FANO HYPERSURFACES IN WEIGHTED PROJECTIVE 4-SPACES

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A *Fano* variety is a projective variety whose anticanonical class is ample. A 2–dimensional Fano variety is called a Del Pezzo surface. In higher dimensions, attention originally centered on smooth Fano 3–folds, but singular Fano varieties are also of considerable interest in connection with the minimal model program. The existence of Kähler–Einstein metrics on Fano varieties has also been explored, see [Bourguignon97] for a summary of the main results. Here again the smooth case is of primary interest, but Fano varieties with quotient singularities and their orbifold metrics have also been studied.

In any given dimension there are only finitely many families of smooth Fano varieties [Campana91, Nadel91, KoMiMo92], but very little is known about them in dimensions 4 and up. By allowing singularities, infinitely many families appear and their distribution is very poorly understood.

A natural experimental testing ground is given by hypersurfaces and complete intersections in weighted projective spaces. These can be written down rather explicitly, but they still provide many more examples than ordinary projective spaces. Experimental lists of certain 3–dimensional complete intersections were compiled by [Fletcher89]. In connection with Kähler–Einstein metrics, the 2–dimensional cases were first investigated in [Demailly-Kollár99] and later in [Johnson-Kollár00].

It is also of interest to study Calabi–Yau hypersurfaces and hypersurfaces of general type in weighted projective spaces. [Fletcher89] contains some lists with terminal singularities.

The aim of this paper is threefold.

First, we determine the complete list of anticanonically embedded quasi smooth Fano hypersurfaces in weighted projective 4-spaces. There are 48 infinite series and 4442 sporadic examples (7). As a consequence we obtain that the Reid–Fletcher list (cf. [Fletcher89, II.6.6]) of 95 types of anticanonically embedded quasi smooth terminal Fano threefolds in weighted projective 4-spaces is complete (11).

Second, we prove that many of these Fano hypersurfaces admit a Kähler–Einstein metric (15). We also study the nonexistence of tigers on these Fano 3–folds (in the colorful terminology of [Keel-McKernan99]).

Third, we prove that there are only finitely many families of quasi smooth Calabi–Yau hypersurfaces in weighted projective spaces of any given dimension (16). This implies finiteness for various families of general type hypersurfaces (18).

Definition 1. For positive integers a_i let $\mathbb{P}(a_0, \dots, a_n)$ denote the *weighted projective n-space* with weights $a_0, \dots, a_n$. (See [Dolgachev82] or [Fletcher89] for the basic definitions and results on weighted projective spaces.) We always assume that any n of the a_i are relatively prime. We frequently write $\mathbb{P}$ to denote a weighted projective n-space if the weights are irrelevant or clear from the context. We use $x_0, \dots, x_n$ to denote the corresponding weighted projective coordinates. $P_i \in \mathbb{P}(a_0, \dots, a_n)$ denotes the point $(x_j = 0 \ \forall j \neq i)$. These are sometimes called the *vertices* of the weighted projective space. (The vertices are uniquely determined if none of the a_i divides another.) The affine chart where $x_i \neq 0$ can be written as

$$\mathbb{C}^n(y_0, \dots, \hat{y}_i, \dots, y_n)/\mathbb{Z}_{a_i}(a_0, \dots, \hat{a}_i, \dots, a_n). \tag{1.1}$$

(Here and later $\hat{}$ denotes an omitted coordinate.) This shorthand denotes the quotient of $\mathbb{C}^n$ by the action

$$(y_0, \dots, \hat{y}_i, \dots, y_n) \mapsto (\epsilon^{a_0} y_0, \dots, \hat{y}_i, \dots, \epsilon^{a_n} y_n),$$

where ϵ is a primitive a_ith root of unity. The identification is given by $y_j^{a_i} = x_j^{a_i}/x_i^{a_j}$. (1.1) are called the *orbifold charts* on $\mathbb{P}(a_0, \dots, a_n)$.

For any i, $\mathbb{P}(a_0, \dots, a_n)$ has an index a_i quotient singularity at P_i. For any $i < j$, if $\gcd(a_0, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_n) > 1$, then $\mathbb{P}(a_0, \dots, a_n)$ has a quotient singularity along $(x_i = x_j = 0)$. These give all the codimension 2 singular subsets of $\mathbb{P}(a_0, \dots, a_n)$.

For every $m \in \mathbb{Z}$ there is a rank 1 sheaf $\mathcal{O}_{\mathbb{P}}(m)$ which is locally free only if $a_i | m$ for every i. A basis of the space of sections of $\mathcal{O}_{\mathbb{P}}(m)$ is given by all monomials in $x_0, \dots, x_n$ with weighted degree m. Thus $\mathcal{O}_{\mathbb{P}}(m)$ may have no sections for some $m > 0$.

2 (Anticanonically embedded quasi smooth Fano hypersurfaces)**.**
Let $X \in |\mathcal{O}_{\mathbb{P}}(m)|$ be a hypersurface of degree m. The adjunction formula

$$K_X \cong \mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}} + X)|_X \cong \mathcal{O}_{\mathbb{P}}(m - (a_0 + \cdots + a_n))|_X$$

holds if X does not contain any of the codimension 2 singular subsets. If this condition is satisfied then X is a Fano variety iff $m < a_0 + \cdots + a_n$. Frequently the most interesting cases are when m is as large as possible. Thus we consider the case $X_d \in |\mathcal{O}_{\mathbb{P}}(d)|$ for $d = a_0 + \cdots + a_n - 1$. Such an X is called *anticanonically embedded.*

In most cases, all hypersurfaces of a given degree d are singular and pass through some of the vertices P_i. In these cases the best one can hope is that a general hypersurface X_d is smooth in the orbifold sense, called *quasi smooth*. At the vertex P_i this means that the preimage of X_d in the orbifold chart $\mathbb{C}^n(y_0,\dots,\hat{y}_i,\dots,y_n)$ is smooth. In terms of the monomials of degree d this is equivalent to saying that

$$\text{For every } i \text{ there is a } j \text{ and a monomial } x_i^{m_i}x_j \text{ of degree } d. \tag{2.1}$$

$j=i$ is allowed, corresponding to the case when the general X_d does not pass through P_i. The condition that X_d does not contain any of the singular codimension 2 subsets is equivalent to

$$\begin{array}{l}\text{If } \gcd(a_0,\dots,\hat{a}_i,\dots,\hat{a}_j,\dots,a_n)>1 \text{ then there is a}\\ \text{monomial of degree } d \text{ not involving } x_i,x_j.\end{array} \tag{2.2}$$

For $n\geq 3$, these are the two most important special cases of the general quasi smoothness condition:

$$\begin{array}{l}\text{For every } I\subset\{0,\dots,n\} \text{ there is an injection}\\ e:I\hookrightarrow\{0,\dots,n\} \text{ and monomials}\\ x_{e(i)}\prod_{j\in I}x_j^{m_{ij}} \text{ of degree } d \text{ for every } i\in I.\end{array} \tag{2.3}$$

Remark 3. The quasi-smoothness condition in [Fletcher89, I.5.1] says that

$$\begin{array}{l}\text{For every } I\subset\{0,\dots,n\} \text{ either (2.3) holds,}\\ \text{or there is a monomial } \prod_{j\in I}x_j^{b_j} \text{ of degree } d.\end{array} \tag{2.3$'$}$$

The two versions are, however, equivalent. We prove this by induction on $|I|$. Indeed, assume that there is a monomial $\prod_{j\in I}x_j^{b_j}$ of degree d and let $I'\subset I$ be all the indices which are involved in at least one such monomial. By induction (2.3) holds for $I\setminus I'$, giving monomials $x_{e(i)}\prod_{j\in I\setminus I'}x_j^{b_{ij}}$ for $i\in I\setminus I'$. By assumption these $e(i)$ are not in I, so we can choose $I'\cup e(I\setminus I')$ as the image of $e:I\to\{0,\dots,n\}$. (A suitable reordering of the values of e may be necessary.)

The computer searches done in connection with [Fletcher89] and [Demailly-Kollár99] looked at values of a_i in a certain range to find the a_i satisfying the constraints (2.1-3). This approach starts with the a_i and views (2.1-3) as linear equations in the unknown nonnegative integers m_i, m_{ij}.

In the cases studied in [Fletcher89] and [Demailly-Kollár99] these searches seemed exhaustive. Aside from one series of examples, the computers produced solutions for low values of the a_i and then did not find any more as the range of the allowable values was extended. This of course does not ever lead to a proof that the lists were complete.

A similar search for log Fano hypersurfaces in weighted projective 4–spaces seems much harder. With some reasonably large bounds, say $a_i \leq 100$, the programs run very slowly and they produce examples where the a_i are near 100. It is quite unlikely that any systematic search of this kind could have discovered the example with the largest a_0:

$(407, 547, 5311, 12528, 18792)$ with monomials

$$x_4^2,\ x_3^3,\ x_1^{59}x_2,\ x_0x_2^7,\ x_0^{91}x_1,$$

or the beautiful pair of sporadic examples with largest a_4:

$(223, 9101, 46837, 112320, 168480)$ with monomials

$$x_4^2,\ x_3^3,\ x_1x_2^7,\ x_0x_1^{37},\ x_0^{1301}x_2, \quad \text{and}$$

$(253, 7807, 48101, 112320, 168480)$ with monomials

$$x_4^2,\ x_3^3,\ x_1^{37}x_2,\ x_0x_2^7,\ x_0^{1301}x_1.$$

The biggest values of a_0 are of some interest in connection with the conjectures of [Shokurov97, 1.3].

Next we describe the computer programs that led to the list of anticanonically embedded quasi smooth Fano hypersurfaces in weighted projective 4-spaces. The programs, written in C, are available at

`www.math.princeton.edu/~jmjohnso`

4 (Preliminary steps)**.**

In order to find all solutions, we change the point of view. We consider (2.1) to be the main constraint with coefficients m_i and unknowns a_i. The corresponding equations can then be written as a linear system

$$(M + J + U)(a_0\ a_1\ a_2\ a_3\ a_4)^t = (\ -1\ \ -1\ \ -1\ \ -1\ \ -1)^t \tag{4.1}$$

where $M = \operatorname{diag}(m_0, m_1, m_2, m_3, m_4)$ is a diagonal matrix, J is a matrix with all entries -1 and U is a matrix where each row has 4 entries $= 0$ and one entry $= 1$. The main advantage is that some of the m_i can be bounded a priori. Assume for simplicity that $a_0 \leq a_1 \leq \cdots \leq a_4$.

Consider for instance m_4. The relevant equation is

$$m_4a_4 + a_{e(4)} = a_0 + a_1 + a_2 + a_3 + a_4 - 1.$$

Since a_4 is the biggest, we get right away that $1 \leq m_3 \leq 3$. Arguing inductively with some case analysis we obtain that

$$3 \leq m_2 \leq 16,\ 2 \leq m_3 \leq 6,\ 1 \leq m_4 \leq 3 \tag{4.2}$$

Thus we have only finitely many possibilities for the matrix U and the numbers m_2, m_3, m_4. Fixing these values, we obtain a linear system

$$(M+J+U)(a_0\ a_1\ a_2\ a_3\ a_4)^t = (\ -1\ \ -1\ \ -1\ \ -1\ \ -1)^t,$$

where the only variable coefficients are m_0, m_1 in the upper left corner of M. Solving these formally we obtain that

$$a_0 = \frac{\alpha_0 m_1 + \beta_0}{\gamma_2 m_0 m_1 + \gamma_0 m_0 + \gamma_1 m_1 + \delta}, a_1 = \frac{\alpha_1 m_0 + \beta_1}{\gamma_2 m_0 m_1 + \gamma_0 m_0 + \gamma_1 m_1 + \delta}.$$

where the $\alpha_i, \beta_i, \gamma_i, \delta$ depend only on U and m_2, m_3, m_4.

We distinguish 3 cases. The first one is the main source of examples. Cases 2 and 3 are anomalies from the point of view of our method. In both cases we ended up experimentally finding strong restrictions on the a_i. Even with hindsight we do not know how to prove these a priori.

Case 1 $(\gamma_2 \neq 0)$. In this case the absolute value of

$$\frac{\alpha_0 m_1 + \beta_0}{\gamma_2 m_0 m_1 + \gamma_0 m_0 + \gamma_1 m_1 + \delta}$$

goes to zero as m_0, m_1 go to infinity. It is not hard to write down the precise condition and a computer check shows that

$$\frac{\alpha_0 m_1 + \beta_0}{\gamma_2 m_0 m_1 + \gamma_0 m_0 + \gamma_1 m_1 + \delta} \geq 1 \quad \Rightarrow \quad \min\{m_0, m_1\} \leq 83.$$

Case 2 $(\gamma_2 = 0, \gamma_0\gamma_1 \neq 0\ )$. It turns out that if this holds then $\gamma_0\gamma_1 > 0$ and a_0, a_1 are bounded by 8 for $\min\{m_0, m_1\} \geq 36$. Moreover, the 3 linear forms

$$\alpha_0 m_1 + \beta_0,\ \ \alpha_1 m_0 + \beta_1,\ \ \gamma_0 m_0 + \gamma_1 m_1 + \delta$$

are dependent. This implies that

$$\alpha_1\gamma_1 a_0 + \alpha_0\gamma_0 a_1 = \alpha_0\alpha_1.$$

A computer search shows that this is possible only if $a_0 = a_1 = 1$.

Case 3 $(\gamma_2 = 0, \gamma_0\gamma_1 = 0)$. It turns out that one of $\alpha_0 m_1+\beta_0, \alpha_1 m_0+\beta_1$ equals $\gamma_0 m_0 + \gamma_1 m_1 + \delta$. Thus $a_0 = 1$ or $a_1 = 1$. Moreover, we also see by explicit computation that one of the following holds:

$$a_2 = a_3 = a_4, \quad a_2 = a_3 = a_4/2 \quad \text{or} \quad a_2 = a_3/2 = a_4/3.$$

5 (Main Computer Search)**.**

Here we discuss the main case when, in addition to the inequalities (4.2) we also assume that $3 \leq m_1 \leq 83$. In this case the system (4.1) reduces to a single unknown m_0. This is very similar to the 4–variable case discussed in [Johnson-Kollár00].

We solve formally for a_0 to get

$$a_0 = \frac{\gamma_0}{m_0\alpha + \beta}$$

where α, β, γ_0 depend only on U and m_1, m_2, m_3, m_4. If $\alpha \neq 0$ then we get a bound on m_0 too, and we are down to finitely many possibilities all together. There are 403455 cases of this. The resulting solutions need considerable cleaning up. Many of them occur multiply and we also have to check the other conditions (2.2-3). Discarding repetitions, we get 15757 cases, out of which 4594 are quasi smooth.

If $\alpha = 0$ then we get a series solution where the a_i are linear functions of a variable m_0. There are 550122 cases of this. Here the main difficulty is that the program does not produce the series in a neat form. Usually one series is put together out of many pieces according to some congruence condition.

6 (Additional Cases)**.**

Assume first that we are in Case 2 of (4). Since $a_0 = a_1 = 1$, the numbers a_1, a_2, a_3, a_4 and $d = a_1 + a_2 + a_3 + a_4$ satisfy the numerical conditions (2.3). This leads to a lower dimensional problem which is easy to solve.

Case 2 of (4) is even easier. We get solutions of the form

$$(1, a, b, b, b), \quad (1, a, b, b, 2b) \quad \text{or} \quad (1, a, b, 2b, 3b).$$

Applying (2.3) to $I = \{2, 3, 4\}$ gives that $b|a$. Thus b divides all but one of the weights, so $b = 1$. This implies that $a \leq 6$. At any case, all these appear also under Case 2 of (4).

At the end we obtain our first main result:

Theorem 7. *The following is a complete list of anticanonically embedded quasi smooth Fano hypersurfaces in weighted projective 4-spaces:*

1. *48 infinite series of the form*

$$X_{2k(b_1+b_2+b_3)} \subset \mathbb{P}(2, kb_1, kb_2, kb_3, k(b_1+b_2+b_3)-1) \quad \textit{for } k = 1, 3, 5, \ldots$$

   The occurring 3-tuples b_1, b_2, b_3 are described in (9).

2. *4442 sporadic examples whose list is available at*

   `www.math.princeton.edu/~jmjohnso`

8 (An error check)**.**

We wrote a program that looked at all 5-tuples satisfying

$$a_0 \leq 100,\ a_1 \leq 200,\ a_2 \leq 200,\ a_3 \leq 400,\ a_4 \leq 600.$$

The program ran for 4 days and produced 3610 quasi smooth examples. These were in complete agreement with the correspondingly truncated list of 4442 sporadic examples.

Remark 9. It turns out that a 3-tuple b_1, b_2, b_3 appears in (7.1) iff $|-2K|$ of $\mathbb{P}(b_1, b_2, b_3)$ has a quasi smooth member. The list of these is implicit in Reid's list of 95 families of singular K3 surfaces in weighted projective 3-spaces. In [Fletcher89, II.3.3] they correspond to those quadruplets (b_1, b_2, b_3, b_4) for which $b_4 = b_1 + b_2 + b_3$. Our 48 3-tuples occured explicitly in [Yonemura90, Tomari00] in connection with the study of simple K3 singularities of multiplicity 2.

One direction of this observation is easy to establish in all dimensions.

Lemma 10. *Assume that $|-2K|$ of $\mathbb{P}(b_1, \dots, b_n)$ has a quasi smooth member. Then the general anticanonically embedded Fano hypersurface in*

$$\mathbb{P}(2, kb_1, \dots, kb_n, k(b_1 + \cdots + b_n) - 1)$$

is quasi smooth for $k = 1, 3, 5, \dots$

We conjecture that conversely, every infinite series is of this form. It is interesting that every quasi smooth hypersurface in $\mathbb{P}(2, kb_1, \dots, kb_n, k(b_1 + \cdots + b_n) - 1)$ has a codimension 2 singular set. Thus the above conjecture would imply that for every $n \geq 4$ there are only finitely many anticanonically embedded quasi smooth Fano hypersurfaces with isolated singularities in weighted projective n-spaces.

It is not hard to check which of the above Fano threefolds have terminal singularities. The families in (7.1) always have nonisolated singularities, and for the remaining cases the conditions of [Fletcher89, II.4.1] work. As a consequence, we obtain the following:

Corollary 11. *The Reid–Fletcher list (cf.* [Fletcher89, II.6.6]*) of 95 families of anticanonically embedded quasi smooth terminal Fano threefolds in weighted projective 4-spaces is complete.* □

Next we study the existence of Kähler–Einstein metrics and the nonexistence of tigers on our Fano hypersurfaces. After some definitions we recall the criterion established in [Johnson-Kollár00]. In the case of Kähler–Einstein metrics this in turn relies on earlier work of [Nadel90, Demailly-Kollár99].

Definition 12. Let X be a normal variety and D a $\mathbb{Q}$-divisor on X. Assume for simplicity that K_X and D are both $\mathbb{Q}$-Cartier. Let $g : Y \to X$ be any proper birational morphism, Y smooth. Then there is a unique $\mathbb{Q}$-divisor $D_Y = \sum e_i E_i$ on Y such that

$$K_Y + D_Y \equiv g^*(K_X + D) \quad \text{and} \quad g_* D_Y = D.$$

We say that (X, D) is *klt* (resp. *log canonical*) if $e_i > -1$ (resp. $e_i \geq -1$) for every g and for every i. See, for instance, [Kollár-Mori98, 2.3] for a detailed introduction.

Definition 13. [Keel-McKernan99] Let X be a normal variety. A *tiger* on X is an effective $\mathbb{Q}$-divisor D such that $D \equiv -K_X$ and (X, D) is not klt. As illustrated in [Keel-McKernan99], the tigers carry important information about birational transformations of log del Pezzo surfaces. They are expected to play a similar role in higher dimensions.

Proposition 14. [Johnson-Kollár00] *Let $X_d \subset \mathbb{P}(a_0, \dots, a_n)$ be a quasi smooth hypersurface of degree $d = a_0 + \dots + a_n - 1$.*

1. *X does not have a tiger if $d \leq a_0 a_1$.*
2. *X admits a Kähler–Einstein metric if $d < \frac{n}{n-1} a_0 a_1$.* □

Corollary 15. *Of the sporadic series of quasi smooth Fano hypersurfaces given in (7.2), there are 1605 types where none of the members have a tiger and 1936 types where every member admits a Kähler–Einstein metric. This information is contained in the list given in (7.2).*

Finally we study the case of Calabi–Yau hypersurfaces and hypersurfaces of general type in weighted projective spaces. For these cases there are finiteness results in all dimensions. The key part is the case of Calabi–Yau hypersurfaces.

Theorem 16. *For any n there are only finitely many types of quasi smooth hypersurfaces with trivial canonical class in weighted projective spaces $\mathbb{P}(a_0, \dots, a_n)$.*

Proof. As in the Fano case, first we look at those hypersurfaces which are quasi smooth at the vertices of $\mathbb{P}(a_0, \dots, a_n)$. This condition is equivalent to a linear system of equations

$$(M + J + U)(a_0, \dots, a_n)^t = (0, \dots, 0)^t \tag{16.1}$$

where $M = \operatorname{diag}(m_0, \dots, m_n)$ is a diagonal matrix, J is a matrix with all entries -1 and U is a matrix where each row has n entries $= 0$ and one entry $= 1$. In the geometric setting the m_i and the a_i are positive integers, but it will be convenient to allow the a_i to be positive real numbers. By the homogenity of the system we may assume that $\sum a_i = 1$.

Assume now that we have an infinite sequence of solutions $(a_0(t), \dots, a_n(t))$ where a priori $M(t), J(t), U(t)$ also vary with t. By passing to a subsequence we may assume that $J(t), U(t)$ are constant and each $a_i(t)$ converges to a value A_i. Thus we can write $a_i(t) = A_i + c_i(t)$ where

$\lim_{t\to\infty} c_i(t) = 0$, $\sum_i A_i = 1$ and $\sum_i c_i(t) = 0$. By passing to a subsequence and rearranging, we can also assume that $I := \{i : c_i(t) < 0\}$ is independent of t and that $A_0/(-c_0(t))$ is the smallest positive number among $\{A_i/(-c_i(t)) : i \in I\}$. The quasi smoothness condition at the vertex P_0 translates into $m_0(t)a_0(t) + a_j(t) = 1$. $\lim_{t\to\infty} a_0(t) = A_0 > 0$ since $c_0(t) < 0$, hence $m_0(t)$ is bounded from above. Thus we may assume that $m_0(t) = m_0$ is constant and $\lim_{t\to\infty} m_0c_0(t) + c_j(t) = 0$. $m_0a_0(t) + a_j(t) = 1$ is equivalent to

$$[m_0A_0 + A_j] + [m_0c_0(t) + c_j(t)] = 1. \tag{16.2}$$

By the above considerations, (16.2) splits into 2 equations

$$m_0A_0 + A_j = 1 \quad \text{and} \quad m_0c_0(t) + c_j(t) = 0. \tag{16.3}$$

Using $\sum_i c_i(t) = 0$ and the second equation in (16.3) we obtain that

$$\sum_{i\in I} c_i(t) = -\sum_{i\notin I} c_i(t) \le -c_j(t) = m_0c_0(t). \tag{16.4}$$

Multiplying by $A_0/c_0(t)$ and using the special choice of $A_0/c_0(t)$ we get that

$$m_0A_0 \le \sum_{i\in I} c_i(t)\frac{A_0}{c_0(t)} \le \sum_{i\in I} A_i. \tag{16.5}$$

Combining with the first equation of (16.3) we get that

$$1 = m_0A_0 + A_j \le A_j + \sum_{i\in I} A_i \le \sum_{i=0}^{n} A_i = 1. \tag{16.6}$$

This implies that all inequalities in (16.4–6) are equalities. Hence $A_k, c_k(t)$ are zero for $k \notin I \cup \{j\}$. By assumption the $a_k(t)$ are positive, so $I \cup \{j\} = \{0, \dots, n\}$. Moreover, the ratios $A_i/c_i(t)$ are all the same for $i \in I$.

These imply that, up to rearranging the indices, the $a_i(t)$ are of the form

$$(A_0(1 - c(t)), \dots, A_{n-1}(1 - c(t)), A_n + c(t)\textstyle\sum_{i=0}^{n-1} A_i).$$

Consider next the equation

$$m_n(A_n + c(t)\textstyle\sum_{i=0}^{n-1} A_i) + A_j(1 - c(t)) = 1,$$

where for notational simplicity we allow $j = -1$ with $A_{-1} = 0$. For large t this implies that $\sum_{i=0}^{n-1} A_i = A_j$, which is not possible for $n \ge 2$. Thus $A_n = 0$ and the solutions become

$$(A_0(1 - c(t)), \dots, A_{n-1}(1 - c(t)), c(t)) \quad \text{where} \quad \textstyle\sum_{i=0}^{n-1} A_i = 1. \tag{16.7}$$

To get quasi smoothness, we need to understand all monomials of degree $\sum a_i$, which amounts to finding all integer solutions of $\sum b_ia_i = 1$.

In our case, for large t there are no solutions with $b_n = 0$ which means that every hypersurface of degree $\sum a_i$ contains the hyperplane $(x_n = 0)$, hence they are all reducible. Thus the solutions (16.7) do not correspond to quasi smooth hypersurfaces. $\square$

Remark 17. The solutions (16.7) do correspond to interesting series of singularities. Namely, for every integer solution of $\sum_{i=0}^{n-1} \frac{1}{m_i} = 1$ they give an infinite series of singularities

$$(x_0^{m_0} + \cdots + x_{n-1}^{m_{n-1}} + x_n^k)x_n = 0 \subset \mathbb{A}^{n+1} \quad \text{for } k = 1, 2, \ldots.$$

These singularities are weighted homogeneous and semi log canonical (see [Kollár et al.92, 16.2.1] for the definition) but not isolated. By adding a general higher degree term, we get isolated log canonical singularities.

Corollary 18. *For any n and $k > 0$ there are only finitely many families of quasi smooth hypersurfaces $X \subset \mathbb{P}(a_0, \ldots, a_n)$ such that $\omega_X \cong \mathcal{O}_X(k)$.*

Proof. Assume that

$$X = (F(x_0, \ldots, x_n) = 0) \subset \mathbb{P}(a_0, \ldots, a_n)$$

is quasi smooth of degree d and $\omega_X \cong \mathcal{O}_X(k)$. Then

$$X^* := (F(x_0, \ldots, x_n) + x_{n+1}^d + \cdots + x_{n+k}^d = 0) \subset \mathbb{P}(a_0, \ldots, a_n, \overbrace{1, \ldots, 1}^{k-times})$$

is also quasi smooth of degree d and $\omega_X \cong \mathcal{O}_X$. Thus we are done by (16). $\square$

Remark 19. The finiteness result (18) is in accordance with the conjectures [Kollár et al.92, 18.16]. On the other hand, (16) seems to be a more special finiteness assertion.

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