CHARACTERIZATIONS OF $\mathbb{P}^n$ IN ARBITRARY CHARACTERISTIC

YASUYUKI KACHI AND JÁNOS KOLLÁR

There are several results which characterize $\mathbb{P}^n$ among projective varieties. The first substantial theorem in this direction is due to [Hirzebruch-Kodaira57], using the topological properties of $\mathbb{CP}^n$. Later characterizations approached $\mathbb{P}^n$ as the projective variety with the most negative canonical bundle. The first such theorem is due to [Kobayashi-Ochiai72] and later it was generalized by [Fujita90]. A characterization using the tangent bundle was given in [Mori79, Siu-Yau80]. Other characterizations using vector bundles are given in the papers [Ye-Zhang90, Sato99, Kachi99].

The results of [Kobayashi-Ochiai72, Fujita90] ultimately relied on the Kodaira vanishing theorem, thus they were restricted to characteristic zero. The aim of this note is to provide an argument which does not use higher cohomologies of line bundles. This makes the proofs work in arbitrary characteristic. Thus (1) generalizes [Fujita90, II.7] and (2) is a characteristic free version of [Kobayashi-Ochiai72]. The proof of [Fujita90] works if we know enough about extremal contractions of smooth varieties. Thus the results have been known in any characteristic for surfaces and they can be read off from the classification of contractions of smooth 3–folds in positive characteristic developed in the papers [Kollár91, Shepherd-Barron97, Megyesi98].

For the rest of the paper all varieties are over an algebraically closed field of arbitrary characteristic.

Theorem 1. *Let X be a smooth projective variety of dimension n and H an ample divisor on X. Then the pair (X, H) satisfies one of the following:*

1. *$(n-1)H + K_X$ is nef.*
2. *$X \cong \mathbb{P}^n$ and H is a hyperplane section.*
3. *X is a quadric in $\mathbb{P}^{n+1}$ and H is a hyperplane section.*
4. *X is the projectivization of a rank n vector bundle over a smooth curve A and $H = \mathcal{O}(1)$.*

With a little bit of work on the scrolls in (1.4) this implies the following characterization of projective spaces and hyperquadrics.

Corollary 2. *Let X be a smooth projective variety of dimension n.*

1. *X is isomorphic to $\mathbb{P}^n$ iff there is an ample divisor H such that $-K_X$ is numerically equivalent to $(n+1)H$.*
2. *X is isomorphic to a hyperquadric $\mathbb{Q}^n \subset \mathbb{P}^{n+1}$ iff there is an ample divisor H such that $-K_X$ is numerically equivalent to nH.* □

The first step of the proof is to identify the family of lines on X. The results of [Mori79] show that there is a family of rational curves $\{C_t\}$ on X such that $(L\cdot C_t) = 1$ and this family has the expected dimension. Focusing on these rational curves leads to other characterizations of $\mathbb{P}^n$ and of $\mathbb{Q}^n$:

Theorem 3. *Let X be a smooth projective variety of dimension n over $\mathbb{C}$ and L an ample line bundle on X.*

1. [Andreatta-Ballico-Wiśniewski93] *Assume that for every $x, y \in X$ there is a rational curve C_{xy} through x, y with $(L\cdot C_{xy}) = 1$. Then $X \cong \mathbb{P}^n$ and $L \cong \mathcal{O}(1)$.*
2. [Kachi-Sato99] *Assume that for every $x, y, z \in X$ there is a rational curve C_{xyz} through x, y, z with $(L \cdot C_{xyz}) = 2$. Then $X \cong \mathbb{P}^n$ or $X \cong \mathbb{Q}^n$ and $L \cong \mathcal{O}(1)$.*

The positive characteristic versions of these theorems are still open. Similarly, concentrating on higher degree curves leads to the following characterization of Veronese varieties:

Theorem 4. [Kachi99] *Let X be a smooth projective variety of dimension n (over an algebraically closed field of arbitrary characteristic) and L an ample divisor on X. Assume that:*

1. *For general $x, y \in X$ there is a connected (possibly reducible) curve C_{xy} through x, y with $(L \cdot C_{xy}) \le d$.*
2. *For every irreducible subvariety $Z \subset X$ of dimension at least 2 we have $(L^{\dim Z} \cdot Z) \ge d^{\dim Z}$.*
3. *Fujita's sectional genus of $\frac{1}{d}L$ is less than 1, that is,*

$$(L^{n-1} \cdot (\tfrac{n-1}{d}L + K_X)) < 0.$$

Then $X \cong \mathbb{P}^n$ and $L \cong \mathcal{O}(d)$.

5 (Proof of (1))**.** As we noted, we may assume that $n \ge 3$. We are done if $K_X + (n-1)H$ is nef.

If $K_X + (n-1)H$ is not nef, then by the cone theorem of Mori (cf. [Kollár95, III.1]) there is a $c > 0$ and a rational curve $C \subset X$ such that

1. $K_X + (n-1+c)H$ is nef,
2. $C \cdot (K_X + (n-1+c)H) = 0$, and
3. $-(C \cdot K_X) \le n+1$.

Thus we see that

$$(C \cdot H) \leq \frac{-(C \cdot K_X)}{n-1+c} \leq \frac{n+1}{n-1+c} < 2.$$

Hence $C \cdot H = 1$ and $-(C \cdot K_X) = n$ or $n+1$. Let $f : \mathbb{P}^1 \to X$ be the normalization of C, $0 \in \mathbb{P}^1$ a point and $x = f(0)$. Then by [Kollár95, II.1]

$$\begin{array}{lll} \dim_{[f]} \operatorname{Hom}(\mathbb{P}^1, X) & \geq -(C \cdot K_X) + n, & \text{and} \\ \dim_{[f]} \operatorname{Hom}(\mathbb{P}^1, X, 0 \mapsto x) & \geq -(C \cdot K_X). & \end{array}$$

We consider 3 separate cases corresponding to the three outcomes (1.2-4).

6 (Case 1: $-(C \cdot K_X) = n+1$)**.**

In order to go from homomorphisms to curves in X, we have to quotient out by the automorphism group. Thus $\operatorname{Hom}(\mathbb{P}^1, X, 0 \mapsto x)$ gives an $(n-1)$-dimensional family of rational curves through $x \in X$ since $\operatorname{Aut}(\mathbb{P}^1, 0)$ is 2–dimensional. This implies that the Picard number of X is 1 (cf. [Kollár95, IV.3.13.3]). Thus $-K_X = (n+1)H$. The following lemma shows that $X \cong \mathbb{P}^n$. We formulate it for singular varieties, the setting needed for the third case.

Lemma 7. *Let X be a normal, projective variety and $x \in X$ a smooth point. Let H be an ample Cartier divisor on X and let $\{C_t : t \in T\}$ be an $(n-1)$-dimensional family of curves through x. Assume that $(C_t \cdot H) = 1$ and $(-K_X \cdot H^{n-1}) > (n-1)(H^n)$. Then $X \cong \mathbb{P}^n$, $H = \mathcal{O}(1)$ and the C_t are lines through x.*

Proof. By Riemann–Roch,

$$h^0(X, \mathcal{O}_X(mH)) = m^n \frac{(H^n)}{n!} + m^{n-1} \frac{-(K_X \cdot H^{n-1})}{2(n-1)!} + O(m^{n-2}).$$

For a section of a line bundle it is

$$\binom{m+n-1}{n} = m^n \frac{1}{n!} + m^{n-1} \frac{n-1}{2(n-1)!} + O(m^{n-2})$$

conditions to vanish at a smooth point $x \in X$ to order m. Comparing the two estimates we see that for large m

$$\begin{array}{lll} H^0(X, \mathcal{O}_X(mH) \otimes I_x^{m+1}) & \geq (\text{const}) \cdot m^n & \text{if } (H^n) > 1, \text{ and} \\ H^0(X, \mathcal{O}_X(mH) \otimes I_x^{m}) & \geq (\text{const}) \cdot m^{n-1} & \text{if } (H^n) = 1. \end{array}$$

Pick any member $D \in |\mathcal{O}_X(mH) \otimes I_x^{m+1}|$ (resp. $D \in |\mathcal{O}_X(mH) \otimes I_x^{m}|$) and take $C_t \not\subset D$. Then $(C_t \cdot D) = m$. On the other hand, $(C_t \cdot D) \geq \sum_{y \in C_t} \operatorname{mult}_y D \geq m+1$ (resp. $\geq m$). This is a contradiction if

$(H^n) > 1$. If $(H^n) = 1$ then $(C_t \cap D) = \{x\}$ and the linear system $|\mathcal{O}_X(mH) \otimes I_x^m|$ is constant along C_t.

By varying $x \in C_t$, we also see that the line bundle $\mathcal{O}_{C_t}(mH)$ has a section with an m-fold zero at a general point of C_t. Since $\deg \mathcal{O}_{C_t}(mH) = m$, this implies that a general C_t is a smooth rational curve.

Set $Y = B_x X$ with exceptional divisor E and projection $\pi : Y \to X$. Set $M = \pi^* H - E$. Then $H^0(\mathcal{O}_Y(mM)) \cong H^0(X, \mathcal{O}_X(mH) \otimes I_x^m)$, hence $h^0(\mathcal{O}_Y(mM)) \geq (\text{const}) \cdot m^{n-1}$. For large m, let $p : Y \dashrightarrow Z$ denote the map induced by $|mM|$. Up to birational equaivalence, p does not depend on m and it has connected 1–dimensional general fibers. Moreover, for any subset $V \subset Y$ of dimension at most $n-2$ there is a member $D \in |mM|$ containing V.

Every curve C_t lifts to a curve C'_t on Y and there is an open subset $T^0 \subset T$ such that the curves $\{C'_t : t \in T^0\}$ form a single algebraic family. (In our case these correspond to the curves C_t which are smooth at x.) By shrinking T we pretend that $T = T^0$. Note that $(C'_t \cdot E) = 1$ and $(C'_t \cdot M) = 0$.

We may assume that Z is proper. Let $V \subset Y$ be the set where p is not defined and choose $V \subset D \in |mM|$.

If a curve C'_t passes through V then $C'_t \subset D$ since $(M \cdot C'_t) = 0$. The curves C'_t cover an open subset of Y, hence a general C'_t is disjoint from D. Thus p is defined everywhere along a general C'_t and C'_t is a fiber of p (at least set theoretically). Hence there are open subsets $Y^0 \subset Y$ and $Z^0 \subset Z$ such that $p^0 : Y^0 \to Z^0$ is proper and flat. $(E \cdot C'_t) = 1$, thus E is a rational section of p. In particular, a general fiber of p is reduced. Thus, possibly after shrinking Y^0, we may assume that p^0 is a $\mathbb{P}^1$-bundle. Set $E^0 := E \cap Y^0$.

Z^0 parametrizes curves in Y and by looking at the image of these curves in X we obtain a morphism $h : Z^0 \to \text{Chow}(X)$. (See [Kollár95, I.3–4] for the definition of Chow and its basic properties.) Let Z' be the normalization of the closure of the image of Z^0 in $\text{Chow}(X)$, $p' : U' \to Z'$ the universal family and $u' : U' \to X$ the cycle map. We want to prove that $U' = Y$.

h induces a morphism between the universal families $h' : Y^0 \to U'$ which sits in a diagram

$$Y^0 \xrightarrow{h'} U' \xrightarrow{u'} X \overset{\pi^{-1}}{\dashrightarrow} Y.$$

The composite is an open immersion hence birational. Thus the intermediate maps are also birational. h' is thus an isomorphism near a general fiber of p^0, and so $p' : U' \to Z$ is a $\mathbb{P}^1$-bundle over an open set.

Z' parametrizes curves wich have intersection number 1 with an ample divisor. Thus Z' parametrizes irreducible curves with multiplicty 1.

Let $E' \subset U'$ be the closure of $h'(E^0)$. E' is a rational section of p' and $u'(E') = \{x\}$. If E' contains a whole fiber of p' then $u'(E') = \{x\}$ implies that every fiber is mapped to a point which is impossible. So, E' is a section of p'.

Let $B \subset U' \setminus E'$ be a curve such that $u(B) \in X$ is a point. Then $S := (p')^{-1}(p'(B))$ is a ruled surface and u' contracts the curves B and $E' \cap S$ to points. By bend–and–break (cf. [Kollár95, II.5.5.2]) this leads to a curve $C^* \subset X$ such that $(C^* \cdot H) < 1$, but this is impossible. Thus u' is quasifinite on $U' \setminus E'$. u' is also birational, hence

$$u' : U' \setminus E' \to X \quad \text{is an open embedding.}$$

Its image contains $X \setminus \{x\}$ and it is not projective, hence $U' \setminus E' \cong X \setminus \{x\}$.

Let us now look at the birational map $\phi := {u'}^{-1} \circ \pi : Y \dashrightarrow U'$. Both U' and Y contain Y^0 and $X \setminus \{x\}$ as open sets, and ϕ is the identity on Y^0 and $X \setminus \{x\}$. Thus ϕ is an isomorphism outside the codimension 2 sets $E \setminus E^0$ and $E' \setminus E^0$. Since Y/X has relative Picard number 1, this implies that ϕ is a morphism by (8). Thus ϕ is an isomorphism since ϕ can not contract a subset of $E \cong \mathbb{P}^{n-1}$ without contracting E.

In particular $E' \cong \mathbb{P}^{n-1}$. $p' : U' \to E'$ is flat with reduced fibers by [Hartshorne77, Ex.III.10.9], hence it is a $\mathbb{P}^1$-bundle. Thus $X \cong \mathbb{P}^n$ by an easy argument (see, for instance, [Kollár95, V.3.7.8]). ☐

The following lemma is essentially in [Matsusaka-Mumford64].

Lemma 8. *Let $Z_i \to S$ be projective morphisms with Z_1 smooth. Let $\phi : Z_1 \dashrightarrow Z_2$ be a birational map of S-schemes which is an isomorphism outside the codimension 2 subsets $E_i \subset Z_i$. Assume that the relative Picard number $\rho(Z_1/S)$ is 1 and $Z_1 \setminus E_1 \to S$ is not quasifinite. Then ϕ is a morphism.*

Proof. We may assume that S is affine. Let H_2 be a relatively ample divisor on Z_2 and H_1 its birational transform. H_1 is also relatively ample because $\rho(Z_1/S) = 1$ and $-H_1$ can not be relatively ample (since H_1 is effective when restricted to a positive dimensional fiber of $Z_1 \setminus E_1 \to S$). Thus $|mH_2|$ and $|mH_1|$ are both base point free for $m \gg 1$ and these are the birational transforms of each other by ϕ. Let $\Gamma \subset Z_1 \times_S Z_2$ be the closure of the graph of ϕ with projections $\pi_i : \Gamma \to Z_i$. If $\pi_1^{-1}(z)$ is positive dimensional then every member of $|mH_2|$ intersects $\pi_2(\pi_1^{-1}(z))$, thus $|mH_1| = \phi^{-1}|mH_2|$ has z as its base point, a contradiction. Thus π_1 is an isomorphism and ϕ is a morphism. ☐

We are left with the cases when $-(C\cdot K_X)=n$. Then we have an at least $(n-2)$-dimesional family of curves through every $x\in X$. If there is an $x\in X$ such that all curves of degree 1 through x cover X, then we obtain that $X\cong\mathbb{P}^n$ and this leads to a contradiction. Thus for every $x\in X$ the curves $\{C_t : x\in C_t\}$ sweep out a divisor B_x. The divisors form an algebraic family for x in a suitable open set $X^0\subset X$.

9 (Case 2: $-(C\cdot K_X)=n$ and $B_{x_1}\cap B_{x_2}\neq\emptyset$ for $x_1,x_2\in X^0$)**.**
By assumption, any two points of X are connected by a chain of length 2 of the form $C_{t_1}\cup C_{t_2}$, thus X has Picard number 1 by [Kollár95, IV.3.13.3]. Thus we see that $-K_X\equiv nH$. The computation of the genus of a general complete intersection curve of members of mH for odd $m\gg 1$ shows that (H^n) is even. Thus $(H^n)\geq 2$.

Pick general x_1,x_2 such that there is no degree 1 curve through x_1 and x_2 and pick curves $C_i\supset x_i$ such that $C_1\cap C_2\neq\emptyset$. We can view $C_1\cup C_2$ as the image of a map $f:\mathbb{P}^1\vee\mathbb{P}^1\to X$ where $\mathbb{P}^1\vee\mathbb{P}^1$ denotes the union of 2 lines in $\mathbb{P}^2$. Let $y_i\in\mathbb{P}^1\vee\mathbb{P}^1$ be a preimage of x_i. By the usual estimates (cf. [Kollár95, II.1.7.2]) we obtain that

$$\dim_{[f]}\operatorname{Hom}(\mathbb{P}^1\vee\mathbb{P}^1, X, y_i\mapsto x_i)\geq 2n+n-2n=n.$$

On the other hand, these maps correspond to pairs of curves $C_1'\cup C_2'$ where C_i' passes through x_i and $C_1\cap C_2\neq\emptyset$. By our assumption these form an $n-2$-dimensional family. The automorphism group $\operatorname{Aut}(\mathbb{P}^1\vee\mathbb{P}^1,y_1,y_2)$ accounts for the missing 2 dimensions. Viewing $\mathbb{P}^1\vee\mathbb{P}^1$ as a reducible plane conic, [Kollár95, II.1.7.3] implies that there is an at least $(n-1)$-dimensional family of degree 2 rational curves $\{A_s\subset X : s\in S\}$ which pass through both of x_1,x_2.

As in Case 1, we obtain that

$$\begin{array}{lll} H^0(X,\mathcal{O}_X(mH)\otimes I_{x_1}^{m+1}\otimes I_{x_2}^{m+1}) & \geq (\text{const})\cdot m^n & \text{if } (H^n)>2, \text{ and}\\ H^0(X,\mathcal{O}_X(mH)\otimes I_{x_1}^{m}\otimes I_{x_2}^{m}) & \geq (\text{const})\cdot m^{n-1} & \text{if } (H^n)=2. \end{array}$$

Pick any member $D\in|\mathcal{O}_X(mH)\otimes I_{x_1}^{m+1}\otimes I_{x_2}^{m+1}|$ (resp. $D\in|\mathcal{O}_X(mH)\otimes I_{x_1}^{m}\otimes I_{x_2}^{m}|$) and take $A_s\not\subset D$. Then $(A_s\cdot D)=2m$. On the other hand, $(A_s\cdot D)\geq\sum_{y\in A_s}\operatorname{mult}_y D\geq 2m+2$ (resp. $\geq 2m$). Thus $(H^n)=2$, $(A_s\cap D)=\{x_1,x_2\}$ and the linear system $|\mathcal{O}_X(mH)\otimes I_{x_1}^{m}\otimes I_{x_2}^{m}|$ is constant along A_s for general s.

We also see that for general A_s, the line bundle $\mathcal{O}_{A_s}(mH)$ has a section with an m-fold zero at two general points of A_s. Since $\deg\mathcal{O}_{A_s}(mH)=2m$, this implies that a general A_s is a smooth rational curve.

Set $Y=B_{x_1x_2}X$ with exceptional divisors E_1,E_2 and projection $\pi: Y\to X$. Set $M=\pi^*H-E_1-E_2$. Then $|mM|=|\mathcal{O}_X(mH)\otimes I_{x_1}^{m}\otimes I_{x_2}^{m}|$. As in Case 1, we obtain an open set $Y^0\subset Y$ and a $\mathbb{P}^1$-bundle $Y^0\to Z^0$ with two sections $E_i\cap Y^0$. Construct $p': U'\to Z'$ and $u: U'\to X$

using Chow(X) as before. Let $z \in Z'$ be any point and $A_z \subset X$ the 1–cycle corresponding to z. A_z has degree 2 and it passes through the points x_1, x_2. We have assumed that there is no degree 1 curve passing through x_1, x_2, thus either A_z is irreducible and $(H \cdot A_z) = 2$ or A_z has 2 different irreducible components $A_z^1 \cup A_z^2$ and $(H \cdot A_z^i) = 1$.

The key point is to establish that the cycle map u is quasifinite on $U' \setminus (E_1' \cup E_2')$. Assuming the contrary, we have a normal surface S and morphisms $p' : S \to Z^* \subset Z'$ and $u : S \to X$ such that every fiber of $p' : S \to Z^*$ is either $\mathbb{P}^1$ or the union of 2 copies of $\mathbb{P}^1$ and u contracts $E_i \cap S$ and another curve B to points.

A version of bend–and–break (10) establishes that in this case $u({p'}^{-1}(z))$ is independent of $z \in Z^*$. This is, however, impossible since $Z' \to$ Chow(X) is finite.

Again using (8) we conclude that $U' \cong Y$ and $Z' \cong E_i' \cong \mathbb{P}^{n-1}$. Therefore

$$\mathcal{O}_Y(\pi^* H - E_1 - E_2) \cong {p'}^* \mathcal{O}_{E_i'}(1) \cong {p'}^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)$$

is generated by global sections. Pushing these sections down to X and varying the points x_i we obtain that $\mathcal{O}_X(H)$ is generated by $n+2$ global sections. Thus $|H|$ gives a finite morphism $X \to \mathbb{P}^{n+1}$ whose image is a quadric since $(H^n) = 2$. Thus X is a smooth quadric.

Lemma 10 (3 point bend–and–break)**.** *Let S be a normal and proper surface which is a conic bundle (that is, there is a morphism $p : S \to A$ such that every (scheme theoretic) fiber is isomorphic to a plane conic). Let $E_1, E_2 \subset S$ be disjoint sections of p and $B \subset S$ a multisection. Assume that every singular fiber F_a of p has 2 components $F_a = F_a^1 \cup F_a^2$, and $(F_a^i \cdot E_i) = 0$ for $i = 1, 2$. Let L be a nef divisor on S such that $(L \cdot F_a^i) = 1$ for every F_a and i and $(L \cdot E_1) = (L \cdot E_2) = (L \cdot B) = 0$. Then $S \cong A \times \mathbb{P}^1$, the E_i are flat sections and B is a union of flat sections.*

Proof. The Picard group of S is generated by the classes F_a^i, E_1 with rational coefficients (cf. [Kollár95, IV.3.13.3]). From the conditions $(L \cdot F_a^i) = 1$ we conclude that $L = E_1 + E_2 + dF$ for some d (and $(E_1^2) = (E_2^2)$). $(L^2) \geq 0$ and $(L \cdot E_i) = 0$, so by the Hodge index theorem $(E_i^2) \leq 0$. Hence we conclude that $d = -(E_i^2) \geq 0$. This implies that

$$(L \cdot B) = (E_1 \cdot B) + (E_2 \cdot B) + \deg(B/A) \cdot (-E_1^2),$$

All terms on the right hand side are nonnegative. Thus $(E_1^2) = (E_2^2) = 0$. Since $(E_1 \cdot E_2) = 0$, this implies that E_1 and E_2 are algebraically equaivalent. Thus there are no singular fibers. $\square$

11 (Case 3: $-(C \cdot K_X) = n$ and $B_{x_1} \cap B_{x_2} = \emptyset$ for general x_1, x_2)**.** Take a general $x_1 \in X$ and a general $x_3 \in B_{x_1}$. If $\dim(B_{x_1} \cap B_{x_3}) = n-2$ then $\dim(B_{x_1} \cap B_{x_2}) = n-2$ for a general x_2, a contradiction. Thus B_{x_1} and B_{x_2} are either disjoint or they coincide for (x_1, x_2) in an open subset of $X \times X$. The algebraic family of divisors B_x thus determines a morphism $p : X \to A$ to a smooth curve A. A general fiber of p is B_x. Let Y be a general fiber of p and $x \in Y$ a general smooth point. We have an $(n-2)$-dimensional family of curves C_t through x and all of these are contained in Y. $(C_t \cdot H) = 1$ and the Picard number of Y is 1 as before. Thus $-K_Y \cong -K_X|_Y \cong nH|_Y$. Let $\pi : \bar{Y} \to Y$ denote the normalization of Y. The curves C_t lift to $\bar{Y}$, and

$$-K_{\bar{Y}} = -K_Y + (\text{conductor of } \pi),$$

where the conductor of π is an effective divisor which is zero iff π is an isomorphism (this is a special case of duality for finite morphisms, cf. [Hartshorne77, Ex.III.7.2]). Thus

$$-(K_{\bar{Y}} \cdot \pi^* H^{n-2}) \geq -(K_Y \cdot H^{n-2}) = n(H|_Y)^{n-1} = n(\pi^* H)^{n-1}.$$

So $\bar{Y} \cong \mathbb{P}^{n-1}$ by (7). Furthermore, we also obtain that $-K_{\bar{Y}} = n\pi^* H +$ (conductor), which implies that the conductor of π is zero. Thus $Y \cong \mathbb{P}^{n-1}$ and $p : X \to A$ is generically a $\mathbb{P}^{n-1}$-bundle. This implies that p has a section (cf. [Serre79, X.6-7]), hence every fiber of p has a reduced irreducible component.

Let $W \subset X$ be a reduced irreducible component of a fiber of p and $x \in W$ a general smooth point. The above argument applies also to W, and we obtain that $W \cong \mathbb{P}^{n-1}$ and W is a connected component of its fiber. This shows that X is a $\mathbb{P}^{n-1}$-bundle over A. □

Acknowledgments . Partial financial support was provided by the NSF under grant numbers DMS-9800807 and DMS-9622394.

References

Andreatta-Ballico-Wiśniewski93. M. Andreatta, E. Ballico and J. Wiśniewski, Two theorems on elementary contractions, Math. Ann. 297 (1993) 191–198

Fujita90. T. Fujita, Classification theories of polarized varieties, London Mathematical Society Lecture Note Series, 155 Cambridge University Press 1990

Hartshorne77. R. Hartshorne, Algebraic Geometry, Graduate Text in Math. 52 Springer–Verlag, Berlin, Heidelberg, New York 1977

Hirzebruch-Kodaira57. F. Hirzebruch and K. Kodaira, On the complex projective spaces, J. Math. Pure Appl. 36 (1957) 201–216

Kachi99. Y. Kachi, Characterization of Veronese varieties and the Hodge index theorem, (preprint), 1999

Kachi-Sato99. Y. Kachi and E. Sato, Polarized varieties whose points are joined by rational curves of small degrees, Illinois J. Math. 43 (1999) 350-390

Kobayashi-Ochiai72. S. Kobayashi and T. Ochiai, Characterization of the complex projective space and hyperquadrics, J. Math. Kyoto U. 13 (1972) 31–47

Kollár91. J. Kollár, Extremal Rays on Smooth Threefolds, Ann. Sci. ENS 24 (1991) 339-361

Kollár95. J. Kollár, Rational curves on algebraic varieties, Springer–Verlag 1995

Matsusaka-Mumford64. T. Matsusaka and D. Mumford, Two fundamental theorems on deformations of polarised varieties, Amer. J. Math. 86 (1964) 668-684

Megyesi98. G. Megyesi, Fano threefolds in positive characteristic, J. Alg. Geom. 7 (1998) 207–218

Mori79. S. Mori, Projective manifolds with ample tangent bundles, Ann. of Math. 110 (1979) 593–606

Sato99. E. Sato, Adjoint bundles of ample vector bundles in any characteristic, J. Math. Kyoto Univ. (to appear)

Serre79. J.-P. Serre, Local fields, Springer Verlag, 1979

Shepherd-Barron97. N. I. Shepherd–Barron, Fano threefolds in characteristic p, Comp. Math. 105 (1997) 237–265

Siu-Yau80. Y.-T. Siu and S. T. Yau, Compact Kähler manifolds of positive bisectional curvature, Inv. Math. 59 (1980) 189-204

Ye-Zhang90. Y.G. Ye and Q. Zhang, On ample vector bundles whose adjunction bundles are not numerically effective, Duke Math. J. 60 (1990) 671–687

Johns Hopkins University, Baltimore MD 21218

kachi@math.jhu.edu

Princeton University, Princeton NJ 08544-1000

kollar@math.princeton.edu