KÄHLER–EINSTEIN METRICS ON LOG DEL PEZZO SURFACES IN WEIGHTED PROJECTIVE 3–SPACES

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A *log del Pezzo* surface is a projective surface with quotient singularities such that its anticanonical class is ample. Such surfaces arise naturally in many different contexts, for instance in connection with affine surfaces [Miyanishi81], moduli of surfaces of general type [Alexeev94], 3 and 4 dimensional minimal model program [Alexeev93]. They also provide a natural testing ground for existence results of Kähler–Einstein metrics. The presence of quotient singularities forces us to work with orbifold metrics, but this is usually only a minor inconvenience. Log del Pezzo surfaces with a Kähler–Einstein metric also lead to Sasakian–Einstein 5–manifolds by [Boyer–Galicki00].

In connection with [Demailly-Kollár99], the authors ran a computer program to find examples of log del Pezzo surfaces in weighted projective spaces. The program examined weights up to a few hundred and produced 3 examples of log del Pezzo surfaces where the methods of [Demailly-Kollár99, §6] proved the existence of a Kähler–Einstein metric.

The aim of this paper is twofold. First, we determine the complete list of anticanonically embedded quasi smooth log del Pezzo surfaces in weighted projective 3-spaces. Second, we improve the methods of [Demailly-Kollár99, 6.10] to prove that many of these admit a Kähler–Einstein metric. The same method also proves that some of these examples do not have tigers (in the colorful terminology of [Keel-McKernan99]).

Higher dimensional versions of these results will be considered in a subsequent paper.

Definition 1. For positive integers a_i let $\mathbb{P}(a_0, a_1, a_2, a_3)$ denote the *weighted projective* 3*-space* with weights $a_0 \le a_1 \le a_2 \le a_3$. (See [Dolgachev82] or [Fletcher89] for the basic definitons and results.) We always assume that any 3 of the a_i are relatively prime. We frequently write $\mathbb{P}$ to denote a weighted projective 3-space if the weights are irrelevant or clear from the context. We use x_0, x_1, x_2, x_3 to denote the corresponding weighted projective coordinates. We let (i, j, k, ℓ) be an unspecified permutation of $(0,1,2,3)$. $P_i \in \mathbb{P}(a_0,a_1,a_2,a_3)$ denotes the point $(x_j = x_k = x_\ell = 0)$. The affine chart where $x_i \neq 0$ can be written as

$$\mathbb{C}^3(y_j, y_k, y_\ell)/\mathbb{Z}_{a_i}(a_j, a_k, a_\ell). \tag{1.1}$$

This shorthand denotes the quotient of $\mathbb{C}^3$ by the action

$$(y_j, y_k, y_\ell) \mapsto (\epsilon^{a_j} y_j, \epsilon^{a_k} y_k, \epsilon^{a_\ell} y_\ell)$$

where ϵ is a primitive a_ith root of unity. The identification is given by $y_j^{a_i} = x_j^{a_i}/x_i^{a_j}$. (1.1) are called the *orbifold charts* on $\mathbb{P}(a_0,a_1,a_2,a_3)$.

$\mathbb{P}(a_0,a_1,a_2,a_3)$ has an index a_i quotient singularity at P_i and an index (a_i,a_j) quotient singularity along the line $(x_k = x_\ell = 0)$.

For every $m \in \mathbb{Z}$ there is a rank 1 sheaf $\mathcal{O}_{\mathbb{P}}(m)$ which is locally free only if $a_i | m$ for every i. A basis of the space of sections of $\mathcal{O}_{\mathbb{P}}(m)$ is given by all monomials in x_0, x_1, x_2, x_3 with weighted degree m. Thus $\mathcal{O}_{\mathbb{P}}(m)$ may have no sections for some $m > 0$.

2 (Anticanonically embedded quasi smooth surfaces)**.**

Let $X \in |\mathcal{O}_{\mathbb{P}}(m)|$ be a surface of degree m. The adjunction formula

$$K_X \cong \mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}} + X)|_X \cong \mathcal{O}_{\mathbb{P}}(m - (a_0+a_1+a_2+a_3))|_X$$

holds iff X does not contain any of the singular lines. If this condition holds then X is a (singular) del Pezzo surface iff $m < a_0+a_1+a_2+a_3$. It is also well understood that from many points of view the most interesting cases are when m is as large as possible. Thus we consider the case $X_d \in |\mathcal{O}_{\mathbb{P}}(d)|$ for $d = a_0+a_1+a_2+a_3-1$. We say that such an X is *anticanonically embedded.*

Except for the classical cases

$$(a_0,a_1,a_2,a_3) = (1,1,1,1),\ (1,1,1,2) \quad \text{or} \quad (1,1,2,3),$$

X is not smooth and it passes through some of the vertices P_i. Thus the best one can hope is that X is smooth in the orbifold sense, called *quasi smooth.* At the vertex P_i this means that the preimage of X in the orbifold chart $\mathbb{C}^3(y_j, y_k, y_\ell)$ is smooth. In terms of the equation of X this is equivalent to saying that

$$\text{For every } i \text{ there is a } j \text{ and a monomial } x_i^{m_i} x_j \text{ of degree } d. \tag{2.1}$$

Here we allow $j = i$, corresponding to the case when the general X does not pass through P_i. The condition that X does not contain any of the singular lines is equivalent to

$$\text{If } (a_i,a_j) > 1 \text{ then there is a monomial } x_i^{b_i} x_j^{b_j} \text{ of degree } d. \tag{2.2}$$

Finally, if every member of $|\mathcal{O}_{\mathbb{P}}(d)|$ contains a coordinate axis ($x_k = x_\ell = 0$) then the general member should be smooth along it, except possibly at the vertices. That is

$$\begin{array}{l}\text{For every } i, j, \text{ either there is a monomial } x_i^{b_i} x_j^{b_j} \text{ of degree } d,\\ \text{or there are monomials } x_i^{c_i} x_j^{c_j} x_k \text{ and } x_i^{d_i} x_j^{d_j} x_\ell \text{ of degree } d.\end{array} \tag{2.3}$$

The computer search done in connection with [Demailly-Kollár99] looked at values of a_i in a certain range to find the a_i satisfying the constraints (2.1-3). This approach starts with the a_i and views (2.1-3) as linear equations in the unknowns m_i, b_i, c_i, d_i. In order to find all solutions, we change the point of view.

3 (Description of the computer program)**.**
We consider (2.1) to be the main constraint, the m_i as coefficients and the a_i as unknowns. The corresponding equations can then be written as a linear system

$$(M + J + U)(a_0\ a_1\ a_2\ a_3)^t = (\ -1\ \ -1\ \ -1\ \ -1)^t \tag{3.1}$$

where $M = \operatorname{diag}(m_0, m_1, m_2, m_3)$ is a diagonal matrix, J is a matrix with all entries -1 and U is a matrix where each row has 3 entries $= 0$ and one entry $= 1$. It is still not easy to decide when such a system has positive integral solutions, but the main advantage is that some of the m_i can be bounded a priori.

Consider for instance m_3. The relevant equation is

$$m_3 a_3 + a_j = a_0 + a_1 + a_2 + a_3 - 1.$$

Since a_3 is the biggest, we get right away that $1 \le m_3 \le 2$. Arguing inductively with some case analysis we obtain that

2. either $2 \le m_2 \le 4$ and $2 \le m_1 \le 10$,
3. or the a_i are in a series $(1, a, b, b,)$ with $a|2b-1$. The latter satisfy (2.2) only for $a = b = 1$.

Thus we have only finitely many possibilities for the matrix U and the numbers m_1, m_2, m_3. Fixing these values, we obtain a linear system

$$(M + J + U)(a_0\ a_1\ a_2\ a_3)^t = (\ -1\ \ -1\ \ -1\ \ -1)^t,$$

where the only variable coefficient is the upper left corner of M. Solving these formally we obtain that

$$a_0 = \frac{\gamma_0}{m_0\alpha + \beta}$$

where α, β, γ_0 depend only on U and m_1, m_2, m_3. a_0 is supposed to be a positive integer, thus if $\alpha \ne 0$ then there are only finitely many possibilities for m_0. Once m_0 is also fixed, the whole system can be solved and we check if the a_i are all positive integers. We get 1362 cases.

If $\alpha = 0$ but $\beta \neq 0$ then the general solution of the system has the form

$$a_0 = \frac{\gamma_0}{\beta}, a_i = \frac{m_0\delta_i + \gamma_i}{\beta} \quad \text{for} \quad i = 1, 2, 3.$$

These generate the series of solutions, 405 of them. Finally, with some luck, the case $\alpha = \beta = \gamma_0 = 0$ never occurs, so we do not have to check further.

The resulting solutions need considerable cleaning up. Many solutions a_0, a_1, a_2, a_3 occur multiply and we also have to check the other conditions (2.2-3). At the end we get the complete list, given in (8).

The computer programs are available at

`www.math.princeton.edu/~jmjohnso/LogDelPezzo`

These log del Pezzo surfaces are quite interesting in their own right. Namely, it turns out that for many of them, members of the linear systems $|-mK_X|$ can not be very singular at any point. First we recall the notions log canonical etc. (see, for instance, [Kollár-Mori98, 2.3] for a detailed introduction).

Definition 4. Let X be a surface and D a $\mathbb{Q}$-divisor on X. Let $g : Y \to X$ be any proper birational morphism, Y smooth. Then there is a unique $\mathbb{Q}$-divisor $D_Y = \sum e_i E_i$ on Y such that

$$K_Y + D_Y \equiv g^*(K_X + D) \quad \text{and} \quad g_* D_Y = D.$$

We say that (X, D) is *klt* (resp. *log canonical*) if $e_i < 1$ (resp. $e_i \leq 1$) for every g and for every i. We say that (X, D) is *canonical* if $e_i \leq 0$ for every g and for every i such that E_i is g-exceptional

Definition 5. [Keel-McKernan99] Let X be a normal surface. A *tiger* on X is an effective $\mathbb{Q}$-divisor D such that $D \equiv -K_X$ and (X, D) is not klt. As illustrated in [Keel-McKernan99], the tigers carry important information about birational transformations of log del Pezzo surfaces.

Remark 6. By a result of Shokurov (cf. [Keel-McKernan99, 22.2]), if the log del Pezzo surface X has Picard number 1 and it has a tiger then $|\mathcal{O}_X(-mK_X)| \neq \emptyset$ for some $m = 1, 2, 3, 4, 6$. The log del Pezzo surfaces in (8) mostly have bigger Picard number. It is quite interesting though that the two results work for almost the same cases.

We use the following sufficient condition to obtain the existence of Kähler–Einstein metrics.

Theorem 7. [Nadel90, Demailly-Kollár99] *Let X be an n dimensional Fano variety (possibly with quotient singularities). Assume that there is an $\epsilon > 0$ such that*

$$(X, \tfrac{n+\epsilon}{n+1} D) \quad \textit{is klt}$$

for every effective $\mathbb{Q}$-divisor $D \equiv -K_X$. Then X has a Kähler–Einstein metric. □

The main result of this note is the following.

Theorem 8. *There is an anticanonically embedded quasi smooth log del Pezzo surface $X_d \subset \mathbb{P}(a_0, a_1, a_2, a_3)$ iff the a_i and d are among the following. The table below also gives our results on the nonexistence of tigers (5) and on the existence of Kähler–Einstein metrics. (Lower case y means that the answer has been previously known.)*

	a_0	a_1	a_2	a_3	d	*tiger*	*KE metric*
Series:	2	$2k+1$	$2k+1$	$4k+1$	$8k+4$	Y	Y
Sporadic:	1	1	1	1	3	y	y
	1	1	1	2	4	y	y
	1	1	2	3	6	y	y
	1	2	3	5	10	y	?
	1	3	5	7	15	y	?
	1	3	5	8	16	y	?
	2	3	5	9	18	?	?
	3	3	5	5	15	N	Y
	3	5	7	11	25	?	Y
	3	5	7	14	28	?	Y
	3	5	11	18	36	?	Y
	5	14	17	21	56	N	Y
	5	19	27	31	81	N	Y
	5	19	27	50	100	N	Y
	7	11	27	37	81	N	Y
	7	11	27	44	88	N	Y
	9	15	17	20	60	N	y
	9	15	23	23	69	N	Y
	11	29	39	49	127	N	Y
	11	49	69	128	256	N	y
	13	23	35	57	127	N	Y
	13	35	81	128	256	N	y

Remark 9. The above results hold for every quasi smooth surface with the indicated numerical data.

Near the end of the list there are very few monomials of the given degree and in many cases there is only one such surface up to isomorphism. In some other cases, for instance for the series, there are moduli.

It is generally believed that the algebraic geometry of any given log del Pezzo surface can be understood quite well. There is every reason to believe that all of the remaining cases of (8) can be decided, though it may require a few pages of computation for each of them.

10 (How to check if (X, D) is klt or not?).

The definition (4) requires understanding all resolutions of singularities. Instead, we use the following multiplicity conditions to check that a given divisor is klt. These conditions are far from being necessary.

Let X be a surface with quotient singularities. Let the singular points be $P_i \in X$ and we write these locally analytically as

$$p_i : (\mathbb{C}^2, Q_i) \to (\mathbb{C}^2/G_i, P_i) \cong (X, P_i),$$

where $G_i \subset GL(2, \mathbb{C})$ is a finite subgroup. We may assume that the origin is an isolated fixed point of every nonidentity element of G_i (cf. [Brieskorn68]). Let D be an effective $\mathbb{Q}$-divisor on X. Then (X, D) is klt if the following three conditions are satisfied.

1. (Non isolated non-klt points) D does not contain an irreducible component with coefficient ≥ 1.
2. (Canonical at smooth points) $\operatorname{mult}_P D \leq 1$ at every smooth point $P \in X$. This follows from [Kollár-Mori98, 4.5].
3. (Klt at singular points) $\operatorname{mult}_{Q_i} D_i \leq 1$ for every i where $D_i := p_i^* D$. This follows from [Kollár-Mori98, 5.20] and the previous case.

In our applications we rely on the following estimate.

Proposition 11. *Let $Z \subset \mathbb{P}(a_0, \dots, a_n)$ be a d-dimensional subvariety of a weighted projective space. Assume that Z is not contained in the singular locus and that $a_0 \leq \cdots \leq a_n$. Let $Z_i \subset \mathbb{A}^n$ denote the preimage of Z in the orbifold chart*

$$\mathbb{A}^n \to \mathbb{A}^n/\mathbb{Z}_{a_i} \cong \mathbb{P}(a_0, \dots, a_n) \setminus (x_i = 0).$$

Then for every i and every $p \in Z_i$,

$$\operatorname{mult}_p Z_i \leq (a_n \cdots a_{n-d})(Z \cdot \mathcal{O}(1)^d).$$

Moreover, if $Z \neq (x_0 = \cdots = x_{n-d-1} = 0)$ then we have a stronger inequality

$$\operatorname{mult}_p Z_i \leq (a_n \cdots a_{n-d+1} a_{n-d-1})(Z \cdot \mathcal{O}(1)^d).$$

Proof. Let $0 \in C(Z) \subset \mathbb{A}^{n+1}$ denote the cone over Z with vertex 0. Z_i can be identified with the hyperplane section $C(Z) \cap (x_i = 1)$. The multiplicty of a point is an upper semi continuous function on a variety, thus it is sufficient to prove that

$$\operatorname{mult}_0 C(Z) \leq (a_n \cdots a_{n-d})(Z \cdot \mathcal{O}(1)^d).$$

This is proved by induction on $\dim Z$.

If $C(Z)$ is not contained in the coordinate hyperplane $(x_i = 0)$, then write

$$Z \cap (x_i = 0) = \sum_j m_j Y_j \subset \mathbb{P}(a_0, \dots, a_n).$$

Next we claim that

$$\begin{aligned} \sum_j m_j (Y_j \cdot \mathcal{O}(1)^{d-1}) &= a_i (Z \cdot \mathcal{O}(1)^d) \quad \text{and} \\ \sum_j m_j \operatorname{mult}_0 C(Y_j) &\geq \operatorname{mult}_0 C(Z). \end{aligned}$$

The first of these is the associativity of the intersection product, and the second is a consequence of the usual estimate for the intersection multiplicty (cf. [Fulton84, 12.4]) applied to $C(Z), (x_i = 0)$ and $d-1$ other general hyperplanes through the origin. (Note that in the first edition of [Fulton84] there is a misprint in (12.4). $\sum_{i=1}^r e_P(V_i)$ should be replaced by $\prod_{i=1}^r e_P(V_i)$.) By the inductive assumption

$$\operatorname{mult}_0 C(Y_j) \leq (a_n \cdots a_{n-d+1})(Y_j \cdot \mathcal{O}(1)^{d-1}),$$

hence

$$\operatorname{mult}_0 C(Z) \leq (a_n \cdots a_{n-d})(Z \cdot \mathcal{O}(1)^d)$$

as claimed.

In most cases, we can even choose $i < n - d$. This is impossible only if $Z \subset (x_0 = \cdots = x_{n-d-1} = 0)$, but then equality holds. □

Corollary 12. *Let $X_d \subset \mathbb{P}(a_0, a_1, a_2, a_3)$ be a quasismooth surface of degree $d = a_0 + a_1 + a_2 + a_3 - 1$. Then X does not have a tiger if $d \leq a_0 a_1$. If $(x_0 = x_1 = 0) \not\subset X$ then $d \leq a_0 a_2$ is also sufficient.*

Proof. Assume that $D \subset X_d$ is a tiger. We can view D as a 1–cycle in $\mathbb{P}(a_0, a_1, a_2, a_3)$ whose degree is

$$(D \cdot \mathcal{O}_X(1)) = (\mathcal{O}_{\mathbb{P}}(d) \cdot \mathcal{O}_{\mathbb{P}}(1) \cdot \mathcal{O}_{\mathbb{P}}(1)) = \tfrac{d}{a_0 a_1 a_2 a_3}.$$

By (11), this implies that the multiplicity of D_i (as in (10.3)) is bounded from above by $\frac{d}{a_0 a_1}$ at any point. Thus (X, D) is klt if $d \leq a_0 a_1$.

If $(x_0 = x_1 = 0) \not\subset X$ then we can weaken this to $d \leq a_0 a_2$, again by (11). □

Using (7) and a similar argument we obtain the following.

Corollary 13. *Let $X_d \subset \mathbb{P}(a_0, a_1, a_2, a_3)$ be a quasismooth surface of degree $d = a_0 + a_1 + a_2 + a_3 - 1$. Then X admits a Kähler–Einstein metric if $d < \frac{3}{2}a_0a_1$. If $(x_0 = x_1 = 0) \not\subset X$ then $d < \frac{3}{2}a_0a_2$ is also sufficient.* □

14 (Proof of (8))**.**

The nonexistence of tigers and the existence of a Kähler–Einstein metric in the sporadic examples follows from (12) and (13). There are 5 cases when we need to use that X does not contain the line $(x_0 = x_1 = 0)$. This is equivalent to claiming that the equation of X contains a monomial involving x_2, x_3 only. In all 5 cases this is already forced by the condition (2.1).

Assume next that X is one of the series $(2, 2k+1, 2k+1, 4k+1)$. Its equation is a linear combination of terms

$$x_0^{4k+2}, x_3^2x_0, x_3x_1x_0^{k+1}, x_3x_2x_0^{k+1}, g_4(x_1, x_2), g_2(x_1, x_2)x_0^{2k+1}.$$

Moreover, the conditions (2.1–3) imply that the first 2 appear with nonzero coefficient and g_4 does not have multiple roots.

$(x_0 = 0)$ intersects X in a curve C whith equation

$(q_{8k+4}(x_1, x_2) = 0) \subset \mathbb{P}^2(2k+1, 2k+1, 4k+1)$, isomorphic to $(q_4(x_1, x_2) = 0) \subset \mathbb{P}^2(1, 1, 4k+1)$.

Thus C has 4 irreducible components $C_1 + C_2 + C_3 + C_4$ meeting at $P_3 = (0 : 0 : 0 : 1)$. This shows that $\frac{1}{2}C$ is not klt at P_3 and $\frac{1}{2}C$ is a tiger on X.

Next we prove that for $k \geq 2$, (X, D) is log canonical for every effective $\mathbb{Q}$-divisor $D \equiv -K_X$. For $k = 1$ we show that $(X, \frac{33}{38}D)$ is log canonical. These are stronger than needed in order to apply (7).

Consider the linear system $\mathcal{O}_{\mathbb{P}}(2(2k+1))$. This is the pull back of $\mathcal{O}(2(2k+1))$ from the weighted projective plane $\mathbb{P}(2, 2k+1, 2k+1)$. The latter is isomorphic to $\mathbb{P}(2, 1, 1)$ which is the quadric cone in ordinary $\mathbb{P}^3$ and the linear system is the hyperplane sections, thus very ample. Hence for every smooth point $P \in X$ there is a divisor $F \in |\mathcal{O}_X(2(2k+1))|$ passing through P and not containing any of the irreducible components of D. So

$$\operatorname{mult}_P D \leq (D \cdot F) = \frac{2(2k+1)(8k+4)}{2(2k+1)^2(4k+1)} = \frac{4}{4k+1} < 1.$$

We are left to deal with the singular points of X. These are at $P_3 = (0 : 0 : 0 : 1)$ and at $P_a = (0 : a : 1 : 0)$ where a is a root of $g_4(x_1, 1)$.

P_3 is the most interesting. Let $p_3 : (S \cong \mathbb{C}^2, Q_3) \to (X, P_3)$ be a local orbifold chart. Intersecting p_3^*D with a general member of the linear system $|x_0^{2k+1}, x_1^2|$ we obtain that

$$\operatorname{mult}_{Q_3} p_3^* D \le \tfrac{4k+1}{2}(D \cdot \mathcal{O}(2(2k+1)) = 2.$$

This is too big to apply (10.3). Let $\pi : S' \to S$ be the blow up of the origin with exceptional divisor E. Then

$$K_{S'} + \alpha E + \pi_*^{-1}(p_3^* D) \equiv \pi^*(K_S + p_3^* D),$$

and $\alpha \le 1$. Using Shokurov's inversion of adjunction (see, for instance [Kollár-Mori98, 5.50]) (X, D) is log canonical at P_3 if $\pi_*^{-1}(p_3^* D)|_E$ is a sum of points, all with coefficient ≤ 1. In order to estimate these coefficients, we write $D = D' + \sum a_i C_i$ where D' does not contain any of the C_i.

We first compute that

$$(C_i \cdot C_j) = \tfrac{1}{4k+1} \quad \text{if } i \ne j, \text{ and} \quad (C_i \cdot \mathcal{O}(1)) = \tfrac{1}{(2k+1)(4k+1)}.$$

From this we obtain that

$$(C_i \cdot C_i) = (C_i \cdot \mathcal{O}(2)) - \sum_{j \ne i}(C_i \cdot C_j) = \tfrac{-(6k+1)}{(2k+1)(4k+1)}.$$

Note that

$$\tfrac{1}{(2k+1)(4k+1)} = (C_i \cdot D) = a_i(C_i \cdot C_i) + (\textstyle\sum_{j \ne i} a_j)(C_i \cdot C_j) + (C_i \cdot D').$$

Multiplying by $(2k+1)(4k+1)$ and using that $\sum a_i \le \operatorname{mult}_{Q_3} p_3^* D \le 2$ and $(C_i \cdot D') \le 2(D \cdot D') \le 2(D \cdot D)$ this becomes

$$1 \le -(6k+1)a_i + (2 - a_i)(2k+1) + 4 \quad \text{which gives} \quad a_i \le \tfrac{4k+5}{8k+2}.$$

Furthermore,

$$\operatorname{mult}_{Q_3} p_3^* D' \le \tfrac{1}{4}(p_3^* D' \cdot \sum_i p_3^* C_i) \le \tfrac{4k+1}{4}(D \cdot \mathcal{O}(2)) = \tfrac{1}{2k+1}.$$

Thus we see that

$$\pi_*^{-1}(p_3^* D)|_E = \sum a_i \pi_*^{-1}(p_3^* C_i)|_E + \pi_*^{-1}(p_3^* D')|_E$$

is a sum of 4 distinct points with coefficient $\le \frac{4k+5}{8k+2}$ and another sum of points where the sum of the coefficients is $\le \frac{1}{2k+1}$. This implies that (X, D) is log canonical at P_3 for $k \ge 2$.

If $k = 1$ then our estimate does not show that (X, D) is log canonical. However, we still get that $\pi_*^{-1}(p_3^* D')|_E$ is a sum of points with coefficients $\le \frac{9}{10} + \frac{1}{3}$. Thus $(X, \frac{30}{37} D)$ is log canonical at P_3.

The points P_a are easier. Only one of the C_i passes through each of them, and the multiplicity of the pull back of D' is bounded by $(2k+1)(D \cdot \mathcal{O}(2)) = \frac{4}{4k+1}$. This shows right away that (X, D) is klt at these points. $\square$

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References

Alexeev93. V. Alexeev: Two two-dimensional terminations. Duke Math. J. 69 (1993) 527–545.

Alexeev94. V. Alexeev: Boundedness and K^2 for log surfaces. Internat. J. Math. 5 (1994) 779–810.

Boyer–Galicki00. C. Boyer and K. Galicki: New Einstein Manifolds in Dimension Five; Manuscript,

Brieskorn68. E. Brieskorn, Rationale Singularitäten komplexer Flächen, Invent. Math. 4 (1968) 336–358

Demailly-Kollár99. J.-P. Demailly and J. Kollár: Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds, (to appear)

Dolgachev82. I. Dolgachev, Weighted projective varieties, in: Group actions and vector fields, Springer Lecture Notes in Math. vol. 956, (1982) 34–71

Fletcher89. A.R. Fletcher: Working with weighted complete intersections ; Preprint MPI/89-35, Max-Planck Institut für Mathematik, Bonn, 1989

Fulton84. W. Fulton, Intersection theory, Springer-Verlag, Berlin – New York, 1984, Second ed. 1998

Keel-McKernan99. S. Keel and J. McKernan: Rational curves on quasi-projective surfaces. Mem. Amer. Math. Soc. 140 (1999), no. 669

Kollár97. J. Kollár: Singularities of pairs, Algebraic Geometry, Santa Cruz, 1995; Proceedings of Symposia in Pure Math. vol. 62, AMS, 1997, 221-287

Kollár-Mori98. J. Kollár and S. Mori: Birational geometry of algebraic varieties, Cambridge Univ. Press 1998

Miyanishi81. M. Miyanishi: Noncomplete algebraic surfaces. Lecture Notes in Mathematics, 857. Springer-Verlag, Berlin-New York, 1981.

Nadel90. A.M. Nadel: Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature; Annals of Math. 132 (1990), 549–596

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