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Rational Curves on Varieties

CAROLINA ARAUJO and JÁNOS KOLLÁR

The aim of these notes is to give an introduction to the ideas and techniques of handling rational curves on varieties. The main emphasis is on varieties with many rational curves, which are the higher dimensional analogs of rational curves and surfaces.

Sections one through five and seven closely follow the lectures of J. Kollár given at the summer school "Higher dimensional varieties and rational points". The notes were written up by C. Araujo and substantially edited later.

Terminology. We follow the terminology of [9]. Thus a variety is an integral separated scheme of finite type over a field (not necessarily algebraically closed). Somewhat inconsistently, by a curve we mean a geometrically reduced (not necessarily irreducible) separated 1-dimensional scheme of finite type over a field.

1. Rational Curves on Surfaces

In this section we study rational curves on surfaces. Our main point is that rational surfaces contain plenty of rational curves (at least when we are over an algebraically closed field).

In the following example we analyze the possible genera of curves lying on smooth surfaces in $\mathbb{P}^3$. This illustrates our point: while surfaces of degree $d \leq 3$ (which are rational) contain many rational curves, those of degree $d \geq 5$ tend not to contain rational curves at all.

Example 1 (Curves on Surfaces in $\mathbb{P}^3$)**.** Let $S = S_d$ be a smooth surface of degree d in $\mathbb{P}^3_{\mathbb{C}}$. By the Noether–Lefschetz Theorem, if $d \geq 4$ and S_d is very general, any curve $C \subset S_d$ is a complete intersection. This means that there is a surface H_m of degree m in $\mathbb{P}^3$, for some m, such that $C = S_d \cap H_m$ scheme-theoretically. (See [20, I] for a survey of different proofs of this result.)

Let $\tilde{C} \to C$ be the normalization of C. We ask the following question: for fixed d and m, what are the possible values of the genus $g(\tilde{C})$ of $\tilde{C}$?

The biggest possible value of $g(\tilde{C})$ is attained when $C = S_d \cap H_m$ is already nonsingular. In this case $g(\tilde{C}) = g(C) = \frac{1}{2}md(m + d - 4) + 1$.

Lower bounds for $g(\tilde{C})$ are much more interesting and difficult to get.

Let H denote the hyperplane class in $\mathbb{P}^3$, so that $H_m \sim mH$.

By Riemann–Roch we have:

$$h^0\big(S, \mathcal{O}_S(mH)\big) = m^2\frac{d}{2} - m\frac{d(d-4)}{2} + \chi(\mathcal{O}_S).$$

By the adjunction formula we have:

$$p_a(C) = m^2\frac{d}{2} + m\frac{d(d-4)}{2} + 1.$$

The "expected" maximal number of singular points of C is

$$h^0\big(S, \mathcal{O}_S(mH)\big) - 1.$$

By "expected" we mean that this is not a theorem but it is frequently correct. Here is an argument to support it.

Requiring that the curve has a singularity at a given point imposes 3 conditions. We have a 2-dimensional family of points. Thus, requiring that the curve has a singularity at some point imposes 1 condition.

Putting these formulas together, we obtain that the expected smallest genus of $\tilde{C}$ is

$$md(d-4) + 2 - \chi(\mathcal{O}_S).$$

If $d \geq 5$, this grows linearly with m. So we expect that, given g, there are only finitely many values of m for which $\big|\mathcal{O}_S(m)\big|$ contains a curve of geometric genus g. It is important to keep in mind that this "conclusion" may not hold for every surface S_d. However, this is essentially correct for very general surfaces (although this is not at all easy to prove). In fact, by [21, Theorem 2.1], on a very general $S_d \subset \mathbb{P}^3_{\mathbb{C}}$, any irreducible curve of type $S_d \cap H_m$ has geometric genus $\geq \frac{1}{2}dm(d-5) + 2$.

If $d = 2$ or 3, not every curve on S_d is a complete intersection (for instance, the lines on smooth quadrics and cubics are not complete intersections). One can show that for $d \leq 3$ the linear system $\left|\mathcal{O}_S(m)\right|$ contains rational curves for every m. Instead of proving this fact here, we only prove that, for $d \leq 3$, S_d contains plenty of rational curves (this argument works over any algebraically closed field).

$d = 1$. In this case $S_1 \cong \mathbb{P}^2$, and for every m there are plenty of rational curves of degree m (i.e., rational curves that can be obtained as $C = S_1 \cap H_m$). These are given by maps $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ sending $(s : t)$ to $\big(\phi_0(s,t) : \phi_1(s,t) : \phi_2(s,t)\big)$, where $\phi_0(s,t)$, $\phi_1(s,t)$ and $\phi_2(s,t)$ are homogeneous polynomials of degree m.

$d = 2$. In this case S_2 is a smooth quadric surface in $\mathbb{P}^3$, and the stereographic projection from a point in S_2 gives a birational equivalence $S_2 \sim \mathbb{P}^2$.

$d = 3$. In this case S_3 is a smooth cubic surface in $\mathbb{P}^3$, which, again, is birational to $\mathbb{P}^2$ (it can be realized as $\mathbb{P}^2$ blown up at 6 points).

Example 2 (Rational curves on $\mathbb{P}^n$)**.** Given finitely many points $P_1, \ldots, P_m$ in $\mathbb{P}^n$, let us write down a rational curve $C \subset \mathbb{P}^n$ passing through these points. Set $P_i = (a_{i0} : \cdots : a_{in})$. Pick any points $c_1, \ldots, c_m \in \mathbb{A}^1$. The Lagrange interpolation formula gives degree $m-1$ polynomials $f_0, \ldots, f_n$ such that $f_j(c_i) = a_{ij}$ for every i, j. These determine a morphism $F : \mathbb{A}^1 \to \mathbb{P}^n$ such that $F(c_i) = P_i$.

Note that if the P_i and c_i are defined over a field k then F has coordinate functions defined over k.

Assume next that we fix not only the points P_i but also tangent directions at each P_i. It is not hard to see that there is a rational curve $C \subset \mathbb{P}^n$ through these points with the preassigned tangent directions.

More generally, we can even specify finitely many terms of the Taylor expansion of $F : \mathbb{A}^1 \to \mathbb{P}^n$ at each point c_i.

The next example shows that any smooth rational variety has an even stronger property.

Example 3 (Constructing curves on rational varieties)**.** Let X be a smooth projective rational variety of dimension n, and $\Phi : \mathbb{P}^n \dashrightarrow X$ a birational map given by polynomial coordinate functions $\phi_0, \ldots, \phi_N$.

Let C be a smooth projective curve with finitely many points $c_i \in C$. For each i let us specify finitely many terms of the Taylor expansion near c_i of a map from C to X. (That is, we choose morphisms $g_i : \operatorname{Spec} \mathcal{O}_{c_i,C}/m_i^{e_i} \to X$, where $\mathcal{O}_{c_i,C}$ is the local ring of C at c_i with maximal ideal m_i.)

Our aim is to find a morphism $C \to X$ that has the specified Taylor expansion at the points c_i. Moreover, if the data are defined over a field k we want $C \to X$ to be defined over k.

The first step is to reduce the problem to $X = \mathbb{P}^n$. In order to do this, we extend each g_i to something bigger. If we are over $\mathbb{C}$, we extend each g_i to a small analytic neighborhood of c_i. Over general fields this corresponds to extending each g_i to the completion $\hat{\mathcal{O}}_{c_i,C}$. The completion of a 1-dimensional smooth local ring is always the power series ring in one variable, thus these can be best given by the pull back maps

$$G_i^* : \mathcal{O}_{P_i,X} \to \hat{\mathcal{O}}_{c_i,C} \cong k[[t]].$$

By choosing the extension general enough, we may assume that the image of $\operatorname{Spec} \hat{\mathcal{O}}_{c_i,C}$ is not contained in the exceptional locus of Φ^{-1}. Thus G_i can be lifted to a morphism $H_i : \operatorname{Spec} k[[t]] \to \mathbb{P}^n$ such that $G_i = \Phi \circ H_i$. At the function level this means that

$$G_i^*(u_j) = W_i(t) \cdot H_i^*(\phi_j) \quad \text{for every } j,$$

where the u_j are homogeneous coordinates on $X \subset \mathbb{P}^N$, $G_i^*(u_j) \in k[[t]]$ and the extra factor $W_i(t)$ comes from the nonuniqueness of the projective coordinates. Let w_i be the order of poles of W_i.

Assume now that we choose a different map $H_i' : \operatorname{Spec} k[[t]] \to \mathbb{P}^n$ that agrees with H_i up to order d_i. By composing with Φ we get $G_i' : \operatorname{Spec} k[[t]] \to X$ and the above formula implies that G_i' agrees with G_i up to order $d_i - w_i$.

This implies that if we solve our original problem for maps $C \to \mathbb{P}^n$ for every e_i then we also obtain a solution for the problem $C \to X$.

Now let us consider the problem for $C \to \mathbb{P}^n$. Choose a line bundle L on C of degree at least $2g(C) - 1 + \sum e_i$. Then the restriction map

$$H^0(C, L) \to \sum_i H^0\big(C, L \otimes \mathcal{O}_{c_i,C}/m_i^{e_i}\big) \cong \sum_i \mathcal{O}_{c_i,C}/m_i^{e_i}$$

is surjective. Thus for every j there is a section $\sigma_j \in H^0(C, L)$ such that the restriction of σ_j to $\mathcal{O}_{c_i,C}/m_i^{e_i}$ agrees with $g_i^*(x_j)$. The map $C \to \mathbb{P}^n$ given by all these σ_j has the required properties.

We should keep this in mind in Section 7, when we define rationally connected varieties, in order to characterize the higher dimensional analogs of rational curves and rational surfaces.

2. Deformation of Morphisms I

In this section we are interested in the following problem: suppose we have a rational curve C on a smooth variety X defined over some base field k. How can we produce more rational curves out of C?

More generally, let X be a smooth quasi-projective variety, Y a projective variety, and $f_0 : Y \to X$ a morphism. We would like to deform the morphism f_0 in order to produce more morphisms from Y to X.

That is, we would like to find a map

$$F : Y \times D \to X \quad \text{such that} \quad F|_{Y\times\{0\}} = f_0,$$

where $(D, 0)$ is a smooth pointed algebraic curve. For $t \in D$ we set $f_t = F|_{Y\times\{t\}} : Y \to X$, and we also require that F be nontrivial, that is, $f_t \neq f_0$ (as morphisms) for general t.

It turns out that the existence of F depends on the choice of D in a very unpredictable manner. To avoid this problem, we need to replace D with a "very small" neighborhood of 0 in D. In general, no affine neighborhood is small enough. Technically, the best is to work with the spectrum of the formal power series ring $\operatorname{Spec} k[[t]]$, but this is not very intuitive.

Assume for simplicity that we are over $\mathbb{C}$. Then we get a geometrically appealing picture by working with $D = \Delta$, where Δ denotes the unit disc in $\mathbb{C}$. So we want to find a map

$$F : Y \times \Delta \to X \quad \text{such that} \quad F|_{Y\times\{0\}} = f_0.$$

Here of course F is complex analytic. We also allow Δ to be replaced by a smaller disc at any time. (All discs in $\mathbb{C}$ are biholomorphic to each other, so this is only a matter of convenience.)

Suppose we have such a map. For each $p \in Y$, $\{ f_t(p)\}_{t\in\Delta}$ is an analytic arc on X. Let $v_p \in T_{f_0(p)}X$ be the derivative of this arc at the point $f_0(p) \in X$ (see Fig. 1). We "expect" that this correspondence

$$p \in Y \mapsto v_p \in T_{f_0(p)}X$$

determines a nonzero element $\partial F/\partial t \in H^0(Y, f_0^* T_X)$. (This is again an expectation only. Given any $F(y,t) : Y \times \Delta \to X$ we can replace it by

$$G(y, s) := F(y, s^2) : Y \times \Delta \to X$$

to get another morphism with $\partial G/\partial s = 0$.) In this case, however, the expectation is completely correct, and we get the following criterion.

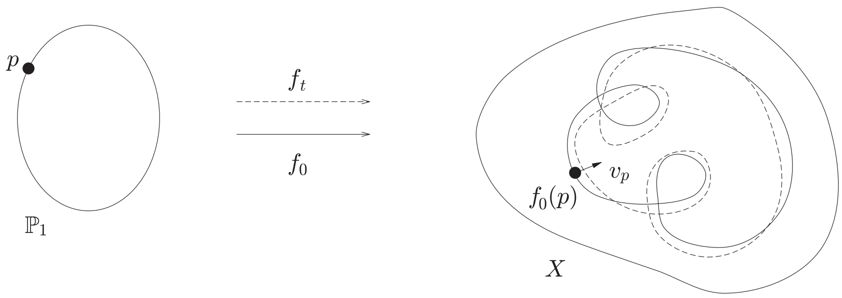

Fig. 1. Deforming a rational curve

If $H^0(Y, f_0^*T_X) = 0$, then f_0 cannot be deformed in a nontrivial way.

4 (The space of morphisms $Y \to X$)**.** Let X and Y be varieties over some base field k. Suppose we have a morphism $f_0 : Y \to X$, and we want to study deformations of f_0. In this situation, it is very useful to consider a space parametrizing all morphisms from Y to X.

For instance, when $Y = X = \mathbb{P}^1$, we would like to say that isomorphisms $\mathbb{P}^1 \to \mathbb{P}^1$ are parametrized by the group PGL_2. More precisely, by this we mean that for any field k, the points of $PGL_2(k)$ are in a "natural" one-to-one correspondence with the k-isomorphisms $\mathbb{P}^1 \to \mathbb{P}^1$.

Similarly, when trying to understand morphisms $Y \to X$, we are looking for a scheme $\mathrm{Hom}\,(Y, X)$, together with a morphism, called *universal morphism,*

$$u : \mathrm{Hom}\,(Y, X) \times Y \to X,$$

both defined over k, such that for every field extension K/k and every K-morphism $f : Y_K \to X_K$ there is a unique point $p \in \mathrm{Hom}\,(Y, X)(K)$ such that $f = u|_{\{p\}\times Y}$. The point p is then denoted by $[f]$.

It turns out that when X is quasi-projective and Y is projective there is such a scheme $\mathrm{Hom}\,(Y, X)$ (see [6] or [12, I.1.10]).

It is also very useful to consider morphisms $Y \to X$ sending a specified point of Y to a specified point of X. More generally, given a closed subscheme B of Y, and $g : B \to X$ a morphism, we want to study morphisms $Y \to X$ extending g. Under the same assumptions as before, there exists a scheme parametrizing such morphisms. We denote it by $\mathrm{Hom}\,(Y, X, g)$. It is clearly a subscheme of $\mathrm{Hom}\,(Y, X)$.

The scheme $\mathrm{Hom}(Y, X)$ is not always a variety. Usually it has countably many components. However, each irreducible component of $\mathrm{Hom}(Y, X)$ is of finite type over k.

These spaces are very complicated in general. However, if

$$H^1(Y, f^*T_X) = 0,$$

$\mathrm{Hom}(Y, X)$ has a very simple description near the point $[f]$. This is the content of the second part of the next theorem.

Theorem 5. *Let Y and X be schemes over a field k, Y projective and without embedded points and X smooth and quasi-projective. Let $f : Y \to X$ be a morphism. Then*

1. *We have* $T_{[f]} \mathrm{Hom}(Y, X) \cong H^0(Y, f^*T_X)$.
2. (Kodaira–Spencer) *If* $H^1(Y, f^*T_X) = 0$, *then* $\mathrm{Hom}(Y, X)$ *is smooth at* $[f]$ *and has dimension* $h^0(Y, f^*T_X)$.
3. (Mumford) *In general,* $\dim_{[f]} \mathrm{Hom}(Y, X) \geq h^0(Y, f^*T_X) - h^1(Y, f^*T_X)$.

Proof. Statements (1) and (3) are special cases of [12, I.2.16], statements (2.16.1) and (2.16.2) respectively.

Now assume that $H^1(Y, f^*T_X) = 0$. It follows from (1) and (3) above that $\dim_{[f]} \mathrm{Hom}(Y, X) = h^0(Y, f^*T_X)$. Therefore, by [12, I.2.17], $\mathrm{Hom}(Y, X)$ is a local complete intersection. Since $\dim_{[f]} \mathrm{Hom}(Y, X) = \dim T_{[f]} \mathrm{Hom}(Y, X)$, we conclude that $\mathrm{Hom}(Y, X)$ is smooth at $[f]$. ■

The theorem says that when $H^1(Y, f^*T_X) = 0$, there is an irreducible curve through $[f]$ on $\mathrm{Hom}(Y, X)$ in every direction of $T_{[f]} \mathrm{Hom}(Y, X)$. This means that any global section of f^*T_X comes from a deformation $F : Y \times \Delta \to X$.

This is not the case if $\mathrm{Hom}(Y, X)$ is not smooth at $[f]$.

3. Deforming Rational Curves

Now we apply the theory presented in the previous section to study deformations of curves on smooth varieties. We are mainly interested in deforming rational curves.

Like before, let X be a smooth quasi-projective variety over a field k, but now assume that $Y = C$ is a proper curve over k. A crucial simplification in this case is that $h^0(C, f^*T_X) - h^1(C, f^*T_X)$ is just the Euler characteristic $\chi(C, f^*T_X)$, which can be computed by Riemann–Roch:

$$\begin{aligned}\chi(C, f^*T_X) &= \deg(f^*T_X) + \chi(\mathcal{O}_C) \dim X \\ &= -\big(f_*[C] \cdot K_X\big) + \big(1 - p_a(C)\big) \dim X.\end{aligned}$$

Hence we obtain that

$$\dim_{[f]} \operatorname{Hom}(C, X) \geq -\big(f_*[C] \cdot K_X\big) + \big(1 - p_a(C)\big) \dim X.$$

Example 6. A smooth projective variety X is called *Fano* if its anti-canonical class $-K_X$ is ample (for instance, we can take for X a smooth hypersurface in $\mathbb{P}^n$ of degree $d \leq n$). In this case $(C \cdot K_X) < 0$ for every curve $C \subset X$, and we expect curves on X to move a lot. For instance, if $f : \mathbb{P}^1 \to X$ is a nonconstant morphism, then

$$\dim_{[f]} \operatorname{Hom}(\mathbb{P}^1, X) \geq -\big(f_*[\mathbb{P}^1] \cdot K_X\big) + \dim X \geq 1 + \dim X.$$

The automorphisms of $\mathbb{P}^1$ account for a 3-dimensional family of maps. Thus, if $\dim X \geq 3$, every rational curve on X moves. (We really need $\dim X \geq 3$ here. Lines on a smooth cubic surface in $\mathbb{P}^3$ provide an example of nonmoving rational curves on a Fano variety of dimension 2.) In general, the larger the degree of the curve, the more it moves.

When X is a Fano variety defined over an algebraically closed field, a result of Mori shows that through any point of X there passes a rational curve (the next exercise shows this in the hypersurface case, when $d \leq n-1$).

Exercise 7. Let $X_d \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d \leq n-1$ over an algebraically closed field. Show that through every point of X_d there is an $(n-1-d)$-dimensional family of lines. Compute the dimension of the family of all lines on X_d and compare it with the above estimate.

Let us study the case $C \cong \mathbb{P}^1$ in more detail. Every vector bundle on $\mathbb{P}^1$ splits into a direct sum of line bundles (see [9, V. Exercise 2.6]). Therefore we can write, for suitable integers a_i,

$$f^*T_X \cong \sum_{i=1}^{\dim X} \mathcal{O}_{\mathbb{P}^1}(a_i).$$

Definition 8. We say that $f : \mathbb{P}^1 \to X$ is *free* if all the a_i are ≥ 0.

We say that $f : \mathbb{P}^1 \to X$ is *very free* if all the a_i are ≥ 1.

Remark 9. Intuitively, a free curve can be moved in every direction, while a very free curve still moves freely in any direction with one point fixed.

In particular, if $Z \subset X$ has codimension ≥ 2, any free curve $f : \mathbb{P}^1 \to X$ has nearby deformations that are disjoint from Z. If $f : \mathbb{P}^1 \to X$ is very free, then it has nearby deformations f_t such that $f_t(0) = f(0)$ and $f_t\big(\mathbb{P}^1 \setminus \{0\}\big)$ is disjoint from Z. Moreover, the tangent spaces of the images $f_t(\mathbb{P}^1)$ at $f(0)$ sweep out an open subset of the tangent space of X at $f(0)$.

These results are easy to obtain from Propositions 10 and 13. See [12, II.3] for detailed proofs.

Notice that a morphism $f : \mathbb{P}^1 \to X$ is free (resp. very free) if and only if $H^1\big(\mathbb{P}^1, f^*T_X(-1)\big) = 0$ (resp. $H^1\big(\mathbb{P}^1, f^*T_X(-2)\big) = 0$). By the Semicontinuity Theorem (see [9, III.12.8]), nearby deformations of a free (resp. very free) rational curve are still free (resp. very free).

Proposition 10. *Let X be an n-dimensional smooth quasi-projective variety over an algebraically closed field k. Let $f : \mathbb{P}^1 \to X$ be a nonconstant morphism.*

1. *If f is free, there exists a variety Y of dimension $n-1$, a point $y_0 \in Y$, and a dominant morphism $F : \mathbb{P}^1 \times Y \to X$ such that $F|_{\mathbb{P}^1 \times \{y_0\}} = f$.*
2. *Assume there exists a variety Y and a dominant morphism $F : \mathbb{P}^1 \times Y \to X$. If k has characteristic 0, $f_y := F|_{\mathbb{P}^1 \times \{y\}}$ is free for general $y \in Y$.*

Proof. (1) Let Z be a suitable neighborhood of $[f]$ in $\mathrm{Hom}\,(\mathbb{P}^1, X)$. Since f is free, any given point of $f(\mathbb{P}^1)$ can be moved in arbitrary direction. This implies that the evaluation morphism $F : \mathbb{P}^1 \times Z \to X$ is submersive at every point of $\mathbb{P}^1 \times \big\{[f]\big\}$, thus it is also dominant. (See [12, II.3.5.3] for a rigorous proof.) If $Y \subset Z$ is a general subvariety of dimension $n-1$ then $F|_{\mathbb{P}^1 \times Y} : \mathbb{P}^1 \times Y \to X$ is still dominant.

(2) Since F is dominant, there is a dense open subset $U \subset \mathbb{P}^1 \times Y$ such that $F : U \to X$ is smooth (here we use the characteristic 0 assumption). Therefore, the induced map $dF : T_{\mathbb{P}^1 \times Y} \to F^*T_X$ is surjective on U.

Let $\pi_2 : \mathbb{P}^1 \times Y \to Y$ denote the second projection, and assume that $y \in \pi_2(U)$. Then

$$\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\dim Y} \cong T_{\mathbb{P}^1 \times Y}\big|_{\mathbb{P}^1 \times \{y\}} \to f_y^* T_X \cong \sum_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(a_i)$$

is surjective on an open set. Therefore all the a_i are ≥ 0. ■

Corollary 11. *Let X be a smooth quasi-projective variety over an algebraically closed field of characteristic* 0*. Let $x \in X$ be a very general point, i.e., x is outside a countable union of proper closed subvarieties of X. Then any morphism $f : \mathbb{P}^1 \to X$ whose image passes through x is free.*

Proof. For each (of countably many) irreducible component Z of $\operatorname{Hom}(\mathbb{P}^1, X)$, consider the evaluation morphism $F_Z : \mathbb{P}^1 \times Z \to X$. Since being free is an open condition, the points of Z parametrizing nonfree morphisms form a closed subscheme $Z' \subset Z$ (notice that we may have $Z' = Z$). By Proposition 10, the restriction $F_Z|_{\mathbb{P}^1 \times Z'} : \mathbb{P}^1 \times Z' \to X$ is not dominant, and thus $\overline{F_Z(\mathbb{P}^1 \times Z')} \neq X$.

Set $V_Z = \overline{F_Z(\mathbb{P}^1 \times Z')} \subsetneq X$. Then any morphism $f : \mathbb{P}^1 \to X$ whose image passes through a point in $X \setminus \bigcup\limits_Z V_Z$ is free. ■

Remark 12. This corollary says little when X is a variety over a countable field k. Indeed, in this case we may have $X(k) = \bigcup\limits_Z V_Z(k)$ (notation as in the proof above). No such example is known, and this leads to the following open problem: *Is there a smooth projective variety X over $\bar{\mathbb{Q}}$ such that through every point of X there is a nonfree rational curve?*

A related question is the following: *Is there a smooth projective variety X over $\bar{\mathbb{Q}}$ such that through every point of X there is a rational curve, but rational curves do not cover $X(\mathbb{C})$?*

Variants of the latter are also interesting (and possibly easier) over finite fields.

We have an analogue of Proposition 10 for very free morphisms. The proof is very similar to the proof of Proposition 10, and we leave it to the reader.

Proposition 13. *Let X be a n-dimensional smooth quasi-projective variety over an algebraically closed field k. Fix $x_0 \in X$. Let $f : \mathbb{P}^1 \to X$ be a nonconstant morphism such that $f(0) = x_0$.*

1. *If f is very free, there exists a variety Y of dimension $n-1$, a point $y_0 \in Y$, and a dominant morphism $F : \mathbb{P}^1 \times Y \to X$ such that $F|_{\mathbb{P}^1 \times \{y_0\}} = f$ and $F|_{\{0\} \times Y} \equiv x_0$.*
2. *Assume there exists a variety Y and a dominant morphism $F : \mathbb{P}^1 \times Y \to X$ such that $F|_{\{0\} \times Y} \equiv x_0$. If k has characteristic* 0 *then $f_y := F|_{\mathbb{P}^1 \times \{y\}}$ is very free for general $y \in Y$.* ■

Example 14. Let X be a smooth Fano variety over a field k. We mentioned in Example 6 that through any point of $X_{\bar{k}}$ there passes a rational curve, defined over $\bar{k}$. When k has characteristic 0, Theorem 10.2 implies that through a very general point of $X_{\bar{k}}$ there passes a free rational curve. In fact, there is a free rational curve through every point, as we see in Theorem 29. On the other hand, so far we know nothing about the existence of rational curves on X that are defined over k. In general this is a very difficult problem that depends on the arithmetic nature of the field k. In the simplest case, when $k = \mathbb{R}$, we consider this problem in Application 19. The more general case of local fields is studied in Section 9.

4. Deformation of Morphisms II

Sometimes it is useful to consider a wider class of deformations than the one considered in the last two sections. Let $f : Y \to X$ be a morphism from a projective variety Y to a smooth quasi-projective variety X. Now, instead of changing only the morphism f, we also want to change Y and X in a predetermined way.

In order to get this setup, let $(R, 0)$ be an irreducible pointed scheme, and $\mathcal{X}$ and $\mathcal{Y}$ schemes over R, with $\mathcal{X} \to R$ smooth and $\mathcal{Y} \to R$ proper and flat. Let $\mathcal{X}_0$ and $\mathcal{Y}_0$ be the fibers over $0 \in R$. Starting with a given morphism $f_0 : \mathcal{Y}_0 \to \mathcal{X}_0$, we would like to obtain deformations $f_r : \mathcal{Y}_r \to \mathcal{X}_r$.

Like before, there exists an R-scheme $\operatorname{Hom}(\mathcal{Y}/R, \mathcal{X}/R)$ parametrizing morphisms $\mathcal{Y} \to \mathcal{X}$. This means that there is a universal morphism

$$u : \operatorname{Hom}(\mathcal{Y}/R, \mathcal{X}/R) \times_R \mathcal{Y} \to \mathcal{X}$$

such that for every R-scheme $S \to R$ and for every S-morphism $f : S \times_R \mathcal{Y} \to S \times_R \mathcal{X}$ there is a unique R-morphism $[f] : S \to \operatorname{Hom}(\mathcal{Y}/R, \mathcal{X}/R)$ such that f is the composition

$$f : S \times_R \mathcal{Y} \xrightarrow{(\mathrm{id}_S, [f], \mathrm{id}_{\mathcal{Y}})} S \times_R \operatorname{Hom}(\mathcal{Y}/R, \mathcal{X}/R) \times_R \mathcal{Y} \xrightarrow{(\mathrm{id}_S, u)} S \times_R \mathcal{X}.$$

The fiber of the map $\operatorname{Hom}(\mathcal{Y}/R, \mathcal{X}/R) \to R$ over the point $0 \in R$ is precisely $\operatorname{Hom}(\mathcal{Y}_0, \mathcal{X}_0)$:

$$\begin{array}{ccc} \operatorname{Hom}(\mathcal{Y}_0, \mathcal{X}_0) & \longrightarrow & \operatorname{Hom}(\mathcal{Y}/R, \mathcal{X}/R) \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spec} k(0) & \longrightarrow & R \end{array}$$

(Here $k(0)$ denotes the residue field of 0 on R.)

The next result is an infinitesimal description of the space $\operatorname{Hom}(\mathcal{Y}/R, \mathcal{X}/R)$ near a point $[f_0]$ for which $H^1(\mathcal{Y}_0, f_0^* T_{\mathcal{X}_0}) = 0$. (For a proof see [12, I.2.17] and the proof of Theorem 5.)

Theorem 15. *Let $(R, 0)$ be an irreducible pointed scheme and $\mathcal{X}$, $\mathcal{Y}$ schemes over R, with $\mathcal{X} \to R$ smooth and $\mathcal{Y} \to R$ proper and flat. Assume that $\mathcal{Y}_0$ has no embedded points and let $f_0 : \mathcal{Y}_0 \to \mathcal{X}_0$ be a morphism. Then*

1. *If $H^1(\mathcal{Y}_0, f_0^* T_{\mathcal{X}_0}) = 0$ then $\operatorname{Hom}(\mathcal{Y}/R, \mathcal{X}/R) \to R$ is smooth in a neighborhood of $[f_0]$, and all fibers in this neighborhood have dimension $h^0(\mathcal{Y}_0, f_0^* T_{\mathcal{X}_0})$.*
2. *In general,*

$$\dim_{[f_0]} \operatorname{Hom}(\mathcal{Y}/R, \mathcal{X}/R) \geq h^0(\mathcal{Y}_0, f_0^* T_{\mathcal{X}_0}) - h^1(\mathcal{Y}_0, f_0^* T_{\mathcal{X}_0}) + \dim R. \quad \blacksquare$$

Here is an example of how this theorem can be applied. Let X be a smooth variety over a field k. Suppose we have a connected union of rational curves $D \subset X$, and we want to deform it into a single rational curve. We represent D as the image of a tree of rational curves C under a morphism $f : C \to X$. Since C itself is reducible, $\operatorname{Hom}(C, X)$ does not give us any new integral curve on X. So we look for an irreducible k-scheme R and an R-scheme $\mathcal{Y}$ (proper and flat over R) such that the general fiber $\mathcal{Y}_r$ is an irreducible rational curve, while the special fiber $\mathcal{Y}_0$ is isomorphic to C. (We can usually take R to be $\mathbb{A}^1_k$ and $\mathcal{Y}$ to be the blow up of $\mathbb{P}^1_k \times \mathbb{A}^1_k$ at suitable points lying above $0 \in \mathbb{A}^1_k$. A concrete example is worked out in Application 17.) We also set $\mathcal{X} = X \times_k R$. If $H^1(C, f^* T_X) = 0$, by Theorem 15, $\mathcal{Z} = \operatorname{Hom}(\mathcal{Y}/R, \mathcal{X}/R) \to R$ is smooth at $[f]$, and nearby fibers have dimension $h^0(C, f^* T_X) > 0$. By choosing a point in one such fiber, $[f_r] \in \mathcal{Z}_r$, we obtain a deformation of f

$$f_r : \mathcal{Y}_r \to \mathcal{X}_r = X \times \{r\},$$

which is a morphism from a rational curve to X.

There is one thing we should be careful about: if k is not algebraically closed, it is possible that nearby fibers $\mathcal{Z}_r$ do not have k-points, and this method does not yield any deformation of D defined over k. We encounter this situation in Application 19.

Warning 16. It is important to keep in mind that in general the deformations obtained in this way do *NOT* give maps from $\mathcal{Y}$ to $\mathcal{X}$, not even over any open subscheme of R. Usually, the only thing we can hope to get is that $\operatorname{Hom}(\mathcal{Y}/R, \mathcal{X}/R) \to R$ is smooth at $[f]$. If this is the situation, we claim that there is a subscheme $[f] \in T \subset \operatorname{Hom}(\mathcal{Y}/R, \mathcal{X}/R)$ such that

1. the projection $T \to R$ is quasifinite and étale, mapping $[f]$ to 0;
2. the map f extends to a morphism

$$F_T : \mathcal{Y} \times_R T \to \mathcal{X} \times_R T.$$

To see this, let $t_1, \dots, t_m$ be a regular sequence of generators for the maximal ideal in the local ring of $[f]$ on $\operatorname{Hom}(\mathcal{Y}_0, \mathcal{X}_0)$. (Such a sequence exists iff $[f]$ is a smooth point.) By flatness, these can be lifted to $T_1, \dots, T_m$, in the local ring of $[f]$ on $\operatorname{Hom}(\mathcal{Y}/R, \mathcal{X}/R)$. Then $(T_1 = \cdots = T_m = 0)$ define T in a neighborhood of $[f]$.

Thus, if for instance $R = \mathbb{A}^1_{\mathbb{C}}$, we have deformations over an Euclidean neighborhood of 0 in $\mathbb{A}^1_{\mathbb{C}}$, realized as an open subset (in the Euclidean topology) of some $T \to \mathbb{A}^1_{\mathbb{C}}$. However, there may be no Zariski neighborhood of 0 in $\mathbb{A}^1_{\mathbb{C}}$ over which a nontrivial deformation exists.

Application 17 (Moving the union of free rational curves)**.** Let X be a smooth variety over an algebraically closed field, and suppose we have two rational curves D_1 and D_2 on X such that $D_1 \cap D_2 \neq \varnothing$. When can we deform $D_1 \cup D_2$ into a rational curve?

Let $R = \mathbb{A}^1$, $\mathcal{X} = X \times \mathbb{A}^1$, and $\mathcal{Y}$ be the blow up of $\mathbb{P}^1 \times \mathbb{A}^1$ at some point on the fiber over $0 \in \mathbb{A}^1$. Then $C = \mathcal{Y}_0$ is the union of two rational curves C_1 and C_2, meeting transversely in a point P, while all other fibers are smooth rational curves. We view $D_1 \cup D_2$ as the image of a morphism $f_0 = f_1 \vee f_2 : C \to \mathcal{X}_0$. (Notice that we do not require that D_1 and D_2 meet transversely, nor that they intersect in a single point. If they meet in more than one point, we choose one of them to be the image of P.)

By Theorem 15, $D_1 \cup D_2$ can be deformed into a rational curve provided that $H^1\big(C, (f_1 \vee f_2)^* T_{\mathcal{X}_0}\big) = 0$. We claim that this is the case whenever f_1 and f_2 are free.

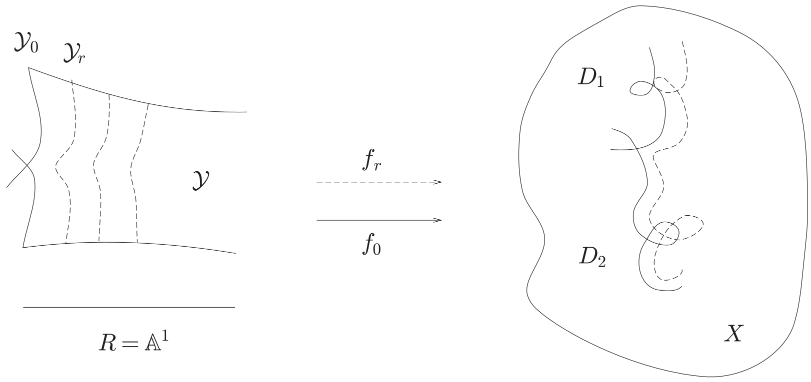

Fig. 2. Deforming the union of 2 free rational curves

Indeed, by taking $E = (f_1 \vee f_2)^* T_{\mathcal{X}_0}$ in exercise 18, we see that if both f_1 and f_2 are free, then $H^1\big(C, (f_1 \vee f_2)^* T_{\mathcal{X}_0}\big) = 0$, and hence $D_1 \cup D_2$ can be deformed into a rational curve. We actually have more than that: by taking $E = (f_1 \vee f_2)^* T_{\mathcal{X}_0}(-Q)$, where Q is any point in $C_1 \setminus C_2$, we see that $H^1\big(C, (f_1 \vee f_2)^* T_{\mathcal{X}_0}(-Q)\big) = 0$. By the Semicontinuity Theorem, $H^1\big(\mathbb{P}^1, f_r^* T_X(-1)\big) = 0$ for nearby deformations $f_r : \mathbb{P}^1 \to X$. Hence:

The union of two free rational curves with nonempty intersection can be deformed into a free rational curve.

Exercise 18. Let $C = \cup_{i=0}^n C_i$ be a reduced projective curve of genus zero (i.e., with $h^1(C, \mathcal{O}_C) = 0$) and E a vector bundle on C. Assume that

1. $H^1\big(C_i, E|_{C_i}(-1)\big) = 0$ for $i = 1, \ldots, n$, and
2. $H^1\big(C_0, E|_{C_0}\big) = 0$.

Then $H^1(C, E) = 0$.

Application 19 (Real Varieties)**.** Let X be a smooth variety defined over $\mathbb{R}$, and assume that $X_{\mathbb{C}}$ contains free rational curves (defined over $\mathbb{C}$). We investigate the existence of free rational curves (defined over $\mathbb{R}$) on $X = X_{\mathbb{R}}$.

Assume there exists a point $P \in X(\mathbb{R})$ and a free rational curve $C \subset X_{\mathbb{C}}$ passing through P. If C is defined over $\mathbb{R}$, we are done. Otherwise, let $\bar{C}$ be its conjugate curve. Then $\bar{C}$ is also a free rational curve, and their union is defined over $\mathbb{R}$.

Set $R = \mathbb{A}^1_{\mathbb{R}}$, $\mathcal{X} = X_{\mathbb{R}} \times \mathbb{A}^1_{\mathbb{R}}$ and $\mathcal{Y} = (x_0^2 + x_1^2 = tx_2^2) \subset \mathbb{P}^2_{\mathbb{R}} \times \mathbb{A}^1_{\mathbb{R}}$. Then $\mathcal{Y}_0$ is the union of two conjugate rational curves meeting transversely in one point, while all other real fibers are smooth rational curves defined over $\mathbb{R}$. The curve $\mathcal{Y}_t$ contains $\mathbb{R}$-points for $t > 0$, while $\mathcal{Y}_t(\mathbb{R}) = \varnothing$ for $t < 0$. We view $C \cup \bar{C}$ as the image of a morphism $f_0 : \mathcal{Y}_0 \to \mathcal{X}_0$.

We have $H^1(\mathcal{Y}_0, f_0^* T_{\mathcal{X}_0}) = 0$. Let $\mathcal{Z} = \operatorname{Hom}(\mathcal{Y}/R, \mathcal{X}/R)$. By Theorem 15, $\mathcal{Z} \to R$ is smooth at $[f_0]$, and nearby fibers have dimension $h^0(\mathcal{Y}_0, f_0^* T_{\mathcal{X}_0}) > 0$. Now comes a subtle point: it is possible that nearby fibers $\mathcal{Z}_r$ have no $\mathbb{R}$-points. We must show that this is not the case. As in Warning 16, let $T \subset \mathcal{Z}$ be a curve through $[f_0]$ such that $T \to R$ is smooth at $[f_0]$. We apply the Inverse Function Theorem for real functions, and conclude that $T(\mathbb{R}) \to \mathbb{R}^1$ is a local homeomorphism at $[f_0]$ (in the Euclidean topology). Thus we obtain many $\mathbb{R}$-points in a neighborhood of $[f_0]$ in T. These points correspond to morphisms $f_t : \mathcal{Y}_t \to X \times \{t\}$ that are defined over $\mathbb{R}$.

Notice that we need that $t > 0$ in order to get a curve on $X_{\mathbb{R}}$ containing real points. The fact that $T \to R$ is étale ensures that we can pick such t.

By taking t near 0, we get that f_t is free.

Remarks 20.

1. The argument above requires the existence of a free rational curve $C \subset X_{\mathbb{C}}$ through a real point. When $X_{\mathbb{C}}$ contains a free rational curve, some dense open subset of $X_{\mathbb{C}}$ is covered by free rational curves. The set of real points, if nonempty, is Zariski dense, and so we can start the argument.
   Problems arise when X has no real points. One might suppose that in this case the question of existence of free rational curves on $X_{\mathbb{R}}$ does not even make sense. There can be no real maps $\mathbb{P}^1 \to X$ if X has no real points. There is, however, a quite interesting variant.
   Even without real points, X may contain empty real conics (i.e., curves obtained from morphisms $\mathbb{P}^2_{\mathbb{R}} \supset (x_0^2 + x_1^2 + x_2^2 = 0) \to X_{\mathbb{R}}$ defined over $\mathbb{R}$). If $X(\mathbb{R}) \neq \varnothing$, the above deformations corresponding to $t < 0$ are such. If $X(\mathbb{R}) = \varnothing$, the problem is open.
2. The argument in Application 19 can be generalized to show that if X is a variety defined over a p-adic field k, and $X_{\bar{k}}$ contains a free rational curve passing through a k-point x, then X_k contains free rational curves (defined over k) through x (see [14]). This is because a p-adic field also admits an Inverse Function Theorem. We return to this in Section 9.

5. Deforming Nonfree Rational Curves I

In Section 3 we studied deformations of free rational curves, and in Section 4 we saw how to deform unions of free rational curves into irreducible ones. Now we are interested in the following problem.

How can we deform a nonfree rational curve on X
when X contains many free rational curves?

One possible way of dealing with this problem was introduced in [17, 1.1.6]. In this section we explain this method. In the next section we describe a different approach to the same problem. The main trick is to use combs, which we define now.

Definition 21. Let k be an arbitrary field. A *comb* with n *teeth* over k is a projective curve with $n+1$ irreducible components $C_0, C_1, \ldots, C_n$ over $\bar{k}$ satisfying the following conditions:

1. The curve C_0 is defined over k.
2. The union $C_1 \cup \cdots \cup C_n$ is defined over k. (Each individual curve may not be defined over k.)
3. The curves $C_1, \ldots, C_n$ are smooth rational curves disjoint from each other, and each of them meets C_0 transversely in a single smooth point of C_0 (which may not be defined over k).

The curve C_0 is called the *handle* of the comb, and $C_1, \ldots, C_n$ are called the *teeth*. A *rational comb* is a comb whose handle is a smooth rational curve. A comb can be pictured as in Fig. 3.

In Section 9 we study combs over nonclosed fields in more detail. For the rest of this section we work over a fixed algebraically closed field $k = \bar{k}$ of arbitrary characteristic.

We start with an arbitrary rational curve D on a smooth quasi-projective variety X over k. Since D is not free, it is possible that we cannot move D on X. Assume that there are several free rational curves $D_1, \ldots, D_p \subset X$ meeting D in distinct points $d_i \in D$. Then there is a chance that we can deform their union $D \cup D_1 \cup \cdots \cup D_p$ into a rational curve.

Set $R = \mathbb{A}^p$, with affine coordinates y_i, and $\mathcal{X} = X \times \mathbb{A}^p$. Fix a birational morphism $\mathbb{P}^1 \to D$ and distinct points $z_1, \ldots, z_p \in \mathbb{P}^1$ mapping to $d_1, \ldots, d_p$. Let $\mathcal{Y}$ be the blowup of $\mathbb{P}^1 \times \mathbb{A}^p$ along the subvarieties

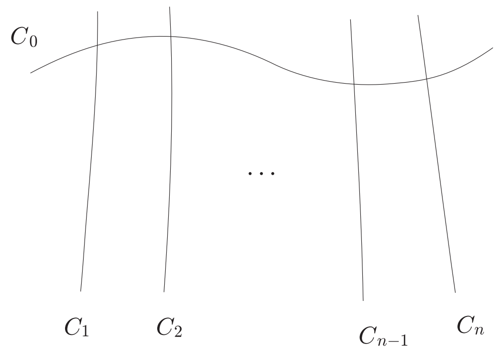

Fig. 3. Comb with n teeth

$\{z_i\} \times \{y_i = 0\}$, $i = 1, \ldots, p$. Then $\mathcal{Y}_0$ is a rational comb with handle C_0 and teeth $C_1, \ldots, C_p$. The general fiber is a smooth rational curve, but there are also fibers $\mathcal{Y}_r$, $r \in \mathbb{A}^p$ that are subcombs of $C_0 \cup C_1 \cup \cdots \cup C_p$. The number of teeth in this subcomb is precisely the number of coordinates of r that equal 0. We view $D \cup D_1 \cup \cdots \cup D_p$ as the image of a morphism $f_0 : \mathcal{Y}_0 \to \mathcal{X}_0$ mapping C_0 birationally onto D and each C_i birationally onto D_i, $1 \leq i \leq p$.

Notice that $\mathrm{Hom}\,(\mathcal{Y}_0, X)$ only gives us reducible or nonreduced curves on X. However, if $\dim_{[f_0]} \mathrm{Hom}\,(\mathcal{Y}/R, \mathcal{X}/R) > \dim_{[f_0]} \mathrm{Hom}\,(\mathcal{Y}_0, X)$, then we can deform some subcurve $D \cup D_{i_1} \cup \cdots \cup D_{i_l}$ of $D \cup D_1 \cup \cdots \cup D_p$ into a rational curve on X. We show that this is the case if p is large enough.

Remark 22. One limitation of this method is that we only know that *some* subcurve of $D \cup D_1 \cup \cdots \cup D_p$ deforms into an irreducible rational curve. In general, we do not even know the number of irreducible components of such a subcurve.

By Theorem 15 and Riemann–Roch, we have

$$(5.1)\quad \dim_{[f_0]} \mathrm{Hom}\,(\mathcal{Y}/R, \mathcal{X}/R) \geq -\big((D \cup D_1 \cup \cdots \cup D_p) \cdot K_X\big) + \dim X + p.$$

Now consider the restriction morphism $\mathrm{Hom}\,(\mathcal{Y}_0, X) \to \mathrm{Hom}\,(\mathbb{P}^1, X)$ sending $[h]$ to $\left[h|_{C_0}\right]$. The fiber over $[f] \in \mathrm{Hom}\,(\mathbb{P}^1, X)$ is isomorphic to

$$\prod_{i=1}^{p} \mathrm{Hom}\left(\mathbb{P}^1, X, 0 \mapsto f(z_i)\right).$$

By Theorem 5, when $[f] \in \operatorname{Hom}(\mathbb{P}^1, X, 0 \mapsto x)$ is free,

$$\dim_{[f]} \operatorname{Hom}(\mathbb{P}^1, X) = -\big(f_*[\mathbb{P}^1] \cdot K_X\big) + \dim X.$$

Since we have to impose $\dim X$ conditions in order to fix the image of the point $0 \in \mathbb{P}^1$, we get

$$\dim_{[f]} \operatorname{Hom}(\mathbb{P}^1, X, 0 \mapsto x) = \dim_{[f]} \operatorname{Hom}(\mathbb{P}^1, X) - \dim X = -\big(f_*[\mathbb{P}^1] \cdot K_X\big).$$

Therefore we obtain that

$$\begin{aligned} (5.2) \qquad \dim_{[f_0]} \operatorname{Hom}(\mathcal{Y}_0, X) \le \dim_{[f_0|C_0]} & \operatorname{Hom}(\mathbb{P}^1, X) + \\ & -\big((D \cup D_1 \cup \cdots \cup D_p) \cdot K_X\big) + (D \cdot K_X). \end{aligned}$$

By inequalities (5.1) and (5.2), we just need to take

$$p > \dim_{[f_0|C_0]} \operatorname{Hom}(\mathbb{P}^1, X) - \dim X + (D \cdot K_X)$$

in order to have $\dim_{[f_0]} \operatorname{Hom}(\mathcal{Y}/R, \mathcal{X}/R) > \dim_{[f_0]} \operatorname{Hom}(\mathcal{Y}_0, X)$.

We have proved:

Theorem 23. *Let X be a smooth projective variety over an algebraically closed field. Let D be a rational curve on X (possibly nonfree), and represent it by a morphism $f : \mathbb{P}^1 \to X$. Assume there are p free rational curves $D_1, \ldots, D_p$ meeting D in distinct points, with $p > \dim_{[f]} \operatorname{Hom}(\mathbb{P}^1, X) - \dim X + (D \cdot K_X)$. Then there is an integer l, $1 \le l \le p$, and distinct indices $\{i_1, \ldots i_l\} \subset \{1, \ldots, p\}$ such that $D \cup D_{i_1} \cup \cdots \cup D_{i_l}$ can be deformed into a rational curve.* ■

Remark 24. Given finitely many points on D, we can actually deform some curve $D \cup D_1 \cup \cdots \cup D_m$ keeping these points fixed. In this case, however, we may need to attach more free rational curves to D before we deform it.

We leave it to the reader to work out the necessary small modifications when D is not a rational curve.

In Section 7, we will see some applications of this theorem.

6. Deforming Nonfree Rational Curves II

Let X be a smooth projective variety over an algebraically closed field, and assume X has very free rational curves. Let $D \subset X$ be a curve.

In the last section we explained the approach of [17] to the problem of moving a curve D on X. First we attach several free rational curves $D_1, \dots, D_p$ to D, and represent $D \cup D_1 \cup \cdots \cup D_p$ as the image of a morphism $f : C \to X$, where C is a comb with p teeth. Then we study deformations of f. There are two limitations of this method.

1. As we noted in Remark 22, by taking p large enough, we can only guarantee that some subcurve $D \cup D_{i_1} \cup \cdots \cup D_{i_l}$ deforms into an irreducible curve. We do not know which subcurve it is.
2. If we look at deformations of the morphism f, the obstruction space $H^1(C, f^*T_X)$ usually never vanishes, no matter how large p is.

Recently, Graber, Harris and Starr [5] introduced a variant to this picture which is very useful in some applications, especially in arithmetic questions.

By attaching very free rational curves $D_1, \dots, D_p$ and by looking at a modified deformation setting (in the Hilbert scheme of X), we can guarantee that, for p large enough, the relevant deformation problem is unobstructed. We devote the rest of this section to explaining this new method.

First of all, instead of looking at the space of morphisms to X, we look at the Hilbert scheme of X. Below we gather some facts about Hilbert schemes. We refer to [6] and [12, I.1 and I.2] for details and proofs.

25 (Hilbert schemes)**.** Let X be a projective scheme over a field k. There is a k-scheme $\mathrm{Hilb}\,(X)$ parametrizing closed subschemes of X. Like for the Hom schemes, this implies that there is a universal subscheme $U \subset \mathrm{Hilb}\,(X) \times X$, flat over $\mathrm{Hilb}\,(X)$, such that for every field extension K/k and every K-subscheme Z of X_K there is a unique point $[Z] \in \mathrm{Hilb}\,(X)(K)$ such that the fiber of U over $[Z]$ is exactly Z.

Now assume that X is smooth, and let Z be a closed subscheme of X given by the ideal sheaf $\mathcal{I}_Z$. Assume that Z is a local complete intersection, that is, $\mathcal{I}_Z$ is locally generated by $\mathrm{codim}\,(Z, X)$ elements. Then $\mathcal{I}_Z/\mathcal{I}_Z^2$ is a locally free sheaf on Z and its dual $\mathrm{Hom}(\mathcal{I}_Z/\mathcal{I}_Z^2, \mathcal{O}_Z)$ is the *normal bundle* N_Z of Z in X. We have:

1. $T_{[Z]}\,\mathrm{Hilb}\,(X) \cong H^0(Z, N_Z)$.

2. If $H^1(Z, N_Z) = 0$, then $\mathrm{Hilb}\,(X)$ is smooth at $[Z]$.
3. In general, $\dim_{[Z]} \mathrm{Hilb}\,(X) \geq h^0(Z, N_Z) - h^1(Z, N_Z)$.

(See [6] or [12, I.2.8].)

Now we go back to our problem.

Let X be a smooth projective variety over an algebraically closed field and $D \subset X$ an irreducible curve. Our goal is to attach rational curves $D_1, \dots, D_p$ to D so that the resulting comb $D \cup D_1 \cup \cdots \cup D_p$ deforms into an irreducible curve.

The first problem we face is that D may be quite singular, in which case we prefer to work with its normalization $\tilde{D} \to D$. On the other hand, here it is essential that D be embedded in X. To achieve both, we take an embedding $\tilde{D} \hookrightarrow \mathbb{P}^3$ and consider the diagonal map $\delta : \tilde{D} \to X \times \mathbb{P}^3$. This is an embedding with image $D' \cong \tilde{D}$ whose projection to the first factor is D. Any moving of D' can be projected to X to give a deformation of D. For most applications this reduction to the smooth curve D' is satisfactory. There may be, however, delicate geometric questions for which one needs to be more careful.

So assume $D \subset X$ is a smooth irreducible curve. Suppose there are disjoint smooth rational curves $D_1, \dots, D_p$, each meeting D transversely in a single point. Then we can deform their union $C = D \cup D_1 \cup \cdots \cup D_p$ into an irreducible curve C' provided that the following conditions hold:

1. The sheaf N_C is generated by global sections.
2. $H^1(C, N_C) = 0$.

Indeed, we have an exact sequence

$$0 \;\to\; N_D \;\to\; N_C|_D \;\to\; Q \;\to\; 0,$$

where Q is a torsion sheaf supported at the points $P_i = D \cap D_i$. If condition 1 holds, we can find a global section $s \in H^0(C, N_C)$ such that, for each i, the restriction of s to a neighborhood of P_i is not in the image of N_D. This means that s corresponds to a first-order deformation of C that smoothes the nodes P_i of C. If condition 2 holds, there is no obstruction, and we can find a global deformation of C that smoothes its nodes P_i.

Remark 26. In many situations, we would like the resulting irreducible curve C' to have some extra properties. For instance, if we start with a smooth rational curve D, we may want C' to be a free (resp. very free) curve. In this case, by the Semicontinuity Theorem, we can just fix a point $P \in D$ (resp. points $P, Q \in D$) and replace condition (2) above by

$2'$. $H^1\big(C, N_C(-P)\big) = 0$ (resp. $H^1\big(C, N_C(-P-Q)\big) = 0$).

Theorem 27. *Let X be a smooth projective variety of dimension at least 3 over an algebraically closed field. Let $D \subset X$ be a smooth irreducible curve and M a line bundle on D. Let $C \subset X$ be a very free rational curve intersecting D and let $\mathcal{C}$ be a family of rational curves on X parametrized by a neighborhood of $[C]$ in* $\operatorname{Hom}(\mathbb{P}^1, X)$.

Then there are curves $C_1, \ldots, C_p \in \mathcal{C}$ such that $D^* = D \cup C_1 \cup \cdots \cup C_p$ is a comb and satisfies the following conditions:

1. *The sheaf N_{D^*} is generated by global sections.*
2. *$H^1(D^*, N_{D^*} \otimes M^*) = 0$, where M^* is the unique line bundle on D^* that extends M and has degree* 0 *on the C_i.*

We have a lot of freedom in choosing curves from $\mathcal{C}$. Namely, as in Remark 9, for a general point $P \in D$ and for a general tangent direction $\ell \subset T_{P,X}$ there is a curve $C = C(P, \ell) \in \mathcal{C}$ passing through P and having tangent direction ℓ at P. Moreover, given a 1-dimensional subscheme $B \subset X$ we can also assume that C intersects $D \cup B$ only at P. (This is the only point where we need $\dim X \geq 3$.) By looking at the proof, we see that it is sufficient that there is a curve in $\mathcal{C}$ having a general normal direction $\bar{\ell} \subset N_{P,D}$ at P (instead of a general tangent direction $\ell \subset T_{P,X}$). This is very useful in the proof of Theorem 32.

For arithmetic applications, one has to be even more careful with the choice of the teeth C_i (see [16]).

Proof. *Step 1.* Here we produce a comb $D' \subset X$ with handle D such that $H^1\big(D, N_{D'}|_D \otimes M\big) = 0$.

Set $D_0 = D$. Suppose we have a comb $D_i = D \cup C_1 \cup \cdots \cup C_i$ such that $d_i = h^1\big(D, N_{D_i}|_D \otimes M\big) \neq 0$. We show how to choose a curve $C_{i+1} \in \mathcal{C}$, disjoint from $C_1, \ldots, C_i$, forming a comb $D_{i+1} = D_i \cup C_{i+1}$ so that $d_{i+1} < d_i$. Then we just repeat this procedure finitely many times until we reach $d_p = 0$.

By Serre duality, $H^1\big(D, N_{D_i}|_D \otimes M\big)$ is dual to

$$\operatorname{Hom}\big(\omega_D^{-1} \otimes M, (\mathcal{I}_{D_i}/\mathcal{I}_{D_i}^2)\big|_D\big),$$

so we would like to understand how $(\mathcal{I}_{D_i}/\mathcal{I}_{D_i}^2)\big|_D$ changes with i.

Pick any $C_{i+1} \in \mathcal{C}$ disjoint from $C_1, \dots, C_i$ and intersecting D transversely in a single point P. The tangent direction of C_{i+1} at P corresponds to a normal direction $l_{C_{i+1}} \subset N_{P,D}$. By explicit computation we get an exact sequence

$$0 \to (\mathcal{I}_{D_{i+1}}/\mathcal{I}^2_{D_{i+1}})\big|_D \to (\mathcal{I}_{D_i}/\mathcal{I}^2_{D_i})\big|_D \to l^\vee_{C_{i+1}} \to 0,$$

where the last map is the composition of the restriction map $(\mathcal{I}_{D_i}/\mathcal{I}^2_{D_i})\big|_D \to N^\vee_{P,D}$ with the dual map $N^\vee_{P,D} \to l^\vee_{C_{i+1}}$.

In particular, we get an inclusion of groups

$$\operatorname{Hom}\left(\omega_D^{-1} \otimes M, (\mathcal{I}_{D_{i+1}}/\mathcal{I}^2_{D_{i+1}})\big|_D\right) \hookrightarrow \operatorname{Hom}\left(\omega_D^{-1} \otimes M, (\mathcal{I}_{D_i}/\mathcal{I}^2_{D_i})|_D\right).$$

So we always have $d_{i+1} \le d_i$.

Since $d_i \neq 0$, there is a nonzero element

$$\phi \in \operatorname{Hom}\left(\omega_D^{-1} \otimes M, (\mathcal{I}_{D_i}/\mathcal{I}^2_{D_i})\big|_D\right).$$

We are done if we can choose C_{i+1} so that ϕ does not come from an element $\tilde{\phi}$ of $\operatorname{Hom}\left(\omega_D^{-1} \otimes M, (\mathcal{I}_{D_{i+1}}/\mathcal{I}^2_{D_{i+1}})\big|_D\right)$.

For each $P \in D$, let ϕ_P denote the composition of ϕ with the restriction map $(\mathcal{I}_{D_i}/\mathcal{I}^2_{D_i})\big|_D \to N^\vee_{P,D}$. Since ϕ is nonzero, there is a dense open subset $V \subset D$ such that ϕ_P has rank 1 for every $P \in V$. Let $\xi_P \in N^\vee_{P,D}$ span the image of ϕ_P.

Choose a general point $P \in V$ not in $C_1 \cup \dots \cup C_i$ and a curve $C_{i+1} \in \mathcal{C}$ through P such that

1. C_{i+1} intersects $D \cup C_1 \cup \dots \cup C_i$ only at P, and
2. The projection of the tangent direction of C_{i+1} at P to $N_{P,D}$, denoted by $l_{C_{i+1}}$, is not contained in the kernel of $\xi_P : N_{P,D} \to k$.

It follows from our exact sequence that $\phi \in \operatorname{Hom}\left(\omega_D^{-1} \otimes M, (\mathcal{I}_{D_i}/\mathcal{I}^2_{D_i})\big|_D\right)$ comes from an element $\tilde{\phi}$ of $\operatorname{Hom}\left(\omega_D^{-1} \otimes M, (\mathcal{I}_{D_{i+1}}/\mathcal{I}^2_{D_{i+1}})\big|_D\right)$ if and only if $\xi_P|_{l_{C_{i+1}}} = 0$. This does not hold by the choice of C_{i+1}, and we are done.

Notice that, once we have a comb D' such that $H^1\big(D, N_{D'}|_D \otimes M\big) = 0$, any other comb D^* obtained from D' by adding extra teeth also satisfies $H^1\big(D, N_{D^*}|_D \otimes M\big) = 0$.

Step 2. Now we extend the comb D' obtained in Step 1 to a comb D^* so that $N_{D^*}|_D$ is generated by global sections.

Let D' be the comb obtained in Step 1. Let M' be any line bundle on D with degree $-(g+1)$, where g is the genus of the curve D. We use the method described above to attach more teeth to D' so that the resulting comb D^* also satisfies $H^1\big(D, N_{D^*}|_D \otimes M'\big) = 0$. By Riemann–Roch, M' embeds in $\mathcal{O}_D(-P)$ for every $P \in D$. So we get that $H^1\big(D, N_{D^*}|_D(-P)\big) = 0$ for every $P \in D$, and this implies global generation of $N_{D^*}|_D$.

Step 3. Finally, we use the fact that the C_i are free to show that N_{D^*} is generated by global sections and that $H^1(D^*, N_{D^*} \otimes M^*) = 0$.

Let $D^* = D \cup C_1 \cup \cdots \cup C_p$ be the comb obtained in Step 2, so that $N_{D^*}|_D$ is generated by global sections and $H^1\big(D, N_{D^*}|_D \otimes M\big) = 0$.

For each i we have an exact sequence

$$0 \to N_{C_i} \to N_{D^*}|_{C_i} \to Q_i \to 0,$$

where Q_i is a torsion sheaf supported at $P_i = D \cap C_i$. Since C_i is free, $H^1\big(C_i, N_{C_i}(-P_i)\big) = 0$, and hence $H^1\big(C_i, N_{D^*}|_{C_i}(-P_i)\big) = 0$.

We also have the exact sequence

$$0 \to \sum_{i=1}^{p} N_{D^*}|_{C_i}(-P_i) \to N_{D^*} \to N_{D^*}|_D \to 0.$$

Therefore, the vanishing of $H^1(D^*, N_{D^*} \otimes M^*)$ follows from the vanishing of $H^1\big(D, N_{D^*}|_D \otimes M\big)$ and $H^1\big(C_i, N_{D^*}|_{C_i}(-P_i)\big)$, $1 \le i \le p$.

Similarly, global generation of N_D^* follows from global generation of $N_{D^*}|_D$ and N_{C_i}, $1 \le i \le p$, plus the vanishing of $H^1\big(C_i, N_{D^*}|_{C_i}(-P_i)\big)$. ■

7. Rationally Connected Varieties

In this section we introduce the concept of rationally connected varieties. After some motivation, we present possible definitions of the concept and show that they are equivalent. Then we study some properties of the class of rationally connected varieties. This is all done over $\mathbb{C}$ (or any uncountable algebraically closed field of characteristic 0). At the end of the section we consider subtleties that arise in positive characteristic.

28 (Motivation)**.** It is rather clear that rational curves form a distinguished class among all algebraic curves. They are the simplest algebraic curves in many aspects (number theoretical, topological, etc.).

In dimension two, smooth rational surfaces are the best analogs of rational curves. Their class enjoys several nice properties:

1. (Castelnuovo's Criterion) A smooth projective surface S over $\mathbb{C}$ is rational if and only if $H^0(S, \omega_S^{\otimes 2}) = H^1(S, \mathcal{O}_S) = 0$.
2. (Deformation Invariance) Let $S \to T$ be a family of smooth projective surfaces over $\mathbb{C}$, T connected. If S_0 is rational, any S_t is rational.
3. (Noether's Theorem) Let S be a smooth projective surface. If there is a morphism $S \to \mathbb{P}^1$ whose general fibers are rational curves, then S itself is rational.

When we move to higher dimensions, and try to classify the "simplest" algebraic varieties, things get much more complicated. As a first try, one could take rational varieties as higher dimensional analogs of rational curves and rational surfaces. However, there are higher dimensional varieties that behave like rational varieties but are not rational. "Behaving like rational varieties" should be understood as having the relevant properties of rational varieties. Smooth cubic 3-folds, for instance, behave like rational varieties in many aspects, but they are not rational (see [3, Theorem 13.12]). Moreover, the analogs of properties 1–3 above all seem to fail for rational varieties of dimension 3 and up. Smooth cubic 3-folds satisfy $H^i(S, \mathcal{O}_S) = H^0(S, \omega_S^{\otimes m}) = 0$ for every $i, m \geq 1$. There are examples of conic bundles over $\mathbb{P}^2$ which are not rational. It is conjectured that deformation invariance also fails.

Another possibility would be to consider *unirational varieties*. A variety of dimension n is unirational if there is a dominant map $\mathbb{P}^n \dashrightarrow X$. However, the class of unirational varieties still seems to be too small. It is conjectured that deformation invariance fails for the class of unirational varieties.

A new concept was proposed in [17], namely the concept of *rationally connected varieties*. The definition was based on the fact that $\mathbb{P}^n$ has many rational curves, and on the expectation that a variety X behaves like $\mathbb{P}^n$ if and only if it has plenty of rational curves. The precise notion is given in Definition-Theorem 29 below.

It seems rather clear by now that in characteristic 0 rationally connected varieties are the right higher dimensional analogs of rational curves and rational surfaces. This class of varieties enjoys many nice properties:

0. (Birational invariance) The class of rationally connected varieties is closed under birational equivalence.
1. (Castelnuovo's Criterion) A smooth projective rationally connected variety X satisfies
$$H^0\big(X, (\Omega^1_X)^{\otimes m}\big) = 0 \quad for \;\; every \;\; m \geq 1,$$
and the converse is conjectured to hold. (This is Theorem 30.)
2. (Deformation Invariance) Being rationally connected is deformation invariant for smooth projective varieties over. (This is Theorem 31.)
3. (Noether's Theorem) If X is a smooth variety, and there is a dominant morphism $X \to Y$ to a rationally connected variety Y such that the general fibers are rationally connected, then X itself is rationally connected. (This is Corollary 33.)

The class of rationally connected varieties contains the class of unirational varieties. It should be said, however, that it is still an open problem to show that these classes are in fact distinct. That is, there is not a single rationally connected variety over $\mathbb{C}$ that is known not to be unirational. Potential candidates abound, for instance general quartics in $\mathbb{P}^4$, any smooth hypersurface of degree n in $\mathbb{P}^n$ for $n \geq 5$, smooth hypersurfaces of bidegree $(n, 2)$ in $\mathbb{P}^k \times \mathbb{P}^2$ for $n \gg k \geq 2$.

Definition-Theorem 29. Let X be a smooth projective variety over $\mathbb{C}$ (or any uncountable algebraically closed field of characteristic 0). We say that X is *rationally connected* if it satisfies the following equivalent conditions.

(1) There is a dense open set $X^0 \subset X$ such that, for every $x_1, x_2 \in X^0$, there is a chain of rational curves connecting x_1 and x_2.

(2) There is a dense open set $X^0 \subset X$ such that, for every $x_1, x_2 \in X^0$, there is a rational curve through x_1 and x_2.

(3) For every $x_1, x_2 \in X$, there is a rational curve through x_1 and x_2.

(4) For every integer $m > 0$, and every $x_1, \ldots, x_m \in X$, there is a rational curve through $x_1, \ldots, x_m$.

(5) For every integer $m > 0$, and every $x_1, \ldots, x_m \in X$, there is a free rational curve through $x_1, \ldots, x_m$.

(6) There is a very free morphism $f : \mathbb{P}^1 \to X$, that is,
$$f^*T_X = \sum_{i=1}^{\dim X} \mathcal{O}_{\mathbb{P}^1}(a_i) \quad \text{with} \quad a_i \geq 1.$$

If X is defined over an arbitrary field k of characteristic 0, we say that X is *rationally connected* if X_K satisfies the conditions above for some (or equivalently for all) uncountable algebraically closed field K extending k.

It is possible that the six conditions above are equivalent over any algebraically closed field of characteristic 0, but this is not known for countable fields, cf. Remark 12.

We of course need to prove that the six versions above are equivalent. The proof is somewhat more transparent if we introduce two other variants:

(1^+) For every $x_1, x_2 \in X$, there is a chain of rational curves connecting x_1 and x_2.

(6^+) For every $x \in X$ there is a very free morphism $f : \mathbb{P}^1 \to X$ through x.

The implications (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are obvious, and so are (1^+) $\Rightarrow$ (1) and (6^+) $\Rightarrow$ (6).

Assume (1) and let us prove (6). The argument also shows (1^+) $\Rightarrow$ (6^+).

Let X^{free} be the set obtained in Corollary 11, so that any rational curve meeting X^{free} is free. It is the intersection of countably many dense open subsets. We repeatedly use two of its properties over $\mathbb{C}$: X^{free} is dense and for every irreducible subvariety $Z \subset X$, the intersection $X^{\text{free}} \cap Z$ is either empty or dense in Z. Thus, if a curve $C \subset X$ intersects X^{free}, a dense set of its deformations also intersect X^{free}.

Pick a point $x \in X^{\text{free}} \cap X^0$ and let x' be any point in X^0. By hypothesis, there is a chain of rational curves $C_1 \cup \cdots \cup C_n$ connecting x and x'. We show that there is in fact a chain of *free* rational curves connecting them.

Set $x_0 = x$, $x_i = C_i \cap C_{i+1}$ for $0 < i < n$, and $x_n = x'$. We replace each curve C_i by a free rational curve C'_i passing through x_i and x_{i+1}.

The curve C_0 is itself free because it intersects X^{free}, so we set $C'_0 = C_0$.

We can deform C'_0 to produce many free rational curves $D_1, \dots, D_k$ meeting C_1. Indeed, view C'_0 as the image of a morphism $f_0 : \mathbb{P}^1 \to X$, with $f_0(0) = x_1$. Let U be a neighborhood of $[f_0]$ in $\operatorname{Hom}(\mathbb{P}^1, X)$ parametrizing free morphisms and consider the evaluation map $F_U : \mathbb{P}^1 \times U \to X$. Since f_0 is free, the point $x_1 \in C'_0$ can be moved in any direction, i.e., $F_U(\{0\} \times U)$ contains a neighborhood of x_1 in X (see [12, II.3.5.3] for a rigorous argument). In particular $F_U(\{0\} \times U)$ contains a neighborhood of x_1 in C_1. Since C_0 meets X^{free}, we can choose deformations $D_1, \dots, D_k$ that intersect C_1 (in distinct points) and also meet X^{free}. By Theorem 23, if we take k large enough, we can deform some subcurve $C_1 \cup D_{i_1} \cup \cdots \cup D_{i_m}$

of $D \cup D_1 \cup \cdots \cup D_k$ into a rational curve C'_1, keeping x_1 and x_2 fixed. Since the D_i meet X^{free}, we can take C'_1 to meet X^{free} as well. (See Fig. 4.)

We do the same with C'_0 replaced with C'_1, and C_1 replaced with C_2, and so on.

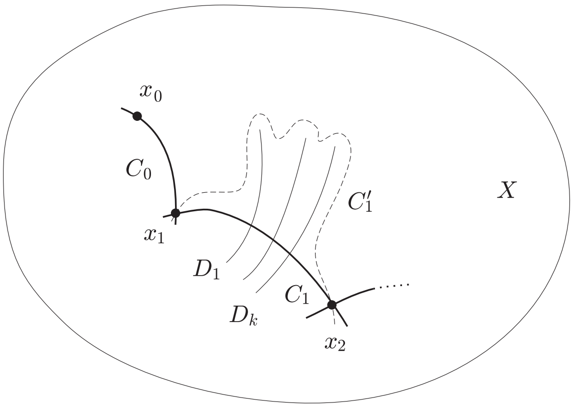

Fig. 4. Obtaining a free rational curve through x_1 and x_2

At the end we get a chain of free rational curves connecting x and x'. As in Application 17, we can deform this chain into a free rational curve, keeping x fixed. (We really would like to deform this chain into a free rational curve, keeping x and x' both fixed, but this is not automatic.)

Now consider the evaluation map $F : \mathbb{P}^1 \times \mathrm{Hom}\,(\mathbb{P}^1, X, 0 \mapsto x) \to X$ sending $\big(t, [f]\big)$ to $f(t)$. We have proved that for every $x' \in X^0$ there is an irreducible component $Z = Z(x') \subset \mathrm{Hom}\,(\mathbb{P}^1, X, 0 \mapsto x)$ such that x' is in the closure of $F\big(\mathbb{P}^1 \times Z(x')\big)$.

The scheme $\mathrm{Hom}\,(\mathbb{P}^1, X, 0 \mapsto x)$ has only countably many irreducible components, hence there is a single irreducible component $Z^* \subset \mathrm{Hom}\,(\mathbb{P}^1, X, 0 \mapsto x)$ such that $F|_{\mathbb{P}^1 \times Z^*}$ is still dominant. By Proposition 13 we conclude that f is very free for general $[f] \in Z^*$.

Now let us prove (6) $\Rightarrow$ (1^+).

Let $f : \mathbb{P}^1 \to X$ be a very free morphism. By Proposition 13, there exists a variety Y of dimension $\dim X - 1$, a point $y \in Y$ and a dominant morphism

$$F : \mathbb{P}^1 \times Y \to X \quad \text{such that} \quad F|_{\mathbb{P}^1 \times \{y\}} = f \quad \text{and} \quad F|_{\{0\} \times Y} = x_0.$$

If x_1 and x_2 are in the image of $F : \mathbb{P}^1 \times Y \to X$, we have a length 2 chain of rational curves connecting x_1 to x_0 and then to x_2. Otherwise, since F is dominant, we still have a 1-parameter family $\{f_t\}$ of deformations of f fixing x_0 that degenerates into a tree of rational curves T_1 connecting x_1 and x_0 (see Fig. 5). Similarly there is a tree of rational curves T_2 connecting x_2 and x_0.

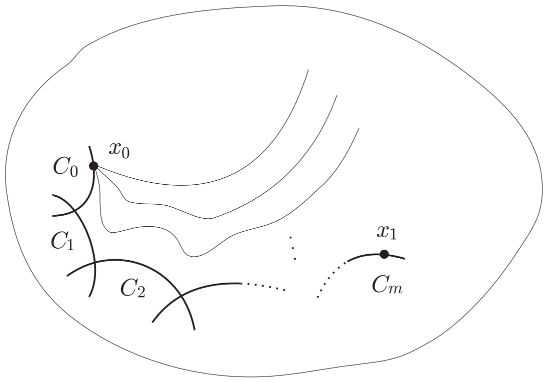

Fig. 5. A rational curve moving with one point fixed, and then degenerating into a union of rational curves.

To see that a 1-dimensional family of rational curves on X can only degenerate into a union of rational curves, we represent this family by a morphism $\phi : S \to X$, where S is a $\mathbb{P}^1$-bundle over a smooth curve T. Let $\bar{T}$ be a smooth compactification of T, and let $\bar{S} \to \bar{T}$ be a $\mathbb{P}^1$-bundle extending $S \to T$. Let $\bar{\phi} : \bar{S} \dashrightarrow X$ be a rational map extending ϕ. If $\bar{\phi}$ is a morphism, then we are done. Otherwise we perform a sequence of blowups of points $\tilde{S} \to \bar{S}$ in order to resolve the indeterminacy of $\bar{\phi}$, and obtain a morphism $\tilde{\phi} : \tilde{S} \to X$ extending ϕ. The result follows from the fact that blowups of points only add copies of $\mathbb{P}^1$ to the fibers of $\bar{S} \to \bar{T}$.

We complete the proof by showing $(6^+) \Rightarrow (5)$.

Assume that for every x_i there is a very free morphism $f_i : \mathbb{P}^1 \to X$ with $f_i(0) = x_i$. By Proposition 13, there are pointed varieties (Y_i, y_i) of dimension $\dim X - 1$, and dominant morphisms

$$F_i : \mathbb{P}^1 \times Y_i \to X \quad \text{such that} \quad F_i|_{\mathbb{P}^1 \times \{y_i\}} = f_i \quad \text{and} \quad F_i|_{\{0\} \times Y_i} = x_i.$$

We may also assume that $F_i|_{\mathbb{P}^1 \times \{y'\}}$ is very free for every $y' \in Y_i$.

Let $X^* \subset X$ be a dense open set contained in all the images $F_i\big(\mathbb{P}^1 \times (Y_i \setminus \{y_i\}\big)$. Let $g_0 : C_0 \cong \mathbb{P}^1 \to X$ be a very free morphism intersecting X^* and pick points $c_1, \dots, c_m \in C_0$ such that $g_0(c_i) \in X^*$. By the choice of X^*, there are $z_i \in Y_i$ such that $F_i\big((\infty, z_i)\big) = g_0(c_i)$. Set $g_i := F_i|_{\mathbb{P}^1 \times \{z_i\}}$ (so $g_i(0) = x_i$ and $g_i(\infty) = g_0(c_i)$).

Let $C_0 \cup C_1 \cup \dots \cup C_m$ be a rational comb such that $C_0 \cap C_i = \{c_i\}$ for $i = 1, \dots, m$. Choose identifications $C_i \cong \mathbb{P}^1$ such that c_i is identified with ∞. Define a morphism

$$G : C \to X \quad \text{by} \quad G|_{C_i} = g_i, \ 0 \le i \le m.$$

Notice that it passes through the points $x_1, \dots, x_m$.

Like in Application 17, the fact that all the g_i are very free implies that $G : C \to X$ can be deformed into a single very free morphism $\mathbb{P}^1 \to X$ passing through all the x_i. ■

Now that we have established our definition, we can show that the class of rationally connected varieties enjoys several good properties.

Let us start with one half of Castelnuovo's theorem.

Theorem 30 [17]. *Let X be a smooth, projective, rationally connected variety over $\mathbb{C}$. Then*

$$H^0\big(X, (\Omega^1_X)^{\otimes m}\big) = 0 \quad \textit{for all } m \ge 1.$$

Proof. Choose $f : \mathbb{P}^1 \to X$ such that f^*T_X is ample (i.e., f is very free). Then $f^*\Omega^1_X$ is a direct sum of line bundles with negative degree, and so is every tensor power. Thus any section of $(\Omega^1_X)^{\otimes m}$ vanishes along $f(\mathbb{P}^1)$. Since X is covered by very free rational curves (this is characterization (6^+) of Definition-Theorem 29), we cannot have a nonzero section. ■

It would be really interesting to know that the converse also holds, but this is known only in dimension 3 (see [17, 3.2]).

Theorem 31 [17]. *Let X be a variety over $\mathbb{C}$, S a connected $\mathbb{C}$-scheme and $p : X \to S$ a smooth projective morphism. If a fiber X_s is rationally connected for some $s \in S$, then every fiber of p is rationally connected.*

This is again not hard to prove. We can use characterization (6) of Definition-Theorem 29 to prove that being rationally connected is an open condition. Then we use characterization (1^+) to prove that it is also a closed condition. For a complete proof we refer to [17, 2.4] or [12, IV.3.11].

The analog of Noether's theorem was recently established by Graber, Harris and Starr.

Theorem 32 [5]**.** *Let X be a smooth projective variety over $\mathbb{C}$ and $f : X \to \mathbb{P}^1$ a morphism. If the general fiber of f is rationally connected, f has a section.* ■

As a consequence of Theorem 32 we have:

Corollary 33. *Let $f : X \to Y$ be a dominant morphism of smooth projective varieties over $\mathbb{C}$. If Y and the general fiber of f are rationally connected, so is X.*

Proof. Let $x_1, x_2 \in X$ be two points such that f is smooth with rationally connected fibers over $f(x_1)$ and $f(x_2)$. We show that they can be connected by a chain of rational curves. Since Y is rationally connected, there is a rational curve joining $f(x_1)$ and $f(x_2)$. Represent it by a morphism $\mathbb{P}^1 \to Y$ birational onto its image, let Z' be the unique irreducible component of $\mathbb{P}^1 \times_Y X$ that dominates $\mathbb{P}^1$ and let $Z \to Z'$ be a resolution of singularities. The fibers of $Z' \to \mathbb{P}^1$ containing x_1, x_2 are already smooth, so we do not need to change them to get Z. Let $f_Z : Z \to \mathbb{P}^1$ denote the projection.

By Theorem 32, f_Z has a section $D \subset Z$. Let $C \subset X$ be the projection of D to X and x_1' (resp. x_2') be the intersection point of C with the fiber $X_{f(x_1)}$ (resp. $X_{f(x_2)}$). By assumption, there are rational curves $C_1 \subset X_{f(x_1)}$ (resp. $C_2 \subset X_{f(x_2)}$) joining x_1 and x_1' (resp. x_2 and x_2'). Then $C_1 \cup C \cup C_2$ is a chain of rational curves through x_1 and x_2 (see Fig. 6). ■

34 (Rational connectedness in positive characteristic)**.** In characteristic p, the conditions of Definition-Theorem 29 are not equivalent, and we have two distinct concepts, rationally connected varieties, and *separably rationally connected varieties*, which we define below.

Let X be a smooth projective variety defined over an arbitrary uncountable algebraically closed field. The following two conditions are equivalent.

1. Through two general points of X there passes a rational curve.
2. There exists a variety Z and a morphism $U = Z \times \mathbb{P}^1 \to X$ such that the induced morphism $U \times_Z U \to X \times X$ is dominant. (A point $(x_1, x_2) \in X \times X$ is in the image of this morphism if and only if there is a point $z \in Z$ such that the image of $\{z\} \times \mathbb{P}^1$ in X contains x_1 and x_2.)

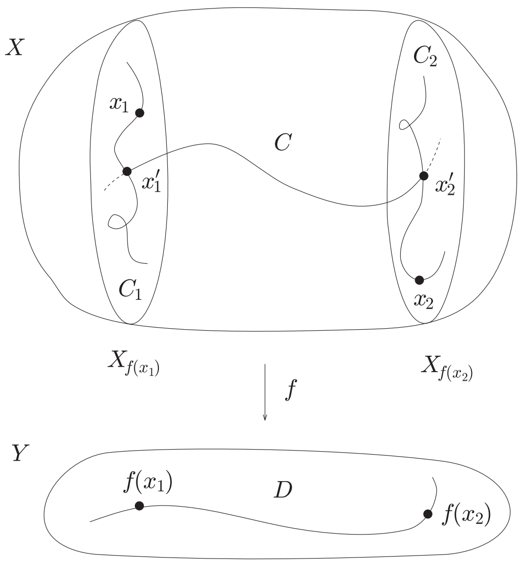

Fig. 6. Joining x_1 and x_2 by a chain of rational curves

In positive characteristic these conditions do not imply the existence of a very free morphism $f : \mathbb{P}^1 \to X$.

It turns out that the following two conditions are equivalent.

3. There is a morphism $f : \mathbb{P}^1 \to X$ such that f^*T_X is ample.
4. There exists a variety Z and a morphism $U = Z \times \mathbb{P}^1 \to X$ such that the induced morphism $U \times_Z U \to X \times X$ is dominant and smooth at the generic point.

This motivates the following definition.

Definition 35. Let X be a smooth projective variety over an arbitrary field k. We say that X is *separably rationally connected* if there is a morphism (defined over $\bar{k}$) $f : \mathbb{P}^1_{\bar{k}} \to X_{\bar{k}}$ such that f^*T_X is ample.

The condition above implies conditions (1) through (6) and also conditions (1^+) and (6^+) in Definition-Theorem 29.

Over fields of characteristic 0, the notions of rationally connected and separably rationally connected varieties coincide. However, there are examples of varieties in positive characteristic satisfying condition 1 above but not 3 (see for instance [12, V.5.19]).

We study separably rationally connected varieties over local fields in Section 9.

8. Genus Zero Stable Curves

In the last sections we saw that it is very useful to be able to deform reducible curves into irreducible ones. Frequently, we also need to degenerate rational curves so that they become reducible. For all of these processes it is very convenient to have a good theory that encompasses all genus zero curves in one object.

One possibility is to work with pairs (C, g), where C is a connected and possibly reducible curve of genus zero (that is, with $h^1(C, \mathcal{O}_C) = 0$) and $g : C \to X$ is a finite morphism. Geometrically these objects are easy to visualize. A technical problem arises, however, when C has $m \geq 3$ irreducible components meeting in one point. Here locally C looks like the m coordinate axes in $\mathbb{A}^m$. There are at least two problems.

1. The dualizing sheaf ω_C is not locally free, making duality less useful.
2. The deformation theory of this object is rather complicated.

[19] observed that it is better to insist that C have only nodes. The price we pay is that we have to allow $g : C \to X$ to map some irreducible components of C to points. This also means that even a reasonably nice curve on X does not correspond to a unique pair (C, g).

Example 36. First let $D \subset \mathbb{P}^3$ consist of 3 lines D_1, D_2, D_3 through a point P with independent directions. Then $h^1(D, \mathcal{O}_D) = 0$. The way we represent this object is by a morphism $g : C \to D$, where C is a rational comb with handle C_0 and teeth C_1, C_2, C_3. The morphism g maps C_i isomorphically onto D_i for $i = 1, 2, 3$ and contracts C_0 to P. We have 3 nodes on C_0.

Next consider the case when $D \subset \mathbb{P}^4$ consists of 4 lines $D_1, \ldots, D_4$ through a point P with independent directions. Again $h^1(D, \mathcal{O}_D) = 0$. As before, we can represent this object by a morphism $g : C \to D$, where C is a rational comb with handle C_0 and teeth C_1, C_2, C_3, C_4. The morphism g maps C_i isomorphically onto D_i for $i = 1, 2, 3, 4$ and contracts C_0 to P. This time we have 4 nodes on C_0, and these four points have moduli, the cross ratio.

We also have 3 degenerate cases when 2 of the nodes on C_0 come together. We represent them by replacing C_0 with two rational curves C_0' and C_0'' meeting transversely in a single point. The curves C_1, C_2, C_3, C_4 are still disjoint from each other. Two of them meet C_0' transversely and do not meet C_0'', while the others meet C_0'' transversely and do not meet C_0'. Each C_i maps isomorphically onto D_i, while C_0' and C_0'' are contracted to P (see Fig. 7).

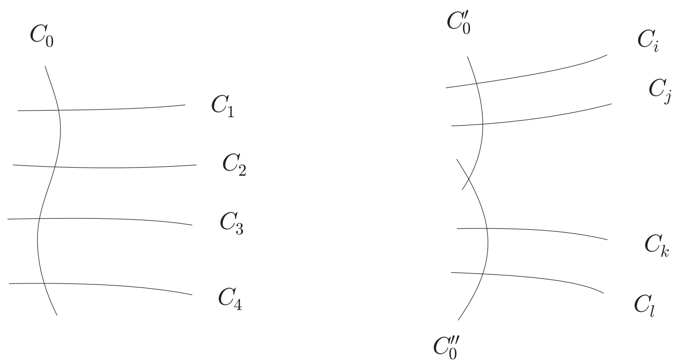

Fig. 7. Stable curves representing 4 lines through a point

Next we define stable curves. Then we give some basic properties of the space parametrizing such objects.

Definition 37. Let X be a scheme over a field k. A *stable curve over* X *(defined over* k*)* is a pair (C, f) where

1. C is a proper connected curve defined over k having only nodes,
2. $f : C \to X$ is a k-morphism, and
3. C has only finitely many automorphisms that commute with f. Equivalently, the following 2 situations are excluded:
   (a) The curve C has an irreducible component $C' \cong \mathbb{P}^1$ that contains at most 2 nodes of C and $f(C')$ is a point.
   (b) The curve C is a smooth elliptic curve and $f(C)$ is a point.

When X is a point, this is the Deligne-Mumford definition of a stable curve [4].

We also need to study cases when all the curves have marked points or they pass through certain points of X. Over an algebraically closed field these are just points, but in the nonclosed case we should keep track of the scheme structure on the points. This is the role of the scheme P in the definition below.

Definition 38. Let X be a scheme over a field k, P a smooth zero dimensional k-scheme and $t : P \to X$ a k-morphism. A *stable curve over X (defined over k) with base point* $t(P)$ is a triple (C, f, σ) where

1. C is a proper connected curve defined over k having only nodes,
2. $\sigma : P \to C$ is a k-embedding into the smooth locus of C,
3. $f : C \to X$ is a k-morphism such that $t = f \circ \sigma$, and
4. C has only finitely many automorphisms that commute with f and σ.

We usually work with families of curves.

Definition 39. A *family of stable curves over X with base point* $t(P)$ is a triple $(\mathcal{C}, f, \sigma)$ where

1. $\mathcal{C} \to U$ is a flat, proper family of connected curves with only nodes over a k-scheme U,
2. $\sigma : P \times_k U \to \mathcal{C}$ is an embedding, and
3. $f : \mathcal{C} \to X \times_k U$ is a U-morphism such that $(\mathcal{C}_u, f_u, \sigma_u)$ is a stable curve (defined over $k(u)$) with base point $t_u(P_u)$ for every $u \in U$.

A basic theorem, due to Kontsevich and Manin [19] (in the context of algebraic stacks) and Alexeev [1] (in the present setting of projective moduli space), asserts that one can parametrize all stable curves with a single scheme. The relevant notion is the concept of coarse moduli spaces, introduced by Mumford. A more general version is proved in Section 10.

Theorem 40. *Let X be a projective scheme over a field k with an ample divisor H, P a smooth zero dimensional k-scheme and $t : P \to X$ a k-morphism. Fix integers g and d. Then there is a projective k-scheme $\overline{M}_g(X, d, t)$ which is a coarse moduli space for all genus g stable curves (C, f, σ) with base point $t(P)$ such that $f_*[C] \cdot H = d$.* ∎

41 (What does this mean?)**.** We want to say that there is a "natural" correspondence between the set of all stable curves and the points of $\overline{M}_g(X, d, t)$. For our purposes this means the following two requirements.

1. There is a bijection of sets

$$\Phi : \left\{ \begin{array}{c} \text{genus } g \text{ stable curves over } \bar{k} \\ (C, f, \sigma) \\ \text{with base point } t(P) \\ \text{such that } f_*[C] \cdot H = d \end{array} \right\} \xrightarrow{\cong} \left\{ \bar{k}\text{-points of } \overline{M}_g(X, d, t) \right\}.$$

2. If $(\mathcal{C}/U, f, \sigma)$ is a family of degree d stable curves with base point $t(P)$, then there is a unique morphism $M_U : U \to \overline{M}_g(X, d, t)$ such that

$$M_U(u) = \Phi(\mathcal{C}_u, f_u, \sigma_u) \quad \text{for every } u \in U(\bar{k}).$$

M_U is called the *moduli map.*

Given a stable curve (C, f, σ) with base point $t(P)$, we denote by $\big[(C, f, \sigma)\big]$ the corresponding point $\Phi\big((C, f, \sigma)\big)$ in $\overline{M}_g(X, d, t)$.

We usually do not care about the degree, and use the scheme $\overline{M}_g(X, t)$ which is the union of all the $\overline{M}_g(X, d, t)$. The larger scheme $\overline{M}_g(X, t)$ is not projective since it has infinitely many connected components.

The requirements above seem somewhat awkward, but one cannot achieve more in general. For instance, (1) usually fails over nonclosed fields. The genus zero curves $(x_0^2 + x_1^2 + x_2^2 = 0) \subset \mathbb{P}^2_{\mathbb{R}}$ and $\mathbb{P}^1_{\mathbb{R}}$ are isomorphic over $\mathbb{C}$ but not over $\mathbb{R}$. Over $\mathbb{Q}$ there are infinitely many nonisomorphic genus zero curves (all isomorphic over $\mathbb{C}$). These examples illustrate that Φ is not injective over nonclosed fields. In general, Φ is also not surjective, but such examples are harder to get.

We need to know a little about the infinitesimal description of $\overline{M}_g(X, t)$ near a stable curve (C, f, σ). As usual, if X is smooth, this is governed by the cohomology groups of the twisted pull back of the tangent bundle, $f^*T_X\big(-\sigma(P)\big)$. The moduli space is usually very complicated, but its local structure is nice in some cases.

Trouble comes from two sources. First, if $H^1\Big(C, f^*T_X\big(-\sigma(P)\big)\Big)$ is large then one can say little beyond a dimension estimate, as in Theorem 5.3. Second, if C has automorphisms commuting with f and σ, then $\overline{M}_g(X, t)$ looks like the quotient of a smooth variety by a finite group. This can be quite complicated, especially in positive characteristic. This accounts for the somewhat messy formulation of Theorem 42.5. Fortunately, this seemingly cumbersome form is more useful for many purposes. A proof is given in Section 10.

Theorem 42. *Let X be a smooth projective variety over a field k. Let P be a smooth zero dimensional k-scheme and $t : P \to X$ a k-morphism. Set $|P| := \dim_k \mathcal{O}_P$. Let (C, f, σ) be a stable curve of genus g with base point $t(P)$. Assume $H^1\big(C, f^*T_X\big(-\sigma(P)\big)\big) = 0$. Then*

1. *There is a unique irreducible component (defined over k)*
$$\mathrm{Comp}\,(C, f, \sigma) \subset \overline{M}_g(X, t)$$
passing through $\big[(C, f, \sigma)\big]$.
2. *The component* $\mathrm{Comp}\,(C, f, \sigma)$ *is geometrically irreducible.*
3. *There is a dense open subset of* $\mathrm{Comp}\,(C, f, \sigma)$ *that corresponds to irreducible curves.*
4. *The dimension of* $\mathrm{Comp}\,(C, f, \sigma)$ *is*
$$-\big(f_*[C] \cdot K_X\big) + (\dim X - 3)(1 - g) - |P|(\dim X - 1).$$
5. *There is a smooth k-scheme U with a rational point $u \in U(k)$ and a family of stable curves*
$$(\mathcal{C}/U, F_U : \mathcal{C} \to X \times U,\ \sigma_U : P \times U \to \mathcal{C})$$
such that $(\mathcal{C}_u, F_u, \sigma_u) \cong (C, f, \sigma)$ and the corresponding moduli map
$$M_U : (U, u) \to \Big(\mathrm{Comp}\,(C, f, \sigma), \big[(C, f, \sigma)\big]\Big)$$
is quasifinite and open.

In fact, if C has no automorphisms commuting with f and σ, we can choose M_U to be an open embedding.

9. Finding Rational Curves over Nonclosed Fields

Let X be a smooth, projective, separably rationally connected variety over a field k. By definition, X has plenty of rational curves over $\bar{k}$, but we know very little about rational curves over k.

It is possible that $X(k) = \varnothing$, and then we certainly do not have any morphism $\mathbb{P}^1 \to X$ defined over k.

Thus assume there exists a point $x \in X(k)$. Can we find a rational curve $f : \mathbb{P}^1 \to X$ defined over k and passing through x? It is quite unlikely that such a curve always exists, but no counterexample is known. Here we discuss a weaker version of the problem, which allows us to settle the above question for certain fields.

Consider the space $\overline{M}_0(X, \operatorname{Spec} k \to x)$ of genus zero stable curves through x and let Z_i be its irreducible components over $\bar{k}$. The Galois group $\operatorname{Gal}(\bar{k}/k)$ acts on the set of all components. If there is a free rational curve (defined over $\bar{k}$) $f : \mathbb{P}^1_{\bar{k}} \to X_{\bar{k}}$, $f(0) = x$, then $\left[(\mathbb{P}^1, f, \operatorname{Spec} k \to 0)\right]$ lies in a unique $Z_i = \operatorname{Comp}(f)$ by Theorem 42. If, in addition, f is defined over k, then $\operatorname{Comp}(f)$ is a geometrically irreducible component of $\overline{M}_0(X, \operatorname{Spec} k \to x)$ defined over k.

In order to answer the above question, we do the following. First we construct geometrically irreducible components Z_i of $\overline{M}_0(X, \operatorname{Spec} k \to x)$ that are defined over k. This is done over any field. For local fields we can then find k-points in Z_i corresponding to irreducible curves through x. For arbitrary fields this seems rather difficult.

The main method is to assemble combs from given morphisms. This is what we define next.

Definition 43. Let X be a k-variety and $x \in X(k)$. Assume that we have $\bar{k}$-morphisms (not necessarily defined over k) $f_i : \mathbb{P}^1_{\bar{k}} \to X_{\bar{k}}$ such that $f_i(0) = x$ for $i = 1, \dots, n$.

Let $C := C_0 \cup \cdots \cup C_n$ be a rational comb (over $\bar{k}$) with handle C_0. Choose identifications $C_i \cong \mathbb{P}^1$ such that the node is identified with 0. Define a morphism

$$F : C \to X \quad \text{by} \quad F|_{C_i} = f_i \quad \text{and} \quad F(C_0) = \{x\}.$$

The pair (C, F) is called a *comb assembled from the f_i*. If $n \geq 3$, (C, F) is a stable curve over $X_{\bar{k}}$, provided none of the f_i are constant.

A comb assembled from the f_i is not unique, but all the combs assembled from the f_i form an irreducible family (parametrized by the space of n points on $C_0 \cong \mathbb{P}^1$).

It is natural to think of the f_i as $\bar{k}$-points of $\overline{M}_0(X, \operatorname{Spec} k \to x)$. In this context, we pick an extra point $p_0 \in C_0(\bar{k})$ distinct from the nodes of C and set $\sigma : \operatorname{Spec} \bar{k} \to p_0$. This makes (C, F, σ) into a stable curve (defined over $\bar{k}$) with base point x, provided that $n \geq 2$ and none of the f_i are constant. We call this triple a *base pointed comb assembled from the f_i*.

Assume next that the

$$f_i : \mathbb{P}^1 \to X, \ f_i(0) = x, \quad i = 1, \dots, n,$$

form a conjugation invariant set over k. We would like to assemble all the f_i into an element of $\overline{M}_0(X, \operatorname{Spec} k \to x)$ that is defined over k. The construction above has to be carried out carefully: even if we choose C to be a comb over k, the morphism F is not defined over k in general. In order to achieve this, we need that the induced scheme structure on the subset of nodes $\{P_i = C_0 \cap C_i\} \subset C_0$ matches the one on $\left\{\left[(C_i, f_i)\right]_{i=1,\dots,n}\right\} \subset \overline{M}_0(X, \operatorname{Spec} k \to x)$. This is the role of the scheme T below.

View each f_i as a $\bar{k}$-point of $\overline{M}_0(X, \operatorname{Spec} k \to x)$. This identifies the collection $\{f_i\}$ with a zero dimensional smooth k-subscheme $T \subset \overline{M}_0(X, \operatorname{Spec} k \to x)$. (See Remark 45 for some further clarification in positive characteristic.) The f_i correspond to a k-morphism $f_T : \mathbb{P}^1_T \to X$ which maps $\{0\} \times T$ to x. This construction is interesting only when none of the f_i corresponds to a k-point of $\overline{M}_0(X, \operatorname{Spec} k \to x)$, so we assume that T has no k-points.

Let $C_0 = \mathbb{P}^1$ and fix an embedding $j : T \hookrightarrow C_0$. This determines a rational comb C over k with teeth $\mathbb{P}^1_T$ whose nodes coincide with $j(T)$. Now define a morphism

$$F : C \to X \quad \text{by} \quad F|_{\mathbb{P}^1_T} = \{f_i\} \quad \text{and} \quad F(C_0) = \{x\}.$$

Notice that C and F are both defined over k. Since T has no k-points, we can pick a point $p_0 \in C_0(k)$ distinct from the nodes of C and set $\sigma : \operatorname{Spec} k \to p_0$. The triple (C, F, σ) is called a *pointed k-comb assembled from the f_i*. If $n \geq 2$, (C, F, σ) is a stable curve over X with base point x, provided none of the f_i are constant.

As before, a k-comb assembled from the f_i is not unique, but all $\bar{k}$-combs assembled from the f_i form an irreducible family defined over k (parametrized by the space of all $\bar{k}$-embeddings $T \cup \operatorname{Spec} k \hookrightarrow \mathbb{P}^1$).

Proposition 44. *Let k'/k be a finite, separable field extension. Let X be a smooth projective variety over k and let $x \in X(k)$ be a point. Let*

$$f_i : \mathbb{P}^1 \to X, \; f_i(0) = x, \quad i = 1, \dots, n,$$

be a set of nonconstant free k'-morphisms together defined over k. Let (C, F, σ) be a pointed k-comb assembled from the f_i, so that (C, F, σ) is a stable curve defined over k with base point x.

Then there is a smooth k-scheme U with a k-point $u \in U(k)$ and a family of stable curves

$$(\mathcal{C}/U, F_U : \mathcal{C} \to X \times U, \; \sigma_U : P \times U \to \mathcal{C})$$

such that $(\mathcal{C}_u, F_u, \sigma_u) \cong (C, F, \sigma)$ *and the corresponding moduli map*

$$M_U : (U, u) \to \Big(\operatorname{Comp}(C, F, \sigma), \big[(C, F, \sigma) \big] \Big)$$

is quasifinite and open.

Moreover, the irreducible component $\operatorname{Comp}(C, F, \sigma) \subset \overline{M}_0(X, \operatorname{Spec} k \to x)$ *is geometrically irreducible and independent of the choice of the comb.*

Proof. Denote by C_0 the handle of C and by $C_1, \ldots, C_n$ its teeth. Let $p_0 = \sigma(\operatorname{Spec} k) \in C_0$.

Set $E = F^* T_X(-p_0)$. Since F maps C_0 to a point, $f^* T_X|_{C_0} = \mathcal{O}_{C_0}^{\oplus n}$. Thus $H^1\big(C_0, E|_{C_0}\big) = 0$. For any other C_i, $E|_{C_i}(-1) \cong f_i^* T_X(-1)$, and this has vanishing H^1 since f_i is free. Thus $H^1\big(C, F^* T_X(-p_0)\big) = 0$ by Exercise 18. The first assertion then follows from Theorem 42, and so does geometric irreducibility of $\operatorname{Comp}(C, F, \sigma)$.

Theorem 42 also implies that each $\big[(C, F, \sigma)\big]$ lies in a unique irreducible component $\operatorname{Comp}(C, F, \sigma) \subset \overline{M}_0(X, \operatorname{Spec} k \to x)$. All $\bar{k}$-combs assembled from the f_i form an irreducible family defined over k. Hence they all lie in the same irreducible component of $\overline{M}_0(X, \operatorname{Spec} \bar{k} \to x)$, and $\operatorname{Comp}(C, F, \sigma)$ does not depend on the choice of the comb. ∎

Remark 45. In positive characteristic we have to deal with the possibility that $f_i : \mathbb{P}^1 \to X$ is not defined over a separable field extension. Since f_i is free, its irreducible component in $\operatorname{Hom}(\mathbb{P}^1, X, 0 \mapsto x)$ is generically smooth, thus it contains a dense set of points defined over separable field extensions.

As an application, we prove a result about separably rationally connected varieties over local fields.

Definition 46. For our purposes *local fields* are:

1. Finite degree extensions of the p-adic fields $\mathbb{Q}_p$;
2. Power series fields over finite fields, $\mathbb{F}_q\big((t)\big)$;
3. $\mathbb{R}$ and $\mathbb{C}$.

For all of these fields we have an analytic implicit function theorem, thus if X is a variety of dimension n over a local field k, and $x \in X(k)$ is a smooth point, then a neighborhood of x in the analytic variety $X(k)$ is biholomorphic to a neighborhood of 0 in k^n.

In particular, if X has *one* smooth k-point then k-points are Zariski dense.

Theorem 47 [14]**.** *Let k be a local field and X a smooth, projective, separably rationally connected variety defined over k. Let $x \in X(k)$ be a point.*

Then there is a nonconstant free morphism defined over k

$$f : \mathbb{P}^1 \to X \quad \text{such that} \quad f(0) = x.$$

Proof. By assumption and by Remark 45, there is such a morphism g over a separable extension of finite degree k'/k. Use g and all of its conjugates to assemble a k-comb $(C, F, \operatorname{Spec} k \to p_0)$ that is a stable curve with base point x.

By Proposition 44, there is a smooth pointed k-scheme (U, u) and a dominant, quasifinite morphism

$$(U, u) \to \big(\operatorname{Comp}(C, F, \operatorname{Spec} k \to p_0), \big[(C, F, \operatorname{Spec} k \to p_0)\big]\big).$$

Since k is a local field, $U(k)$ is Zariski dense in U, thus k-points are Zariski dense in $\operatorname{Comp}(C, F, \operatorname{Spec} k \to p_0)$. By Theorem 42.3, there exists a k-point $\big[(C', F', \sigma')\big]$ that corresponds to an irreducible curve $F' : C' \to X$ through x. Since C' comes with a k-point $\sigma'(\operatorname{Spec} k)$, there is an isomorphism $\mathbb{P}^1 \to C'$ defined over k. So we obtain a nonconstant k-morphism $\mathbb{P}^1 \to X$ whose image passes through x. ■

The construction of Proposition 44 can be made simultaneously for all points. This is the content of the next proposition.

Proposition 48. *Let X be a smooth, projective, separably rationally connected variety over a field k.*

There is an X-scheme U and a family of stable curves

$$(\mathcal{C}/U, F_U : \mathcal{C} \to X \times U,\ \sigma_U : \operatorname{Spec} k \times U \to \mathcal{C})$$

such that for each $x \in X$ the corresponding moduli map

$$M_{U_x} : U_x \to \overline{M}_0\big(X_{k(x)}, \operatorname{Spec} k(x) \to x\big)$$

is quasifinite and open and maps U_x onto an open subset of

$$\overline{M}_0\big(X_{k(x)}, \operatorname{Spec} k(x) \to x\big)$$

parametrizing free stable curves.

Proof. Let $x_g \in X$ be the generic point. As in Proposition 44, we obtain a $k(X)$-scheme U and a family of stable curves

$$\big(\mathcal{C}/U, F_U : \mathcal{C} \to X_{x_g} \times U,\ \sigma_U : \operatorname{Spec} k(X) \times U \to \mathcal{C}\big)$$

such that for each $x \in X$ the corresponding moduli map is quasifinite and open and maps U onto an open subset of $\overline{M}_0\big(X_{k(X)}, \operatorname{Spec} k(X) \to x_g\big)$ parametrizing free stable curves.

By specialization, for x in an open set $X^0 \subset X$ we obtain a family of stable curves

$$\big(\mathcal{C}/U_x, F_{U_x} : \mathcal{C}_{U_x} \to X_{k(x)} \times U_x, \operatorname{Spec} k(x) \times U_x \to \mathcal{C}_{U_x}\big)$$

such that the corresponding moduli map is quasifinite and open and maps U_x onto an open subset of $\overline{M}_0\big(X_{k(x)}, \operatorname{Spec} k(x) \to x\big)$ parametrizing free stable curves.

Repeat the argument with x_g replaced with the generic points of $X \setminus X^0$. Continuing in this way we eventually take care of all points of X.

It is quite likely that one can choose U to be irreducible and $U \to X$ smooth but the above construction gives neither. ■

10. Moduli Spaces of Stable Curves

The aim of this rather technical section is to prove Theorems 50 and 51, which are generalizations of Theorems 40 and 42. The main outlines of the proofs follow the general philosophy of constructing moduli spaces. The closest example is the special case explained in [8]. There are two key differences in the setup.

First, in [8] the marking of curves over $\mathbb{C}$ was done by choosing n smooth points of a curve. In the general setting it is more natural to let the marking be an embedding of a zero dimensional scheme Q into the smooth locus of a curve. The theory is interesting only when Q has embedding dimension at most 1, and the global theory works best only if it is smooth.

Second, we also want to account for the possibility that some of the marked points have a fixed image. For existence questions this is not much different, but the local structure has to be worked out by a somewhat different construction.

We also drop the "convexity" condition of [8], and replace it with the assumption $H^1\big(C, f^*T_{X_k}(-\sigma(P_k))\big) = 0$, which is essentially the local version of it.

First we generalize the notion of stable curves, defined in Section 8, to the relative setting. This means that instead of working with schemes over a field k, we fix a Noetherian scheme S, and work with schemes over S.

Definition 49. Fix a Noetherian scheme S. Let X be a scheme over S, let $P \to S$ and $Q \to S$ be finite, flat morphisms and let $t : P \to X$ be a morphism. A *stable curve over* X *with base point* $t(P)$ *and marking* Q is a quadruple (C, f, σ, ρ) where

1. $C \to U$ is a flat, proper family of connected curves with only nodes over an S-scheme U,
2. $f : C \to X \times_S U$ is a U-morphism,
3. $\sigma : P \times_S U \to C$ and $\rho : Q \times_S U \to C$ are embeddings into the smooth locus of $C \to U$ with disjoint images,
4. $f \circ \sigma$ is the morphism $P \times_S U \to X \times_S U$ induced by t, and
5. for every point $\operatorname{Spec} k \to U$, C_k has only finitely many automorphisms commuting with f_k and fixing the closed points of $\sigma_k(P_k)$ and $\rho_k(Q_k)$.

Note: Condition 5 is equivalent to saying that every rational component of $C_{\bar{k}}$ that is mapped to a point by $f_{\bar{k}}$ has at least 3 nodes or marked points on it. We allow marking by a nonreduced P_k, but a node plus a single point in P_k with a length 3 nilpotent structure is not enough for stability, although it is enough to kill all commuting automorphisms. The reason for imposing this stronger condition is that without it we get a nonseparated moduli problem.

As a simple example, consider a smooth curve B of genus $g \geq 2$ defined over a field k. Set $X = B$ and $P = \operatorname{Spec} k[t]/(t^3)$. Take $U = \mathbb{A}^1$ and consider the family $C = B \times U \to U$ with a section $S = \{b\} \times U$. Denote by $\mathcal{I}_S$ the ideal sheaf of S in C. Let σ map $P \times U$ onto the subscheme of C with ideal sheaf $\mathcal{I}_S^3$. Now let $f' : C' \to X \times U$ be the blow up of $(b, 0)$, let $\sigma' : P \times U \to C'$ be the map induced by σ and consider the triple (C', f', σ'). This shows that the identity map $B \to X$ with $P \hookrightarrow B$ degenerates to $B' \to X$ where $B' = B \cup \mathbb{P}^1$ and P is embedded into $\mathbb{P}^1$.

In the sequel we assume that $\deg(P/S)$ and $\deg(Q/S)$ are actually constants. This is the case if S is connected, so this is not a significant restriction. We use the shorthand $|P| = \deg(P/S)$ and $|Q| = \deg(Q/S)$.

The next theorem is a generalization of Theorem 40.

Theorem 50. *Fix a Noetherian base scheme S. Let X be a projective scheme over S with an ample divisor H. Let $P \to S$ and $Q \to S$ be finite and flat morphisms over S of relative embedding dimension at most 1 and let $t : P \to X$ be a morphism. Fix integers g and d. Then there is a separated algebraic space $\overline{M}_{g,Q}(X, d, t)$ of finite type over S which is a coarse moduli space for all genus g stable curves (C, f, σ, ρ) with base point $t(P)$ and marking Q such that $\deg_C f^*H = d$.*

Moreover, the space $\overline{M}_{g,Q}(X, d, t)$ is projective over S if the maps $P \to S$ and $Q \to S$ are smooth.

It is quite likely that $\overline{M}_{g,Q}(X, d, t)$ is always quasi-projective over S, but the proof does not give this. The reason is that the projectivity criterion of [11] used in [8] does not have a good quasi-projective analog.

As explained in [8, 5.1], one can use mH to embed X into $\mathbb{P}^M_S$ and deduce the existence of $\overline{M}_{g,Q}(X, d, t)$ from the existence of $\overline{M}_{g,Q}(\mathbb{P}^M_S, md, t)$. Thus we can restrict ourselves to the case when X is smooth over S. This is the only case in which the local structure of $\overline{M}_{g,Q}(X, d, t)$ is easy to describe. Such a description is given in the next theorem, which is a generalization of Theorem 42. As before, we work with the space $\overline{M}_{g,Q}(X, t)$, which is the union of all the $\overline{M}_{g,Q}(X, d, t)$, and we denote by $\big[(C, f, \sigma, \rho)\big]$ the point in $\overline{M}_{g,Q}(X, t)$ corresponding to a stable curve (C, f, σ, ρ).

Theorem 51. *Let S be a normal scheme and X a smooth projective scheme over S. Let $P \to S$ and $Q \to S$ be finite and flat S-schemes of relative embedding dimension at most 1 and let $t : P \to X$ be a morphism. Let k be a field, $\operatorname{Spec} k \to s \in S$ a point and*

$$(C \to \operatorname{Spec} k, f : C \to X_k,\ \sigma : P_k \to C,\ \rho : Q_k \to C)$$

a stable curve of genus g and degree d with base point $t_k(P_k)$ and marking Q_k over k. Assume $H^1\big(C, f^*T_{X_k}\big(-\sigma(P_k)\big)\big) = 0$. Then

1. *There is a unique irreducible component* $\operatorname{Comp}(C, f, \sigma, \rho) \subset \overline{M}_{g,Q}(X, t)$ *passing through* $\big[(C, f, \sigma, \rho)\big]$.
2. *If k is the residue field of s in S, then the unique irreducible component of the fiber of $\overline{M}_{g,Q}(X, d, t) \to S$ over s containing $\big[(C, f, \sigma, \rho)\big]$ is geometrically irreducible.*
3. *There is a dense open subset of* $\operatorname{Comp}(C, f, \sigma, \rho)$ *that corresponds to irreducible stable curves.*

4. *The dimension of* $\mathrm{Comp}\,(C, f, \sigma, \rho)$ *is*

$$-\big(f_*[C]\cdot K_X\big) + (\dim X - 3)(1-g) + \dim S + |Q| - |P|(\dim X - 1).$$

5. *Assume* $k/k(s)$ *is a finite field extension, where* $k(s)$ *denotes the residue field of* s *in* S*. Then there is an* S*-scheme* U*, smooth over* S*, with a point* $\mathrm{Spec}\, k \overset{\cong}{\to} u \in U$ *and a family of stable curves*

$$(C/U, F_U : C \to X_U,\ \sigma_U : P_U \to C,\ \rho_U : Q_U \to C)$$

such that $(C_u, F_u, \sigma_u, \rho_u) \cong (C, f, \sigma, \rho)$ *and the corresponding moduli map*

$$M_U : (U, u) \to \Big(\mathrm{Comp}\,(C, f, \sigma, \rho), \big[(C, f, \sigma, \rho)\big]\Big)$$

is quasifinite and open.

If C has no automorphism commuting with f, σ and ρ, we can choose M_U to be an open embedding.

52 (Outline of the proof)**.** Let (C, f, σ, ρ) be a stable curve with base point $t(P)$, where $h : C \to U$ is a U-scheme and U is a connected S-scheme. The starting point, as in [8, 2.3], is the observation that $\omega_{C/U}\big(\sigma(P) + \rho(Q)\big) \otimes f^*\mathcal{O}_X(3H)$ is h-ample on C/U. Let L be the above line bundle raised to the $\big(2 + |P|\big)$-th power. Then $L\big(-\sigma(P)\big)$ is h-very ample and $R^1h_*L = R^1h_*L\big(-\sigma(P)\big) = 0$. Let $N+1$ be the rank of the vector bundle h_*L (where N depends on the genus of C, $|P|$, $|Q|$ and d). After replacing U with an open affine cover, we may assume that $h_*L \cong \mathcal{O}_U^{N+1}$. Thus we have an embedding $\delta : C/U \hookrightarrow \mathbb{P}^N_U$. This is of course not unique, but defined only up to automorphisms of $\mathbb{P}^N_U$.

Let us now look at the diagonal map

$$(f, \delta) : C/U \hookrightarrow X \times_U \mathbb{P}^N.$$

Our strategy has 3 main parts:

1. Identify all possible lifts (f, δ) with some subset of a Hilbert scheme. This is straightforward if $P = \varnothing = Q$, but needs an extra twist otherwise.
2. Construct $\overline{M}_{g,Q}(X, d, t)$ as the quotient of the above subset of a Hilbert scheme by $PGL(N+1)$.
3. Read off properties of $\overline{M}_{g,Q}(X, d, t)$ through the properties of the Hilbert scheme.

53 (Construction of a related Hilbert scheme)**.**

In order to deal with the base point $t(P)$, let us fix an embedding $t' : P_U \hookrightarrow \mathbb{P}^N_U$ that is nondegenerate (that is, over any algebraically closed field K, the image $t'(P_K)$ spans a subspace of dimension $|P|-1$). First we consider only those liftings δ such that $t' = \delta \circ \sigma$. In this case the image of (f,δ) passes through the image of (t,t'), so it is more natural to view it as an embedded curve in the scheme $B_{t'} = B_{t'}(X \times_U \mathbb{P}^N)$ obtained by blowing up the image of (t,t').

We need to check a few things since the scheme $B_{t'}(X \times_U \mathbb{P}^N)$ is singular if P is not smooth. Let us consider a single fiber of $X \times_U \mathbb{P}^N \to U$. We assume that P has embedding dimension at most 1 and (t,t') is an embedding, thus in suitable local analytic coordinates $z_1,\dots,z_k$ (where $k = \dim X/U + N$) we can write

$$C = (z_1 = \cdots = z_{k-1} = 0) \quad \text{and} \quad P = (z_1 = \cdots = z_{k-1} = z_k^d = 0).$$

The key chart of the blow up is given by coordinates $z'_1,\dots,z'_{k-1},z_1,\dots,z_k$ and relations

$$z_i = z'_i z_k^d \quad \text{for } i = 1,\dots,k-1.$$

This chart is smooth with coordinates $z'_1,\dots,z'_{k-1},z_k$ and the birational transform of C is $C' = (z'_1 = \cdots = z'_{k-1} = 0)$. Thus $B_{t'}$ is smooth along C'. (All the other local charts of the blow up are singular.)

Inside $\operatorname{Hilb}\left(B_{t'}(X \times_U \mathbb{P}^N)\right)$ consider the open subset

$$W = W_{U,t',g}(X,d,t)$$

parametrizing 1-dimensional subschemes D such that

1. D is a connected curve of genus g contained in the smooth locus of $B_{t'}$ and with at worst nodes,
2. the natural projection $D \to \mathbb{P}^N$ is a nondegenerate embedding,
3. D is smooth at the points of E, and $E \cap D$ projects isomorphically onto the image of (t,t').

We still have to add in the marking by Q. To do this consider the universal family $\mathbf{D} \to W_{U,t',g}(X,d,t)$ and let Q_W denote the pull back of Q to $W_{U,t',g}(X,d,t)$. Consider the open subset

$$J_{U,t',g,Q}(X,d,t) \subset \operatorname{Hom}_W(Q_W, \mathbf{D}),$$

parametrizing curves D and homomorphisms $\rho : Q \to D$ such that

4. the map ρ is an embedding into the smooth locus and its image is disjoint from $E \cap D$,
5. the natural projection $D \to X$ is a stable curve of degree d with base point $t(P)$ and marking Q,
6. on every irreducible component of every geometric fiber of $D \to U$, the degree of $\mathcal{O}_{\mathbb{P}^N}\big(2 + |P|\big)$ is the same as the degree of
$$\omega_{D/U} \otimes \mathcal{O}_X(3H) \otimes \mathcal{O}_{B_{t'}}(E) \otimes \mathcal{O}_D\big(\rho(Q)\big).$$
(The various pull backs to D have been omitted from the notation.)

Since we are looking only at those morphisms in $\mathrm{Hom}_W(Q_W, \mathbf{D})$ that land in the smooth locus of $\mathbf{D} \to W$, we conclude from the general case of Theorem 5 given in [12, I.2.17] that
$$J_{U,t',g,Q}(X, d, t) \to \mathrm{Hilb}\,\big(B_{t'}(X \times_U \mathbb{P}^N)\big)$$
is smooth of relative dimension $|Q|$.

If the genus of C is zero, this is the subscheme we want. In general, based on our construction we also want to achieve that

7. the pull back of $\mathcal{O}_{\mathbb{P}^N}\big(2 + |P|\big)$ to D is isomorphic to
$$\omega_{D/U} \otimes \mathcal{O}_X(3H) \otimes \mathcal{O}_{B_{t'}}(E) \otimes \mathcal{O}_D\big(\rho(Q)\big).$$

As explained in [8, 2.2], the latter is a locally closed condition and defines $J^*_{U,t',g,Q}(X, d, t) \subset J_{U,t',g,Q}(X, d, t)$, which is open if $g = 0$ and has codimension g in general.

Ultimately we do not want to fix the embedding $t' : P \to \mathbb{P}^N$. To achieve this, let $\mathrm{Hom}^1(P, \mathbb{P}^N)$ denote the open subset of all nondegenerate embeddings. We have a universal embedding
$$T : P \times_S \mathrm{Hom}^1(P, \mathbb{P}^N) \to \mathbb{P}^N \times_S \mathrm{Hom}^1(P, \mathbb{P}^N).$$
Out of T and t we get a diagonal embedding
$$(t, T) : P \times_S \mathrm{Hom}^1(P, \mathbb{P}^N) \to X \times_S \mathbb{P}^N \times_S \mathrm{Hom}^1(P, \mathbb{P}^N).$$
Observe that the automorphism group $\mathrm{Aut}\,(\mathbb{P}^N)$ acts on both sides and (t, T) is equivariant. Let $B \to X \times_S \mathbb{P}^N \times_S \mathrm{Hom}^1(P, \mathbb{P}^N)$ denote the blow up of the image. We can now construct
$$\begin{aligned} W_g(X, d, t) &:= W_{\mathrm{Hom}^1(P,\mathbb{P}^N),(t,T),g}(X, d, t) \\ &\subset \mathrm{Hilb}\,\big(X \times_S \mathbb{P}^N \times_S \mathrm{Hom}^1(P, \mathbb{P}^N)\big) \end{aligned}$$

and

$$J_{g,Q}(X,d,t) := J_{\mathrm{Hom}^1(P,\mathbb{P}^N),(t,T),g,Q}(X,d,t) \to W_g(X,d,t).$$

The pullback of $J_{g,Q}(X,d,t)$ by the morphism $U \to S$ and the embedding t' is precisely our earlier $J_{U,t',g,Q}(X,d,t)$. As before we get that:

Claim 54. With the above notation,

$$J_{g,Q}(X,d,t) \to W_g(X,d,t)$$

is smooth of relative dimension $|Q|$. ■

The automorphism group $\mathrm{Aut}\,(\mathbb{P}^N)$ acts on B, hence it also acts on $\mathrm{Hilb}\,(B)$ and on $J_{g,Q}(X,d,t)$. From our construction we have the key observation:

Claim 55. Let K be any algebraically closed field and $\mathrm{Spec}\, K \to S$ a morphism.

1. If $g = 0$, there is a one-to-one correspondence between the $PGL(N+1,K)$ orbits on $J_{g,Q}(X_K,d,t_K)$ and genus zero stable curves over K of degree d with base point $t(P_K)$ and marking Q_K.
2. If $g \geq 0$, there is a one-to-one correspondence between the $PGL(N+1,K)$ orbits on $J_{g,Q}(X_K,d,t_K)$ and genus zero stable curves over K of degree d with base point $t(P_K)$ and marking Q_K plus the choice of a line bundle which has degree zero on every irreducible component of the curve. ■

Remark 56. The slight mystery is why we need K algebraically closed. This is already visible in the much simpler example of the hyperelliptic plane curves $ay^2 = f(x)$, where we can even fix f for simplicity. The multiplicative group K^* acts on the family by $(x,y) \mapsto (x,ty)$ and over an algebraically closed field we have a single orbit. On the other hand, if $K = \mathbb{Q}$, the $\mathbb{Q}^*$-action on the $\mathbb{Q}$-points has infinitely many orbits, parametrized by the square free integers. Upon deeper analysis one sees that all of this is caused by the automorphism $(x,y) \mapsto (x,-y)$ of the hyperelliptic plane curves.

This completes the first step of the plan, and next we need to take the quotient $J(X,d,t)/\,\mathrm{Aut}\,(\mathbb{P}^N)$.

57 (Construction of $\overline{M}_g(X,d,t)$ as a quotient space)**.**

As we noted above, we would like to find a morphism

$$q : J_{g,Q}(X,d,t) \to \overline{M}_{g,Q}(X,d,t)$$

with the property that for any algebraically closed field K and for any morphism $\operatorname{Spec} K \to S$ the fibers of

$$q_K : J_{g,Q}(X_K,d,t_K) \to \overline{M}_{g,Q}(X_K,d,t_K)$$

are precisely the $PGL(N+1,K)$ orbits on $J_{g,Q}(X_K,d,t_K)$.

At this point it is very convenient to use the general quotient theorems about algebraic group actions established in [13] and [10]. First we need to check a technical condition about the $\operatorname{Aut}(\mathbb{P}^N)$-action on $J_{g,Q}(X,d,t)$, namely that it is *proper*. Properness of the group action is equivalent to the separatedness of the family considered in [8, 4.2]. We need the following:

Proposition 58. *In the setting of Theorem* 50 *assume that S is the spectrum of a complete DVR with quotient field K and residue field k. Let $(C_K, f_K, \sigma_K, \rho_K)$ be a stable curve over K.*

1. *There is at most one way to extend it to a stable curve $(C_S, f_S, \sigma_S, \rho_S)$ over S.*
2. *If $P \to S$ and $Q \to S$ are smooth, then there is a scheme S', which is also the spectrum of a DVR, and a local map $S' \to S$ such that the pull back of $(C_K, f_K, \sigma_K, \rho_K)$ can be extended to $\left(C_{S'}, f_{S'}, \sigma_{S'}, \rho_{S'}\right)$.*

Proof. For the first part, we plan to follow the proof of the uniqueness of canonical models (see for instance [18, 3.52]).

If we have two extensions, they are dominated by a common resolution. So it is enough to show how to recover C_S from a resolution of it. The notion of resolution here is slightly different from the usual one since the general fiber may also be singular. Thus we allow 3 types of operations:

1. blowing up a smooth point of the special fiber, introducing a (-1)-curve,
2. blowing up an isolated singular point of the special fiber, introducing one or two (-2)-curves, and
3. blowing up a nonisolated singular point of the special fiber, introducing two (-1)-curves in the normalization of C_S.

We would like to consider the pair $\left(C_S, \sigma_S(P_S) + \rho_S(Q_S)\right)$, but it is not canonical. Instead we do the following. Let D_i be an irreducible component of $\sigma_S(P_S)+\rho_S(Q_S)$ with central fiber $(D_i)_k = \operatorname{Spec} k'[t]/(t^{d_i})$ for some $k' \supset k$ and $d_i \geq 1$. (Here we use that S is complete.) Set $D = \sum \frac{1}{d_i} D_i$ as a $\mathbb{Q}$-divisor on C_S. The reason for introducing D is to assure that every irreducible component of D has local intersection number 1 with C_k at every geometric intersection point. Our stability assumption 49.5 was made to assure that $\omega_{C_S/S}(D)$ is relatively ample.

We also see that under a blowup $C'_S \to C_S$, the exceptional curves appear in $\omega_{C'_S/S}(D)$ with coefficient 0, unless we blow up a smooth point not on D, when the coefficient is 1. Thus if $g : C^m_S \to C_S$ is any resolution, the global sections of $\left(\omega_{C^m_S/S}(D)\right)^{\otimes n}$ all pull back from $\left(\omega_{C_S/S}(D)\right)^{\otimes n}$ for $n \geq 1$. Thus C_S is the image of C^m_S under the morphism given by the global sections of $\left(\omega_{C^m_S/S}(D)\right)^{\otimes n}$ for some $n \gg 1$.

Assume next that the maps $P \to S$ and $Q \to S$ are smooth. In this case the existence proof of [8, 4.2] works, except that one should use the semi stable reduction theorem in arbitrary characteristic as proved in [2]. ■

Thus we conclude that $\overline{M}_{g,Q}(X,d,t)$ exists as an algebraic space that is separated and of finite type over S. If $P \to S$ and $Q \to S$ are smooth, the properness of $\overline{M}_{g,Q}(X,d,t)$ over S follows from Proposition 58. The projectivity over S can be checked as in [8, 3–4], but it is actually not important for any of what follows.

59 (Local structure of $\overline{M}_{g,Q}(X,d,t)$)**.**

We have constructed $\overline{M}_{g,Q}(X,d,t)$ as a quotient of $J_{g,Q}(X,d,t)$, and so we expect to read off the local properties of $\overline{M}_{g,Q}(X,d,t)$ from the local properties of $J_{g,Q}(X,d,t)$. The latter is actually much nicer, and parts 1,2,3 and 5 of Theorem 51 follow from the corresponding properties of $J_{g,Q}(X,d,t)$.

Proposition 60. *Let S be a normal scheme and X a smooth projective scheme over S. Let P,Q be finite and flat S-schemes of relative embedding dimension at most* 1 *and let $t : P \to X$ be a morphism. Let k be a field,* $\operatorname{Spec} k \to s \in S$ *a point in S and*

$$(C, f : C \to X_k,\ \sigma : P_k \to C,\ \rho : Q_k \to C)$$

a stable curve of genus g and degree d with base point $t_k(P_k)$ and marking Q_k over k. Let L be the line bundle on C defined as in Paragraph 52. Let $\delta : C \hookrightarrow \mathbb{P}^N_k$ be a nondegenerate embedding by all sections of L

and let $\left[(C,f,\sigma,\delta,\rho)\right]$ *be the corresponding point in* $J_{g,Q}(X,d,t)$*. Assume* $H^1\big(C,L(-\sigma(P_k))\big)=0$ *and* $H^1\big(C,f^*T_{X_k}(-\sigma(P_k))\big)=0$*. Then*

1. *The map* $J_{g,Q}(X,d,t)\to \operatorname{Hom}^1(P,\mathbb{P}^N)$ *is smooth at* $\left[(C,f,\sigma,\delta,\rho)\right]$.
2. *There is a unique irreducible component*

$$\operatorname{Comp}(C,f,\sigma,\delta,\rho) \quad \textit{of} \quad J_{g,Q}(X,d,t)$$

   passing through $\left[(C,f,\sigma,\delta,\rho)\right]$.
3. *The dimension of* $\left[(C,f,\sigma,\delta,\rho)\right]$ *is*

$$-\big(f_*[C]\cdot K_X\big)+(\dim X-3)(1-g)$$
$$+|Q|-|P|(\dim X-1)+\dim S+g+(N+1)^2-1.$$

4. *A general point of* $\operatorname{Comp}(C,f,\sigma,\delta,\rho)$ *corresponds to a smooth stable curve.*

Proof. Recall from Claim 54 that the map $J_{g,Q}(X,d,t)\to W_g(X,d,t)$ is smooth of relative dimension $|Q|$, thus the main point is to prove the analogous statements for the map $W_g(X,d,t)\to \operatorname{Hom}^1(P,\mathbb{P}^N)$.

The local structure of the Hilbert scheme is governed by the normal bundle (see Paragraph 25). Thus the key to prove Proposition 60 is to understand the normal bundle of the image of

$$C \xrightarrow{(f,\delta)} X\times_k \mathbb{P}^N \dashrightarrow B_{t'}(X\times_k\mathbb{P}^N).$$

Going from the embedding into $X\times_k\mathbb{P}^N$ to the blowup is straightforward, and the important part is to describe the normal bundle of the image of C in $X\times_k\mathbb{P}^N$.

The argument has three main ingredients.

5. General remarks about the "second exact sequence".
6. Computing the normal bundle of a curve in $\mathbb{P}^N$.
7. Relating the normal bundle of $C\subset X\times\mathbb{P}^N$ to f^*T_X.

61 (The second exact sequence)**.**

Let Z be a closed subscheme of Y with ideal sheaf $\mathcal{I}_Z$. The so-called "second exact sequence" (see [9, II.8.4.A]) is

$$\mathcal{I}_Z/\mathcal{I}_Z^2 \xrightarrow{\delta} \Omega^1_Y|_Z\to\Omega^1_Z\to 0.$$

In general it is not left exact, but it is easily seen to be left exact at points where Z and Y are smooth. If such points are dense in Z, Z is reduced and $\mathcal{I}_Z/\mathcal{I}_Z^2$ is locally free, then δ can have no kernel at all, so we also have left exactness. This holds if, for instance, Y is smooth and Z is a reduced local complete intersection subscheme.

If $C \subset \mathbb{P}^N$ is a reduced local complete intersection curve, we can compute the degree of its normal bundle from the second exact sequence as

$$\deg N_{C,\mathbb{P}^N} = (N+1)\deg C + 2p_a(C) - 2.$$

62 (Normal bundles in $\mathbb{P}^N$).

Let $Z \subset \mathbb{P}^N$ be a reduced local complete intersection subscheme. By the previous remarks, the natural map

$$T_{\mathbb{P}^N}|_Z \to N_{Z,\mathbb{P}^N}$$

is a surjection at smooth points of Z. The tangent bundle of $\mathbb{P}^N$ is the quotient of a direct sum of $N+1$ copies of $\mathcal{O}_{\mathbb{P}^N}(1)$ (see [9, II.8.13]), hence we get

$$\mathcal{O}_{\mathbb{P}^N}(1)^{N+1}\big|_Z \to N_{Z,\mathbb{P}^N}.$$

If $Z = C$ is a curve, we get a surjection of the first cohomologies

$$H^1\big(C, \mathcal{O}_{\mathbb{P}^N}(1)\big|_C\big)^{N+1} \twoheadrightarrow H^1(C, N_{C,\mathbb{P}^N}).$$

Thus if C is embedded by the sections of a line bundle L, this allows us to get vanishing results about the normal bundle of C in terms of $H^1(C, L)$.

The normal bundle of a diagonal embedding is described by the following lemma. We are interested in the case $Y = \mathbb{P}^N$.

Lemma 63. *Let X, Y be smooth varieties over a field k. Let C be a proper, reduced, local complete intersection curve. Let $f : C \to X$ be a morphism and $g : C \hookrightarrow Y$ an embedding. Let $C' \subset X \times Y$ be the image of the diagonal embedding. Then there is an exact sequence*

$$0 \to f^*T_X \to N_{C',X\times Y} \to N_{C,Y} \to 0.$$

Proof. The middle term of the second exact sequence

$$0 \to \mathcal{I}_{C'}/\mathcal{I}_{C'}^2 \to \Omega^1_{X\times Y}\big|_{C'} \to \Omega^1_{C'} \to 0$$

can be identified as

$$\Omega^1_{X\times Y}\big|_{C'} \cong f^*\Omega^1_X + \Omega^1_Y\big|_C.$$

Since g is an embedding, the restriction $\Omega^1_Y\big|_C \to \Omega^1_C$ is surjective, which implies that the composition

$$\mathcal{I}_{C'}/\mathcal{I}^2_{C'} \to \Omega^1_{X\times Y}\big|_{C'} \to f^*\Omega^1_X$$

is surjective and its kernel is the same as the kernel of $\Omega^1_Y\big|_C \to \Omega^1_C$, which is just $\mathcal{I}_C/\mathcal{I}^2_C$ by the second exact sequence applied to $C \subset Y$. Putting all these together we get the exact sequence

$$0 \to \mathcal{I}_C/\mathcal{I}^2_C \to \mathcal{I}_{C'}/\mathcal{I}^2_{C'} \to f^*\Omega^1_X \to 0,$$

which is the dual of the claimed sequence. ■

64 (The local structure of $J(X,d,t)$)**.**

We can now put things together to prove parts 1,2 and 3 of Proposition 60.

Let N_C be the normal bundle of C in $X_k \times \mathbb{P}^N$, embedded by (f,δ), and let N'_C be the normal bundle of the birational transform of C in $B_{t'}(X_k \times \mathbb{P}^N)$. Then $N'_C \cong N_C\big(-\sigma_k(P_k)\big)$. Twisting the exact sequence of Lemma 63 we obtain an exact sequence

$$0 \to f^*T_X\big(-\sigma(P_k)\big) \to N'_C \to N_{C,\mathbb{P}^N}\big(-\sigma(P_k)\big) \to 0.$$

We have proved that $N_{C,\mathbb{P}^N}\big(-\sigma(P_k)\big)$ is generically a quotient of $N+1$ copies of $L\big(-\sigma(P_k)\big)$. This implies that $H^1(C, N'_C) = 0$. By Theorem 5.2, this implies Proposition 60.1.

The scheme $\mathrm{Hom}^1(P,\mathbb{P}^N)$ is smooth over S. Therefore, in a neighborhood of $\mathrm{Comp}\,(C,f,\sigma,\delta,\rho)$ the composition $W_g(X,d,t) \to \mathrm{Hom}^1(P,\mathbb{P}^N) \to S$ is smooth over the normal scheme S, hence $W_g(X,d,t)$ is normal. A normal scheme is locally irreducible, proving Proposition 60.2.

The dimension of the fiber of

$$W_g(X,d,t) \to \mathrm{Hom}^1(P,\mathbb{P}^N)$$

at $\big[(C,f,\sigma,\delta,\rho)\big]$ is $H^0(C,N'_C)$. This is computed as

$$\begin{aligned}
&h^0(C,N'_C) = \chi(C,N'_C)\\
&= \deg N'_C + (\dim X + N - 1)(1-g)\\
&= \deg f^*T_X\big(-\sigma(P_k)\big) + \deg N_{C,\mathbb{P}^N}\big(-\sigma(P_k)\big) + (\dim X + N - 1)(1-g)\\
&= -\big(f_*[C]\cdot K_X\big) - |P|(\dim X + N - 1) + (N+1)\deg L + 2g - 2\\
&\quad +(\dim X + N - 1)(1-g)\\
&= -\big(f_*[C]\cdot K_X\big) + (\dim X - 3)(1-g) - |P|(\dim X - 1)\\
&\quad +(N+1)\deg L + N(1-g) - |P|\cdot N.
\end{aligned}$$

In order to get the dimension of $W_g(X,d,t)$, we have to add the dimension of $\mathrm{Hom}^1(P,\mathbb{P}^N)$, which is $|P|\cdot N + \dim S$. Finally using that $N = \deg L - g$ we get Proposition 60.3.

Proposition 60.4 is a special case of the next Lemma. ■

Lemma 65. *Let Y be a smooth variety over k and $C\subset Y$ a reduced curve with at worst nodes. Assume that $N_{C,Y}$ is generated by global sections and $H^1(C,N_{C,Y})=0$.*

Then $\mathrm{Hilb}\,(Y)$ *has a unique irreducible component containing $[C]$. This component is smooth at $[C]$ and a nonempty open subset of it parametrizes irreducible and smooth curves in Y.*

Proof. It is enough to prove that a general 1-parameter deformation has no nodes near a given node of C.

This is a local question, so choose local (analytic or formal) coordinates so that C is given by

$$y_1y_2 = y_3 = \cdots = y_n = 0.$$

The coordinates $y_3,\dots,y_n$ generate a well-defined subspace of $N_{C,Y}\big|_P$. Assume that our 1-parameter deformation corresponds to a section of $N_{C,Y}$ that is not in that subspace at P. Then we have equations

$$y_1y_2 + tg(t,\mathbf{y}) = y_3 + tg_3(t,\mathbf{y}) = \cdots = y_n + tg_n(t,\mathbf{y}) = 0,$$

where $g(0,0)\neq 0$ by our assumption. We can choose new coordinates $y_i + tg_i(t,\mathbf{y})$ for $i=3,\dots,n$, and call them again y_i, to get a simpler system

$$y_1y_2 + t\big(a + G(t,\mathbf{y})\big) = y_3 = \cdots = y_n = 0,$$

where $a \neq 0$ and $G(0,0) = 0$. The singular points are given by equations $y_3 = \cdots = y_n = 0$ and

$$y_2 + t\frac{\partial G}{\partial y_1} = y_1 + t\frac{\partial G}{\partial y_2} = 0.$$

We can view these as equations that can be substituted back into $y_1y_2 + t\bigl(a + G(t,\mathbf{y})\bigr) = 0$ to get a new equation for the supposed singular point:

$$ta = -tG(t,\mathbf{y}) - t^2\frac{\partial G}{\partial y_1}\frac{\partial G}{\partial y_2}.$$

The latter has no solutions for $t \neq 0$ and $t, y_1, \ldots, y_n$ small since $a \neq 0$ and $G(0,0) = 0$. ■

66 (The case $g \geq 1$)**.** The only remaining problem is that we have worked with $J_{g,Q}(X,d,t)$ but, as noted in 53.7, we would like to conclude similar results about $J^*_{g,Q}(X,d,t)$. The latter is a subscheme of $J_{g,Q}(X,d,t)$, but it is better to view $J_{g,Q}(X,d,t)$ as Pic^0 of the universal curve over $J^*_{g,Q}(X,d,t)$. By [7], Pic^0 of a flat family of curves is smooth, hence we conclude that $J_{g,Q}(X,d,t) \to J^*_{g,Q}(X,d,t)$ is smooth of relative dimension g. Thus the smoothness results about J translate into smoothness results about J^*.

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Carolina Araujo
Mathematics Department
Princeton University
Princeton
NJ 08544

caraujo@math.princeton.edu

János Kollár
Mathematics Department
Princeton University
Princeton
NJ 08544

kollar@math.princeton.edu