3. MODULI OF POLARIZED SCHEMES

János Kollár

The aim of the theory of moduli spaces is to construct schemes which parametrize all objects of a certain type. There does not seem to be any general introductory text devoted to moduli problems. One reason is that any approach to the theory of moduli spaces gets very technical, and most of the difficulties may be inevitable.

Our aim here is to give a relatively short introduction that is geared to do just as much as we need later. The books [Mumford-Fogarthy82; Viehweg95] have different aims, but they contain many topics that we discuss here. The main topics that we completely skip are Artin's theorems on the existence of coarse moduli spaces [Artin69a,b,70,74a] and Mumford's geometric invariant theory [Mumford-Fogarthy82].

To have a concrete question in mind, assume, for instance, that we want to understand all proper schemes and how they fit together to families. This concept can be formalized as a functor.

??.1 Moduli Functors

??.1.1 Definition. Fix a base scheme S. The *moduli functor of proper schemes* is the contravariant functor

$$\begin{aligned} &\mathcal{MS} : S\text{-schemes} \to \text{Sets}, \quad \text{given by} \\ &\mathcal{MS}(X/S) := \{\text{flat and proper morphisms } f : Y \to X\}/\{\text{isomorphims}\}, \end{aligned}$$

where two families $(f_1 : Y_1 \to X)$ and $(f_2 : Y_2 \to X)$ are isomorphic iff there is an isomorphism $h : Y_1 \to Y_2$ such that $f_1 = h \circ f_2$. If $g : X_1/S \to X_2/S$ is a morphism then

$$\begin{aligned} &g^* : \mathcal{MS}(X_2/S) \to \mathcal{MS}(X_1/S) \quad \text{is given by} \\ &(f : Y \to X_2) \mapsto (g^*f : Y \times_{X_2} X_1 \to X_1). \end{aligned}$$

Studying this functor is equivalent to understanding all flat families of proper schemes.

It is also of interest to study the cases when we consider families of schemes with additional structure. For instance this may be a sheaf, a divisor or a distinguished subscheme. For us the most important case is the following:

??.1.2 Definition. Fix a base scheme S. The *moduli functor of polarized proper schemes* is the functor

$$\mathcal{MP}: S\text{-schemes} \to \text{Sets}, \quad \text{given by}$$

$$\mathcal{MP}(X/S) := \left\{ \begin{array}{c} \text{pairs } (f: Y \to X, L), \text{ where} \\ f \text{ is a flat and proper morphism,} \\ L \text{ is an } f\text{-ample line bundle on } Y \end{array} \right\} \Big/ \{\text{equivalence}\},$$

where two families $(f_1 : Y_1 \to X, L_1)$ and $(f_2 : Y_2 \to X, L_2)$ are *equivalent* iff there is an isomorphism $h : Y_1 \to Y_2$ such that $f_1 = h \circ f_2$ and there is a line bundle M on X such that $L_1 \cong h^* L_2 \otimes f_1^* M$.

??.1.3 Remarks. There are several versions of this definition, not equivalent to each other. The two most important ones are the follwoing:

(??.1.3.1) Numerical equivalence. One can say that two families are equivalent iff $Y_1/X \cong Y_2/X$ and there are natural numbers n_1, n_2 such that for every point $x \in X$ the line bundles $L_1^{n_1}|Y_x$ and $L_2^{n_2}|Y_x$ are numerically equivalent. (For very singular schemes it is not clear that this gives something reasonable since the definition does not control infinitesimal properties of the base.)

Considering powers of the L_i is a minor point and the general case can usually be reduced to considering line bundles modulo numerical equivalence. Considering numerical equivalence instead of isomorphism is a more significant change which may require different considerations (see, for instance, [Viehweg95, 7.5]). In some other settings, the difference is minor [Kollár85, 4.1]. If $L|Y_x$ is a power of the canonical bundle, then none these problems occur. We formalize this observation in (??.8).

(??.1.3.2) Sheafifying the definition. The definition (??.1.2) has the disadvantage that it does not satisfy faithfully flat descent (see [BLR90, 6.1] for a very lucid discusion). This is essentially caused by similar problems with the naive definition of the relative Picard functor (see [Grothendieck62, 232] or [BLR90, 8.1]).

One can go around this problem by appropriate sheafification of $\mathcal{MP}$. The notion of canonical polarization (??.8) also provides a natural solution in many cases.

The functor of all polarized smooth varieties is too large. For instance, there are infinitely many families of plane curves, but there is only one family for each degree. In a similar manner, it is possible to decompose $\mathcal{MP}$ as a disjoint union of smaller subfunctors. The role of the degree is played by the notion of the Hilbert polynomial:

??.1.4 Definiton. Let $(f : Y \to X, L) \in \mathcal{MP}(X)$. Then $x \mapsto \chi(Y_x, L^k|Y_x)$ is a locally constant function on X. If it is constant (e.g. when X is connected) then the common value is denoted by $\chi(Y/X, L^k)$. This is a polynomial in k which is called the *Hilbert polynomial* of $(f : Y \to X, L)$.

??.1.5 Definition. Fix a base scheme S. Let $P(x)$ be a polynomial. The *functor of polarized schemes with Hilbert polynomial P* is given as

$$\mathcal{M}_P : S\text{-schemes} \to \text{Sets}, \quad \text{where}$$

$$\mathcal{M}_P(X/S) := \left\{ (f : Y \to X, L) \in \mathcal{MP}(X/S) \mid \chi(Y/X, L^k) = P(k) \text{ for every } k \right\}.$$

??.1.6 Remark. If X is a connected scheme then $\mathcal{MP}(X)$ is the disjoint union of the $\mathcal{M}_P(X)$ as P runs through the set of all polynomials. Similarly, any subfunctor $\mathcal{F} \subset \mathcal{MP}$ can be written as a disjoint union of the functors $\mathcal{F}_P$ defined in the obvious way.

In practice understanding all flat families is rather hopeless. Fortunately, in most cases one can make additional restrictions on the families that one needs to consider. For instance, one could consider only smooth morphisms. Another example of a functor with additional structure is the Hilbert functor. Instead of considering all schemes, we concentrate on subschemes of a given scheme:

??.1.7 Definition. (??.1.7.1) Let X/S be a scheme. The *Hilbert functor*

$$Hilb(X/S) : \{\text{schemes over } S\} \to \{\text{sets}\}$$

is defined as follows:

$$Hilb(X/S)(Z) = \left\{\begin{matrix}\text{subschemes } V \subset X \times_S Z \\ \text{which are proper and flat} \\ \text{over } Z\end{matrix}\right\}.$$

(??.1.7.2) Let X/S be a scheme, $\mathcal{O}(1)$ a relatively ample line bundle and P a polynomial. As in (??.1.5) one can define subfunctors

$$Hilb_P(X/S) : \{\text{schemes over } S\} \to \{\text{sets}\},$$

such that for any connected scheme Z

$$Hilb(X/S)(Z) = \cup_P Hilb_P(X/S)(Z).$$

??.2 Coarse and Fine Moduli Spaces

One approach to study a functor as above is to try to collect all information about it into a single scheme:

??.2.1 Definition. Let $\mathcal{F}$: S-schemes $\to$ Sets be a contravariant functor. We say that a scheme $\mathbf{F}/S$ *represents* $\mathcal{F}$ (or that $\mathbf{F}/S$ is a *fine moduli space* for $\mathcal{F}$) if there is a natural isomorphism

$$\mathcal{F}(*/S) \cong \mathrm{Hom}_S(*, \mathbf{F}).$$

The most important example of a representable functor is the Hilbert functor. Similar examples are the Quot functors or the Picard functor. These resuts are due to [Grothendieck62]. More detailed proofs can be found in [Mumford66; Kollár95].

??.2.2 Theorem. [Grothendieck62] *Let X/S be a projective scheme, $\mathcal{O}(1)$ a relatively ample line bundle and P a polynomial. The functor $Hilb_P(X/S)$ is represented by a morphism*

$$\mathrm{Univ}_P(X/S) \xrightarrow{u} \mathrm{Hilb}_P(X/S), \quad \textit{where} \quad \mathrm{Univ}_P(X/S) \subset X \times_S \mathrm{Hilb}_P(X/S).$$

$\mathrm{Hilb}_P(X/S)$ *is projective over* S. $\square$

Unfortunately fine moduli spaces rarely exist when one tries to parametrize varieties, as shown by the next example.

??.2.3 Example. Let $\mathcal{F}$ be a functor which is represented by a separated scheme $\mathbf{F}$. Let $Z_1, Z_2 \in \mathcal{F}(X/S)$ be two elements; X of finite type and reduced. For a point $x \in X$ let $j_x : x \to X$ denote the injection. Show that $Z_1 = Z_2$ iff $j_x^* Z_1 = j_x^* Z_2$ for every closed point $x \in X$.

Let $S = \operatorname{Spec} \mathbb{C}$, Y a variety with a nontrivial automorphism of finite order and E an elliptic curve. Show that there are two smooth and proper morphisms $f_i : Z_i \to E$ such that every closed fiber is isomorphic to Y but the two families are not isomorphic.

Thus the existence of nontrivial automorphisms precludes the existence of a fine moduli space.

In practice the following weakening of (??.2.1) is the most useful:

??.2.4 Definition. Let $\mathcal{F}$: S-schemes $\to$ Sets be a contravariant functor. We say that a scheme $\mathbf{F}/S$ *coarsely represents* $\mathcal{F}$ (or that $\mathbf{F}/S$ is a *coarse moduli space* for $\mathcal{F}$) if:

(??.2.4.1) There is a natural transformation

$$T : \mathcal{F}(*/S) \to \mathrm{Hom}_S(*, \mathbf{F}).$$

(??.2.4.2) If Z is an S-scheme and there is a natural transformation

$$T_Z : \mathcal{F}(*/S) \to \mathrm{Hom}_S(*, Z),$$

then there is a morphism $h : \mathbf{F} \to Z$ such that $T_Z = h \circ T$.

(??.2.4.3) If K is an algebraically closed field then

$$T : \mathcal{F}(\operatorname{Spec} K/S) \to \mathrm{Hom}_S(\operatorname{Spec} K, \mathbf{F}) \quad \text{is an isomorphism.}$$

It is technically convenient to have a further weakening of this notion:

??.2.5 Definition. Notation as above. We say that a scheme $\mathbf{F}/S$ is a *categorical moduli space* for $\mathcal{F}$ if the conditions (??.2.4.1–2) are satisfied.

It is important to note that the various moduli spaces depend only on the values of a functor on local schemes. To be precise:

??.2.6 Proposition. *Let* $\mathcal{F}$: S*-schemes* $\to$ *Sets be a contravariant functor and*

$$\mathcal{F}^{loc} : \{\textit{local } S\textit{-schemes}\} \to \textit{Sets}$$

be its restriction. Then $\mathbf{F}/S$ *is a categorical (resp. coarse) moduli space for* $\mathcal{F}$ *iff it is a categorical (resp. coarse) moduli space for* $\mathcal{F}^{loc}$.

Proof. Any natural transformation $T : \mathcal{F}(*) \to \mathrm{Hom}(*, W)$ restricts to $T^{loc} : \mathcal{F}^{loc}(*) \to \mathrm{Hom}(*, W)$. Conversely, asume that we have $T^{loc} : \mathcal{F}^{loc}(*) \to \mathrm{Hom}(*, W)$. Giving a morphism $f : X \to W$ is equivalent to giving a compatible set of morphisms $f_x : \operatorname{Spec} \mathcal{O}_{x,X} \to W$ where x runs through all points of X. Thus T^{loc} extends to $T : \mathcal{F}(*) \to \mathrm{Hom}(*, W)$. This shows the assertion for categorical moduli spaces.

A categorical moduli space is coarse iff condition (??.2.4.3) is satisfied. This concerns only the local schemes $\operatorname{Spec} K$. $\square$

??.3 Separatedness of Moduli Functors

Let X be a scheme. The valuative criteria of separatedness and of properness express these properties in terms of the functor $\mathrm{Hom}(*, X)$. Using these versions as guide posts, we can define the above notions for functors as well. For simplicity we define these only for subfunctors of $\mathcal{MP}$; the general case should be clear.

Unfortunately it seems impossible to define these notions for an abstract functor. The main reason is that we consider families up to isomorphism, and so we loose the automorhism groups of the families. At this point it would be very natural to introduce the notion of *stacks*. For the case of canonically polarized varieties the advantages are relatively minor, therefore we stay with moduli functors. See [Artin69b,70,74a; Deligne-Mumford69] for various discussions about stacks.

??.3.1 Definition. A subfunctor $\mathcal{F} \subset \mathcal{MP}$ is said to be *separated* if the following condition is satisfied:

Let T be the spectrum of a DVR with generic point $j : t_g \to T$ and $(f_i : Z_i \to T, L_i) \in \mathcal{F}(T)$ two families. Then every isomorphism $h_g : j^*(Z_1, L_1) \to j^*(Z_2, L_2)$ over the generic point extends to an isomorphism $h : Z_1/T \to Z_2/T$ such that $L_1 \cong h^* L_2$.

??.3.2 Examples of nonseparatedness. Even for the moduli functor of smooth and proper varieties, separatedness fails. There are three distinct ways that this can happen:

(??.3.2.1) Consider for example the family $Y = \mathbb{P}^1 \times \mathbb{C}/\mathbb{C}$ with coordinates $(x : y; t)$. Let $h : Y \dashrightarrow Y$ be given by $(x : y; t) \mapsto (tx : y; t)$. This is an isomorphism over $\mathbb{C} - \{0\}$ but it does not even give a birational map of the central fibers. Similar examples can be constructed from ruled surfaces. By a theorem of [Matsusaka-Mumford64], this problem only occurs if the fiber over the closed point is ruled.

(??.3.2.2) Let Z be any smooth variety of dimension at least 2. Consider the family $Y' = Z \times \mathbb{C}/\mathbb{C}$ and let $s_i : \mathbb{C} \to Y'$ be two sections such that $\operatorname{im} s_1$ and $\operatorname{im} s_2$ intersect transversally at the central fiber and they are disjoint otherwise. Let $Y_1 \to Y'$ be obtained by first blowing up $\operatorname{im} s_1$ and then the birational transform of $\operatorname{im} s_2$. We construct Y_2 by first blowing up $\operatorname{im} s_2$ and then the birational transform of $\operatorname{im} s_1$. These two families are isomorphic over $\mathbb{C} - \{0\}$, but not over $\mathbb{C}$. The natural map $Y_1 \to Y \leftarrow Y_2$ is birational on the central fiber, but is not an isomorphism there.

(??.3.2.3) The third example, observed in [Atiyah58], is more subtle. Let Y' be a threefold with an ordinary node $p \in Y'$ and $f' : Y' \to \mathbb{C}$ a proper morphism which is smooth outside p. Assume in addition that ${f'}^{-1}(0)$ also has an ordinary node at p. Y' has two small resolutions; these give two families $f_i : Y_i \to \mathbb{C}$, and the f_i are smooth and proper. The two central fibers are isomorphic, but the isomorphism does not come from an isomorphism of the families Y_i. This situation gives rather unusual nonseparated algebraic spaces as moduli spaces; see [Artin74b].

The examples (??.3.2.2–3) can not occur for polarized schemes:

??.3.3 Theorem. [Matsusaka-Mumford64] *Let $f_i : X_i \to S$ be proper morphisms and L_i f_i-ample line bundles on X_i. Let $U_i \subset X_i$ be open sets; $Z_i := X_i - U_i$. Assume that:*

(??.3.3.1) There is an isomorphism $g_U : (U_1/S, L_1|U_1) \cong (U_2/S, L_2|U_2)$*, and*
(??.3.3.2) $\operatorname{depth}_{Z_i} X_i \geq 2$ *for* $i = 1, 2$.
Then g_U *extends to an isomorphism* $g : X_1 \cong X_2$ *such that* $L_1 \cong g^* L_2$.

??.3.3.3 Remark. Condition (??.3.3.2) is satisfied if, for instance, the X_i are normal and the Z_i have codimension at least 2.

Proof. Let $j_i : U_i \to X_i$ be the injections. By (??.3.3.2) we obtain that $(j_i)_*(L_i^m|U_i) = L_i^m$ for every m (in fact this holds for any line bundle). Thus

$$(f_i|U_i)_*(L_i^m|U_i) = (f_i)_*(L_i^m) \quad \text{for } i = 1, 2. \tag{??.3.3.4}$$

Pick m such that L_i^m is f_i-very ample. Choosing a finite set of generating sections for $(f_1|U_1)_*(L_1^m|U_1) \cong (f_2|U_2)_*(L_2^m|U_2)$ we obtain two projective embeddings

$$\phi_i : X_i \hookrightarrow \mathbb{P}^N_S \quad \text{such that} \quad \phi_1(U_1) = \phi_2(U_2).$$

By (??.3.3.4) we see that $\phi_i(X_i)$ is the closure of $\phi_i(U_i)$ for $i = 1, 2$. Thus $X_1 \cong \phi_1(X_1) = \phi_2(X_2) \cong X_2$. $\square$

??.3.4 Corollary. *Let* T *be the spectrum of a DVR with closed point* $0 \in T$. *Let* $(f_i : Z_i \to T, L_i) \in \mathcal{MP}(T)$ *be two families, and assume that the central fibers* $f_i^{-1}(0)$ *are* S_1. *Let* $U_i \subset Z_i$ *be open sets which contain the generic fiber and every generic point of the central fiber. Let* $h_U : (U_1/T, L_1) \to (U_2/T, L_2)$ *be an isomorphism. Then* h_U *extends to an isomorphism* $h : Z_1/T \to Z_2/T$ *such that* $L_1 \cong h^* L_2$. $\square$

??.3.5 Example. Let us see what happens with (??.2) in the presence of a polarization. Assume for simplicty that Z is a smooth surface. For $t \neq 0$ the fiber of our family is Z blown up at the two points $s_i(t)$. Let E_1, E_2 be the two exceptional curves and L the polarizing ample line bundle.

Check that if $L \cdot E_1 < L \cdot E_2$ then Y_2 is the only polarized extension, and if $L \cdot E_1 > L \cdot E_2$ then Y_1 is. If $L \cdot E_1 = L \cdot E_2$ then the correct polarized extension is singular; it is obtained by blowing up $\operatorname{im} s_1 \cup \operatorname{im} s_2$.

??.4 Boundedness of Moduli Functors

In practice one prefers to work with moduli functors which are "not too big", for instance, where the coarse moduli scheme is of finite type. Even when there is no coarse moduli space, the following definition works:

??.4.1 Definition. Let $\mathcal{F}$ be a subfunctor of $\mathcal{MP}$. We say that $\mathcal{F}$ is *bounded* if there is a scheme of finite type X and a family $(f : Y \to X, L) \in \mathcal{MP}(X)$ with the following property:

For any $(g : U \to V, M) \in \mathcal{F}(V)$ there is an open cover $\cup V_i = V$ and morphisms $c_i : V_i \to X$ such that $(g : U \to V, M)|V_i$ is equivalent to $c_i^*(f : Y \to X, L)$.

In this case we say that $(f : Y \to X, L)$ is a *bounding family* for $\mathcal{F}$.

If in addition $(f : Y \to X, L) \in \mathcal{F}(X)$, then $(f : Y \to X, L)$ is called a *locally versal family* for $\mathcal{F}$.

??.4.2 Remarks. The definition makes sense for subfunctors of $\mathcal{M}$, but it may not be the right one. For instance, with the above definition, the functor

$$\mathcal{E} := \{\text{families of smooth curves of genus } 1\}$$

is not bounded. In general one should allow $\{V_i\}$ to be an étale cover of V, or one may use an even finer topology.

The first major general theorem about boundedness is due to [Matsusaka72]. Its proof is rather difficult, and we do not need it. Several recent results give more detailed information; see [!!] for an overview.

??.4.3 Theorem. [Matsusaka72] *Fix a polynomial P. The functor*

$$\mathcal{M}_P(smooth) := \{(f : Y \to X, L) \in \mathcal{M}_P | f \text{ is smooth}\}$$

is bounded. □

??.4.4 Remark. $\mathcal{M}_P$ itself is not bounded, not even for 1–dimensional schemes. Let C_n be a curve such that $\operatorname{red} C_n \cong \mathbb{P}^1$ and whose structure sheaf sits in an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-n) + \mathcal{O}_{\mathbb{P}^1}(n) \to \mathcal{O}_{C_n} \to \mathcal{O}_{\mathbb{P}^1} \to 0.$$

Let L_n be a line bundle on C_n such that $L_n | \operatorname{red} C_n \cong \mathcal{O}_{\mathbb{P}^1}(1)$. Then $\chi(C_n, L_n^k) = 3k + 3$, but the family is not bounded.

It is, however, possible, that $\mathcal{M}_P(reduced)$ is bounded. Partial results can be found in [Kollár85; Matsusaka86].

The following definition is designed to give all bounded subfunctors of $\mathcal{M}_P$:

??.4.5 Definition. For any integer k_0, let $\mathcal{M}_P(k_0)$ be the subfunctor of $\mathcal{M}_P$ consisting of those $(f : Y \to X, L) \in \mathcal{M}_P(X)$ such that for $k \geq k_0$

$$R^i f_* L^k = 0 \quad \text{for } i > 0, \quad L^k \quad \text{is } f\text{-very ample, and} \quad f_* L^k \quad \text{is locally free.}$$

??.4.5.1 Remark. If $R^i g_* M^k = 0$ for $i > 0$ then $g_* M^k$ is automatically locally free, and under these conditions being g-very ample is an open condition.

The following characterization of boundedness is frequently chosen as its definition:

??.4.6 Proposition. *Let $\mathcal{F}$ be a subfunctor of $\mathcal{M}_P$. The following are equivalent:*
(??.4.6.1) $\mathcal{F}$ is bounded.
(??.4.6.2) $\mathcal{F}$ is a subfunctor of $\mathcal{M}_P(k_0)$ for some k_0.

Proof. Let $f : Y \to X$ show that $\mathcal{F}$ is bounded. There is $k_0 > 0$ such that for every $k \geq k_0$

$$R^i f_* L^k = 0 \quad \text{for } i > 0, \quad L^k \quad \text{is } f\text{-very ample, and} \quad f_* L^k \quad \text{is locally free.}$$

These properties are preserved after an arbitrary base change, thus (??.4.6.1) implies (??.4.6.2). We do not need the converse; see e.g. [Viehweg85, 1.38]. □

??.4.7 Reduction to $\mathcal{M}_P(1)$. There are some natural transformations of $\mathcal{MP}$ to itself which allow us to reduce some questions to $\mathcal{M}_P(1)$.

If k is a natural number and $P(x)$ a polynomial then set $P_k(x) := P(kx)$. Then

$$(f : Y \to X, L) \mapsto (f : Y \to X, L^k)$$

defines a natural transformation $T_k : \mathcal{M}_P \to \mathcal{M}_{P_k}$. If $k \geq k_0$ then this restricts to a natural transformation $T_k : \mathcal{M}_P(k_0) \to \mathcal{M}_{P_k}(1)$.

T_k is neither injective nor surjcetive. For us the main problem is that it is not injective. This is caused by k-torsion line bundles on Y. On the nice side, T_k is étale if all the residue characteristics are relatively prime to k (cf. [Grothendieck62, 236-2.6]). Unfortunately it is not separated. A simple example is the following:

Let $g : S \to C$ be an elliptic surface with an m-fold fiber $mE \subset S$ over $0 \in C$. Then $\mathcal{O}_S(E)$ is a line bundle which is trivial over $C - 0$ but not over C. Furthermore, $\mathcal{O}_S(E)^m$ is equivalent to $\mathcal{O}_S$.

Most likely this problem does not occur if all fibers of f are geometrically reduced (which is a very reasonable assumption). Also, we are mainly interested in functors $\mathcal{F}$ where the polarization is given by the relative dualizing sheaf. In such cases $T_k : \mathcal{F} \to \mathcal{M}_{P_k}(1)$ is injective.

[Viehweg95, 1.38] notes that

$$(T_k, T_{k+1}) : \mathcal{M}_P(k_0) \to \mathcal{M}_{P_k}(1) \times \mathcal{M}_{P_{k+1}}(1)$$

is an injection. Unfortunately in this case it is not clear that the image is a locally closed subfunctor (??.5.1), but this is ok if all fibers are geometrically reduced.

There are some theorems which are easier for $\mathcal{M}_P(1)$ than for $\mathcal{M}_P(k_0)$ (see especially (??.7.3)). The above tricks can then be used to get similar results for $\mathcal{M}_P(k_0)$, at least when all fibers are geometrically reduced.

??.5 Locally Closed Moduli Functors

Even if a subfunctor $\mathcal{F} \subset \mathcal{MP}$ is bounded and separated, it may give us rather random families. The following definition formulates the notion that the families $\mathcal{F}(X)$ be characterized by a reasonable algebro-geometric property.

??.5.1 Definition. A subfunctor $\mathcal{F} \subset \mathcal{MP}$ is called *locally closed* iff the following condition is satisfied:

For every $(f : Y \to X, L) \in \mathcal{MP}(X)$ there is a locally closed subscheme $j : X' \hookrightarrow X$ such that if $g : Z \to X$ is any morphism then

$$g^*(f : Y \to X, L) \in \mathcal{F}(Z) \quad \text{iff there is a factorization} \quad g : Z \to X' \xrightarrow{j} X.$$

We say that $\mathcal{F} \subset \mathcal{MP}$ is *open* iff $X' \subset X$ is open for every X.

??.5.2 Remark. This definition mixes together two notions which, from the technical point of view, better to keep separate.

If we require (??.5.1) only for those schemes which are spectra of Artin rings, we obtain a notion that can best be expressed in terms of an obstruction theory (see

[Artin69b]). The global case is related to the existence of an algebraic approximation to $\mathcal{F}$ (see [Artin69b]).

??.5.3 Examples. (??.5.3.1) $\mathcal{MP}(smooth)$ and $\mathcal{MP}(reduced)$ are open in X.

(??.5.3.2) The moduli functor $\mathcal{MP}(S_r)$ consisting of those morphisms whose fibers satisfy Serre's condition S_r is open in X. Similarly, $\mathcal{MP}(Gorenstein)$ consisting of those morphisms whose fibers are Gorenstein is open in X.

(??.5.3.3) Fix an integer k and let $\mathcal{F}(X)$ be the set of morphisms $(f : Y \to X, L)$ with reduced and Gorenstein fibers such that L is equivalent to $\omega^k_{Y/X}$. Then $\mathcal{F}$ is locally closed. This follows from (!!).

??.5.4 Remark. The above examples are rather simple since they give open subfunctors. In such cases it is sufficient to pose restrictions on the closed fibers. The functor of canonically polarized surfaces (!!) is only locally closed, and therefore the characterization should involve infinitesimal conditions as well. These results turn out to be rather techincal.

Another example of an open subfunctor is the following:

??.5.5 Lemma. *$\mathcal{M}_P(k_0)$ is an open subfunctor of $\mathcal{M}_P$.*

Proof. Pick $(f : Y \to X, L) \in \mathcal{M}_P(X)$ and fix k. By semicontinuity, there is a largest open subset $X_k \subset X$ such that $R^i f_* L^k|X_k = 0$ for $i > 0$. Then $f_* L^k|X_k$ is locally free and relative very ampleness is an open condition over X_k. Thus there is a unique largest open set $X'_k \subset X$ such that the required conditions for k are satisfied on X'_k.

$X'_k = X$ for $k \gg 1$, thus $X' = \cap_k X'_k$ is a finite intersection. Therefore $X' \subset X$ is open. $\square$

??.6 Complete Moduli Functors

??.6.1 Definition. Let $\mathcal{F}$ be a subfunctor of $\mathcal{MP}$. $\mathcal{F}$ is said to be *proper* or *complete* if it is separated, bounded and the following condition is satisfied:

If T is the spectrum of a DVR with generic point $j : t_g \to T$ and $(Z_g, L_g) \in \mathcal{F}(T_g)$, then there are

(??.6.1.1) T', the spectrum of another DVR;

(??.6.1.2) a finite and dominant morphism $h : T' \to T$;

(??.6.1.3) $(Z', L') \in \mathcal{F}(T')$;

such that $h^*(Z_g, L_g) = j'^*(Z', L')$.

Informally speaking: Every family $f_g : (Z_g, L_g) \to t_g$ can be extended to a family over the closed point, at least after a finite base change.

??.6.2 Remark. The moduli functor of smooth and proper varieties is not complete. For instance, let Y be a smooth variety and $f : Y \to \mathbb{C}$ a proper morphism which is smooth outside the central fiber and such that the central fiber has only ordinary nodes. Assume also that $\dim Y$ is even. Lefschetz theory (see, for instance, [!!]) tells us that $f' : Y' := f^{-1}(\mathbb{C} - \{0\}) \to \mathbb{C} - \{0\}$ can not be extended to a smooth and proper morphism, even after a base change and even if we allow the extension to be a differentiable manifold.

Thus, if we would like to get proper moduli spaces, we are forced to allow some singular schemes as limits of smooth varieties. It is not at all clear which singular schemes should be allowed, and there does not seem to be a unique choice. We come back to this problem in (!!).

??.7 Moduli Spaces up to a Group Action

As we saw, in most cases one can not hope for fine moduli spaces when we consider moduli functors of schemes. We also observed in (??.2.3) that this problem is connected with the existence of nontrivial automorphisms. This raises the possibility that maybe with a suitable group action twist, a fine moduli space can be constructed.

We use the basic definitions and results about group actions surveyed in (!!).

??.7.1 Tentative definition. Let G be a group scheme over S and X a scheme over S with a G-action. For any scheme Y, $\operatorname{Hom}_S(Y, G)$ acts on $\operatorname{Hom}_S(Y, X)$; the set of orbits is denoted by $\operatorname{Hom}_S(Y, X)/G$. This gives a functor

$$Y \mapsto \operatorname{Hom}_S(Y, X)/G,$$

which can be viewed as an analog of the Hom functor.

One problem with this definition is that it is not local. For instance, let $X = \mathbb{C}^{n+1} - \{0\}$ and $G = \mathbb{C}^*$ acting by multiplication. Then $X/G \cong \mathbb{P}^n$, so one would expect that there is a morphism $I \in \operatorname{Hom}(\mathbb{P}^n, \mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*$ corresponding to the identity morphism of $\mathbb{P}^n$. Such a map exists locally, but not globally.

There are two ways of overcoming this problem; they are discussed in (??.7.4–10). For the moment we just ignore it by considering only the case when Y is local.

??.7.2 Definition. Let G be a group scheme. A *G-equivariant family* in $\mathcal{MP}$ is a G-morphism of G-schemes $f : Y \to X$ and a polarization L on Y such that $p_X^* L$ is equivalent to $m_X^* L$ (as polarizations of $G \times Y \to G \times X$) where $p_X : G \times X \to X$ is the second projection and $m_X : G \times X \to X$ is multiplication.

Fine moduli spaces up to a group action usually exist:

??.7.3 Theorem. *For every P there is a group scheme G and a G-equivariant family $(f : U_{P,G}(1) \to M_{P,G}(1), L) \in \mathcal{M}_P(1)$ such that there is an isomorphism of functors*

$$\mathcal{M}_P(1)(*) \cong \operatorname{Hom}_S(*, M_{P,G}(1))/G,$$

where we view both sides as functors on the category of local S-schemes.

??.7.3.1 Complement. (??.7.3.1.1) G is not unique. Our construction gives $G = PGL(P(1), S)$.

(??.7.3.1.2) I do not know if the analogous result for $\mathcal{M}_P(k_0)$ holds or not. See (??.4.7) for ways of overcoming this problem.

Proof. Pick $(f : Y \to X, L) \in \mathcal{M}_P(1)(X)$. By assumption $f_* L$ is locally free, hence free if X is local. Choose a trivialization $t : f_* L \cong \mathcal{O}_X^{m+1}$. t gives us $m+1$ sections $t_0, \ldots, t_m \in H^0(Y, L)$ and a morphism

$$s(t) : Y/X \to \mathbb{P}^m_S \times_S X \quad \text{given by} \quad y \mapsto (t_0(y) : \cdots : t_m(y), f(y)).$$

$s(t): Y/X \to \mathbb{P}^m_X/X$ is a closed embedding. Its image gives a family in $\mathrm{Hilb}_P(\mathbb{P}^m_S/S)(X)$. Thus we obtain a morphism

$$\sigma(t): X \to \mathrm{Hilb}_P(\mathbb{P}^m_S/S).$$

Set $G = PGL(m+1, S)$. A different trivialization $t': f_*L \cong \mathcal{O}_X^{m+1}$ gives a different morphism $\sigma(t'): X \to \mathrm{Hilb}_P(\mathbb{P}^m_S/S)$. The two trivializations differ by an element of $\mathrm{Hom}_S(X, GL(m+1))$ and the two morphisms $\sigma(t)$ and $\sigma(t')$ differ by the corresponding element in $\mathrm{Hom}_S(X, PGL(m+1))$. Thus $(f: Y \to X, L)$ gives a well defined element of $\mathrm{Hom}_S(X, \mathrm{Hilb}_P(\mathbb{P}^m_S/S))/G$.

There is a universal family

$$(u: \mathrm{Univ}_P(\mathbb{P}^m_S/S) \to \mathrm{Hilb}_P(\mathbb{P}^m_S/S), \mathcal{O}(1)) \in \mathcal{M}_P(\mathrm{Hilb}_P(\mathbb{P}^m_S/S)),$$

but it need not be in $\mathcal{M}_P(1)$. By (??.5.5) there is a largest open subscheme $M' \subset \mathrm{Hilb}_P(\mathbb{P}^m_S/S)$ such that the induced family over M' is in $\mathcal{M}_P(1)$.

This is still not our required $M_P(1)$ since the embedding $\sigma(t)$ is by a complete linear system. Let $v \in M'$ be a point and $U_v \subset \mathbb{P}^m_S$ the corresponding subscheme. $H^0(U_v, \mathcal{O}(1)) = m+1$ by the definition of M', thus the injection $U_v \to \mathbb{P}^m_S$ is by a complete linear system iff U_v is not contained in any hyperplane. The latter is an open condition, thus there is an open subscheme

$$M_{P,G}(1) \subset M' \subset \mathrm{Hilb}_P(\mathbb{P}^m_S/S)$$

with the following properties:

(??.7.3.2) If $h: X \to \mathrm{Hilb}_P(\mathbb{P}^m_S/S)$ is a morphism and $(u_X: Y \to X, \mathcal{O}(1))$ the induced family of subschemes $Y \subset \mathbb{P}^m_X$, then h factors through $M_{P,G}(1)$ iff the following conditions are satisfied:

(??.7.3.2.1) $(u_X: Y \to X, \mathcal{O}(1)) \in \mathcal{M}_P(1)$, and

(??.7.3.2.2) $t(h): p_*\mathcal{O}_{\mathbb{P}^m_Y}(1) \to (u_X)_*\mathcal{O}_Y(1)$ is an isomorphism.

For such a family $\sigma(t(h)) = h$, which shows (??.7.3). $\square$

In order go from local to global schemes, the straightforward approach is to sheafify everything:

??.7.4 Definition. Let $\mathcal{G}$ be a contravariant functor from schemes to sets. The *sheafification* (in the Zariski topology) of $\mathcal{G}$ is the functor $\mathcal{G}^*$ defined as follows:

Fix a scheme X. Then $X \supset U \mapsto \mathcal{G}(U)$ is a presheaf on X. Let $\mathcal{G}^*(X)$ denote the the global sections of the associated sheaf.

??.7.5 Remarks. (??.7.5.1) There is a natural transformation $\mathcal{G}(X) \to \mathcal{G}^*(X)$ which is, in general, neither injective nor surjective.

(??.7.5.2) In studying moduli functors, it usually makes more sense to sheafify in the étale topology or, especially in positive characteristic, in the fppf topology (see [BLR90, 8.1] for a detailed discussion concerning the Picard functor). None of these sheafifications, however, make the moduli functors representable. For the purposes of (??.7.6) the Zariski topology suffices.

Essentially automatically we obtain the following global version of (??.7.3):

??.7.6 Theorem. *For every P there is a group scheme G and a G-equivariant family $(f : U_{P,G}(1) \to M_{P,G}(1), L) \in \mathcal{M}_P(1)$ such that there is an isomorphism of functors*

$$\mathcal{M}^*_P(1)(Z) \cong \operatorname{Hom}^*_S(Z, M_{P,G}(1))/G. \quad \square$$

A better way to globalize is to solve the problem of the choice of a trivialization $t : f_*L \cong \mathcal{O}_X^{m+1}$ by using a universal trivialization. The main step is the following construction:

??.7.7 Definition. Let X be a scheme and E a vector bundle of rank r on X. Let $p : PF(E) \to E$ denote the associated *projective frame bundle*. This is constructed as follows:

Set $Z = \operatorname{Proj}_X(E^* \otimes \mathcal{O}^r_X)$ and $p : Z \to X$ the projection. A point $z \in Z$ can be identified with r vectors in $E|p(z)$, up to scalars. Let $PF(E) \subset Z$ be the open subset where these r vectors are linearly independent. $PF(E) \to X$ is a Zariski locally trivial principal $PGL(r, S)$-bundle (!!).

If M is a line bundle on X then we obtain a natural isomorphism $PF(E) \cong PF(E \otimes M)$.

In order to obtain the universal frame, note that $p_*\mathcal{O}_Z(1) = E^* \otimes \mathcal{O}^r_X$, thus

$$H^0(Z, p^*E(1)) = H^0(X, E \otimes p_*\mathcal{O}_Z(1)) = H^0(X, (E \otimes E^*) \otimes \mathcal{O}^r_X).$$

$E \otimes E^*$ has a natural section (the identity morphism $E \to E$), and this gives r natural sections $t_i : \mathcal{O}_Z(-1) \to p^*E$. Over $PF(E)$ these sections are independent, thus we obtain an isomorphism

$$t : \mathcal{O}_{PF(E)}(-1)^r \cong p^*E|PF(E). \tag{??.7.7.1}$$

??.7.8 Construction. Pick $(f : Y \to X, L) \in \mathcal{M}_P(1)(X)$. By assumption f_*L is locally free. Using (??.7.7) we obtain

$$PF(f : Y \to X, L) := PF(f_*L).$$

It is called the *projective frame bundle* associated to the family $(f : Y \to X, L)$. The universal trivialization t (??.7.7.1) gives a morphism

$$u : PF(f : Y \to X, L) \to M_{P,G}(1).$$

??.7.9 Definition. Let G be an algebraic group scheme and X a G-scheme. Let $P(X, G)(*)$ denote the (contravariant) functor of Zariski locally trivial principal G-bundles over X. I.e., $P(X, G)(V)$ consists of pairs $(g : W \to V, h : W \to X)$ where $g : W \to V$ is a Zariski locally trivial principal G-bundle and h a G-morphism.

Two pairs $(g : W \to V, h : W \to X)$ and $(g' : W' \to V, h' : W' \to X)$ are identified if there is an isomorphism of G-schemes $q : W \to W'$ such that $g = g' \circ q$ and $h = h' \circ q$.

There is a natural inclusion $\operatorname{Hom}(Z, X) \to P(X, G)(Z)$ given by

$$[f : Z \to X] \mapsto [G \times Z \xrightarrow{(id_G, f)} G \times X \xrightarrow{m_X} X].$$

Ths shows that if $P(X, G)(*)$ and $P(X', G)(*)$ are isomorphic iff $X \cong X'$ as G-schemes.

??.7.10 Theorem. *The construction in (??.7.8) gives a natural transformation*

$$PF : \mathcal{M}_P(1)(*) \to P(M_{P,G}(1), G)(*),$$

which is an isomorphism on local schemes. □

The analogous result for subfunctors follows at once:

??.7.11 Corollary. *Let $\mathcal{F}$ be a locally closed subfunctor of $\mathcal{M}_P(1)$; $G = PGL(P(1), S)$. There is a G-equivariant family $(f : U_{F,G} \to F_G, L) \in \mathcal{F}(F_G)$ and a natural transformation*

$$PF : \mathcal{F}(*) \to P(F_G, G)(*),$$

which is an isomorphism on local schemes.

Proof. By assumption there is a unique largest subscheme $F_G \subset M_{P,G}(1)$ such that the induced family is in $\mathcal{F}$. F_G is G-invariant since it is unique. □

The natural transformation PF is not an isomorphism in general. If $g : W \to V$ is in $P(F_G, G)(V)$ then we obtain a family of polarized schemes

$$h^*(f : U_{F,G} \to F_G, L) \in \mathcal{F}(W).$$

It is also G-equivariant (as defined in (??.7.2)) but this is not enough to descend the family to V. (This is a problem about faithfully flat descent, alluded to in (??.1.3.2).

In some cases, these problems can be settled. This is discussed in the next section.

??.8 Canonical Polarizations

The following notion was introduced in [Kollár90] (in a somewhat sloppy way).

??.8.1 Definition. Let $\mathcal{F}$ be a subfunctor of $\mathcal{MP}$. $\mathcal{F}$ is said to be *functorially (or canonically) polarizable* (of level k) if the following conditions are satisfied:

(??.8.1.1) For every $(f : Y \to X, L) \in \mathcal{F}(X/S)$ there is an $(f : Y \to X, L^c) \in \mathcal{MP}(X/S)$ such that $(f : Y \to X, L^k)$ is equivalent to $(f : Y \to X, L^c)$.

(??.8.1.2) For any morphism $g : X' \to X$ there is an isomorphism $\sigma(g) : h^*(L^c) \cong (h^*L)^c$ where $h : X' \times_X Y \to Y$ is the second projection.

(??.8.1.3) These choices are functorial in the following sense:

If $X'' \xrightarrow{g'} X' \xrightarrow{g} X$ are morphisms and $h' : X'' \times_X Y \to X' \times_X Y$ the induced morphism then

$$\sigma(g' \circ g) = \sigma(g') \circ h'^* \sigma(g).$$

If the condition is satisfied, then L^c is called a *functorial (or canonical) polarization* of $f : Y \to X$. If the choice of L^c is specified, then we say that $\mathcal{F}$ is *functorially (or canonically) polarized* (of level k).

In the sequel, the level is assumed to be one, unless the contrary is stated explicitly.

??.8.2 Remark. The above definition has two parts. First of all, in every equivalence class it chooses a polarizing line bundle L^c. Second, this choice is done not just

up to isomorphism but up to canonical isomorphism. In particular the following is satisfied:

Let G be an algebraic group and $f : Y \to X$ a G-morphism of G-schemes. Assume that L is G-invariant up to equivalence. Then L^c is a G-line bundle.

Thus the notion of functorial polarization is the analog of the notion of G-linearization in [Mumford-Fogarthy82,1.3].

There are several examples of functorially polarizable moduli functors.

??.8.3 Examples.

(??.8.3.1) If $L|Y_x = \omega_{Y_x}$ for every $x \in X$ then $L^c = \omega_{Y/X}$ is a canonical polarization.

(??.8.3.2) If every family $(f : Y \to X, L) \in \mathcal{F}$ has a natural section $s : X \to Y$ then we can take $L^c = L \otimes (s^*L)^{-1}$. This applies, for instance, for polarized abelian varieties or for pointed varieties.

(??.8.3.3) Let $(f : Y \to X, L) \in \mathcal{M}_P$. Pick r such that $k := P(r) > 0$. We use the notation of (??.8.4).

Set $L^c = L^k \otimes f^*(\det R^{\cdot} f_*(L^r))^{-1}$. If M is a line bundle on X then there is a canonical ismorphism $(L \otimes f^*M)^c = L^c$. Thus L^c is a level k functorial polarization.

??.8.4 Definition. Let $g : U \to V$ be a proper morphism, F a coherent sheaf on U, flat over V. By [Mumford68, §5] there is a finite complex of locally free sheaves

$$K^{\cdot} := 0 \to K^0 \to K^1 \to \cdots \to K^n \to 0,$$

such that $H^i(K^{\cdot}) \cong R^i g_* F$ (and similarly for any base change). The line bundle

$$\det R^{\cdot} g_* F := \Big(\prod_{i \text{ even}} \det K^i \Big) \otimes \Big(\prod_{i \text{ odd}} \det K^i \Big)^{-1}$$

is independent of the choices made (this is left as an exercise).

The construction is especially simple if $R^i g_* F = 0$ for $i > 0$. Then $g_* F$ is locally free, and so $\det R^{\cdot} g_* F = \det g_* F$. This is the main case that we will use.

One of the main advantages of canonical polarizations is the next theorem:

??.8.5 Theorem. *Let $\mathcal{F}$ be a locally closed and canonically polarized subfunctor of $\mathcal{MP}$). Then faithfully flat descent holds for $\mathcal{F}$.*

Proof. This follows from the general theory of faithfully flat descent (see, for instance, [BLR90, 6.1]). The only case that we need is when $g : W \to V$ is a Zariski locally trivial principal G-bundle, where the following straightforward proof works:

Let $(f : U \to W, L^c) \in \mathcal{F}(W)$. Since $g : W \to V$ is Zariski locally trivial, there is an open cover $V = \cup V_i$ and sections $s_i : V_i \to W$. We obtain the families $s_i^*(f : U \to W, L^c)$ over V_i. Over $V_i \cap V_j$ the two sections differ by G-action only. That is, there are $\phi_{ij} : V_i \cap V_j \to G$ such that $s_i = \phi_{ij} \cdot s_j$. Multiplication by ϕ_{ij} is an automorphism of $g^{-1}(V_i \cap V_j)$. By (??.8.1.2) we get isomorphisms

$$\sigma(\phi_{ij}) : (f : U \to W, L^c)|g^{-1}(V_i \cap V_j) \to (f : U \to W, L^c)|g^{-1}(V_i \cap V_j),$$

which restrict to isomorphisms

$$\sigma'(\phi_{ij}) : s_j^*(f : U \to W, L^c)|(V_i \cap V_j) \to s_i^*(f : U \to W, L^c)|(V_i \cap V_j).$$

The condition (??.8.1.3) ensures that these patch the polarized families $s_i^*(f : U \to W, L^c)$ to a single family over V. □

As a consequence we obtain that PF is an isomorphism for canonically polarized moduli functors:

??.8.6 Theorem. *Let $\mathcal{F}$ be a locally closed and canonically polarized subfunctor of $\mathcal{M}_P(1)$. There is a G-equivariant family $(f : U_{F,G} \to F_G, L) \in \mathcal{F}(F_G)$ and a natural isomorphism*

$$PF : \mathcal{F}(*) \cong P(F_G, G)(*). \quad \square$$

Canonical polarizations also give natural line bundles on the base space of a family:

??.8.7 Definition. Let $\mathcal{F}$ be a canonically polarized subfunctor of $\mathcal{MP}$. For any $(f : Y \to X, L) \in \mathcal{F}(X)$ one can define a series of line bundles on X by

$$\lambda_k(f : Y \to X, L) = \det R^{\cdot} f_*((L^c)^{\otimes k}).$$

These line bundles are functorial in the sense that if $g : X' \to X$ is a morphism then we get canonical isomorphisms

$$\lambda_k(g^*(f : Y \to X, L)) \cong g^*(\lambda_k(f : Y \to X, L)).$$

??.9 Coarse Moduli and Geometric Quotients

The main advantage of (??.7.11) and (??.8.6) is that they allow us to construct moduli spaces as quotients of schemes by group actions. Before we state the main result along these lines, we need a theorem comparing good properties of a functor $\mathcal{F}$ with good properties of the corresponding G-space F_G.

We should remember that the notion of separatedness is not defined for an arbitrary functor. In order to give a general result, we need to translate the notion of separatedness to functors of the form $P(X, G)(*)$. This is based on recognizing isomorphisms:

??.9.1 Definition. Let $m_X : G \times X \to X$ be a G-scheme, Z a scheme and $(h_i : W_i \to Z, g_i : W_i \to X) \in P(X, G)(Z)$, $i = 1, 2$. Then the set of ismorphisms between the two principal G-bundles is defined by

$$\operatorname{Isom}(W_1, W_2) := \{\phi \in \operatorname{Hom}_Z^G(W_1, W_2) | h_1 = \phi \circ h_2 \text{ and } g_1 = \phi \circ g_2\}.$$

For principal G-bundles of the form $PF(f : Y \to X, L)$ this is equivalent to isomorphisms of the families. Thus the notion of separatedness for moduli functors translates to the following:

??.9.2 Definition. Let $m_X : G \times X \to X$ be a G-scheme. We say that the functor $P(X, G)(*)$ is *separated* if the following holds:

Let T be the spectrum of a DVR with generic point $j : t_g \to T$ and $(h_i : W_i \to T, q_i : W_i \to X) \in P(X, G)(T)$, $i = 1, 2$. Then the restriction map

$$\operatorname{Isom}(W_1, W_2) \to \operatorname{Isom}(j^*W_1, j^*W_2) \quad \text{is an isomorphism.}$$

??.9.3 Theorem. *Let G be an algebraic group scheme over S and X a G-scheme. Then*

(??.9.3.1) $P(X,G)(*)$ is separated iff the G action on X is proper.

(??.9.3.2) $P(X,G)(*)$ is complete iff X is complete mod G.

Proof. The G action on X is proper iff for every commutative diagram

$$\begin{array}{ccc} t_g & \xrightarrow{\phi_g} & G\times X \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle (m_X,p_2)} \\ T & \xrightarrow{f} & X\times X \end{array}$$

there is a unique morphism $\phi : T\to G\times X$ which keeps the diagram commutative. f is given by a pair of morphisms $f_1,f_2 : T\to X$ and the commutativity of the diagram is equivalent to $(f_1|t_g)=\phi_g\cdot(f_2|t_g)$. ϕ keeps commutativity iff $\phi|t_g=\phi_g$ and $f_1=\phi\cdot f_2$.

Each f_i determines an element $[f_i]\in P(X,G)(T)$ by $W_i = G\times T$ and $g_i : W_i\to X$ given by $g_i=m_X(id_G,f_i)$. Then $f_1=\phi\cdot f_2$ iff multiplication by ϕ is an isomorphism of W_1 and W_2 and $(f_1|t_g)=\phi_g\cdot(f_2|t_g)$ iff multiplication by ϕ_g is an isomorphism of j^*W_1 and j^*W_2. This shows (??.9.3.1).

(??.9.3.2) is similarly straightforward. □

The following theorem reduces the construction of moduli spaces to the construction of quotient spaces by group actions:

??.9.4 Theorem. *Let G be an algebraic group scheme over S and X a G-scheme.*

(??.9.4.1) A categorical moduli space for $P(X,G)(*)$ exists iff a categorical quotient of X by G exists and they are naturally isomorphic. It is denoted by $q : X\to Z$.

(??.9.4.2) If $q : X\to Z$ is a geometric quotient of X by G then it is a coarse moduli space for $P(X,G)(*)$.

(??.9.4.3) $q : X\to Z$ is a fine moduli space for $P(X,G)(*)$ iff $q : X\to Z$ is a Zariski locally trivial principal G-bundle.

Proof. The first part is equivalent to the following:

??.9.4.4 Claim. *Notation as above. For any scheme W there is an isomorphism*

$$\left\{\begin{matrix}\textit{natural transformations}\\ T : P(X,G)(*)\to \operatorname{Hom}(*,W)\end{matrix}\right\}\leftrightarrow\left\{\begin{matrix}G\textit{-equivariant morphisms}\\ \\ t : X\to W\end{matrix}\right\}.$$

Proof. Let T be a natural transformation. Setting $*=X$ and using the identity morphism $id_X : X\to X$ we get a G-equivariant morphism $t:=T(id_X) : X\to W$.

Conversely, fix $t : X\to W$ and let $(h : U\to V, g : U\to X)\in P(X,G)(V)$. Then $g\circ t : U\to W$ is a G-morphism. Since V is a categorical quotient of U, $g\circ t$ descendes to a morphism $T(t) : V\to W$. □

Assume that $q : X\to Z$ is a geometric quotient of X. If K is an algebraically closed field then $P(X,G)(\operatorname{Spec} K)$ is the set of $G(K)$-orbits on on $X(K)$. By property (!!) of geometric quotients, this is the same as $Z(K)$. Thus Z is a coarse moduli space for $P(X,G)(*)$.

Finally, assume that X is a Zariski locally trivial principal G-bundle over Z. If $p : U \to Z$ is any morphism then $U \times_Z X$ is a Zariski locally trivial principal G-bundle over U. Thus Z is a fine moduli space for $P(X,G)(*)$.

Conversely, assume that Z is a fine moduli space for $P(X,G)(*)$. Corresponding to the identity morphism of Z there is a Zariski locally trivial principal G-bundle $h : X' \to Z$ together with a G-morphism $g : X' \to X$. We are done if g is an isomorphism.

By assumption $P(X,G)(*)$ and $P(X',G)(*)$ are naturally isomorphic. This implies that $X \cong X'$. $\square$

??.9.4.5 Remark. The converse of (??.9.4.2) is a very interesting question which seems to be true in many important cases. The definition of geometric quotient is much more restrictive then the definiton of coarse moduli space. Conditions (!!) have no counterparts in (??.2.4), but in some cases they can be derived from the rest.