BOUNDING SINGULAR SURFACES OF GENERAL TYPE

VALERY ALEXEEV AND SHIGEFUMI MORI

Contents

0. Introduction

0.1. The purpose of this paper is to give simpler proofs for several theorems contained in articles [2], [3] of one of the authors, and at the same time make these proofs effective. We give explicit formulas for the bounds in the theorems. It has to be admitted, however, that these bounds are unrealistically high.

Notation 1. Let X be a normal surface defined over an algebraically closed field of arbitrary characteristic, and let $B=\sum b_jB_j$ be an $\mathbb{R}$-divisor on X with $b_j\ge 0$ such that K_X+B is $\mathbb{R}$-Cartier. For a resolution of singularities $f:Y\to X$ we, as it is customary, call the coefficients a_i in the following formula *the log discrepancies.*

$$K_Y+\sum b_jf_*^{-1}B_j+\sum F_i=f^*(K_X+\sum b_jB_j)+\sum a_iF_i,$$

where F_i are the exceptional divisors of the morphism f.

We say that the pair $(X,\sum b_jB_j)$ is ε-log canonical (resp. ε-log terminal) if all $a_i\ge\varepsilon$ and $b_j\le 1-\varepsilon$ (resp. if the inequalities are strict) for any resolution $f:Y\to X$. For $\varepsilon=0$ we get the usual definition of a log canonical (resp. Kawamata log terminal) pair.

We will also use the abbreviations lc and lt.

If we are given a normal surface Z birational to X, we can define B^Z as follows. Take a resolution $f:Y\to X$ which dominates Z, say via $g:Y\to Z$.

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Then write $K_Z + B^Z = g_* f^*(K_X + B)$. It is easy to see that B^Z does not depend on the choice of Y.

We will write $A \equiv B$ (resp. $A \sim B$) if A is numerically (resp. linearly) equivalent to B.

1. Bounds for Picard groups

1.1. The main result of this section is (1.8). We start with a few easy introductory lemmas. The next one is used very often in this paper.

Lemma 1.2. *Let X be a nonsingular projective surface and $B = \sum b_j B_j$ be an $\mathbb{R}$-divisor on X with $0 \le b_j \le 1 - \varepsilon < 1$. If E is an irreducible curve on X with $(K+B)\cdot E \le 0$ and $E^2 < 0$ then $E \simeq \mathbb{P}^1$ and $E^2 \ge -2/\varepsilon$.*

Proof. Follows from

$$\begin{aligned} -2 \le 2p_a(E) - 2 = (K+E)E = \\ \varepsilon E^2 + (K + (1-\varepsilon)E)\, E \le \\ \varepsilon E^2 + (K+B)E \le \varepsilon E^2 < 0 \quad \square \end{aligned}$$

The following is a special case of (2.8). However (1.8) needs only this easy form.

Lemma 1.3. *Let X be a nonsingular projective surface F_n ($n \ge 0$) and $B = \sum b_j B_j$ be an $\mathbb{R}$-divisor on X with $0 \le b_j \le 1$. Assume that $-(K+B)$ is nef. Then $\sum b_j \le 4$.*

Proof. Let F, S be a fiber and a section of X such that $S^2 = -n$. We can always add a curve to B with coefficient 0 without changing the situation. Thus we may set $B_0 = S$. From $0 \ge (K+B)\cdot F$, we see

$$2 \ge \sum b_j (B_j \cdot F) \ge \sum_{(B_j \cdot F) \ne 0} b_j.$$

Since $(K \cdot S) = n - 2$, we get the following from $0 \ge (K+B)\cdot S$.

$$2 \ge n(1 - b_0) + \sum b_j (B_j \cdot S) \ge \sum_{(B_j \cdot S) > 0} b_j.$$

Since $(E \cdot F) = 0$ implies $(E \cdot S) > 0$ for an arbitrary irreducible curve E, we have the lemma. $\square$

The following is a special case of (4.2.1) in [10], which is however enough as the starting point of (1.8) and allows us to to focus on the main case (3).

Lemma 1.4. *Let X be a nonsingular projective surface and $B = \sum b_j B_j$ be an $\mathbb{R}$-divisor on X with $0 \le b_j \le 1 - \varepsilon < 1$. Assume that $K + B \equiv 0$. Then one of the following is true:*

1. *$B = 0$, $K_X \equiv 0$ and X is either a K3, or an Enriques, or an Abelian, or a hyperelliptic surface (in particular $\rho(X) \le 22$ by the classification),*

2. *$X \simeq \mathbb{P}_C(\mathcal{E})$, where C is an elliptic curve and $\mathcal{E}$ is a locally free sheaf which is an extension of an invertible sheaf $\mathcal{L}$ by $\mathcal{O}_C$ such that $0 \le deg\mathcal{L} \le 2$,*
3. *X is rational, and either $X \simeq \mathbb{P}^2$ or there exists a birational morphism $g : X \to \mathbb{F}_n$ with $n \le 2/\varepsilon$.*

Remark 1.5. For the study of $K+B \equiv 0$, it is essential by (1.4) to study the birational map $g : X \to \overline{X} = \mathbb{F}_n$ and $\overline{B} = g_*(B)$. Let Y be any nonsingular projective surface decomposing g as $\alpha : X \to Y$ and $\beta : Y \to \overline{X}$. We note that $\beta^*(K_{\overline{X}} + \overline{B}) = \alpha_*(K_X + B)$ by $K_X + B \equiv 0$ and $\beta^*(K_{\overline{X}} + \overline{B})$ has an effective boundary.

Proof. If $B = 0$ and $K \equiv 0$ then X is one of (1) by the classification of surfaces. Otherwise, $K \equiv -B \neq 0$. Assuming $X \neq \mathbb{P}^2$, we contract (-1)-curves on X and its contractions until we get a birational morphism $g : X \to \overline{X}$ to a model $\overline{X}$ which is a $\mathbb{P}^1$-bundle over a nonsingular curve C. (Since $K_X \succ g^*g_*K_X$, we have $g_*B \neq 0$ and we see the above assertion.) If $g(C) = 0$ then $\overline{X} = \mathbb{F}_n$ and $n \le 2/\varepsilon$ by Lemma 1.2 applied to $\overline{X}$. This is the case (3). Assume therefore that $g(C) \ge 1$.

Let us denote by $\overline{B}_j$ (resp. $\overline{B}$) the images of B_j (resp. B) on $\overline{X}$, and omit B_j's (and $\overline{B}_j$'s) with $b_j = 0$ in the rest of this proof.

Case 1. There exists a curve $\overline{D}$ on $\overline{X}$ with $\overline{D}^2 < 0$.
By Lemma 1.2 $\overline{D}$ is a nonsingular rational curve. Since it does not lie in a fiber of $\overline{X} \to C$, $g(C) = 0$. So this does not occur.

Case 2. $\overline{D}^2 \ge 0$ for all curves $\overline{D}$ on $\overline{X}$. By $\overline{B} \equiv -K_{\overline{X}}$, we have

$$0 \ge 8 - 8g(C) = K_{\overline{X}}^2 = \overline{B}^2 \ge 0.$$

It follows that all $\overline{B}_j^{\,2} = K_{\overline{X}}\overline{B}_j = \overline{B}_j\overline{B}_k = 0$ and that $g(C) = 1$. By the arithmetic genus formula, $p_a(\overline{B}_j) = 1$. The normalization of each $\overline{B}_j$ is irrational, otherwise C would be rational too. Hence, each $\overline{B}_j$ is a nonsingular elliptic curve, and different $\overline{B}_j$ do not intersect. It is then easy to see that $g : X \to \overline{X}$ is an isomorphism by Remark 1.5 and Lemma 1.6.

Let S be a section of $\pi : \overline{X} \to C$ such that S^2 (≥ 0) is the smallest. From the standard exact sequence

$$0 \to \mathcal{O}_C \to \pi_*\mathcal{O}(S) \to \mathcal{O}_S(S) \to 0,$$

we see that $h^0(S) \ge S^2$. We claim that $S^2 \le 2$. Indeed if $S^2 > 2$ then $|S - F| \neq \emptyset$ by $h^0(\mathcal{O}_F(S)) = h^0(\mathcal{O}_F(1)) = 2$, which would produce a section S' with $(S')^2 < S^2$. Thus $S^2 \le 2$ and we are done by $X \simeq \mathbb{P}(\pi_*\mathcal{O}(S))$.

□

Th next two results are technical remarks for the reader's convenience.

Lemma 1.6. *Let $\sigma_P : Y \to \overline{X}$ be a blow up of a nonsingular surface $\overline{X}$ at $P \in \overline{X}$ and let $\overline{B} \subset \overline{X}$. Then*

$$\sigma_P^*(K_{\overline{X}} + \overline{B}) = K_Y + \sigma_{P*}^{-1}(\overline{B}) + (m_P(\overline{B}) - 1)E,$$

where E is the exceptional divisor and $m_P(\overline{B})$ is the multiplicity of $\overline{B}$ at P. Furthermore $m_Q(\sigma_{P*}^{-1}(\overline{B})) \le m_P(\overline{B})$ for every $Q \in E$.

Proof. The formula is a direct computation. The second assertion follows from $E \cdot \sigma_{P*}^{-1}(\overline{B}) = m_P(\overline{B})$. □

The last assertion can be used to switch the order of two successive blow ups if $m_i < m_{i+1}$ in the next corollary.

Corollary 1.7. *Let $g : X \to \overline{X}$ be a composition of point blow ups of nonsingular surfaces. Let $\overline{B} \subset \overline{X}$. Then we can decompose g as*

$$g : X = Y_{r+1} \to Y_r \to \cdots \to Y_2 \to Y_1 = \overline{X}$$

so that $Y_{i+1} \to Y_i$ is the blow up at $P_i \in Y_i$ and the multiplicities m_i at P_i of the birational transform of $\overline{B}$ to Y_i satisfy the condition $m_1 \ge m_2 \ge \cdots \ge m_r$.

Here is the main result of this section.

Theorem 1.8. *Let X be a nonsingular projective surface and $B = \sum b_j B_j$ be an $\mathbb{R}$-divisor on X with $0 \le b_j \le 1 - \varepsilon < 1$. Assume that $K + B \equiv 0$ and $\varepsilon < 1/\sqrt{3}$. Then the following are true:*

1. $\rho(X) \le 128/\varepsilon^5$,
2. *if, in addition, $b_j \ge \delta > 0$ then $\rho(X) \le \max(22, 8/\varepsilon^3\delta^2)$.*

Proof. By (1.4) we can assume that X is rational and there exists a birational morphism g from X to $\overline{X} = \mathbb{F}_n$. As before, denote by $\overline{B}_j$ the images $g(B_j)$.

We fix a positive number ν_1 and divide g into two parts: $g_1 : X_1 \to \overline{X}$ and $g_2 : X \to X_1$ by Corollary 1.7. The morphism g_1 is a composition of blowups at points P where the birational transform of $\overline{B}$ has multiplicity $\operatorname{mult}_P \overline{B} = \sum \bar{b}_j \operatorname{mult}_P \overline{B_j} \ge \nu_1$, and the morphism g_2 is a composition of blowups at points with multiplicity $< \nu_1$. Let $\nu_2 \in [0, \nu_1]$ be such that for blowups of g_2 we have all the multiplicities $\le \nu_2$. Later the values ν_1, ν_2 will be specified in such a way that $\nu_2 < \varepsilon$.

We first bound $\rho(X_1/\overline{X})$. At each blowup the expression $(\sum \bar{b}_j \overline{B}_j)^2$ decreases by at least ν_1^2. On X one has

$$\left(\sum \bar{b}_j \overline{B}_j\right)^2 \ge \sum \bar{b}_j^{\,2} \overline{B}_j^{\,2} \ge$$
$$\left(\sum \bar{b}_j^{\,2}\right)(-2/\varepsilon) \ge \left(\sum \bar{b}_j\right)(1-\varepsilon)(-2/\varepsilon)$$

Lemma 1.3 immediately implies that if $-\left(K_{\mathbb{F}_n} + \sum \bar{b}_j \overline{B}_j\right)$ is nef, then $\sum \bar{b}_j \le 4$. Therefore, on the surface X

$$\left(\sum \bar{b}_j \overline{B}_j\right)^2 \ge 4(1-\varepsilon)(-2/\varepsilon) = 8 - \frac{8}{\varepsilon}$$

On the other hand, on the surface $\overline{X}$:

$$\left(\sum \bar{b}_j \overline{B}_j\right)^2 = K_{\mathbb{F}_n}^2 = 8$$

We conclude that

$$\rho\left(X_1/\overline{X}\right) \le 8/\varepsilon\nu_1^2.$$

We next bound $\rho(X/X_1)$. Let Y be an arbitrary intermediate blowup of $X \to X_1$. We write the image B_Y of B as $B_Y = \overline{B}_Y + F_Y$, where $\overline{B}_Y$ is the birational transform of $\overline{B}$ and F_Y is the sum of curves which are exceptional for $X_1 \to \overline{X}$. Let F_i be an irreducible component of F_Y, and the corresponding coefficients in $K_Y + B_Y$ by f_i. Note that F_Y is a simple normal crossing divisor. Since $m_P(\overline{B}_Y) \le \nu_2 < \varepsilon$ and $f_i < 1-\varepsilon$, the blowups of $X \to X_1$ are the blowups at the nodes of $\operatorname{Supp} F_Y$ by Lemma 1.6. Let $P = F_1 \cap F_2$ be one of such points. We have $f_1, f_2 \le 1-\varepsilon$, $\operatorname{mult}_P \overline{B} \le \nu_2$. The coefficient of the new curve F_3 appearing after the blowup will be at most $(1-\varepsilon) - (\varepsilon - \nu_2)$. So, by $\nu_2 < \varepsilon$, the new coefficients f_i appearing at future blowups are strictly decreasing. At the same time they all have to be nonnegative by Remark 1.5, so the number of blowups over P can be easily bounded. The following two lemmas were suggested to us by J.Kollár.

Lemma 1.9. *The number of blowups in the case $f_1 = 1-a$, $f_2 = 1-b$ and arbitrary $\nu_2 (< \varepsilon)$ is at most $1/(a-\nu_2)(b-\nu_2) - 1$.*

Proof. Easy by descending induction. Note that

$$\frac{1}{(a-\nu_2)(b-\nu_2)} - 1 =$$
$$\left(\frac{1}{(a+b-2\nu_2)\,(a-\nu_2)} - 1\right) + \left(\frac{1}{(a+b-2\nu_2)\,(b-\nu_2)} - 1\right) + 1$$

Thus the estimate for the number at $P = F_1 \cap F_2$ is reduced to those for $F_1' \cap F_3$ and $F_2' \cap F_3$. Thus by Remark 1.5, it is enough to check the lemma in the case where no further blowups are allowed. In this case, we have $a-\nu_2, b-\nu_2 \in (0,1]$ and hence the lemma holds by $0 \le 1/(a-\nu_2)(b-\nu_2) - 1$. □

Applying these lemmas we see that

$$\rho\left(X/X_1\right) \le \left(1/\left(a-\nu_2\right)\left(b-\nu_2\right) - 1\right) \times N,$$

where N is the number of nodes of $\cup F_i$ on X_1, and $N \le \rho(X_1/\overline{X}) - 1$.

Adding $\rho(X_1/\overline{X})$ and $\rho(X/X_1)$ together we obtain the following estimate

$$\begin{aligned} \rho(X) \quad &\le \tfrac{8}{\varepsilon\nu_1^2} + \left(\tfrac{8}{\varepsilon\nu_1^2} - 1\right) \times \left(\tfrac{1}{(\varepsilon-\nu_2)^2} - 1\right) + 2 \\ &= \tfrac{8}{\varepsilon\nu_1^2(\varepsilon-\nu_2)^2} + \left(3 - \tfrac{1}{(\varepsilon-\nu_2)^2}\right) \le \tfrac{8}{\varepsilon\nu_1^2(\varepsilon-\nu_2)^2} \end{aligned}$$

by $0 < \varepsilon - \nu_2 \le \varepsilon \le 1/\sqrt{3}$.

For the statement (1) of the theorem we take $\nu_1 = \nu_2 = \varepsilon/2$. For (2) we can take $\nu_1 = \delta$ and $\nu_2 = 0$ because in this case $m_P(\overline{B}) < \delta$ means $m_P(\overline{B}) = 0$. □

Corollary 1.10. *Let X be a projective surface and $B = \sum b_j B_j$ be an $\mathbb{R}$-divisor on X. Let ε be a real number such that $0 < \varepsilon < 1/\sqrt{3}$. Assume that the pair (X, B) is mrs ε-lc and that $-(K_X + B)$ is ample. Denote by $\pi : \widetilde{X} \to X$ the minimal resolution of singularities. Then*

$$\rho\left(\widetilde{X}\right) \leq 128/\varepsilon^5.$$

Proof. Ampleness is an open condition, therefore changing the coefficients b_j slightly we can assume that they are rational, rather than only real, numbers. Let D be a general member of the linear system $-N(K_X + B)$ for a large divisible N. Then $K_X + B + (1/N)D \equiv 0$ and the pair $(X, B + (1/N)D)$ is again ε-lc.

Now apply (1.8.1) to

$$K_{\widetilde{X}} + B^{\widetilde{X}} = \pi^* \left(K_X + B + (1/N)\, D\right).$$

Remark 1.11. Theorem 1.8 was proved in [3] 6.3 in more general situation, when $-(K+B)$ is only nef instead of being numerically trivial. The explicit bound was not given. It can be deduced from [3] but turns out to be worse than that of (1.8). More importantly, in [3] it was proved using a much more combinatorially involved technique: the Diagram Method. The Diagram Method was also applied for proving bounds for Picard numbers of surfaces that do not follow in any obvious way from (1.8), see for example [1]. In other situations where both techniques are applicable the Diagram Method sometimes produces better bounds.

2. DCC sets and Shokurov's Log Adjunction

2.1. In this section, we will explain DCC sets and Shokurov's log adjunction and prove an application.

Below we give the necessary definitions and list several facts about sets satisfying the descending chain condition (DCC).

Definition 2.2. A subset $\mathcal{A}$ of $\mathbb{R}$ is said to satisfy the descending chain condition if any strictly decreasing subsequence of elements of $\mathcal{A}$ is finite. We also say that $\mathcal{A}$ is a DCC set.

Lemma 2.3. *Let $\mathcal{A} \subset \mathbb{R}$ be an arbitrary subset. Then $\mathcal{A}$ satisfies the DCC iff every infinite sequence $\{a_n\}$ of elements of $\mathcal{A}$ contains an infinite nondecreasing subsequence.*

Proof. Elementary. □

Definition 2.4. A sum of n sets $\mathcal{A}_1, \mathcal{A}_2 ... \mathcal{A}_n$ is defined as

$$\sum_{i=1}^{n} \mathcal{A}_i = \{a_1 + a_2 + ... a_n \,|\, a_i \in \mathcal{A}_i\}$$

Next, we define

$$\mathcal{A}_\infty = \{0\} \cup \bigcup_{n=1}^{\infty} \sum_{i=1}^{n} \mathcal{A}$$

If each $\mathcal{A}_i$ satisfies DCC, then so does $\sum_{i=1}^n \mathcal{A}_i$. The proof immediately follows from Lemma 2.3. If, in addition, $\mathcal{A}$ consists only of nonnegative numbers then clearly $\mathcal{A}_\infty$ also satisfies DCC since positive numbers in a DCC set have a minimum.

Definition 2.5. For a set $\mathcal{A} \subset [0,1]$ we define the *derivative set*

$$\mathcal{A}' = \{\frac{n-1+a_\infty}{n} | n \in \mathbb{N}, a_\infty \in \mathcal{A}_\infty \cap [0,1]\} \cup \{1\}$$

It is easy to see that if $\mathcal{A}$ is a DCC set then so is $\mathcal{A}'$.

2.6. The derivative set appears very naturally in the following situation.

Lemma 2.7 (Shokurov's Log Adjunction Formula). *Let X be a projective surface and $B = B_0 + \sum b_j B_j$ be an $\mathbb{R}$-divisor on X. Assume that the pair (X,B) is lc. Denote by $\pi : \widetilde{X} \to X$ the minimal resolution of singularities and by $\widetilde{B}_0$ the birational transform $\pi_*^{-1}(B_0)$. Then*

1. *there exists a natural formula*
$$\pi^* (K+B)|_{\widetilde{B}_0} \equiv K_{\widetilde{B}_0} + \sum d_k P_k,$$
where P_k are nonsingular points on $\widetilde{B}_0$ and
$$d_k = 1 \quad or \quad \frac{n_k - 1 + \sum_j a_{j,k} b_j}{n_k}$$
for some integers $n_k > 0$ and $a_{j,k} \geq 0$. Thus if $b_j \in \mathcal{A}$ for all j then $d_k \in \mathcal{A}'$.
2. *$K_{\widetilde{B}_0} + \sum_k d_k P_k$ is lc.*
3. *If $P_k \in \widetilde{B}_0 \cap \pi^{-1}(B_\ell)$, then $d_k = (n_k - 1 + \sum_j a_{j,k} b_j)/n_k$ and $a_{\ell,k} > 0$.*

Proof. It follows from the classification of log canonical surface singularities that $\widetilde{B}_0$ intersects the exceptional divisors F_i of π transversally. Now using the adjunction formula for the divisor $\widetilde{B}_0$ on $\widetilde{X}$ we see the existence of the above formula with some $0 \leq d_k \leq 1$. The precise form of the coefficients is an exercise in linear algebra. It can be found e.g. in [11]. □

The essential case of $\rho(X) = 1$ in following result holds true in all dimensions in characteristic 0 ([6],18.24). Here we derive it as a corollary to (2.7).

Lemma 2.8. *Let X be a normal projective $\mathbb{Q}$-factorial surface such that (X,B) is lc and $K+B \equiv 0$, where $B = \sum_j b_j B_j$. Then $\sum b_j \leq \rho(X) + 2$.*

Proof. Set $\sum(X,B) = \sum b_j - \rho(X)$. Assume that $B \neq 0$. Since $K_X \equiv -B \neq 0$, we can apply K-MMP to get $g : X \to Y$ such that $\sum(X,B) \leq \sum(Y, g_*B)$ and K_Y is lc and such that

1. $\rho(Y) = 2$ and there is a surjective morphism $\pi : Y \to C$ such that a general fiber F is $\mathbb{P}^1$, or
2. $\rho(Y) = 1$ and $-K_Y$ is ample.

(Since $-B \equiv K_X \succ g^* g_* K_Y$, we have $g_* B \neq 0$ and hence we have the above cases.) We treat the two cases.

Case 1. This case is reduced to two easy cases.

Subcase 1. *Supp $g_* B$ contains no fibers of π.* By the adjunction, we have

$$0 = (K + B) \cdot F = -2 + \sum b_j (B_j \cdot F) \geq -2 + \sum b_j.$$

Subcase 2. *Supp $g_* B \supset F_0$, a fiber of π.* Then by $(K_X + g_* B) \cdot F = 0$, we have $g_* B \cdot F_0 > 0$ and hence $\delta g_* B + F_0$ is ample if $0 < \delta \ll 1$. Then $D = K + (1 - \delta^2) g_* B - \delta F_0$ is lc and $-D$ is ample. Thus Y has two D-extremal rays R_1 and R_2 such that π contracts R_1. If the contraction $\rho : Y \to Z$ of R_1 is of fiber type, then let G be a general fiber of ρ and we have $\sum b_j \leq 4$ by $(K + B) \cdot (F + G) = 0$. If ρ is birational, then we have $\sum(Y, g_* B) \leq \sum(Z, \rho_* b_* B)$. So this case is reduced to the next case.

Case 2. *X is $\mathbb{Q}$-factorial, $\rho(X) = 1$, K_X is lc, $-K_X$ is ample.*
We treat two cases.

Subcase 1. *There exists an i (say $i = 1$) such that $b_i = 1$.* We apply Lemma 2.7 to $B_1 \subset X$, and get

$$0 \geq -2 + \sum d_k \geq -2 + \sum_{j \neq 1} b_j,$$

which proves $\sum b_j \leq 3$.

Subcase 2. *$b_j < 1$ for all j.* By renumbering $B_1, B_2, \cdots$, we may assume $(-K \cdot B_i)$ is non-decreasing in i. We try to replace B_i $(i \geq 2)$ with a multiple of B_1 keeping $K + B \equiv 0$. By doing so, $\sum(X, B)$ does not decrease and it is enough to prove $\Sigma(X, B) \leq 2$ after the replacement. One of the following changes occurs.

1. The number of irreducible components of Supp B decreases.
2. b_1 becomes 1.
3. There is a birational morphism $f : Y \to X$ such that $E = f^{-1}(P)_{red}$ is an irreducible divisor, f induces $Y - E \simeq X - \{P\}$ and
$$f^*(K_X + B) = K_Y + f_*^{-1} B + E \equiv 0.$$

The first case is settled by the induction and the second is already treated in the above subcase. So we only need to treat the third case. If we set $D = K_Y + f_*^{-1} B + (1 - \delta) E$ with $0 < \delta \ll 1$, then D is lc, $D \equiv -\delta E$ and Y has a D-extremal ray R. Since $(D \cdot E) > 0$, the contraction $g : Y \to Z$ of R does not contract E. Since $K_Z + g_* f_*^{-1} B + g_* E \equiv 0$ is lc, Z is treated in the above subcase.

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3. Chains of coefficients

3.1. The main purpose of this section is to prove the following, which is an effective version of (5.3) in [2].

Theorem 3.2. *Let $\mathcal{A} \subset [0,1]$ be an arbitrary DCC set. Then there exists a constant $\Delta = \beta(\mathcal{A})$ depending only on $\mathcal{A}$ (defined below) so that the following holds. Let X be a normal projective surface, B_j be divisors on X, and let b_j, x_j be positive real numbers. Assume that*

1. *X is a (singular) $\mathbb{Q}$-factorial Del Pezzo surface (i.e. $-K_X$ is ample) and $\rho(X) = 1$,*
2. *$b_j > 0$ and $b_j \in \mathcal{A}$,*
3. *$1 - \Delta < x_j \leq 1$,*
4. *at least one x_j is strictly less than 1,*
5. *the pair $(X, \sum x_j b_j B_j)$ is lc.*

Then the divisor $K_X + \sum x_j b_j B_j \not\equiv 0$.

3.3. We first make a basic computation for the derivative set. Then we introduce the functions necessary to define $\beta(\mathcal{A})$ used in the theorem.

Lemma 3.4 (Basic computation for the derivative set). *Let*

$$d_i = \frac{n_i - 1 + \sum n_{i,j} b_j}{n_i}$$

be an element of the derivative set $\mathcal{A}'$. Assume that all $b_j > 0$ and that they appear in d_i with positive coefficients. Consider real numbers x_j with $1 - \Delta < x_j \leq 1$, with at least one x_j being strictly less than 1. Then

$$(1 - \Delta)\, d_i < \frac{n_i - 1 + \sum n_{i,j} x_j b_j}{n_i} < d_i$$

Proof. Evident. $\square$

3.5. We now introduce the functions that will be used in the statements below. For a subset $\mathcal{A} \subset \mathbb{R}$ and $t \in \mathbb{R}$, we set $\mathcal{A}_{>t} = \{x \in \mathcal{A} | x > 0\}$. We assume everywhere that $\mathcal{A}$ satisfies the DCC and contains only nonnegative numbers. t is a nonnegative real number.

$$mf_1\,(\mathcal{A}, t) = \min(cA_\infty)_{>t} - t$$

$$mf_2\,(\mathcal{A}, t) = \min\{x > 0 \mid (1-x)\, a_\infty = t,\, a_\infty \in \mathcal{A}_\infty\} = \frac{mf_1\,(\mathcal{A}, t)}{t + mf_1\,(\mathcal{A}, t)}$$

We set by definition $mf_1(\emptyset, t) = mf_2(\emptyset, t) = +\infty$.

$$\alpha\,(\mathcal{A}, \varepsilon) = \alpha\,(\varepsilon) = mf_2\left(\mathcal{A} \cup \{1\}, [2/\varepsilon]^{[128/\varepsilon^5]} \times ([2/\varepsilon] + 2)^2\right)$$
$$\beta\,(\mathcal{A}) = \alpha\,(\alpha\,(\alpha\,(\alpha\,(mf_2\,(\mathcal{A}', 2)))))$$

The following are the very basic properties of the functions we introduced above.

Lemma 3.6. *Assume that a DCC set $\mathcal{A}$ contains 1. Then the following hold.*

1. $0 < mf_1(\mathcal{A}, t) < 1$ *and* $0 < mf_2(\mathcal{A}, t) < 1/(1+t)$ *for all* $t > 0$.
2. *If* $t_1 - t_2 \in \mathbb{N}$, *then* $mf_k(\mathcal{A}, t_1) \le mf_k(\mathcal{A}, t_2)$ *for* $k = 1, 2$.
3. *Let* $a, \Delta > 0$ *and* $\lambda \ge 0$ *be such that* $\Delta \le mf_2(\mathcal{A}, a)/(1+\lambda)$. *Then for any finite number of arbitrary* $b_j, x_j, y \in \mathbb{R}$ *such that* $0 \le y \le \Delta$, $0 < b_j \in \mathcal{A}$ *and* $1 - \Delta < x_j \le 1$ *for all* j, $x_j < 1$ *for some* j, *we have* $\sum x_j b_j \ne a + a\lambda y$.
4. *If* $\varepsilon < 1$, *then* $\alpha(\varepsilon) < \varepsilon^{128}/16$.

Proof. (1) is obvious. By the definition of mf_1, we have $mf_1(\mathcal{A}, t+1) \le mf_1(\mathcal{A}, t)$, whence

$$mf_2(\mathcal{A}, t+1) = \frac{mf_1(\mathcal{A}, t+1)}{t + 1 + mf_1(\mathcal{A}, t+1)} \le \frac{mf_1(\mathcal{A}, t)}{t + mf_1(\mathcal{A}, t)} = mf_2(\mathcal{A}, t).$$

Thus (2) is proved. For (3), assume that $\sum x_j b_j = a + a\lambda y$. Then $a + a\lambda\Delta > (1-\Delta)\sum b_j$ by $y \le \Delta$ and $x_j > 1 - \Delta$. Thus

$$\Delta\left(\sum b_j + a\lambda\right) > \sum b_j - a.$$

By $x_j \le 1$ for all j and $x_j < 1$ for some j, we have $\sum b_j > a + a\lambda y \ge a$. Whence

$$\Delta \cdot (1+\lambda) \sum b_j > \sum b_j - a > 0,$$

and $\Delta > mf_2(\mathcal{A}, a)/(1+\lambda)$, a contradiction. It remains to prove (4). By (1), we have $mf_2(\mathcal{A}, t) < 1/t$ and hence

$$\alpha(\varepsilon) < \frac{1}{16 \cdot [2/\varepsilon]^{128}} < \frac{1}{16}\varepsilon^{128},$$

by $[2/\varepsilon] \ge 2/\varepsilon - 1 > 1/\varepsilon$. □

The following two results treat easy cases of (3.2).

Lemma 3.7. *In the situation of (3.2) assume, in addition, that* $\varepsilon < 1/\sqrt{3}$, *that the pair* $(X, 0)$ *is mrs* ε*-lc for a fixed* $\varepsilon > 0$ *and that* $\Delta = \alpha(\mathcal{A}, \varepsilon)$. *Then* $K + \sum x_j b_j B_j \not\equiv 0$.

Proof. Let $\pi : \widetilde{X} \to X$ be the minimal resolution of singularities of X and F_i be the corresponding exceptional curves. For a sufficiently divisible $m > 1/\varepsilon$, $-mK_X$ is very ample and let $D \in |-mK_X|$ be a general member so that D is nonsingular and disjoint from the singular locus of X. The log divisor

$$\pi^*(K_X + D) = K_{\widetilde{X}} + \sum f_i F_i + \frac{1}{m}\pi^* D$$

has nonnegative coefficients f_i and it is numerically trivial. The additional condition of our lemma means precisely that all $f_i \le 1 - \varepsilon$. Therefore, the pair $\left(\widetilde{X}, \pi^* D/m + \sum f_i F_i\right)$ belongs to one of the types listed in Lemma 1.4.

Since $\sum x_j b_j B_j \ne \emptyset$, it cannot be a surface as in (1.4.1). If it is in (1.4.2), then we restrict the numerically trivial divisor $K_X + \sum x_j b_j B_j$ to a general fiber F of the $\mathbb{P}^1$-fibration to get $\sum x_j b_j (B_j \cdot F) = 2$. By (3.6.2) we have $\Delta \le mf_2(A \cup \{1\}, 2)$, and by (3.6.3) we have a contradiction. Therefore we can assume that X is rational and there is a birational morphism $\widetilde{X} \to \mathbb{F}_n$, $n \le 2/\varepsilon$.

Let t be a positive integer that makes $-tK_X$ into an ample Cartier divisor. Then

$$-tK_X(K_X + \sum x_j b_j B_j) = 0$$

and, consequently,

$$\sum \big(- tK_X B_j \big) x_j b_j = tK_X^2 \tag{1}$$

The coefficients tK_X^2, $-tK_X B_j$ in the latter formula are positive integers.

The integer t is bounded from above by the determinant of the matrix $(-F_i F_k)$. This square matrix has the dimension at most $\rho(\widetilde{X}) < 128/\varepsilon^5$ by corollary (1.10) applied with $B = \emptyset$. Each diagonal element is positive and bounded by $2/\varepsilon$ by Lemma 1.2, other elements are non-positive. It follows that

$$t \le [2/\varepsilon]^{[128/\varepsilon^5]}$$

We also have the following bound for K_X^2:

$$K_X^2 \le ([2/\varepsilon] + 2)^2$$

Indeed, $\widetilde{X}$ has a birational morphism to $\mathbb{F}_n$ with $n \le 2/\varepsilon$. The preimages of the curves in the linear system $|s_n + nf|$ on $\mathbb{F}_n$ form a free system of curves $\{C_t\}$ on $\widetilde{X}$. We have

$$-\pi^*\big(K_X + \sum x_j b_j B_j\big) C_t \le -K_{\widetilde{X}} C_t = -K_{\mathbb{F}_n}(s_n + nf) = n + 2$$

Therefore, by Fano's argument (see for example [8]) one has

$$\big(K_X + \sum x_j b_j B_j\big)^2 \le (n+2)^2 \le ([2/\varepsilon] + 2)^2$$

Thus

$$tK_X^2 \le [2/\varepsilon]^{[128/\varepsilon^5]} \times ([2/\varepsilon] + 2)^2 .$$

By (3.6.2), we have $\Delta \le mf_2(\mathcal{A}, tK_X^2)$ and (3.6.3) contradicts the equation (1). $\square$

Lemma 3.8. *In the situation of (3.2) assume that $\Delta = mf_2(\mathcal{A}', 2)$. Then the following hold.*

1. *If $x_j b_j = 1$ for some j then $K + \sum x_j b_j B_j \not\equiv 0$.*
2. *Let $P \in X$ and $J_1 = \{j | B_j \ni P\}$. If $x_a < 1$ for some $a \in J_1$ and if $x_j b_j < 1$ for all $j \in J_1$, then $K + \sum x_j b_j B_j$ is not maximally lc at P i.e. for some small $\mu > 0$ the divisor $K + \sum_{j \in J_1} (x_j b_j + \mu) B_j$ is still log canonical at P.*

Proof. Assume first that $x_0b_0 = 1$. We apply the log adjunction to B_0 and get $K_{\widetilde{B}_0} + \sum y_i d_i Q_i \equiv 0$, where $0 < d_i \in \mathcal{A}'$, $1 - \Delta < y_i \leq 1$ for all i and $y_b < 1$ for some b (3.4). Then $\deg K_{\widetilde{B}_0} = -2$ and $\sum y_i d_i = 2$ contradicting (3.6.3). This proves (1).

If $K_X + \sum x_j b_j B_j$ is maximally lc at P, then by (1) there exists a partial resolution $f : (Y, E) \to (X, P)$ such that the exceptional set of f is an irreducible curve E with log discrepancy 0, i.e.

$$f^*\left(K_X + \sum x_j b_j B_j\right) = K_Y + \sum x_j b_j f_*^{-1}(B_j) + E$$

Restricting this inequality to E and applying the log adjunction formula (2.7), one gets

$$K_{\widetilde{E}} + \sum y_i d_i Q_i \equiv 0,$$

where $0 < d_i \in \mathcal{A}'$, $1 - \Delta < y_i \leq 1$ for all i by (3.4). Furthermore for any Q_k lying over $E \cap f_*^{-1}(B_a)$ we have $y_k < 1$. So $\sum y_i d_i = 2$, and it is a contradiction by (3.6.3). □

The following is the key lemma for the proof of (3.2).

Lemma 3.9. *In the situation of (3.2) assume, in addition, that $b_0 = 1$ and that $\Delta = mf_2(\mathcal{A}', 2)/2$. Then if $K + \sum x_j b_j B_j \equiv 0$, then for the birational transform of B_0 on the minimal desingularization $\pi : \widetilde{X} \to X$ one has $\widetilde{B}_0^2 \geq 0$.*

Proof. First of all, $x_j b_j < 1$ for all j (3.8.1), and we see that the pair $(X, B_0 + \sum_{j>0} x_j b_j B_j)$ is lc. Otherwise, we would get a maximally lc surface $(X, y_0 B_0 + \sum_{j>0} x_j b_j B_j)$ with $x_0 < y_0 < 1$ contradicting the previous lemma 3.8.

We apply the log adjunction to $K + B_0 + \sum_{j>0} x_j b_j B_j \equiv (1 - x_0) B_0$ and get

$$\begin{aligned} 0 \leq (1 - x_0)\, B_0^2 &= \left(K_X + B_0 + \sum_{j>0} x_j b_j B_j\right)|_{B_0} \\ &\equiv K_{\widetilde{B}_0} + \sum y_i d_i Q_i, \qquad (2) \end{aligned}$$

where $0 < d_i \in \mathcal{A}'$, $1 - \Delta < y_i \leq 1$ for all i and $y_a < 1$ for some a by $\rho(X) = 1$ (3.4).

Note that we have $x_0 < 1$. If $\widetilde{B}_0^2 < 0$ then we claim that $\widetilde{B}_0$ is a (-1)-curve and $B_0^2 < 1/x_0$. Indeed,

$$0 > K_X B = \left(\pi^* K_X\right) \widetilde{B}_0 \geq K_{\widetilde{X}} \widetilde{B}_0.$$

Therefore, $\widetilde{B}_0$ is a (-1)-curve and the claim follows from

$$\begin{aligned} x_0 B_0^2 &= -\left(K_X + \textstyle\sum_{j>0} x_j b_j B_j\right) B_0 \\ &\leq -K_X B_0 \leq -K_{\tilde{X}} \tilde{B}_0 = 1. \end{aligned}$$

From the inequality (2) we see that

$$\sum y_j d_j = 2 + (B_0^2) \cdot (1 - x_0)$$

and

$$(B_0^2) \cdot (1 - x_0) \leq (1 - x_0)/x_0 \leq 2(1 - x_0)$$

by $x_0 > 1 - \Delta > 1/2$. This contradicts (3.6.3). □

Proof. Proof of (3.2) Set $\varepsilon_n = \alpha^n(mf_2(\mathcal{A}', 2))$ for $n \geq 0$. We note $\varepsilon_n \leq \varepsilon_0 \leq 1/3$ by (3.6.1).

Let $n \geq 1$ and assume that a surface (X, B_j, b_j, x_j) with $K + \sum b_j x_j B_j \equiv 0$ and $\mathrm{Min}\{x_j\} > 1 - \varepsilon_n$ as in (3.2) exists (suppressing $\Delta = \varepsilon_n$). We will keep variable n for the clarity of argument and set $n = 4$ at the end of the proof.

Then by (3.7) and $\varepsilon_0 \leq 1/3$, $(X, 0)$ is not mrs ε_{n-1}-lc. Thus let $f : Y \to X$ be a partial resolution $f : (Y, E) \to (X, P)$ dominated by the minimal desingularization for which the exceptional set of f is an irreducible curve E with the minimal log discrepancy

$$a(E) = a(E, K_X + \sum x_j b_j B_j) \leq a(E, K_X) < \varepsilon_{n-1}.$$

We have

$$f^*\left(K_X + \sum x_j b_j B_j\right) = K_Y + \sum x_j b_j B_j + (1 - a\,(E))\,E$$

The Picard number $\rho(Y) = 2$, and there exists a second extremal ray $g : Y \to X_1$. For the new curve $B_0 = E$ set $b_0 = 1$ and $x_0 = 1 - a(E)$. The morphism g can collapse some curves. However, we claim that there will be a component of $\bigcup B_j \cup E$ with $x_j < 1$ which is not contracted to a point. Indeed, if $a(E) \neq 0$, this is the curve E. If one has $a(E) = 0$ then the the corresponding pair is maximally lc. Lemma 3.8 guarantees that in this case a curve with $x_j < 1$, for example B_1, does not pass through P, because $mf_2(\mathcal{A}', 2) > \Delta$ by (3.6.4). Therefore the preimage of B_1 on Y has a positive self-intersection, and this curve is not contracted by the morphism g.

We claim that $\dim X_1 = 2$. Otherwise the morphism g is a generically $\mathbb{P}^1$-fibration. Restricting the divisor

$$K_Y + \sum x_j b_j B_j + \big(1 - a(E)\big)E \equiv 0$$

on a general fiber, we get a contradiction because

$$mf_1(\mathcal{A} \cup \{1\}, 2) > mf_2(\mathcal{A}', 2) > \Delta.$$

Thus from a surface

$$(X, B_j, b_j, x_j) \text{ with } K + \sum b_j x_j B_j \equiv 0 \text{ and } \min\{x_j\} > 1 - \varepsilon_n$$

as in (3.2), we obtained another

$$(X^{(1)}, B_j^{(1)}, b_j^{(1)}, x_j^{(1)}) \text{ with } K^{(1)} + \sum b_j^{(1)} x_j^{(1)} B_j^{(1)} \equiv 0 \text{ and } \min\{x_j^{(1)}\} > 1 - \varepsilon_{n-1}$$

as in (3.2) such that $X^{(1)}$ is dominated by the minimal resolution $\widetilde{X}$ of X, $b_1^{(1)} = 1$ and $(\widetilde{B}_1^{(1)})^2 \geq 0$.

If we have such an $(X^{(a)}, B_j^{(a)}, b_j^{(a)}, x_j^{(a)})$, we can obtain the next as long as $a < n$ (so that $\min\{x_k^{(a)}\} > 1 - \varepsilon_1$).

The point in the procedure is that we have a sequence of birational morphisms $\widetilde{X} \to \widetilde{X}^{(1)} \to \cdots \to \widetilde{X}^{(a)} \to X^{(a+1)}$ so that the birational transform of an arbitrary curve $E \subset X^{(i)}$ $(i \leq a)$ with $(\widetilde{E}^2) \geq 0$ is a curve on $X^{(a+1)}$. Thus we have obtained

$$(X^{(n)}, B_j^{(n)}, b_j^{(n)}, x_j^{(n)}) \text{ with } K^{(n)} + \sum b_j^{(n)} x_j^{(n)} B_j^{(n)} \equiv 0$$

such that $b_i^{(n)} = 1$ $(i = 1, \cdots, n)$, $x_i^{(n)} > 1 - \varepsilon_1 > 1 - 1/128$ $(i = 1, \cdots n-1)$, and $1 - x_n^{(n)} > 1 - \varepsilon_0 \geq 2/3$. On the other hand,

$$3 \geq \sum b_i^{(n)} x_i^{(n)} \geq \sum_{i=1}^{n} b_i^{(n)} x_i^{(n)} > (1 - \frac{1}{128})(n-1) + \frac{2}{3}$$

by (2.8). Now we set $n = 4$ and get a contradiction.

Thus if we set $\Delta = \varepsilon_4 = \beta(\mathcal{A})$, then (3.2) holds. ☐

4. A lower bound for $(K+B)^2$ and the boundedness

In this section, we give versions of a few theorems of [3]. Among them, (4.7) and (4.12) are the main results.

We begin with a result from the log minimal model theory of surfaces, followed by two easy lemmas.

Lemma 4.1. *Assume that $K + B$ is lc and big on a normal $\mathbb{Q}$-factorial projective surface X. Let $B' \prec B$ be an effective $\mathbb{R}$-divisor, and let $t_0 > 0$ be the largest real number such that $B - t_0 B' \succ 0$. Then one of the following holds true.*

1. *$K + B - t_0 B'$ is big,*
2. *there exists $t_0' \in (0, t_0]$ such that $K + B - xB'$ is big iff $x < t_0'$. There exists a birational morphism $f : X \to X'$ to a normal $\mathbb{Q}$-factorial projective surface X' such that $D = f_*(K + B - t_0' B_0')$ is nef and $K + B - t_0' B_0' \succ f^* D$, and one of the following holds.*
   (a) *($(D \not\equiv 0$ or $f_*(B - t_0' B') \neq 0)$ and $\rho(X') \geq 2$) There exist a morphism $\pi : X' \to C$ onto a nonsingular projective curve C and an $\mathbb{R}$-divisor L on C such that deg $L \geq 0$ and $D \equiv \pi^* L$ and such that every irreducible curve $G \subset f_*(B - t_0' B')$ in a fiber of π satisfies $G = \pi^{-1}\pi(G)$ as sets.*
   (b) *($(D \not\equiv 0$ or $f_*(B - t_0' B') \neq 0)$ and $\rho(X') = 1$) $-K_{X'}$ is ample and $D \equiv 0$.*
   (c) *($D \equiv 0$ and $f_*(B - t_0' B') = 0$).*

Furthermore if we have $t_0' < t_0$ in Case (a), then a general fiber of π is $\mathbb{P}^1$.

Lemma 4.2. *Let $g : X \to Y$ be a birational morphism of normal projective surfaces and let $B = \sum_{j \in J} b_j B_j \succ \overline{B} \succ 0$ be effective $\mathbb{R}$-divisors on X. Assume that*

1. *$K_X + B$ is big and lc,*
2. *$g_*(K_X + \overline{B})$ is nef and lc,*
3. *$K_X + \overline{B} \succ g^* g_*(K_X + \overline{B})$.*

Let $C_{>0}$ and $C_{<0}$ be effective $\mathbb{R}$-divisors without common components such that $K_X + C_{>0} - C_{<0} = g^* g_*(K_X + \overline{B})$, and let $J_1 = \{j | B_j \not\subset \operatorname{Supp} C_{<0}\}$. Then $K_X + \sum_{j \in J_1} b_j B_j$ is big and lc.

Proof. First we note that the irreducible components of $\operatorname{Supp} B$ are all $\mathbb{Q}$-Cartier because $K + B$ is lc. Since the sum of a big divisor and a nef divisor is big,

$$\mu(K + B) + (1 - \mu) g^* g_*(K + \overline{B}) = K + \mu B + (1 - \mu) C_{>0} - (1 - \mu) C_{<0}$$

is big for all real $\mu \in (0, 1]$. If $0 < \mu \ll 1$, then

$$K + \mu B + (1 - \mu) C_{>0} - (1 - \mu) C_{<0} \prec K + \sum_{j \in J_1} b_j B_j.$$

□

Lemma 4.3. *Let $f : X \to X'$ be a birational morphism of a nonsingular projective surface X to a normal surface X' and let $C_{>0}$ and $C_{<0}$ be two effective $\mathbb{R}$-divisors without common components such that $f_* C_{<0} = 0$, $f_*(K + C_{>0})$ is $\mathbb{Q}$-Cartier, and $K + C_{>0} - C_{<0} = f^* f_*(K + C_{>0})$. Then f factors through a birational morphism $g : X \to Y$ to a nonsingular surface Y such that $\operatorname{Supp} C_{<0} = \operatorname{Exc}(g)$, the exceptional set of g.*

Proof. It is enough to prove that $\operatorname{Supp} C_{<0}$ contains a (-1)-curve if $C_{<0} \neq 0$. We have

$$C_{<0} \cdot K = (C_{<0})^2 - (C_{>0}) \cdot (C_{<0}) \leq (C_{<0})^2 < 0.$$

Hence there exists an irreducible component E of $C_{<0}$ such that $E \cdot K < 0$. Since $E^2 < 0$ by $f_* E = 0$, E is a (-1)-curve. □

The following is an effective version of (7.4) of [3].

Theorem 4.4. *For $\theta, \varepsilon > 0$, let $N = [128/\varepsilon^5 + 4/\theta]$ Then we have the following.*

Let $K + B = \sum_{j \in J} b_j B_j$ be big and lc on a nonsingular projective surface X such that $\theta \leq b_j \leq 1 - \varepsilon$ for all j. Then there is a subset $J' \subset J$ such that $|J'| \leq N$ and $K + \sum_{j \in J'} b_j B_j$ is big and lc.

Proof. We use the induction on the number of irreducible components of $\operatorname{Supp} B$. We apply Lemma 4.1 to subtract B_1 from B. In the case (4.1.1), we have $K + \sum_{j \neq 1} b_j B_j$ is big and lc, and we are done by the induction. So let $t'_0 \in (0, b_0]$ and a birational morphism $f : X \to X'$ be as in Case (4.1.2). In particular, we have $D = f_*(K + B - t'_0 B_0)$ is nef and $K + B - t'_0 \succ f^* D$. If we write $f^*f_*(K+B-t'_0B_0) = K+C_{>0}-C_{<0}$ as in (4.2), then $K+\sum_{j\in J_1} b_jB_j$ is big, where $J_1 = \{j | B_j \not\subset \operatorname{Supp} C_{<0}\}$. We treat two cases.

Case 1. $D \equiv 0$ (4.1.2). We will give a uniform bound of $|J_1|$. By Lemma 4.3, there exists a birational morphism $g : X \to Y$ to a nonsingular projective surface Y such that $\operatorname{Supp} C_{<0} = \operatorname{Exc}(g)$, the exceptional set of g. Let $h : Y \to X'$ be the induced morphism. Since

$$K_Y + g_*C_{>0} = g_*(K + C_{>0}) = h^*D \equiv 0$$

and since $C_{>0} \prec B$, we have $\rho(Y) \le 128/\varepsilon^5$ (1.8). By

$$J_1 = \{0\} \cup \{j|B_j : f\text{-exc., not } g\text{-exc.}\} \cup \{j|B_j \text{ not } f\text{-exc.}\},$$

we have

$$\begin{aligned} |J_1| &\le 1 + (\rho(Y) - \rho(X')) + (\rho(X') - 2 + \tfrac{4}{\theta}) \\ &= \rho(Y) + \tfrac{4}{\theta} - 1 \\ &\le \tfrac{128}{\varepsilon^5} + \tfrac{4}{\theta} - 1 \end{aligned}$$

by (2.8).

Case 2. There exist a surjection $\pi : X' \to C$ to a curve C and an $\mathbb{R}$-divisor L on C such that $\deg L > 0$ and $D \equiv \pi^*L$ (4.1.2). We treat two subcases.

Subcase 1. $(\pi \circ f)_*(B_j) \ne 0$ for every $j \in J_1$. Let F be a general fiber of $\pi \circ f$. By $(K + g_*C_{>0}) \cdot F = 0$, we have $-t'_0(B_0 \cdot F) + \sum b_j(B_j \cdot F) \le 2$. Hence $|J_1| \le 1 + 2/\theta$. We are done in this subcase.

Subcase 2. $B_1 \not\subset \operatorname{Supp} C_{<0}$ and $(\pi \circ f)_*(B_1) = 0$. Let $m_1 \in \mathbb{R}$ be such that $m_1B_1 \equiv D$ (4.1.2). If $m_1 > b_1$, then let $\overline{B} = B - t'_0B_0 - t'_1B_1$ for any $t'_1 \in (b_1, m_1)$. Then $g_*(K + \overline{B})$ is nef and $K_X + \overline{B} \succ g^*g_*(K_X + \overline{B})$. Then by (4.2), $K_X + \sum_{j\ne1} b_jB_j$ is big and we are done by induction. If $m_1 \le b_1$, then set $\overline{B} = B - t'_0B_0 - m_1B_1$. Then $g_*(K + \overline{B}) \equiv 0$ and we can use the same argument as Case 1 except that $|J_1| \le 128/\varepsilon^5 + 4/\theta$.

$\square$

4.5. We rephrase Theorem 3.2 using (4.1) into the following effective version of (7.5) of [3].

Theorem 4.6. *Let $\mathcal{A} \subset [0, 1]$ be an arbitrary DCC set such that $1 \in \mathcal{A}$. Let X be a normal projective surface, B_j be divisors on X, and let b_j be positive real numbers. Assume that*

1. *$b_j \in \mathcal{A}$,*
2. *the pair $(X, \sum b_jB_j)$ is lc,*
3. *$K_X + \sum b_jB_j$ is big.*

Then the divisor $K_X + (1 - \beta(\mathcal{A})) \sum b_jB_j$ is big (cf (3.5)).

Proof. Under the notation and assumptions of the theorem, let $B = \sum b_jB_j$. Let $\pi : Y \to X$ be a projective birational morphism such that $B^Y = \pi_*^{-1}B + \sum E_i$ and $\cup E_i = \operatorname{Exc}(\pi)$ in the formula $\pi^*(K + B) = K_Y + B^Y$, and such that $K_Y + B^Y$ is lt. Since $\pi_*(K_Y + (1-x)(\pi_*^{-1}B + \sum E_i)) \prec K + (1-x)B$, it is enough to prove the theorem for $K_Y + B^Y$. Hence we may assume that our X is $\mathbb{Q}$-factorial ignoring Y.

Assume that there is a positive real number $\delta \leq \beta(\mathcal{A})$ such that $K + (1-x)B$ is big iff $x < \delta$. We note that $\beta(\mathcal{A}) < 1/16$ (3.6.4). We apply (4.1) to $K + B$ with $B' = B$ and we are in case (4.1.2). Thus there is a birational morphism $f : X \to X'$ to a normal $\mathbb{Q}$-factorial surface X' such that $D = f_*(K + (1-\delta)B)$ is nef and $K + (1-\delta)B \succ f^*D$, and one of the following holds.

1. there exist a morphism $\pi : X' \to C$ onto a nonsingular projective curve C with a general fiber $F \simeq \mathbb{P}^1$ such that $(D \cdot F) = 0$.
2. $D \equiv 0$.

In Case (1), we have $(1-\delta)(\sum b_j f_*(B_j) \cdot F) = 2$ which contradicts (3.6.3) because $\beta(\mathcal{A}) < mf_2(\mathcal{A}, 2)$ (3.6.4). Finally Case (2) is disproved by (3.2). □

The following takes the same form as (7.6) of [3]. However it can be made effective in the sense of (4.12).

Theorem 4.7. *Fix $C > 0$ and a DCC set $\mathcal{A}$ such that $1 \in \mathcal{A}$. Then there exists a bounded class of surfaces with divisors (Z, D) such that for every surface X with $K + B = K + \sum b_j B_j$ nef big lc such that $0 < b_j \in \mathcal{A}$ and $(K+B)^2 \leq C$ there exists a diagram*

$$X \xleftarrow{f} Y \xrightarrow{g} Z,$$

where

1. *Y is the minimal resolution of X,*
2. *$D = g(\operatorname{Supp} B^Y \cup \operatorname{Exc}(f))$ where $\operatorname{Exc}(f)$ is the exceptional set of f and B^Y is defined by $K_Y + B^Y = f^*(K + B)$.*

Proof. We make auxiliary constructions of $\mathbb{R}$-divisors B', B'', B''' on Y.

(Construction of B') By (4.6), there exists a natural number $m = m(\mathcal{A})$ such that $m > 3/\min \mathcal{A}_{>0}$ and $K + \sum(b_j - 3/m)B_j$ is big. Let $B^Y = \sum_{j \in J} b_j B'_j + \sum_{i \in I} e_i E_i$ with $\cup E_i = \operatorname{Exc}(f)$ and $0 \leq e_i \leq 1$. Take $b'_j, e'_i \in \{k/m | k = 1, 2, \cdots m\}$ be such that

$$b'_j \in [b_j - \frac{3}{m}, b_j - \frac{2}{m}), \quad e'_i \in [e_i, e_i + \frac{1}{m}].$$

We note that $\operatorname{Exc}(f)$ is an SNC divisor by the classification of lc surface singularities. Thus there is a log resolution $\sigma : Y' \to Y$ of $K_Y + B^Y$ such that

$$\sigma^*(\sum(b_j - b'_j)B'_j + \sum(e_i - e'_i)E_i) \succ \sigma_*^{-1}(\sum(b_j - b'_j)B'_j + \sum(e_i - e'_i)E_i).$$

(The proof is left to the reader.) Hence if we set $B' = \sum b'_j B'_j + \sum e'_i E_i$ then $\sigma^*(K_Y + B^Y) \succ \sigma^*(K_Y + B') + \sigma_*^{-1}(B^Y - B')$ and $K_Y + B'$ is lc. Since $K_Y + B' \succ f^*(K + \sum(b_j - 3/m)B_j)$, $K_Y + B'$ is big.

(Construction of B'') Again by (4.6), there exist a natural number $\ell = \ell(m) > m$ depending only on m and $b''_j, e''_i \in \{k/\ell | k = 1, 2, \cdots \ell - 1\}$ with $b''_j < b'_j$ and $e''_i < e'_i$ such that if we set $B'' = \sum b''_j B'_j + \sum e''_i E_i$ then $K_Y + B''$ is lc big.

(Construction of B''') By (4.4), there exist a natural number $N = N(\ell)$ (depending only on ℓ) and subsets $I' \subset I$ and $J' \subset J$ such that $|I'| + |J'| \le N$ and $K_Y + B'''$ is lc big, where $B''' = \sum_{J'} b''_j B'_j + \sum_{I'} e''_i E_i$. We note that $K_Y + B'''$ has the properties:

1. the coefficients of B''' are in $\{1/\ell, \cdots, (\ell - 1)/\ell\}$,
2. $\#\{$irreducible component of $B'''\} \le N$, and
3. $f_*(K_Y + B''') \prec K + B$.

Now let $g : Y \to Z$ be the log canonical model of $K_Y + B'''$. Since $g^*(K_Z + g_* B''') \prec K_Y + B'''$, K_Z is lt and hence Z is rational. Let $H = K_Z + g_* B'''$, which is ample. Let $\pi : \widetilde{Z} \to Z$ be the minimal resolution. Then by (1), (2) above, the boundary of $\pi^*(K_Z + g_* B''')$ has only coefficients in $[0, 1 - 1/\ell]$ and it has at most N components. Thus if $U \subset Z$ is the complement of the set of Du Val singular points $\notin g_* B'''$, then there are at most N g-exceptional irreducible curves $E_k \subset \pi^{-1}(U)$ and each E_k satisfies $E_k \simeq \mathbb{P}^1$ and $(E_k^2) \ge -2\ell$ (1.2).

Claim 4.8. There exists a natural number $t \le \ell(2\ell)^N$ such that tK_Z/ℓ and tD/ℓ are Cartier for every irreducible component D of $g_* B'''$. In particular tH is Cartier.

For the claim we can ignore points not in U. We work only on H since the argument is the same. On $\pi^{-1}(U)$, we can write $\pi^{-1}(D) \equiv \pi_*^{-1}(D) + \sum a_k E_k$ with $a_k \in \mathbb{Q}$. If we set $t = \ell \cdot |det(E_k \cdot E_{k'})|$, then we have $t \le (2\ell)^N$ and $t(\pi_*^{-1}(D) + \sum a_k E_k)$ is a Cartier divisor relatively trivial for π. Since Z has only rational singularities, π_* induces a Cartier divisor tD. This proves the claim.

Claim 4.9. $H^2 \le (f_* g^* H)^2 \le C$ and $-3tC \le K_Z \cdot H \le C$.

Since $f_* g^* H \prec K + B$ are both nef we have

$$H \cdot K_Z \le H^2 \le (g^* H)^2 \le (f_* g^* H)^2 \le (K + B)^2 \le C.$$

Assume that $(K_Z + 3tH) \cdot H < 0$. Then $(K_{\widetilde{Z}} + 3t\pi^* H) \cdot \pi^* H < 0$ and $K_{\widetilde{Z}} + 3t\pi^* H$ is not nef. If we take an extremal rational curve C of it then $(K_{\widetilde{Z}} + 3t\pi^* H) \cdot C < 0$ and it contradicts $K_{\widetilde{Z}} \cdot C \ge -3$. Thus the claim is proved.

Let $D'' \subset Y$ be the sum of (-2)-curves which are f-exceptional, and let D' be the closure of $(\mathrm{Supp}(B^Y) \cup \mathrm{Exc}(f)) - \mathrm{Supp}(F_2)$. We note that $D = g(D') \cup g(D'')$.

Claim 4.10. $(g_* D' \cdot H) \le (1 + 3t)C / \min\{1/3, \mathcal{A}\}$, $\sum_{E \subset D''} (g_* E \cdot H)^2 \le 2C$

Let $B^Y = f_*^{-1} B + \sum_E a_E E$. Then by $(K_Y + B^Y) \cdot E = 0$, we have $0 \ge (K_Y + a_E E) \cdot E$ and $a_E \ge 1/3$ by $(E^2) \le 3$. Thus the first inequality follows from $(g_*B^Y \cdot H) \le (1+t)C$ (4.9). Let $F = g^*(H) + 1/2\sum_{E\subset D''}(H \cdot g_*E)E$. Since $(F^2) \le (f_*F)^2 = (f_*g^*H)^2 \le C$ (4.9), we have the second by $(F^2) \ge (H^2) + 1/2\sum(H\cdot g_*E)^2$. Thus the claim is proved.

Now by [7], [4] and [9], there exists a uniform $M = M(C,t)$ so that MtH is very ample (4.9). By (4.10), (Z, D) is also bounded. □

4.11. If we use the mrs lc version, the arguments in the proof of Theorem 4.7 become cleaner though we need the mrs lc version of (4.1). Here we stay with the usual lc version, and only state the effective bound of $(K+B)^2$ with a sketch of a "lc" proof.

Theorem 4.12 (An effective bound of $(K+B)^2$)**.** *Let X be a normal projective surface, B_j divisors on X, and let b_j be positive real numbers. Assume that*

1. *$K_X + \sum b_jB_j$ is nef big lc,*
2. *b_j belong to a DCC set $\mathcal{A}$.*

Then

$$\left(K_X + \sum b_jB_j\right)^2 \ge \frac{1}{\ell\cdot(2\ell)^N},$$

where $N = 128\ell^5 + 4\ell$ and $\ell = \lceil 1/(\beta(\mathcal{A})\cdot \min \mathcal{A}_{>0})\rceil$.

Proof. Let $f : Y \to X$ be a log resolution of $K+B$. We set $B' = f_*^{-1}B + \sum_{f\text{-exc. } E} E$. Then we follow the proof of (4.7) from the construction of B'' till the proof of Claim (4.9), where we set $\ell = \lceil 1/(\beta(\mathcal{A})\cdot \min\mathcal{A}_{>0})\rceil$. Thus we get $(K+B)^2 \ge (H^2) \ge \ell/t^2 \ge 1/\ell(2\ell)^N$ with $N = 128\ell^5+4\ell$. □

5. A DCC set for klt surfaces

5.1. Once we obtain a bounded family $\{(Z,D)\}$ as in (4.7), we would like to reconstruct (X,B) from (Z,D) in the bounded family. For this it is enough to obtain Y by blowing up in some bounded manner since then $\mathrm{Exc}(f) \subset g^{-1}(D)$. In order to do this systematically, we study the family of the maps $g : Y \to Z$ as follows.

5.2. Let $\mathcal{A}$ be a DCC set and let $\Phi = \{(Z,D)\}$ be a bounded family of normal surfaces Z with reduced Weil divisors D. Consider a set Ψ of the set (Y, B^Y, g, Z, D) consisting of a pair (Y, B^Y) and a birational morphism $g : Y \to Z$ to a $(Z,D) \in \Phi$ such that

1. Y is a nonsingular surface and $K_Y + B^Y$ is klt,
2. $\mathrm{Supp}\, B^Y \subset g^{-1}(D) \cup \mathrm{Exc}(g)$,
3. $(K_Y + B^Y)^2 > 0$ and (K_Y+B^Y) is nef on $C \subset g^{-1}(D)\cup\mathrm{Exc}(g)$,
4. there are no (-1)-curves $C \subset g^{-1}(D)\cup\mathrm{Exc}(g)$ with $(K_Y+B^Y)\cdot C = 0$,
5. the coefficient in B^Y of every curve $C \subset g^{-1}(D)\cup\mathrm{Exc}(g)$ with $(K_Y + B^Y)\cdot C > 0$ is in $\mathcal{A}\cup\{0\}$.

We note that we have this situation in (4.7).

The main purpose of this section is to prove the following.

Theorem 5.3 (Alexeev [3] (8.5)). *Under the notation and the assumptions of (5.2), let Ψ' be an arbitrary infinite sequence of Ψ. Then there exist an infinite subsequence $\{\psi^s = (Y^s, B^{Y^s}, g^s, Z^s, D^s)\} \subset \Psi'$ and a bounded (flat) family of blowups $\pi^s : V^s \to Z^s$ dominated by Y^s via $h^s : Y^s \to V^s$ such that for every $s < t$ we have*

$$(K_{Y^s} + B^{Y^s})^2 \le (K_{V^s} + B^{V^s})^2 \le (K_{Y^t} + B^{Y^t})^2. \tag{3}$$

Since Ψ in (5.2) arises from klt surfaces $(X, B = \sum b_j B_j)$ with $K + B$ ample and $b_j \in \mathcal{A}$ by (4.7), we have the following.

Corollary 5.4. *Let $\mathcal{A}$ be a DCC set. Consider klt surfaces $(X, B = \sum b_j B_j)$ such that $K + B$ is ample and $b_j \in \mathcal{A}$. Then $\{(K + B)^2\}$ is a DCC set.*

The rest of this section is devoted to the proof of the theorem. Readers mainly interested in knowing how this is used can skip to the next section.

In order to prove (5.3), we would like to simplify the maps g by changing (Z, D) uniformly.

Lemma 5.5. *Let the notation and the assumptions be as in (5.2). We can change the bounded family Φ without changing (Y, B^Y) in Ψ so that we can assume that D is an SNC divisor at every point P blown up by g.*

Proof. Take an arbitrary $(Y, B^Y, g, Z, D) \in \Psi$. Then Y dominates the minimal resolution Z' of Z by $g' : Y \to Z'$. Let $D' \subset Z'$ be the union of the exceptional set of $Z' \to Z$ and the inverse image of D. Furthermore as long as there is a point in Z' blown up by g' at which D' is not an SNC divisor, we blow up such a point. Repeating this until we have a birational morphism $g'' : Y \to Z''$ such that the inverse image D'' of D' is an SNC divisor at every point Z'' blown up by g''. It is obvious that such (Z'', D'') again form a bounded family Φ'. ☐

We say that a reduced curve C on a nonsingular surface has a *reducible ODP* at P if $C = C_1 \cup C_2$ in a neighborhood of P for some curves C_1 and C_2 smooth at P and intersecting transversally at P.

Lemma 5.6. *Let the notation and the assumptions be as in (5.2) and assume that Ψ has the extra property in (5.5) Then for each $(Y, B^Y, g, Z, D) \in \Psi$,*

6. *$g : Y \to Z$ is obtained by repeatedly blowing up a point at which the total transform of D has a reducible ODP as a set.*

Proof. Let $P \in Z$ be a point blow up by g. Assume that D is smooth at P. Then $\mathrm{Supp}(B^Z)$ is also smooth at P. Hence by (1.6), we have $g^*(K_Z + B^Z) \prec K_Y + g_*^{-1}(B^Z)$ in a neighborhood of $g^{-1}(P)$. On the other hand we have $K_Y + B^Y \prec g^*(K_Z + B^Z)$ because $K_Y + B^Y$ is nef on $g^{-1}(D)$. This means $K_Y + B^Y = g^*(K_Z + B^Z)$ in a neighborhood of $g^{-1}(P)$. Then

$$g^{-1}(P) \cdot (K_Y + B^Y) = g^{-1}(P) \cdot g^*(K_Z + B^Z) = 0$$

Hence $g^{-1}(P)$ does not contain any (-1)-curves by the condition (4), which is a contradiction. Since this argument applies to any subsequent blowups, the lemma is proved. □

Remark 5.7 (Reduction to the case of a fixed (Z, D))**.** Φ breaks up into a finite number of families Φ_i $(i = 1, \cdots, r)$ which are projective flat families parameterized by irreducible algebraic varieties so that each irreducible component of D comes from an irreducible component of the total space and each ordinary double point of D forms a section of the flat family. Then Ψ breaks up into $\Psi_i (i = 1, \cdots, r)$ correspondingly. For the infinite sequence Ψ' of Ψ in (5.3), $\Psi'_i = \Psi' \cap \Psi_i$ is still infinite for some i (say $i = 1$).

Take a generic $(\overline{Z}, \overline{D}) \in \Phi_1$. Then for each $\psi = (Y, B^Y, g, Z, D)$, we can make the blow up $\overline{g} : \overline{Y} \to \overline{Z}$ corresponding to $g : Y \to Z$ as above, and we can write $B^{\overline{Y}}$ using the same coefficients as B^Y. Consider $\overline{\Psi'_1} = \{(\overline{Y}, B^{\overline{Y}}, \overline{g}, \overline{Z}, \overline{D}) | \psi \in \Psi'_1\}$. If we prove (5.3) for $\overline{\Psi}'_1$, then from $\overline{\pi} : \overline{V} \to \overline{Z}$ we can construct a bounded family of blowups $\pi^s : V^s \to Z^s$ and (5.3) holds for Ψ'. In other words, we can assume Ψ consists of one member to prove (5.3).

So from now on, (Z, D) will be fixed during the proof and one blow up V will be constructed which works as V^s for every s. We note however that the notation like $K_{V^s} + B^{V^s}$ is used to denote $h^s_*(K_{Y^s} + B^{T^s})$.

With these notation and assumptions, the following is the key technical result.

Theorem 5.8 (Alexeev [3] (8.5))**.** *There exist an infinite subsequence* $\{\psi^s = (Y^s, B^{Y^s}, g^s, Z, D)\} \subset \Psi'_1$ *and a blowup* $\pi : V \to Z$ *dominated by* Y^s*, say via* $h^s : Y^s \to V^s$ *such that for every* $s < t$ *we have*

$$h^{t*}(K_{V^s} + B^{V^s}) \prec K_{Y^t} + B^{Y^t}. \tag{4}$$

Proof. Proof of (5.3) using (5.8)

By the definition of B^{V^s}, we have $h^s_*(K_{Y^s} + B^{Y^s}) = K_{V^s} + B^{V^s}$, this implies $(K_{Y^s} + B^{Y^s})^2 \leq (K_{V^s} + B^{V^s})^2$. Since $K_{Y^t} + B^{Y^t}$ and $K_{V^s} + B^{V^s}$ are nef over D by the condition (3), so is $(K_{Y^t} + B^{Y^t}) + h^{t*}(K_{V^s} + B^{V^s})$ on $(g^t)^{-1}(D)$. By inequality (4), $(K_{Y^t} + B^{Y^t}) - h^{t*}(K_{V^s} + B^{V^s})$ is an effective divisor supported on $(g^t)^{-1}(D)$. Hence $(K_{Y^t} + B^{Y^t})^2 - h^{t*}(K_{V^s} + B^{V^s})^2 \geq 0$, which is inequality (3). □

Remark 5.9 (Reformulation of inequality (4))**.** For any valuation v of the function field of Z we can talk about the log discrepancy $a_\ell(v, Y) = a_\ell(v, K_Y + B^Y)$ for any log divisor in the usual way. We also set

$$a_\ell(v, Y^\infty) = \limsup_{(Y, \cdots) \in \Psi'_1} a_\ell(v, K_Y + B^Y) \in \mathbb{R} \cup \{\infty\}.$$

We identify a divisor and its valuation for simplicity of notation. Note that if v is the valuation of an irreducible component B_j of $B^Y = \sum b_j B_j$ then $a_\ell(v, Y) = 1 - b_j$.

We say that a valuation v is D-toric if either v is the valuation of an irreducible component of D or an exceptional divisor which is obtained by successively blowing up a point at which the total total transform of D has a reducible ODP as a set. In the proof, we consider only D-toric valuations. Then inequality (4) is equivalent to "$a_\ell(v, V^s) \geq a_\ell(v, Y^t)$ for all divisors v on Y^t" and even to

$$a_\ell(v, V^s) \geq a_\ell(v, Y^t) \text{ for all divisors } v \text{ on } Y^t \text{ with } 1 - a_\ell(v, Y^t) \in \mathcal{A} \cup \{0\}.$$

Indeed let us assume inequality (5.9). If we write $K_{Y^t} + B^{Y^t} - (h^t)^*(K_{V^s} + B^{V^s}) = L_1 - L_2$ with effective divisors L_1 and L_2 without common components, then inequality (5.9) means that $1 - a_\ell(C, Y^t) \notin \mathcal{A} \cup \{0\}$ for every curve $C \subset \operatorname{Supp} L_2$. From the conditions (3) and (5), we see that the intersection matrix of components of $\operatorname{Supp} L_2$ is negative definite and $(K_{Y^t} + B^{Y^t}) \cdot L_2 = 0$. Then

$$(L_2^2) \geq (L_2 \cdot L_2 - L_1) = L_2 \cdot (h^t)^*(K_{V^s} + B^{V^s}) \geq 0,$$

and $L_2 = 0$. Thus we have inequality (4).

Remark 5.10 (A consequence of klt)**.** Suppose V is chosen. To check inequality (5.9) for a given s, we need to check the inequality only for a finite number of v's. To be precise, there exists a finite set Λ_s of valuations such that $a(v, V^s) > 1$ if $v \notin \Lambda_s$. (Note that $K_{Y^s} + B^{Y^s}$ is klt and $h^s : Y^s \to V^s$ is an isomorphism except at reducible ODP's of $h^s_*(B^{Y^s})$ by the condition (6). Hence $K_{V^s} + B^{V^s}$ is klt.)

Therefore if $v \notin \Lambda_s$ then inequality (5.9) holds for these s and v no matter what t is.

Remark 5.11 (Finite number of divisors on V can be ignored)**.** When we have $\pi : V \to Z$ and an infinite subsequence of Ψ' with all Y^s dominating V, we need to check inequality (5.9) only for V-exceptional valuations (i.e. valuations whose center on V is a point).

More generally given a finite set Ξ of divisors on V, we can replace the sequence by an infinite subsequence so that inequality (5.9) holds for all $s < t$ and $v \in \Xi$. Indeed we can assume Ξ consists of one valuation v_0. By replacing the sequence by an infinite subsequence, we can treat inequality (5.9) in two cases.

1. $1 - a_\ell(v_0, Y^s) \notin \mathcal{A}$ for all s,
2. $1 - a_\ell(v_0, Y^s) \in \mathcal{A}$ for all s.

In the first case inequality (5.9) holds for v_0. In the second case, we take an infinite subsequence again and assume that $a_\ell(v_0, Y^s)$ is a non-increasing sequence since $\mathcal{A}$ is a DCC set. Then inequality (5.9) holds for v_0.

In the proof, we will mean an infinite subsequence by a subsequence.

5.12. The construction of π will be done locally for each reducible ODP P of D. Let D_1 and D_2 be the irreducible components of D such that $D = D_1 \cup D_2$ near P. Each D_i is defined by a local equation $\phi_i = 0$ near

P. It is easy to see that D-toric valuations v centered at P corresponds to $e(v) = (v_1, v_2) \in \mathbb{N}^2$ such that $gcd(v_1, v_2) = 1$ bijectively by $v_i = v(\phi_i)$. We make the additional correspondence $D_1 = (1,0), D_2 = (0,1)$.

We note that $\phi = (\phi_1, \phi_2) : (Z, P) \to (\mathbb{A}^2, 0)$ is étale at P and everything near P can be computed by the torus embedding method.

Set $N = \mathbb{Z}^2$ and $\mathbb{A}^2 \supset \mathbb{G}_m^2$ corresponds to the cone $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \supset \mathbb{R}^2_{\geq 0}$.

The theory of torus embeddings says the following.

For any finite set $E \subset \mathbb{R}^2_{\geq 0}$, let $< E >$ be the cone $\sum_{e \in E} \mathbb{R}_{\geq 0} e$ spanned by E. For any finite set E of primitive elements of $N \cap \mathbb{R}^2_{\geq 0}$, let $\Sigma(E)$ be the subdivision of $\mathbb{R}^2_{\geq 0}$ by the rays $\mathbb{R}_{\geq 0} e$ $(e \in E)$. That is, if all the elements $e_1, \cdots, e_r$ of E are numbered in the decreasing order of slopes then $< e_i, e_{i+1} >$ $(i = 0, \cdots, r)$ are the 2-dimensional cones of $\Sigma(E)$ where $e_0 = (0,1)$ and $e_{r+1} = (1,0)$ and the slope of a non-zero $v = (c,d) \in \mathbb{R}^2_{\geq 0}$ is $d/c \in \mathbb{R} \cup \{\infty\}$.

To the decomposition $\Sigma(E)$, associated an algebraic variety $T(E)$ and a proper birational morphism $\pi(E) : T(E) \to \mathbb{A}^2$. If we set

$$E(Y^s) = \{e(v) | v \text{ is an irreducible component of } (g^s)^{-1}(P)\},$$

then $Y^s \simeq T(E(Y^s)) \times_{\mathbb{A}^2} Z$ over a neighborhood of P. (This can be proved by reconstructing the blow-ups of g^s on $\mathbb{A}^2$.) In this sense, we identify (Z, P) with $(\mathbb{A}^2, 0)$ and also v with $e(v)$ from now on.

The following formulas for log discrepancies a_ℓ are the basis of our proof.

Lemma 5.13. 1. *Let $u_i \in \mathbb{Z}^2_{\geq 0}$ $(i = 1, 2, 3)$ be primitive vectors such that $u_3 = \lambda_1 u_1 + \lambda_2 u_2$ for some $\lambda_i \in \mathbb{Q}_{\geq 0}$. then for all $s \leq \infty$ we have*

$$a_\ell(u_3, Y^s) \geq \lambda_1 a_\ell(u_1, Y^s) + \lambda_2 a_\ell(u_2, Ys).$$

2. *Let $E = \{v_1, \cdots, v_q\} \subset \mathbb{Z}^2_{>0}$ be primitive vectors in the decreasing order of slopes and let $v \in \mathbb{Z}^2_{>0}$ be a primitive vector such that $v = \lambda v_a + \mu v_{a+1}$ for some $a \in [1, q-1]$ and $\lambda, \mu \in \mathbb{Q}_{\geq 0}$. Then for all $s \leq \infty$ we have*

$$a_\ell(v, T(E)^s) = \lambda a_\ell(v_a, Y^s) + \mu a_\ell(v_{a+1}, Y^s).$$

Proof. If $\lambda_1 = 0$ or $\lambda_2 = 0$ in (1), then $u_3 = u_1$ or u_2 and the formula is trivial. So we can assume $\lambda_1, \lambda_2 > 0$ in (1) and similarly $\lambda, \mu > 0$ in (2). We can also assume $s < \infty$.

For (1), we look at $W = T(u_1, u_2, u_3)$. It contains three $\mathbb{Q}$-Cartier divisors C_i corresponding to $\mathbb{R}_{\geq 0} u_i$, and C_3 is proper by $\lambda_1, \lambda_2 > 0$. From the relation $\lambda_1 u_1 + \lambda_2 u_2 - u_3 = 0$, we see that ([Torus Embedding])

$$(C_1 \cdot C_3) : (C_2 \cdot C_3) : (C_3^2) = \lambda_1 : \lambda_2 : (-1).$$

We have $K_W + \sum C_i \sim 0$ in a neighborhood of C_3 since W is a toric variety. Since $K_{Y^s} + B^{Y^s}$ is nef,

$$K_W + B^{W^s} = K_W + \sum (1 - a_\ell(u_i, Y^s)) C_i \quad (\text{in a nbd of } C_3)$$

is also nef. Hence we have

$$\begin{aligned} 0 &\le (K_W + \sum (1 - a_\ell(u_i, Y^s))C_i) \cdot C_3 \\ &= (-C_3^2) \cdot (a_\ell(u_3, Y^s) - \sum_{i=1}^{2} \lambda_i \cdot a_\ell(u_i, Y^s)). \end{aligned}$$

For (2), we repeat the argument in (1) with $u_1 = v_a, u_2 = v_{a+1}$ and $u_3 = v$. We note that if we set $\pi : W = T(u_1, u_2, u_3) \to T(E)$, then

$$\pi^*(K_{T(E)} + B^{T(E)^s}) = K_W + \sum (1 - a_\ell(u_i, T(E)^s))C_i \quad \text{(in a nbd of } C_3)$$

is trivial on the π-exceptional C_3. Hence we get the equality. □

Proof. Proof of (5.8) We do the construction locally at each reducible ODP point $P = D_1 \cap D_2$ of D one by one in three steps. In each step, we will construct a blow up $\pi : V \to Z$ and a subsequence $(Y^s, B^{Y^s}, g^s, Z, D)$ of Ψ_1' such that Y^s dominates V and inequality (5.9) holds for all valuations centered at points Q (over P) which are not in subsequent cases. Since we can work on V instead of Z if there are still points on V to treat in a subsequent case, this is enough for (5.8) (cf. (5.11)).

1. P with $a_\ell(D_1, Y^\infty) = a_\ell(D_2, Y^\infty) = 0$,
2. P with $a_\ell(D_1, Y^\infty) = 0, a_\ell(D_2, Y^\infty) > 0$,
3. P with $a_\ell(D_1, Y^\infty) > 0, a_\ell(D_2, Y^\infty) > 0$.

Step (1). Let $v_0 = (1,1)$. Assume first that $a_\ell(v_0, Y^\infty) > 0$. In this case we can pass to a subsequence and assume that $a_\ell(v_0, Y^s) > a_\ell((1,0), Y^s) + a_\ell((0,1), Y^s)$ for all s. Then by (5.13.2), $g^s : Y^s \to Z$ blows up P for all s. Let $\pi(v_0) : T(v_0) \to Z$ be the blow up at P, E the exceptional divisor v_0 and $D_i' = \pi_*^{-1}(D_i)$. We set $V = T(v_0)$. Since $a_\ell(v_0, Y^\infty) > 0$, the two ODP's $E \cap D_i'$ are to be treated in Step (2).

Assume next that $a_\ell(v_0, Y^\infty) = 0$. In this case, we take $V = Z$. To see that $V = Z$ suffices, let us first check $a_\ell(w, Y^\infty) = 0$ for all w. By symmetry, we can set $w = (w_1, w_2)$ with $w_1 \ge w_2$. Then from

$$(1,1) = \frac{1}{w_1}(w_1, w_2) + \frac{w_1 - w_2}{w_1}(0,1),$$

we indeed see $a_\ell(w, Y^\infty) = 0$ by (5.13.1).

Now take any s, then we claim inequality (5.9) holds for all $t \gg s$. By (5.10), we need to check inequality (5.9) only for those v in the finite set Λ_s. Since $K_{Y^s} + B^{Y^s}$ is klt, $a_\ell(v, Z^s) > 0$ for all v. Since $a_\ell(v, Y^\infty) = 0$ for all $v \in \Lambda_s$, we have $a_\ell(v, Z^s) > a_\ell(v, Y^t)$ for all $v \in \Lambda_s$ an all $t \gg s$. Therefore we have the claim, and there exists a required subsequence.

Step (2). Take a natural number n such that $a_\ell(D_2, Y^\infty) \ge 2/n$. By passing to a subsequence, we can assume $a_\ell(D_2, Y^s) > 1/n$ for all s. The following claim is the main step of the proof.

Claim 5.14. There exists a primitive $v_0 = (k_0, 1) \in \mathbb{N}^2$ so that we have the following by passing to a subsequence: if $v = \lambda(1,0) + \mu v_0$ $(\lambda, \mu \in \mathbb{Q}_{>0})$ is primitive and if $s < t$ and $1 - a_\ell(v, Y^t) \in \mathcal{A}$, then

$$\lambda \cdot a_\ell((1,0), Y^s) + \mu \cdot a_\ell(v_0, Y^s) \geq a_\ell(v, Y^t).$$

Let us see first that (5.14) implies Step (2). Let

$$E_j = \{(1,1), (2,1), \cdots, (j,1)\}$$

for $j = 0, \cdots, k_0$. $T(E_j)$ is nonsingular and has exceptional curves C_i corresponding to $(i,1)$ $(i = 1, \cdots, j)$ and $D'_i = \pi(E_j)^{-1}_*(D_i)$. We note $T(E_0) = Z$, and if we blow up the point $P_j = D'_1 \cap C_j$, then we get $T(E_{j+1})$.

By (5.13.1), we have $a_\ell((i,1), Y^\infty) \geq a_\ell((0,1), Y^\infty) > 0$. Let $j \in [0, k_0 - 1]$ be such that all Y^s dominates $T(E_j)$ ($j = 0$ always suffices). Note that all the reducible ODP's on $\pi(E_j)^{-1}(P)$ are in Case (3) except for P_j. So if there exists infinitely many s such that $Y^s \to T(E_j)$ is isomorphic over P_j, then we pass to such a subsequence and we are done with $V = T(E_j)$. So by passing to a subsequence, we can assume $Y^s \to T(E_j)$ blows up P_j that is $Y^s \to T(E_{j+1})$ for all s. We may therefore assume all the Y^s dominates $T(E_{k_0})$. We set $V = T(E_{k_0})$, on which P_{k_0} is the only point to be considered in our step. Let v be an arbitrary valuation centered at P_{k_0} on V.

Then v is in the interior of $< v_0, (1,0) >$ and we can compute $a_\ell(v, V^s)$ by (5.13.2). Hence by (5.14), we have

$$a_\ell(v, V^s) \geq a_\ell(v, Y^t)$$

if $s < t$ and if $1 - a_\ell(v, Y^t) \in \mathcal{A}$. So Step 2 follows from (5.14).

Proof. Proof of (5.14) We see that $a_\ell((k,1), Y^\infty)$ is non-decreasing in k by (5.13.1). Let $a_\ell(Y^\infty)$ be its limit ($\in \mathbb{R} \cup \{\infty\}$). We treat (5.14) in two subcases.

Subcase (1). There exists $v_0 = (k_0, 1)$ such that $a_\ell(Y^\infty) \leq a_\ell(v_0, Y^s)$ for infinitely many s.

By passing to a subsequence, we may assume $a_\ell(Y^\infty) \leq a_\ell(v_0, Y^s)$ for all s. Take any s and $v = \lambda \cdot (1,0) + \mu \cdot v_0$ $(\lambda, \mu > 0)$. Write v as $v = (c, \mu)$. If $c \geq k_s = 1/a_\ell((1,0), Y^s)$ or $\mu \geq n$, then $a_\ell(v, Y^s) \geq 1$ by (5.13.1). So inequality (5.9) holds for all t with these s and v. There are only finitely many $v = (c,d)$ to be considered. Applying (5.13.1) to $v, (k,1)$ and $(1,0)$ for $k \gg 0$, we see

$$a_\ell(v, Y^\infty) \leq \mu \cdot a_\ell(Y^\infty) \leq \mu \cdot a_\ell(v_0, Y^s).$$

Hence $a_\ell(v, Y^\infty) < \lambda \cdot a_\ell((1,0), Y^s) + \mu \cdot a_\ell(v_0, Y^s)$. Since there are only finitely many v to be considered, we have (5.14) for all $t \gg s$. Hence we have the required subsequence.

Subcase (2). For every $k \geq 0$, there exists $t_0(k)$ such that $a_\ell(Y^\infty) > a_\ell((k,1), Y^s)$ for all $s > t_0(k)$.

Let $k \in \mathbb{N}$. For primitive $(c,d) \in \mathbb{N}^2$ with $c \leq k$ and $d \leq n$, we have

$$a_\ell((c,d), Y^t) < d \cdot a_\ell(Y^\infty) \text{ if } t > t_0(k). \tag{5}$$

This is because $a_\ell((c,d), Y^t) < d \cdot a_\ell((k,1), Y^t) < d \cdot a_\ell(Y^\infty)$ by (5.13.1).

Now set

$$\mathcal{B} = \left\{ \frac{a_\ell((c,d),Y^s)}{d} \mid \begin{array}{r} s \text{ and primitive } (c,d) \text{ with} \\ a_\ell((c,d),Y^s)/d < a_\ell(Y^\infty), \\ 1 - a_\ell((c,d),Y^s) \in \mathcal{A} \text{ and } d \le n \end{array} \right\}.$$

Since d is bounded and $\mathcal{A}$ is a DCC set, $\mathcal{B}$ is an ACC set. So $\mathcal{B}$ has a maximal element $\max \mathcal{B} < a_\ell(Y^\infty)$. Let $v_0 = (k_0, 1)$ be such that $\max \mathcal{B} < a_\ell(v_0, Y^\infty)$.

By passing to a subsequence, we have $\max \mathcal{B} < a_\ell(v_0, Y^s)$ for all s. We will show this v_0 works for (5.14).

Take any s and a primitive $v = \lambda\cdot(1,0)+\mu\cdot v_0$ $(\lambda,\mu > 0)$. Write $v = (c,\mu)$. Then as in Subcase (1), we need to check (5.14) only for finitely many v, i.e. $c < k_s$ and $\mu \le n$. Take any t such that $t > t_0(k_s)$ and $1 - a_\ell(v, Y^t) \in \mathcal{A}$. Then by (5.14), $a_\ell(v,Y^t)/\mu < a_\ell(Y^\infty)$ an we see $a_\ell(v,Y^t)/\mu \in \mathcal{B}$. Thus $a_\ell(v,Y^t) \le \mu\cdot\max\mathcal{B} < \mu\cdot a_\ell(v_0,Y^s)$. This proves (5.14), and Step (2) is finished. □

Step (3). Let n be a natural number such that $a_\ell(D_1,Y^s), a_\ell(D_2,Y^s) \ge 1/n$ for all s by passing to an infinite subsequence. Then $a_\ell((c,d),Y^s) > 1$ if $c+d > n$. Therefore the center of valuation (c,d) on Y^s is a point if $c+d>n$. In other words $g^s : Y^s \to Z$ has at most $(n-1)(n-2)/2$ exceptional curves for every s.

Starting with $V_i = Z$ for $i = 0$, we construct blow-ups $\pi_i : V_i \to Z$ dominated by all Y^s inductively until we get V as follows.

If there is a point $P_i \in \pi_i^{-1}(P)$ blown up by $Y^s \to V_i$ for infinitely many s, then we pass to a subsequence and assume that P_i is blown up by all $Y^s \to V_i$. Then we set $\pi_{i+1} : V_{i+1} \to Z$ to be the blow up of V_i at P_i.

Since $Y^s \to Z$ has at most $(n-1)(n-2)/2$ exceptional curves, this process must stop, say at $i = b$ for some $b \le (n-1)(n-2)/2$.

Then we set $\pi = \pi_b : V = V_b \to Z$. For $s \gg 0$, $Y^s \to V$ is an isomorphism over $\pi^{-1}(P)$. Therefore we are done since there are no exceptional curves $\subset (g^s)^{-1}(P)$ for $Y^s \to V$ to check inequality (5.9) with. This proves Theorem (5.8). □

Remark 5.15. Several important ideas of the proof given here are in Proposition 5 of [12].

6. DCC sets and the boundedness

The purpose of this section is to prove the following two theorems in [3], which were conjectured by Kollár [5].

Theorem 6.1 (Alexeev [3] (8.2)). *Fix a DCC set $\mathcal{A}$. Then the following set*

$$K_{slc}(\mathcal{A}) = \left\{ (K_X+B)^2 \middle| \begin{array}{r} (X, B = \sum b_j B_j) \text{ is an slc surface,} \\ K+B \text{ ample, } b_j \in \mathcal{A} \end{array} \right\}$$

is a DCC set.

Theorem 6.2 (Alexeev [3] (9.2)). *Fix a constant C and a DCC set $\mathcal{A}$. Consider all the slc surfaces $(X, B = \sum b_j B_j)$ such that $K + B$ is ample, $b_j \in \mathcal{A}$ and $(K+B)^2 = C$. Then $F_{slc}(C, \mathcal{A}) = \{(X, B)\}$ is bounded.*

Proof. Proof of (6.1 Define $K_{lc}(\mathcal{A})$ (resp. $K_{klt}(\mathcal{A})$) with (X, B) lc (resp. klt) similarly to $K_{slc}(\mathcal{A})$ in (6.1). Then $K_{klt}(\mathcal{A})$ is a DCC set by (5.4). The main point in our proof is the reduction step.

The following allows us to reduce the lc case to the klt case.

6.3. Let $(X, B = \sum b_j B_j)$ be an lc surface and $f : Y \to X$ the minimal resolution and $B^Y = f^*(K_X + B) - K_Y$ as usual. Let $\overline{f} : Y \to \overline{X}$ be the contraction of the fractional part of B^Y. This $\overline{X}$ is guaranteed to exist and is projective (XX). Let $\sigma : \overline{X} \to X$ be the induced birational morphism, $\overline{K} = K_{\overline{X}}$ and $\overline{B} = \overline{f}^*(K+B) - \overline{K}$. Since $\overline{K}$ is klt by the construction $\overline{f}^*(\overline{K}) \prec K_Y + B^Y$, $\overline{X}$ is $\mathbb{Q}$-factorial and we can choose an effective $\mathbb{Q}$-divisor $\overline{E}$ such that $\operatorname{Supp} \overline{E} = \operatorname{Supp} [\overline{B}]$, $[\overline{E}] = 0$ and $\overline{K} + \overline{B} - \lambda \overline{E}$ is ample for all rational numbers $\lambda \in [0, 1]$. Set $F = \overline{f}^* \overline{E} \subset \operatorname{Supp} B^Y$.

The main idea here is that $(\overline{X}, \overline{B} - \lambda \overline{E})$ with $0 < \lambda \ll 1$ is a "good klt approximation" of (X, B).

Suppose $K_{lc}(\mathcal{A})$ is not a DCC set and choose an infinite sequence of lc surfaces $(X^s, B^s = \sum b^s_j B^s_j)$ such that $K_{X^s} + B^s$ is ample, $b^s_j \in \mathcal{A}$ and $(K_{X^s} + B^s)^2$ is strictly decreasing in s. Then for each s, we do the process (6.3) and get a klt surface $(\overline{X}^s, \overline{B}^s - \lambda \overline{E}^s)$. Choose a sequence of $\lambda^s \in (0, 1)$ in such a way that $[s \cdot \lambda^s \overline{E}^s] = 0$ and $(\overline{K}^s + \overline{B}^s - \lambda^s \overline{E}^s)^2 > (\overline{K}^{s+1} + \overline{B}^{s+1} - \lambda^{s+1} \overline{E}^{s+1})^2$ for all s. This is possible because $(K_{X^s} + B^s)^2$ is strictly decreasing in s. Then the set $\mathcal{A}'$ of all the coefficients of $\overline{B}^s - \lambda^s \overline{E}^s$ for all s is again a DCC set. Therefore having a strictly decreasing sequence

$$(\overline{K}^s + \overline{B}^s - \lambda^s \overline{E}^s)^2 \in K_{klt}(\mathcal{A}'),$$

we have a contradiction. Hence $K_{lc}(\mathcal{A})$ is a DCC set.

Let $(X, B = \sum b_j B_j)$ be an slc surface with $K + B$ ample and $b_j \in \mathcal{A}$. Then every irreducible component $(X_m, B_m = \sum b_{mj} B_{mj})$ of its normalization is an lc surface such that $K_{X_m} + B_m$ is ample and $b_{mj} \in \mathcal{A} \cup \{1\}$. Therefore by

$$(K+B)^s = \sum_m (K_{X_m} + B_m)^2,$$

Thus we see $K_{slc}(\mathcal{A}) \subset (K_{lc}(\mathcal{A} \cup \{1\}))_\infty$ by the notation of (2.4), and $K_{slc}(\mathcal{A})$ is a DCC set. □ □

Proposition 6.4. *Fix a constant C and a DCC set $\mathcal{A}$. Then*

$$F_{lc}(C, \mathcal{A}) = \left\{ lc \text{ surface } (X, B = \sum b_j B_j) | \begin{matrix} K + B \text{ ample, } b_j \in \mathcal{A}, \\ (K+B)^2 = C \end{matrix} \right\}$$

is a bounded family.

Proof. By (4.7), we have the following.

For each (X,B) in $F_{lc}(C,\mathcal{A})$, the minimal resolution $f : Y \to X$ has a birational morphism $g : Y \to Z$ to a normal surface and $(Z, D = g(\operatorname{Exc}(f) \cup \operatorname{Supp} f_*^{-1}B))$ forms a bounded family when (X,B) varies in $F_{lc}(C,\mathcal{A})$.

Let us take any infinite sequence (X^s, B^s) from $F_{lc}(C,\mathcal{A})$ and we have $f^s : Y^s \to X^s$, $g^s : Y^s \to Z^s$, B^{Y^s} and D^s as above. If we show that (X^s, B^s) is bounded, then $F_{lc}(C,\mathcal{A})$ is bounded.

We apply (6.3) to each (X^s,B^s) and get an effective $\mathbb{Q}$-divisor $F^s \subset B^{Y^s}$ such that $K_{Y^s} + B^{Y^s} - \lambda^s F^s$ with $\lambda^s \in (0,1)$ is nef big klt. For each given s, we note that there are only finitely many blowups $V \to Z^s$ dominated by Y^s, say via $h : Y^s \to V$. Therefore we can choose $\lambda^s \in (0,1)$ such that $[s\lambda^s F^s] = 0$ and $\{h_*(K_{Y^s} + B^{Y^s} - \lambda^s F^s)\}^2 > C$ for every V with $\{h_*(K_{Y^s}+B^{Y^s})\}^2 > C$. (Note that $\{h_*(K_{Y^s}+B^{Y^s})\}^2 \geq (K_{Y^s}+B^{Y^s})^2 = C$.)

Now apply (5.3) to klt surfaces $(\overline{X}^s, \overline{B}^s - \lambda^s \overline{E}^s)$ with $f^s : Y^s \to X^s$ and $g^s : Y^s \to Z^s$. Then, after passing to a subsequence, we have a bounded family of blowups $\pi^s : V^s \to Z^s$ dominated by Y^s, say via $h^s : Y^s \to V^s$, such that

$$\{h^s_*(K_{Y^s} + B^{Y^s} - \lambda^s F^s)\}^2 \leq (K_{Y^t} + B^{Y^t} - \lambda^t F^t)^2 \text{ if } s < t.$$

Since we have $(K_{Y^t} + B^{Y^t} - \lambda^t F^t)^2 \leq (K_{Y^t} + B^{Y^t})^2 = C$, we have

$$\{h^s_*(K_{Y^s} + B^{Y^s} - \lambda^s F^s)\}^2 \leq C \text{ for all } s.$$

By our choice of λ^s, this means for all s that

$$(K_{Y^s} + B^{Y^s} - \lambda^s F^s)^2 = \{h^s_*(K_{Y^s} + B^{Y^s} - \lambda^s F^s)\}^2 = C,$$
$$(\overline{f}^s)^*(\overline{K}^s + \overline{B}^s - \lambda^s \overline{E}^s) = (h^s)^* h^s_*(K_{Y^s} + B^{Y^s} - \lambda^s F^s).$$

Thus every curve C in $\operatorname{Exc}(h^s)$ satisfies $C \cdot (\overline{f}^s)^*(\overline{K}^s + \overline{B}^s - \lambda^s \overline{E}^s) = 0$, and V^s dominates $\overline{X}^s$ and hence X^s. If we write $e^s : V^s \to X^s$, then $\operatorname{Exc}(e^s) \cup \operatorname{Supp} B^{V^s} \subset (\pi^s)^{-1}(D^s)$ and $(X^s, \operatorname{Supp} B^s)$ is bounded.

Passing to a subsequence and renumbering B^s_j, we may assume that

1. B^s has the same number of components for all s,
2. $(K_{X^s} + \sum x_j B^s_j)^2$ is a quadratic form $q(x)$ in x_j's whose coefficients are constant in s,
3. for each j, b^s_j is non-decreasing in s.

Take any $s < t$, and we claim $b^s_j = b^t_j$ for all j. Set $E^i = \sum (b^t_j - b^s_j) \cdot B^i_j$ for $i = s,t$. Then from (2) above, we have

$$0 = q(b^t) - q(b^s) = (K_{X^t} + B^t) \cdot E^t + (K_{X^s} + B^s) \cdot E^s \geq 0.$$

Since $K_{X^t} + B^t$ are all ample, we have the claim. Thus (X^s, B^s) is bounded, and $F_{lc}(C,\mathcal{A})$ is bounded. □

Proof. Proof of (6.2) If $\amalg(X_m, B_m)$ is the normalization of $(X,B) \in F_{slc}(C,\mathcal{A})$, then $(K_{X_m} + B_m)^2$ are in the DCC set $K_{lc}(\mathcal{A} \cup \{1\})$ and their sum is equal to C. Therefore there are only finitely many possible values C_i for $(K_{X_m} + B_m)^2$, and by (6.4) (X_m, B_m) are bounded. Since the conductor curve $\Delta \subset \cup[B_m]$ is bounded, the way to patch Δ to itself is bounded, and (X, B) are bounded. $\square$

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