REAL ABELIAN VARIETIES WITH MANY LINE BUNDLES

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Let $L \subset \mathbb{C}^n$ be a lattice of rank $2n$. The quotient $A = \mathbb{C}^n/L$ is a compact complex manifold, called a *complex torus*. In some cases A can be embedded into a complex projective space, these are called *Abelian varieties*. See, for instance, [Griffiths-Harris78, p.300ff] or [Mumford70, Ch.I] for a general discussion of the theory. Two quotients $\mathbb{C}^n/L$ and $\mathbb{C}^n/L'$ are biholomorphic iff there is a $B \in GL(n, \mathbb{C})$ such that $BL = L'$.

If L is invariant under complex conjugation then the complex conjugation descends to A, giving a real structure to A. Let $A(\mathbb{R})$ denote the set of real points, that is, the fixed points of complex conjugation. The connected component of $A(\mathbb{R})$ containing the origin is $A(\mathbb{R})^0 = \mathbb{R}^n/(L \cap \mathbb{R}^n)$ which is a real torus of dimension n.

[Bochnak-Kucharz00] posed the following question:

Question 1. When is every topological $\mathbb{C}$-line bundle on $A(\mathbb{R})^0$ the restriction of a holomorphic line bundle on A?

As explained in [BCR87, 13.3.1], at least in the algebraic case, this is equivalent to the following:

Question 2. When can one approximate every continuous map

$$A(\mathbb{R})^0 \to S^2 = (x^2 + y^2 + z^2 = 1) \subset \mathbb{R}^3$$

by a regular map?

It turns out that a nice answer appears when we ask Question (1) only up to torsion. [Bochnak-Kucharz00] answered this in case $\dim A \geq 6$ and A is obtained from a complex Abelian variety by Weil's restriction of scalars. (This is defined as follows. Start with an arbitrary lattice $L \subset \mathbb{C}^n$. We can identify $\mathbb{C}^n = \mathbb{R}^{2n}$. Consider the lattice

$$L + \bar{L} \subset \mathbb{R}^{2n} + \mathbb{R}^{2n}$$

and redefine complex conjugation by the formula $\overline{(x, y)} = (\bar{y}, \bar{x})$.)

The proof in [Bochnak-Kucharz00] first deals with simple Abelian varieties using the inequality for the Picard number $\rho(A) \leq 2 \dim A$ and then goes on to investigate the behaviour of products.

In this note we answer Question (1) in general by working directly with the periods of a complex torus.

Theorem 3. *Let $L \subset \mathbb{C}^n$ be a conjugation invariant lattice of rank $2n$ and $A = \mathbb{C}^n/L$. Assume that $n \geq 3$. Then the following are equivalent.*

1. *The restriction map*
$$\mathrm{Rest} : \left\{ \begin{array}{c} \textit{holomorphic line} \\ \textit{bundles on } A \end{array} \right\} \to \left\{ \begin{array}{c} \textit{topological } \mathbb{C}\textit{-line} \\ \textit{bundles on } A(\mathbb{R})^0 \end{array} \right\}$$
is surjective modulo torsion.
2. *There is a $B \in GL(n, \mathbb{R})$ and an imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$ such that $BL \subset \mathbb{Q}(\sqrt{-d})^n \subset \mathbb{C}^n$.*
3. *There is a real elliptic curve with complex multiplication E such that A is isogeneous to E^n.*

The following proposition gives many cases when Rest is surjective. We believe that (4) covers almost all such cases.

Proposition 4. *Let $\mathcal{O} \subset \mathbb{Q}(\sqrt{-d})$ denote the ring of integers. Assume that $L \cap \mathbb{R}^n = \mathbb{Z}^n$ and $L \subset \mathcal{O}^n$. Then* Rest *is surjective.*

Remark 5. Observe that (3.1) does not assume that A is algebraic but it follows as a consequence. At first this is not too surprizing since algebraicity is equivalent to the existence of an ample line bundle. However, in dimension 2 many nonalgebraic tori satisfy (3.1).

It is also worth noting that a holomorphic line bundle on A with real structure restricts to a 2-torsion $\mathbb{C}$-bundle on $A(\mathbb{R})$, so (3.1) is connected with holomorphic line bundles on A which do not have a real structure.

(4) is closely related to [Bochnak-Kucharz93, Cor.1.6]. That result considers products of real elliptic curves and it deals with restrictions to $A(\mathbb{R})$ instead of $A(\mathbb{R})^0$. These account for the difference in the statements.

If $n = 1$ then every $\mathbb{C}$-line bundle on $A(\mathbb{R})$ is trivial, so the restriction map is always onto.

If $n = 2$ then the $\mathbb{C}$-line bundles on $A(\mathbb{R})^0$ form a group $\cong \mathbb{Z}$ so we need only one line bundle on A with nontrivial restriction. If (Id, Z) is the usual period matrix for L, then $\det Z \in \mathbb{Q}$ is the only condition. This can be satisfied even for A nonalgebraic.

There are some other cases when Rest is surjective. One such example can be constructed as follows. Let $e_1, \dots, e_n$ be a basis of $\mathbb{R}^n$. Set $f_1 = \frac{1}{m}\sqrt{-d}e_1$ and $f_i = m\sqrt{-d}e_i$ for $i > 1$.

Let Ω be the period matrix for L. (That is, columns of Ω are generators of L in some basis of $\mathbb{C}^n$.) A class $[\tau] \in H^2(A, \mathbb{Z})$ can be thought of as a skew symmetric, integral $2n \times 2n$ matrix M. The classical

Appell–Humbert theorem (cf. [Mumford70, I.6]) says that there is a line bundle with Chern class $[\tau]$ iff

$$\Omega M^{-1}\Omega^t = 0.$$

The first question asks for complex tori where the upper left $n\times n$ block of M can be arbitrary. Because of the inverse in the period relation, this is not easy to disentangle. However, looking at the proof of the Riemann condition given in [Griffiths-Harris78, p.303] we see that a suitable normalization of the period matrix reduces the question to a simple linear algebra lemma (10).

The proof of Theorem 3 compares three bases of $H^2(A,\mathbb{C})$, so we start with the definitions.

Notation 6. If $w \in L$ then $\bar{w} \in L$, so $w+\bar{w}$ and $w-\bar{w}$ are both in L. This implies that $L\cap\mathbb{R}^n$ and $L\cap\sqrt{-1}\mathbb{R}^n$ are rank n sublattices of L. Choose generators

$$e_1,\dots,e_n \in L\cap\mathbb{R}^n \quad\text{and}\quad f_1,\dots,f_n \in L\cap\sqrt{-1}\mathbb{R}^n.$$

Let u_i (resp. $\sqrt{-1}v_i$) be the corresponding coordinates on $\mathbb{R}^n$ (resp. $\sqrt{-1}\mathbb{R}^n$). We can choose complex coordinates

$$z_1 = x_1+\sqrt{-1}y_1,\dots,z_n = x_n+\sqrt{-1}y_n$$

such that $v_i = y_i$.

We can then write $u_i = \sum_j \sigma_{ij}x_j$, for an invertible real matrix $\Sigma = (\sigma_{ij})$.

(The usual normalization seems to be to identify the u-coordinates with the x-coordinates, but the current choice will work better for us. Thus Σ is the inverse of the usual period matrix.)

The 2-forms

$$du_i\wedge du_j :\ i<j \tag{6.1}$$

give a basis of $H^2(A(\mathbb{R})^0,\mathbb{Q})$, and, similarly, the 2-forms

$$du_i\wedge du_j : i<j,\quad du_i\wedge dv_j : i,j,\quad\text{and}\quad dv_i\wedge dv_j : i<j \tag{6.2}$$

give a basis of $H^2(A,\mathbb{Q})$.

Another basis of $H^2(A,\mathbb{C})$ is given by

$$dz_i\wedge dz_j : i<j,\quad dz_i\wedge d\bar{z}_j : i,j,\quad\text{and}\quad d\bar{z}_i\wedge d\bar{z}_j : i<j. \tag{6.3}$$

Finally, we also have the closely related basis

$$dx_i\wedge dx_j : i<j,\quad dx_i\wedge dy_j : i,j,\quad\text{and}\quad dy_i\wedge dy_j : i<j. \tag{6.4}$$

An element of $H^2(A,\mathbb{C})$ has *Hodge type* $(1,1)$ iff it is in the subspace spanned by

$$dz_i\wedge d\bar{z}_j \quad \forall\ i,j.$$

Since

$$dz_i \wedge d\bar{z}_j = dx_i \wedge dx_j + dy_i \wedge dy_j - \sqrt{-1}(dx_i \wedge dy_j + dx_j \wedge dy_i)$$

we conclude that

Lemma 7.

$$\sum_{i<j} \alpha_{ij} dx_i \wedge dx_j + \sum_{i,j} \beta_{ij} dx_i \wedge dy_j + \sum_{i<j} \gamma_{ij} dy_i \wedge dy_j$$

has type $(1,1)$ *iff*

$$\alpha_{ij} = \gamma_{ij} \quad \text{and} \quad \beta_{ij} = \beta_{ji} \quad \text{for very } i < j. \qquad \square$$

8 (Proof of Theorem 3)**.** By the Appell–Humbert theorem (which is a special case of Kodaira's theorem on $(1,1)$ classes, cf. [Griffiths-Harris78, p.191]), holomorphic line bundles on A correspond to those elements in $H^2(A,\mathbb{Z})$ whose Hodge type is $(1,1)$.

If (3.1) holds then for every $i<j$ there is a 2-form

$$\phi_{ij} = du_i \wedge du_j + \sum_{p,q} a_{ijpq} du_p \wedge dv_q + \sum_{p<q} b_{ijpq} dv_p \wedge dv_q \in H^2(A,\mathbb{Q})$$

which has type $(1,1)$. Let us rewrite ϕ_{ij} in terms of the basis (6.4). The only terms involving $dx_i \wedge dx_j$ come from

$$\begin{aligned} du_i \wedge du_j &= \sum_p \sigma_{ip} x_p \wedge \sum_q \sigma_{jq} x_q \\ &= \sum_{p<q} \pm(\sigma_{ip}\sigma_{jq} - \sigma_{iq}\sigma_{jp}) dx_p \wedge dx_q. \end{aligned}$$

The terms $du_p \wedge dv_q$ give $\sum_j \sigma_{pj} dx_j \wedge dy_q$ and $dv_p \wedge dv_q = dy_p \wedge dy_q$.

Taking into account (7) we conclude that

$$\sigma_{ip}\sigma_{jq} - \sigma_{iq}\sigma_{jp} = \pm b_{ijpq} \quad \forall i,j,p,q.$$

The only part we need is that the b_{ijpq} are rational. Thus Σ is a matrix such that the determinant of every 2×2 submatrix is rational. Thus $\Sigma = \sqrt{d}\Theta$ for some $\Theta \in GL(n,\mathbb{Q})$ and $d \in \mathbb{Q}$ by (10). d is positive since Σ is real.

Thus $\sqrt{d}L \subset \mathbb{Q}(\sqrt{-d})^n$, giving (3.2).

If (3.2) holds then

$$dz_p \wedge d\bar{z}_q + d\bar{z}_p \wedge dz_q = 2(dx_p \wedge dx_q + dy_p \wedge dy_q)$$

is a $(1,1)$-class in $H^2(A,\mathbb{Q})$, proving (3.1).

Finally, (3.2) is equivalent to A being isogeneous to the nth power of $E = \mathbb{C}/(\mathbb{Z}+\sqrt{-d}\mathbb{Z})$. The latter are all the elliptic curves with complex multiplication up to isogeny (cf. [Silverman86, VI.5.5]). $\square$

9 (Proof of Proposition 4)**.** The ring of integers of an imaginary quadratic field is

$$\mathcal{O} = \begin{cases} \mathbb{Z}[\sqrt{-d}] & \text{if } d \not\equiv -1 \mod 4, \text{ and} \\ \mathbb{Z}[\frac{1+\sqrt{-d}}{2}] & \text{if } d \equiv -1 \mod 4. \end{cases}$$

In the first case, L is contained in the lattice

$$\mathbb{Z}[\sqrt{-d}]^n = \langle e_1, \dots, e_n, \sqrt{-d}e_1, \dots, \sqrt{-d}e_n \rangle$$

and so

$$dx_1, \dots, dx_n \quad \text{and} \quad \tfrac{1}{\sqrt{d}}dy_1, \dots, \tfrac{1}{\sqrt{d}}dy_n$$

are in $H^1(A(\mathbb{C}), \mathbb{Z})$. Therefore the 2–forms

$$dx_i \wedge dx_j + dy_i \wedge dy_j$$

are in $H^2(A(\mathbb{C}), \mathbb{Z})$ and they have type $(1,1)$. By the Appell–Humbert theorem, each of these classes corresponds to a complex line bundle on $A(\mathbb{C})$ and their restrictions generate $H^2(A(\mathbb{R}), \mathbb{Z})$.

If $d \equiv -1 \mod 4$ then L is contained in the lattice

$$\mathbb{Z}[\tfrac{1+\sqrt{-d}}{2}]^n = \langle e_1, \dots, e_n, \tfrac{e_1+\sqrt{-d}e_1}{2}, \dots, \tfrac{e_n+\sqrt{-d}e_n}{2} \rangle$$

Thus $H^1(A(\mathbb{C}), \mathbb{Z})$ contains the forms

$$dx_i \pm \tfrac{1}{\sqrt{d}}dy_i \quad \text{and} \quad \tfrac{2}{\sqrt{d}}dy_i.$$

Since $d+1$ is divisible by 4, the 2–forms

$$\begin{aligned} &(dx_i - \tfrac{1}{\sqrt{d}}dy_i) \wedge (dx_j + \tfrac{1}{\sqrt{d}}dy_j) + \tfrac{d+1}{4}\tfrac{2}{\sqrt{d}}dy_i \wedge \tfrac{2}{\sqrt{d}}dy_j \\ &= dx_i \wedge dx_j + dy_i \wedge dy_j + \tfrac{1}{\sqrt{d}}(dx_i \wedge dy_j + dx_j \wedge dy_i). \end{aligned}$$

are in $H^2(A(\mathbb{C}), \mathbb{Z})$ and they have type $(1,1)$. □

Lemma 10. *Fix integers $m < n$. Let M be an invertible $n \times n$ matrix such that the determinant of every $m \times m$ submatrix is rational. Then $M = c \cdot$ (rational matrix) for some $c \in \mathbb{C}$. (And clearly $c^m \in \mathbb{Q}$.)*

Proof. The proof is by induction on $n - m$. We start with $m = n - 1$. The determinant theoretic adjoint $adj(M)$ is a rational matrix, thus M^{-1} is a rational matrix times $\det M^{-1}$. Hence M is a rational matrix times a constant c. This implies that c^m is rational, hence $N = \sqrt[m]{d} \cdot$ (rational matrix) where we may assume that d is an mth power free integer. In this case d is unique.

If $n - m \geq 2$ then let $N \subset M$ be an $(n-1) \times (n-1)$ submatrix. Its determinant is either zero, or N is invertible. In the latter case $N = \sqrt[m]{d} \cdot$ (rational matrix) and we have a unique choice for d. We claim that the same d works for every N. Indeed, two $(n-1) \times (n-1)$ submatrices N, N' have an $(n-2) \times (n-2)$ submatrix in common which can not be identically zero for $n-1 \geq 3$ since N, N' are invertible. Thus $(\sqrt[m]{d})^{-1}M$ is matrix where all $(n-1)\times(n-1)$ submatrices have rational determinant. □

Acknowledgments . We thank J. Bochnak and W. Kucharz for several very helpful discussions and comments. The first author was supported by a Marie Curie Postdoctoral Fellowship (number HPMF-CT-1999-00019) at the Department of Mathematics at the Vrije Universiteit, Amsterdam. Partial financial support for the second author was provided by the NSF under grant number DMS-0096268.

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