^{-d} F(\lambda^{w_2} y, \lambda^{w_3} z, \lambda^{w_4} t).$$

Set

$$U^\lambda := \{(y,z,t) | F^\lambda(y,z,t) \leq 0 \subset (y^{2u_2} + z^{2u_3} + t^{2u_4} = 1)\}.$$

U^λ is a semi-algebraic set and its boundary is the real algebraic curve

$$C^\lambda := \{(y,z,t) | F^\lambda(y,z,t) = 0 \subset (y^{2u_2} + z^{2u_3} + t^{2u_4} = 1)\}.$$

We have established the following:

Proposition 4.13. *C^λ is a deformation of the real algebraic curve C^0 inside the smooth real algebraic surface $(y^{2u_2} + z^{2u_3} + t^{2u_4} = 1)$.* $\square$

4.14 (Deformations of singular real curves).

The deformations of real algebraic curves can be understood in two steps (cf. [Viro90]). Put small discs around the singularities. Outside the discs all small deformations are topologically trivial and inside the discs we have a local problem involving real curve singularities. Here we have the advantage that $(y^{2u_2} + z^{2u_3} + t^{2u_4} = 1)$ is a compact *affine* algebraic surface, thus we can choose the local deformations independently and they can always be patched together.

Thus we can describe the possible cases for C^λ, and thereby the topological types of the corresponding 3-dimensional terminal singularities, if we can describe the deformations of the occurring real plane curve singularities. By looking at the equations we see that the only singularities that we have to deal with are the 2-variable versions of the Du Val singularities:

A_n: $y^2 \pm z^{n+1} = 0$;
D_n: $y^2 z \pm z^{n-1} = 0$;
E_6: $y^3 + z^4 = 0$;
E_7: $y^3 \pm yz^3 = 0$;
E_8: $y^3 + z^5 = 0$.

For all of these cases, a complete list of the topological types of real deformations is known [Chislenko88]. The list can also be found in [Viro90, Figs. 16–28], which contains many further examples.

Example 4.15. Consider for example the cD_4 cases. The various possibilities for f_3 are easy to enumerate. The most interesting is $f_3 = yzt$. Here C^0 is the intersection of $(yzt = 0)$ with $(y^6 + z^6 + t^6 = 1)$. We have 6 singular points of type $u^2 - v^2 = 0$. At each of them we can choose a deformation $u^2 - v^2 \pm \epsilon = 0$. This gives 2^6 possibilities. The symetries of the octahedron act on the configurations so it is easy to get a complete list.

At the end we get 7 possible topological types for $L(X(\mathbb{R})$ where $X = (x^2 + yzt + f_{\geq 4}(y, z, t) = 0)$:

$$M_2, M_1 \uplus S^2, M_1, S^2, 2S^2, 3S^2, 4S^2.$$

It turns out that these exhaust all the cases given by cD_4.

Example 4.16. Consider the cE_6-type points

$$x^2 + y^3 + yg_{\geq 3}(z, t) \pm z^2t^2 + h_{\geq 5}(z, t).$$

Using the methods of (2.6) these can be brought to the form

$$x^2 + y^3 \pm z^2t^2 + ya(z) + yb(t) + c(z) + d(t).$$

The weighted tangent cone, $(x^2+y^3\pm z^2t^2=0)$ is singular along the z and t-axes. In order to understand the singularity type of C^λ, say along the positive z-axis, set $t=\epsilon$. We get an equation

$$x^2+y^3\pm z^2\epsilon^2+ya(z)+yb(\epsilon)+c(z)+d(\epsilon).$$

$\operatorname{mult}_0 a\geq 3$ and $\operatorname{mult}_0 c\geq 5$, thus all the terms involving z can be absorbed into z^2, and we obtain the equivalent form

$$x^2\pm z^2+y^3+yb(\epsilon)+d(\epsilon).$$

The cubic $y^3+yb(\epsilon)+d(\epsilon)$ has 3 real roots if $4b(\epsilon)^3+27d(\epsilon)^2<0$ and 1 real root if $4b(\epsilon)^3+27d(\epsilon)^2>0$.

C^0 is homeomorphic to S^1 and it has 4 singular points (along the z and t halfaxes). C^λ is a smooth curve which has an oval near a singular point of C^0 if the corresponding cubic has 3 real roots and no ovals if only 1 real root. Thus C^λ has at most 5 connected components. The ovals give the very "thin" components mentioned in (4.5).

We can also determine the location of the ovals relative to the "main component" of C^λ. In deforming $z^2-y^3=0$, the oval can appear only in a neighborhood of the negative y-axis. Putting all this together, we get the following possibilities for $L(X(\mathbb{R}))$:

$$\begin{array}{ll} rS^2,\ 1\leq r\leq 5 & \text{in the } +z^2t^2\text{-case, and}\\ M_r,\ 0\leq r\leq 4 & \text{in the } -z^2t^2\text{-case.}\end{array}$$

5. The topology of terminal quotient singularities

Let $0\in X$ be 3-fold terminal singularity and $\pi:\tilde X\to X$ its index one cover. As we proved, $X=\tilde X/\frac{1}{n}(a_1,\dots,a_m)$ where n is the index of X and the a_i are integers. We use this representation to determine the topology of X in terms of the already known topology of $\tilde X$.

The main question is to determine the real points of $\mathbb{A}^m/\frac{1}{n}(a_1,\dots,a_m)$. Let $\sigma:\mathbb{C}^m\to\mathbb{C}^m$ be the corresponding action of $1\in\mathbb{Z}_n$.

The answer depends on the parity of n. First we discuss the odd index cases which are easier.

Proposition 5.1. *Assume that n is odd and set $Y=\mathbb{A}^m/\frac{1}{n}(a_1,\dots,a_m)$. Then the induced map $\mathbb{R}^n\to Y(\mathbb{R})$ is a homeomorphism.*

Proof. Let $R\subset\mathbb{R}[[x_1,\dots,x_m]]$ denote the ring of invariant functions. A point $P\in\mathbb{A}^m$ maps to a real point of X iff $f(P)\in\mathbb{R}$ for every $f\in R$. Let ϵ be a primitive nth root of unity. If $P=(p_1,\dots,p_m)$ is real then

$$\sigma^b(p_1,\dots,p_m)=(\epsilon^{ba_1}p_1,\dots,\epsilon^{ba_m}p_m)$$

is also real iff $\sigma^b(P)=P$. This shows that the quotient map $\mathbb{R}^n\to X(\mathbb{R})$ is injective.

Let $Q \in X(\mathbb{R})$ be a point. Then $\pi^{-1}(Q) \subset \mathbb{A}^m$ has an odd number of closed points over $\mathbb{C}$ (usually n of them) and as a scheme it is defined over $\mathbb{R}$. Thus it has a real point, hence $\mathbb{R}^n \to X(\mathbb{R})$ is also surjective. $\square$

Corollary 5.2. *Let $0 \in X$ be a 3-fold terminal singularity of odd index and $\pi : \tilde{X} \to X$ its index one cover. Then $\pi : \tilde{X}(\mathbb{R}) \to X(\mathbb{R})$ is a homeomorphism.* $\square$

The even index case is more subtle. For purposes of induction we allow the case when n is odd. Consider the action $\frac{1}{n}(a_1, \ldots, a_m)$ on $\mathbb{A}^m$. Write $n = 2^s n'$ where n' is odd. Let η be a primitive 2^{s+1}-st root of unity and $j : \mathbb{R}^m \to \mathbb{C}^m$ the map

$$j(x_1, \ldots, x_m) = (\eta^{a_1} x_1, \ldots, \eta^{a_m} x_m).$$

(If n is odd then $\eta = -1$, hence $j(\mathbb{R}^m) = \mathbb{R}^m$.) Write $a_i = 2^c a_i'$ such that a_i' is odd for some i. If $s > c$, let $\tau : \mathbb{C}^m \to \mathbb{C}^m$ be the $\mathbb{Z}_2$-action

$$\tau(x_1, \ldots, x_m) = ((-1)^{a_1'} x_1, \ldots, (-1)^{a_m'} x_m).$$

For $s = c$ let τ be the identity. Note that both $\mathbb{R}^n$ and $j(\mathbb{R}^n)$ are τ-invariant.

Proposition 5.3. *Set $Y = \mathbb{A}^m / \frac{1}{n}(a_1, \ldots, a_m)$. Define j and τ as above. Then $Y(\mathbb{R})$ is the quotient of $\mathbb{R}^m \cup j(\mathbb{R}^m)$ by τ.*

Proof. The proof is by induction on m and n. We can assume that the action is faithful, that is $\sum b_i a_i = 1$ is solvable in integers. Indeed, for non faithful actions we get the same quotient from a smaller group action. The definitions of j and τ are set up such that they do not change if we change the group this way.

Set $\Phi := \prod_i x_i^{b_i}$. By induction on m we know that (5.3) holds on each coordinate hyperplane. Thus we have to deal with points $P = (p_1, \ldots, p_m)$ such that each $p_i \neq 0$.

Assume that $\pi(P)$ is real. Let ϵ be a primitive $2n$th root of unity. p_i^n is real, hence $p_i = \epsilon^{c_i} \cdot$ (real number) for some $c_i \in \mathbb{Z}$. Thus $\Phi(P) = \epsilon^c \cdot$ (real number). $\Phi(\sigma(P)) = \epsilon^2 \Phi(P)$, hence by replacing P by $\sigma^r(P)$ for some r we may assume that $\Phi(P) \in \mathbb{R}$ or $\Phi(P) \in \eta \cdot \mathbb{R}$.

Assume first that $\Phi(P)$ is real. For each i the function $\Phi^{n-a_i} x_i$ is invariant, hence has a real value at P. Thus $P \in \mathbb{R}^n$. If $\Phi(P) \in \eta \cdot \mathbb{R}$ then the same argument shows that $p_i \in \eta^{a_i} \cdot \mathbb{R}$, thus $P \in j(\mathbb{R}^m)$.

This shows that $\mathbb{R}^m \cup j(\mathbb{R}^m) \to Y(\mathbb{R})$ is surjective. It is also τ-invariant. Finally, if $P = (p_1, \ldots, p_m) \in \mathbb{R}^m \cup j(\mathbb{R}^m)$ then

$$\sigma^s(P) = (\epsilon^{ba_1} p_1, \ldots, \epsilon^{ba_m} p_m) \in \mathbb{R}^m \cup j(\mathbb{R}^m)$$

iff $\sigma^s(P) = P$ or $\sigma^s(P) = \tau(P)$. Thus $\mathbb{R}^m \cup j(\mathbb{R}^m) \to Y(\mathbb{R})$ is $2:1$ for $s > c$ and $1:1$ for $s = c$. □

Definition 5.4. Let $F \in \mathbb{R}[[x_1, \dots, x_m]]$ be a power series, homogeneous of grade d under the grading $\frac{1}{n}(a_1, \dots, a_m)$. Let η be as above. Define the *companion* F^c of F with respect to the action $\frac{1}{n}(a_1, \dots, a_m)$ by

$$F^c(x_1, \dots, x_m) := \eta^{-d} F(\eta^{a_1} x_1, \dots, \eta^{a_m} x_m).$$

Note that $F^c \in \mathbb{R}[[x_1, \dots, x_m]]$.

Corollary 5.5. *Let $0 \in X$ be a 3-fold terminal singularity of even index and $\pi : \tilde{X} \to X$ its index one cover. Then*

$$L(X(\mathbb{R})) \sim L(\tilde{X}(\mathbb{R}))/(\tau) \uplus L(\tilde{X}^c(\mathbb{R}))/(\tau).$$

Proof. We use the notation of (5.3). Let $F = 0$ be the equation of $\tilde{X}$ and let $W := (F = 0) \cap (\mathbb{R}^m \cup j(\mathbb{R}^m))$. Then $X(\mathbb{R})$ is the quotient of W by τ.

$(F = 0) \cap \mathbb{R}^m = \tilde{X}(\mathbb{R})$. $(F = 0) \cap j(\mathbb{R}^m)$ can be identified with the set of real zeros of $F(\eta^{a_1} x_1, \dots, \eta^{a_m} x_m) = 0$. (The normalizing factor η^{-d} does not change the set of zeros.)

In the terminal case the group action is fixed point free outside the origin on $(F = 0)$, thus $(F = 0) \cap \mathbb{R}^m$ and $(F = 0) \cap j(\mathbb{R}^m)$ intersect only at the origin. □

As a byproduct we obtain the following:

Corollary 5.6. *Let $0 \in X$ be a 3-fold terminal singularity of index > 1. Then $0 \in X(\mathbb{R})$ is not an isolated point.*

Proof. Let $\tilde{X}$ be the index 1 cover. We are done unless $0 \in \tilde{X}(\mathbb{R})$ is an isolated point. This happens only in cases $cA/2, cAx/2$ (and maybe for $cAx/4$) where the equation of $\tilde{X}$ is $F = x^2 + y^2 + f(z, t)$ and $f(z, t)$ is positive on $\mathbb{R}^2 \setminus \{0\}$.

Let us compute F^c. In the $cA/2$ ase we get $-x^2 - y^2 + f(iz, t)$ and this has nontrivial solutions in the (x, y, t)-hyperplane. In the $cAx/2$ case we get $x^2 - y^2 + f(iz, it)$ and this has nontrivial solutions in the (x, y)-plane.

In the $cAx/4$ case already $\tilde{X}$ has nontrivial $\mathbb{R}$-points. Indeed, here f has grade 2, thus every t-power in it has an odd exponent. Thus $f(z, t)$ is not positive on the t-axis. □

5.7 (Orientability of index 2 quotients).

We have seen in (4.2) that every real algebraic hypersurface is orientable, and so are their quotients by odd order groups (5.1). With index 2 quotients, the question of orientabililty is interesting.

Consider a quotient $(F = 0)/\frac{1}{2}(w_1, \dots, w_m)$. Let σ be the corresponding $\mathbb{Z}_2$ action. We can orient $X := (F = 0)$ by choosing an orientation of $\mathbb{R}^m$ and at each smooth point of X we choose the normal vector pointing in the direction where F is positive. σ preserves the orientation of $\mathbb{R}^m$ iff $\sum w_i$ is even. The parity of $w(F)$ determines the sign in $\sigma(F) = \pm F$. Thus σ preserves the induced orientation of X iff $w(F) + \sum w_i$ is even. If $w(F) + \sum w_i$ is odd, the induced orientation is not preserved. If σ fixes a connected component of (the nonsingular part of) $X(\mathbb{R})$, then the corresponding quotient is not orientable. If, however, σ only permutes the connected components of $X(\mathbb{R})$ then the quotient is still orientable. Thus we obtain:

Lemma 5.8. *Let $0 \in X := (F = 0)/\frac{1}{2}(w_1, \dots, w_m)$ be an isolated singular point, with index one cover $\tilde{X}$ and companion $\tilde{X}^c$. Then $L(X(\mathbb{R}))$ is nonorientable iff $w(F) + \sum w_i$ is odd and σ fixes at least one of the connected components of $L(\tilde{X}(\mathbb{R}))$ or $L(\tilde{X}^c(\mathbb{R}))$.* □

Example 5.9 (The topology of $cA/2$ points).

The simplest case is $\mathbb{A}^3/\frac{1}{2}(1,1,1)$. The link of $\mathbb{A}^3$ is the sphere $(x^2 + y^2 + z^2 = \epsilon^2)$. We act by the antipodal map, and the quotient is $\mathbb{RP}^2$. The quotient of the purely imaginary subspace also gives real points, thus $L(X(\mathbb{R})) \sim 2\mathbb{RP}^2$.

In the $cA_{>0}/2$ cases write

$$X := (x^2 \pm y^2 + f(z,t) = 0)/\frac{1}{2}(1,1,1,0)$$

with cover $\tilde{X} := (x^2 \pm y^2 + f(z,t) = 0)$. Only even powers of z occur in $f(z,t)$, thus we can write $f(z,t) = G(z^2,t)$. As in (4.4) we factor it as $G(z^2,t) = \pm h(z,t) \prod_{i=1}^r f_i(z,t)$.

The companion cover is $\tilde{X}^c = (x^2 \pm y^2 - G(-z^2,t) = 0)$. We have to be careful since the product decomposition of G is not preserved. A factor of h may become indefinite and it can also happen that two factors become conjugate over $\mathbb{R}$. Thus we write $-G(-z^2,t) = \pm h'(z,t) \prod_{j=1}^{r'} f'_j(z,t)$.

By (5.5), $L(X(\mathbb{R})) = L(\tilde{X}(\mathbb{R}))/\tau \uplus L(\tilde{X}^c(\mathbb{R}))/\tau$. In all these cases (5.8) shows that if τ fixes a connected component, the quotient is not orientable. Thus if $L(\tilde{X}(\mathbb{R}))$ or $L(\tilde{X}^c(\mathbb{R}))$ is connected, the quotient is not orientable. This holds in all the $cA^-_{>0}(r > 0)$ cases. In the $cA^-_{>0}(0)$ case the equation is $(x^2 = y^2 + h)$, and each halfspace of $(x \neq 0)$ contains a unique connected component. Thus τ interchanges the two connected components and the quotient is orientable.

In the $cA^+_{>0}$ case $(\pm h(z,t) \prod_{i=1}^r f_i(z,t) = 0)$ is negative on r connected regions $P_1, \dots, P_r \subset \mathbb{R}^2$, and $L(\tilde{X}(\mathbb{R}))$ consist of r copies of S^2, one for each P_j. The involution τ fixes the t-axis pointwise, thus if one of the half t-axes is contained in some P_j, then τ fixes the S^2 over that region. The other copies of S^2 are interchanged. The same holds for $\tilde{X}^c$.

Along the t-axis $G(z^2,t)$ and $-G(-z^2,t)$ have opposite signs. Thus among the 4 pairs

$$\begin{array}{ll}(\text{positive half } t\text{-axis}, G(z^2,t)) & (\text{positive half } t\text{-axis}, -G(-z^2,t)) \\ (\text{negative half } t\text{-axis}, G(z^2,t)) & (\text{negative half } t\text{-axis}, -G(-z^2,t))\end{array}$$

there are two where the function is negative along the half axis.

We obtain the following list of possibilities. (We use the notation $K_r := S^2\#r\mathbb{RP}^2$, thus K_2 is the Klein bottle.)

$\tilde{X}$	$\tilde{X}^c$	$L(X(\mathbb{R}))$
cA_0	cA_0	$\mathbb{RP}^2 \uplus \mathbb{RP}^2$
$cA^-_{>0}(r>0)$	$cA^-_{>0}(r'>0)$	$K_r \uplus K_{r'}$
$cA^-_{>0}(r>0)$	$cA^-_{>0}(0)$	$K_r \uplus S^2$
$cA^-_{>0}(0)$	$cA^-_{>0}(0)$	$S^2 \uplus S^2$
$cA^+_{>0}(r>0)$	$cA^+_{>0}(r'>0)$	$2\mathbb{RP}^2 \uplus \frac{r+r'-2}{2}S^2$
$cA^+_{>0}(r>0)$	$cA^+_{>0}(0,+)$	$2\mathbb{RP}^2 \uplus \frac{r-2}{2}S^2$
$cA^+_{>0}(r>0)$	$cA^+_{>0}(0,-)$	$K_2 \uplus \frac{r}{2}S^2$
$cA^+_{>0}(0,-)$	$cA^+_{>0}(0,+)$	K_2

Thus $X(\mathbb{R})$ is not orientable, except in the fourth case. This indeed occurs:

Let $\tilde{X} := (x^2 - y^2 + z^{4m} + t^{2n} = 0)$ and $X := \tilde{X}/\frac{1}{2}(1,1,1,0)$ with companion $\tilde{X}^c = (-x^2 + y^2 + z^{4m} + t^{2n} = 0)$. $\tilde{X}^c \cong \tilde{X}$ and $L(\tilde{X}(\mathbb{R})) \sim S^2 \uplus S^2$. τ interchanges the two copies of S^2. Thus $X(\mathbb{R})$ is orientable and $L(X(\mathbb{R})) \sim S^2 \uplus S^2$.

References

AK92. S. Akbulut and H. King, Topology of Real Algebraic Sets, MSRI Publ. vol.25, Springer 1992

AGV85. V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko, Singularities of Differentiable Maps I–II, Birkhäuser 1985

BCR87. J. Bochnak, M. Coste and M-F. Roy, Géométrie algébrique réelle, Springer 1987

Chislenko88. Yu. S. Chislenko, Decomposition of simple singularities of real functions, Funct. Anal. Appl. 2 (1988) 297-310

CKM88. H. Clemens, J. Kollár and S. Mori, Higher Dimensional Complex Geometry , Astérisque vol.166, 1988

Durfee79. A. Durfee, Fifteen characterizations of rational double points and simple critical points, L'Ens. Math. 25 (1979) 131-163

Kollár87. J. Kollár, The structure of algebraic threefolds - an introduction to Mori's program, Bull. AMS 17 (1987) 211-273

Kollár97a. J. Kollár, Real Algebraic Surfaces, Notes of the 1997 Trento summer school lectures, (preprint)

Kollár97b. J. Kollár, Real Algebraic Threefolds II. Minimal Model Program (preprint)

Markushevich85. D. G. Markushevich, Canonical singularities of three dimensional hypersurfaces, Izv. AN SSSR 49 (1985) 334-368

Matsumura80. H. Matsumura, Commutative Algebra (second edition), Benjamin - Cummings, 1980

Mori85. S. Mori, On 3-dimensional terminal singularities, Nagoya Math. J. 98 (1985) 43-66

Nash52. J. Nash, Real algebraic manifolds, Ann. Math. 56 (1952) 405-421

Reid80. M. Reid, Canonical 3-folds, in Géométrie Algébrique Angers, A. Beauville ed., Sijthoff & Noordhoff, 1980, pp. 273–310

Reid85. M. Reid, Young person's guide to canonical singularities, in Algebraic Geometry, Bowdoin 1985, Proc. Symp. Pure Math. vol.46, pp. 345-414

Riesler93. J.-L. Riesler, Construction d'hypersurfaces réelles (Sém. Bourbaki #763), Astérisque, 216 (1993) 69-86

Viro90. O. Ya. Viro, Real algebraic plane curves, Leningrad Math. J. 1 (1990) 1059-1134

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