LOW DEGREE POLYNOMIAL EQUATIONS: ARITHMETIC, GEOMETRY AND TOPOLOGY

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Polynomials appear in mathematics frequently, and we all know from experience that low degree polynomials are easier to deal with than high degree ones. It is, however, not clear that there is a well defined class of "low degree" polynomials. For many questions, polynomials behave well if their degree is low enough, but the precise bound on the degree depends on the concrete problem.

My interest is to investigate polynomials through their zero sets. That is, using sets of the form

$$\{(x_1,\dots,x_n)|f(x_1,\dots,x_n)=0\}.$$

I intentionally refrain from specifying where the coordinates x_i are. They could be rational, real or complex numbers, but in some cases the x_i will be polynomials in a new variable t. My focus is on the polynomial f.

Consider, for instance, a polynomial

$$f := a_0 + \sum_{i=1}^{n} a_i x_i^k, \quad \text{where} \quad a_i \in \mathbb{Z}\setminus\{0\}.$$

Specifying where the coordinates are, leads us to various branches of mathematics:

Arithmetic. Choose $x_i \in \mathbb{Q}$. The solutions of these Fermat-type equations have been much studied, some cases going back to Diophantus, but we still know very little if $n > 2$.

Topology. Choose $x_i \in \mathbb{R}$ or $x_i \in \mathbb{C}$. The set of solutions is a topological manifold, and various topological properties can be related to algebraic properties of f. For instance, the dimension and the homology can be computed in terms of n, k. (Over $\mathbb{R}$ we also need to know the signs of the a_i.)

Complex manifolds. Choose $x_i \in \mathbb{C}$. The set of solutions is a complex analytic manifold. The holomorphic function theory of this complex manifold can be understood in terms of polynomials. This is especially true in the compact versions of this problem.

Finite fields. We can also look at solutions of $f = 0$ in finite fields. Centuries ago this was done by studying $f \equiv 0 \mod p$. Recently, algebraic geometry over finite fields found many connections with coding theory, combinatorics and computer science.

I like to think of any of the zero sets as a snapshot of the polynomial f. They all show something about f. Certain snapshots reveal more than others:

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Do zero sets determine a polynomial? For instance, $x_1^{2k}+\cdots+x_n^{2k}+1=0$ has no solutions in $\mathbb{Q}$, not even in $\mathbb{R}$. Thus the zero set gives essentially no information. The situation is very different over algebraically closed fields. If $f,g\in\mathbb{C}[x_1,\dots,x_n]$, then

$$\{\mathbf{x}\in\mathbb{C}^n\,|\,f(\mathbf{x})=0\}=\{\mathbf{x}\in\mathbb{C}^n\,|\,g(\mathbf{x})=0\}\qquad\Leftrightarrow\qquad\begin{array}{c}f \text{ and } g \text{ have the same}\\ \text{irreducible factors.}\end{array}$$

(This is an easy special case of the Nullstellensatz of [Hilbert1893].) If we want to go further, we must study solutions of $f=0$ in any commutative ring R with a unit. This approach was first adopted by Grothendieck in [EGA60-67], though in retrospect, [Weil46] and [Rilke30,vol.2.p.175] clearly pointed in this direction. We obtain that if $f,g\in\mathbb{Z}[x_1,\dots,x_n]$ are two polynomials, then

$$\begin{array}{c}\{\mathbf{x}\in R^n|f(\mathbf{x})=0\}=\{\mathbf{x}\in R^n|g(\mathbf{x})=0\}\\ \text{(for every commutative ring } R)\end{array}\qquad\Leftrightarrow\qquad f=\pm g.$$

Thus studying solutions in all commutative rings determines the polynomial up to a sign. This approach is very powerful, but rather technical. Therefore I will stick to studying solutions in fields for the rest of the lecture.

It turns out that there is a collection of basic questions in arithmetic, algebraic geometry and topology all of which give the same class of "low degree" polynomials. The aim of this lecture is to explain these properties and to provide a survey of the known results.

1. Introductory Remarks

We start with the observation that in some cases the degree alone does not provide a good measure of the complexity of a polynomial equation. In order to develop the correct picture, we have to understand which polynomials behave in an atypical manner.

1.1 High degree polynomials that behave like low degree ones.

There are at least three situations when the zero set of a high degree polynomial shares some of the properties of zero sets of low degree polynomials:

1.1.1 Reducible equations. If $f=gh$, then the set $(f=0)$ is the union of the sets $(g=0)$ and $(h=0)$. Thus we can restrict ourselves to the case when f is irreducible.

1.1.2 Low degree in certain variables. Let us consider an extreme case, when f has degree 1 in the variable x_n. Then f can be written as

$$f=f_1(x_1,\dots,x_{n-1})+x_nf_2(x_1,\dots,x_{n-1}).$$

The substitution $x_n=-f_1/f_2$ shows that the set $(f=0)$ behaves like the vector space of the first $(n-1)$ variables $\{(x_1,\dots,x_{n-1})\}$. This is completely true if f is linear, but in general the correspondence breaks down if $f_2=0$. The latter equation involves one fewer variable, and therefore it is considered easier. Roughly speaking, f should be viewed as complicated as a linear equation. In general, if f has low degree in certain variables then it behaves like a low degree equation.

1.1.3 Very singular equations. Consider for instance the equation $x_1^d - x_2^{d-1} = 0$. Its degree in both variables is high. Nonetheless, the substitution

$$x_1 = t^{d-1},\ x_2 = t^d$$

shows that solutions of $x_1^d - x_2^{d-1} = 0$ are parametrized by the values of the variable t. The same happens for any polynomial $f(x_1, x_2)$ of degree d all of whose partials up to order $d-2$ vanish at a certain point. In general, a high degree equation f behaves as a low degree equation if many of the partial derivatives of f simultaneously vanish at many points.

While all of these cases do occur, there are relatively few polynomials that behave this way. For instance, all polynomials $f(x_1, x_2)$ of degree $\leq d$ form a vectorspace V_d of dimension $\binom{d+2}{2}$. The set of polynomials which are exceptional for any of the above 3 reasons is a subset of codimension $d-1$ for $d \geq 2$.

This remark shows that for most polynomials the degree is a good measurement of complexity. In order to run computer experiments, it is desirable to have a class of polynomials with very few nonzero coefficients which are nonetheless "general". A good set of examples to keep in mind is the following.

1.2 Test Examples. The equations $\sum_i c_i x_i^d = c_0$ have been much studied. Unfortunately, they are sometimes too special. It seems that the inhomogeneous version is much more indicative of the general case. Fix natural numbers $d_i : i = 1, \dots, n$ and $c_0, \dots, c_n$ such that $\prod_i c_i \neq 0$. Then

$$\sum_{i=1}^{n} c_i x_i^{d_i} = c_0 \quad \text{has "low degree" iff} \quad \sum_{i=1}^{n} \frac{1}{d_i} \geq 1. \tag{1.2.1}$$

We see in (5.5) that the above condition does coorespond to the eventual definition (4.1). Moreover, I claim that the behaviour of these examples correctly predicts the broad features of the theory. You have to trust me that this purely experimental assertion is valid.

As a first example, let us see what a simple minded constant count gives about solutions of the equations (1.2.1) over $\mathbb{Q}$.

1.3 Heuristic claim. Fix natural numbers $d_i : i = 1, \dots n$ and rational numbers $c_i : i = 0, \dots n$. I claim that usually

$$\sum_{i=1}^{n} c_i x_i^{d_i} = c_0 \quad \text{has many solutions in } \mathbb{Q} \text{ iff} \quad \sum_{i=1}^{n} \frac{1}{d_i} \geq 1. \tag{1.3.1}$$

Unfortunately there are large classes of equations where this is false. For instance, $\sum x_i^2 = -1$ has no solutions in $\mathbb{Q}$, not even in $\mathbb{R}$. Looking at $x_1^2 - x_2^2$ modulo 4, we see that $x_1^2 - x_2^2 = 2$ has no rational solutions. There are several approaches to correct these problems; we encounter two of them later. For the moment I ignore these counterexamples, and give a proof of (1.3.1).

It is easier to look for integral solutions, so we homogenize the equation in the following (somewhat unusual) way. Set d_0 to be the least common multiple of $d_1, \dots, d_n$ and let $d_0 = d_i b_i$. Look at the equation

$$\sum_{i=1}^{n} c_i y_i^{d_i} = c_0 y_0^{d_0}. \tag{1.3.2}$$

There is a correspondence between solutions of (1.2.1) and of (1.3.2) given by

$$(x_1, \dots, x_n) \mapsto (1, x_1, \dots, x_n) \quad \text{and} \quad (y_0, y_1, \dots, y_n) \mapsto (y_1/y_0^{b_1}, \dots, y_n/y_0^{b_n}).$$

This shows that finding all rational solutions of (1.2.1) is equivalent to finding all integral solutions of (1.3.2).

Set $f = -c_0 y_0^{d_0} + \sum_{i=1}^n c_i y_i^{d_i}$. There is a constant C, depending on f, such that

$$|f(y_0, \dots, y_n)| \le C \cdot (\max_i |y_i|^{d_i}). \tag{1.3.3}$$

Fix a large N and let the y_i run through the set of integers in $[-N^{1/d_i}, N^{1/d_i}]$. We get

$$\text{const} \cdot N^{\sum_{i=0}^n (1/d_i)} \quad \text{values of } f \text{ in the interval} \quad [-C \cdot N, C \cdot N].$$

If these values are uniformly distributed, we obtain the asymptotic

$$\#\{\sum_i c_i y_i^{d_i} = c_0 y_0^{d_0}, |y_i| \le N^{1/d_i}\} \sim \text{const} \cdot N^{-1+\sum_{i=0}^n (1/d_i)} \quad \text{as } N \to \infty.$$

If $\sum_{i=1}^n (1/d_i) \ge 1$ then $\sum_{i=0}^n (1/d_i) > 1$, and the number of solutions grows as a power of N. If $\sum_{i=1}^n (1/d_i) < 1$ then $\sum_{i=0}^n (1/d_i) \le 1$ because of the special choice of d_0, thus there should be few solutions. □

For which other polynomials f does this counting method work? The main part is the estimate (1.3.3). This works if f is weighted homogeneous of degree 1 with weights $1/d_i$. That is, if we declare $\deg x_i = 1/d_i$ then $\deg f \le 1$.

There are some examples where the above simpleminded counting method does work, for instance, for equations of the form

$$f(x_1, \dots, x_n) - f(y_1, \dots, y_n) = 0.$$

The above argument gives a lower bound

$$\#\{f(x_1, \dots, x_n) = f(y_1, \dots, y_n), |x_i|, |y_i| \le N\} \ge \text{const} \cdot N^{2n-d}.$$

This is interesting only if $d < n$ since the trivial solutions $x_i = y_i$ always give a lower bound $\text{const} \cdot N^n$.

In the rest of the lecture I aim to explain the various properties that lead to this class of equations, starting with the 2-variable case in section 2. This is called the theory of algebraic curves. Most of the theory was well-established in the 19th century, with the exception of the arithmetic aspects.

Section 3 is devoted to the 3-variable case, which corresponds to the theory of algebraic surfaces. The geometric aspects have been established around the turn of the century, many of the topological results are recent and most of the arithmetical questions are open.

Much less is known in higher dimensions. The open questions involve deep problems in algebraic geometry, number theory and differential topology. I am confident that these problems constitute a very interesting direction of research for a long time to come.

2. Two Variable Polynomials = Algebraic Curves

Let us consider a 2 variable polynomial $f(x,y)=\sum a_{ij}x^iy^j$ of degree d. Let C_{aff} denote its zeros, that is,

$$C_{\mathit{aff}} := \{(x,y) | f(x,y)=0\}.$$

(The subscript *aff* refers to the fact that we are in affine 2-space $\mathbb{A}^2$.) This is not a set since I have not specified where the coordinates x,y are. If the coefficients a_{ij} are in a field F, then for any larger field $E\supset F$ we can look at solutions of $f=0$ in E. The resulting set is

$$C_{\mathit{aff}}(E) := \{(x,y)\in E^2 | f(x,y)=0\}\subset E^2.$$

A common case is when $a_{ij}\in\mathbb{Q}$, and for the larger field E we choose $\mathbb{Q},\mathbb{R}$ or $\mathbb{C}$.

$C_{\mathit{aff}}(\mathbb{Q})$ is just a set of points, but $C_{\mathit{aff}}(\mathbb{R})\subset\mathbb{R}^2$ naturally appears as a curve (that is, a 1-dimensional topological space). $C_{\mathit{aff}}(\mathbb{C})\subset\mathbb{C}^2$ is a Riemann surface: a complex manifold locally like $\mathbb{C}$.

In studying the manifolds $C_{\mathit{aff}}(\mathbb{R})$ or $C_{\mathit{aff}}(\mathbb{C})$ it is frequently inconvenient that they are not compact. The usual way to deal with this problem is to introduce the projective plane $\mathbb{P}^2$ with homogeneous coordinates $(x_0:x_1:x_2)$. Its relationship to the old affine coordinates is $x=x_1/x_0, y=x_2/x_0$. If the coordinates x_i are in a field E, we obtain the corresponding projective plane $\mathrm{E}\mathbb{P}^2$. The most frequently used ones are $\mathbb{Q}\mathbb{P}^2,\mathbb{R}\mathbb{P}^2$ and $\mathbb{C}\mathbb{P}^2$.

The homogenization of f is given by

$$\bar f(x_0,x_1,x_2) := x_0^d f(x_1/x_0,x_2/x_0).$$

The corresponding zero set

$$C(E) := \{(x_0:x_1:x_2)\in \mathrm{E}\mathbb{P}^2 | \bar f(x_0,x_1,x_2)=0\}\subset \mathrm{E}\mathbb{P}^2$$

turns out to be more convenient for most purposes.

Based on the real picture, algebraic geometers say that C is an *algebraic curve.* Thus we prefer to call $\mathbb{C}$ the complex line (the complex plane is of course $\mathbb{C}^2$). This leads to occasional confusion, but this is not the time to change 150 year-old terminology.

In what follows I collect certain properties of algebraic curves defined by equations of degree at most 2. In all cases I would like the properties to hold only for curves defined by equations of degree at most two (assuming the genericity conditions of (1.1)).

All of the characterizations listed here are standard results of the theory of algebraic curves and Riemann surfaces. One of the most accessible introductions to algebraic geometry is [Shafarevich94] (or any of the other editions). For algebraic curves see [Fulton69]. The analytic theory of Riemann surfaces is treated in [Siegel69; Gunning76]. For the arithmetic aspects I found [Serre73; Silverman86] especially useful.

Characterizations of "low degree " curves.

I start with the algebraic geometry condition, not because it is the most obvious for curves, but because this provides the neatest definition in higher dimensions.

2.1 Algebraic geometry. *There is a one-to-one map given by rational functions* $g : \mathbb{CP}^1 \to C(\mathbb{C})$.

In this case C is called a *rational curve.*

Let $(s : t)$ be the homogeneous coordinates on $\mathbb{CP}^1$. If $f = a_0x_0 + a_1x_1 + a_2x_2$ is linear and $a_2 \neq 0$, we can choose

$$g : (s : t) \mapsto (a_2 s : a_2 t : -(a_0 s + a_1 t)) .$$

For $\deg f = 2$ assume for simplicity that $f = a_0x_0^2 + a_1x_1^2 + a_2x_2^2$. (This can always be achieved after a linear change of coordinates.) We can take

$$g : (s : t) \mapsto \left(a_1 s^2 - a_0 t^2 : -2a_0 st : \sqrt{-a_0/a_2}(a_1 s^2 + a_0 t^2)\right) .$$

(In case you wonder where this came from, let $h : C \to L$ be the projection of C from the point $P = (\sqrt{a_2} : 0 : \sqrt{-a_0}) \in C$ to the $(x_2 = 0)$ line (Mercator projection). g is the inverse of h.)

The fact that no such g exists for higher degree equations is harder.

2.2 Topology. $C(\mathbb{C})$ *is homeomorphic to the sphere* S^2.

The maps g from (2.1) also provide a homeomorphism; the hard part is again to see that this cannot be done for higher degree equations. The precise result is that if C is defined by a degree d equation then $C(\mathbb{C})$ is homeomorphic to a sphere with $\frac{1}{2}(d-1)(d-2)$ handles.

2.3 Hard Arithmetic. $C(\mathbb{Q})$ *is "large"*

For this to make sense, we should start with a curve

$$C = (\bar{f}(x_0, x_1, x_2) = 0) \subset \mathbb{P}^2,$$

where $\bar{f}$ has rational coefficients.

Unfortunately it is not easy to pin down what "large" exactly means. First of all, if $n \geq 4$ then $C(\mathbb{Q})$ is finite by [Faltings83]. Unfortunately, $C(\mathbb{Q})$ is often infinite for $n = 3$ and frequently empty for $n = 2$.

To get the right answer, we have to develop a good measure of the size of a solution. This is most conveniently done in projective coordinates.

Any point $P \in \mathbb{QP}^2$ can be represented as a triple $P = (x_0 : x_1 : x_2)$ where $x_0, x_1, x_2 \in \mathbb{Z}$ are relatively prime. This representation is unique up to sign, thus $H(P) := \max\{|x_0|, |x_1|, |x_2|\}$ is well-defined. It is called the *height* of P. One defines the counting function

$$N(C,H) := \#\{P = (x_0 : x_1 : x_2) \in \mathbb{QP}^2 | \bar{f}(x_0, x_1, x_2) = 0 \text{ and } H(P) \leq H\}.$$

Roughly speaking, we look for rational solutions of $f(x,y) = 0$ where the numerators and denominators are bounded.

This nearly gives the right answer. If $n = 2$ then $C(\mathbb{Q})$ is either empty or $N(C,H)$ grows like const $\cdot H$; if $n = 3$ then $N(C,H)$ grows slower than any power of H [Néron65].

In order to deal with the case when $C(\mathbb{Q})$ is empty, we have to count solutions in various algebraic number fields. It is not hard to generalize the notion of height to the case when the coordinates of P are in an algebraic number field $E \supset \mathbb{Q}$ (see [Silverman86,VIII.5] for a short and clear summary). We obtain a similar counting function $N_E(C,H)$. This finally gives the correct generalization:

2.3.1 Theorem. C is a rational curve iff $N_E(C,H)$ grows polynomially with H for a suitable number field E.

2.4 Complex manifolds. *$C(\mathbb{C})$ has genus zero.*

Global holomorphic differential forms on a compact Riemann surface have been much studied, starting with the works of Euler, Abel and Riemann. On a Riemann surface we have only 1-forms, these are locally given as $f(z)dz$ where z is a local coordinate and $f(z)$ is holomorphic. Such forms are automatically closed, thus the integral

$$\int_\gamma f(z)dz \quad \text{over a closed loop} \quad \gamma \subset C(\mathbb{C})$$

depends only on the homology class $[\gamma] \in H_1(C(\mathbb{C}), \mathbb{Z})$. Since the fundamental studies of Riemann, these give the basic approach to finer understanding of Riemann surfaces.

By definition, the *genus* is the dimension of the vector space of global holomorphic differential forms. If there are no such forms, the above integrals give no information. Fortunately, this happens precisely when other descriptions are very simple.

2.5 Easy Arithmetic. *There are many solutions over function fields.*

Here we look at the behaviour of the sets $C(F)$ where $F = \mathbb{C}(t)$ is the field of rational functions in one variable. Of course $f = \sum a_{ij}(t)x^i y^j$ and the coefficients $a_{ij}(t)$ themselves are rational functions. The field $\mathbb{C}(t)$ shares many properties of $\mathbb{Q}$, but the results are easier to state and the proofs are much simpler. (The difference between $\mathbb{Q}$ and $\mathbb{C}(t)$ becomes apparent when studying their Galois cohomology.)

The advantage of $\mathbb{C}(t)$ is that there are two ways of looking at solutions over $\mathbb{C}(t)$.

(2.5.1.1) The algebraic way. Just handle everything as quotients of polynomials in $\mathbb{C}[t]$.

(2.5.1.2) The geometric way. An equation $f(x,y)=0$ with coefficients in $\mathbb{C}(t)$ can be viewed as an equation $\tilde{f}(x,y,t)=0$ with coefficients in $\mathbb{C}$. This defines an algebraic surface $S\subset\mathbb{C}^3$ and we have a distinguished coordinate projection to the t-axis $p:S\to\mathbb{C}_t$.

A solution $(x(t),y(t))$ of $f(x,y)=0$ can be identified with a map

$$h:\mathbb{C}_t\to S\quad\text{given by}\quad t\mapsto(x(t),y(t),t).$$

h is a section of $p:S\to\mathbb{C}_t$ and every (rational) section arises as above.

The first indication that we can expect nicer results is the following theorem, which can be proved by a straightforward generalization of the counting argument (1.3). The first proof is in [Noether1871]. Later algebraic proofs, more suited to generalizations, are in [Baker22,vol.VI.p.147] and [Tsen36].

2.5.2 Theorem. If $\deg f\le 2$ then $f=0$ has a solution in $\mathbb{C}(t)$.

We may also want to know that there are many solutions. A natural approach is to look for solutions $(x(t),y(t))$ where certain values $(x(t_k),y(t_k))$ are specified in advance. This is possible only if the points $(x(t_k),y(t_k),t_k)$ lie on the surface S, that is, if $\sum a_{ij}(t_k)x(t_k)^iy(t_k)^j=0$. In this case we say that the pair $(x(t_k),y(t_k))$ is a solution of $f(x,y)=0$ at t_k.

As an easy exercise in the theory of algebraic surfaces we get a very strong characterization:

2.5.3 Theorem. There is a finite set $B\subset\mathbb{C}$ such that if $t_1,\dots,t_s\in\mathbb{C}\setminus B$ are arbitrary points and (x_k,y_k) any solution of f at t_k then there is a solution $(x(t),y(t))$ of $f=0$ such that $(x(t_k),y(t_k))=(x_k,y_k)$ for $k=1,\dots,s$.

One can reformulate the theorem to specify not just the value of $(x(t),y(t))$ at t_k but also the beginning of its Taylor expansion. With a little more care, the exceptional set B can also be eliminated (5.1).

2.5.4 Remark. More generally all of this works if $\mathbb{C}(t)$ is replaced with any finite degree extension of $\mathbb{C}(t)$. These are exactly the fields of meromorphic functions on compact Riemann surfaces.

2.6 Low degree equations. *C can be described by an equation of degree at most 2.*

This is of course our starting point, but in higher dimensions this becomes a rather nontrivial question.

It is worthwhile to note the following arithmetic implication:

2.6.1 Proposition. If $\deg f\le 2$, then $f(x,y)=0$ always has a solution over a degree 2 field extension.

In order to see this, pick a,b,c and consider $f(x,y)=ax+by+c=0$. Eliminating x or y we are left with a quadratic equation in one variable.

Final remarks about curves.

It should be made clear that the above properties by no means exhaust the known characterizations of curves of degree 1 and 2. Some of the others do not seem to have higher dimensional analogs. I just give a few examples:

2.7 Bad characterizations.

2.7.1 Simply connectedness.

$\pi_1(C(\mathbb{C})) = \{1\}$ iff $\deg f \leq 2$. It turns out that any smooth hypersurface $X = (f(x_0, \ldots, x_n) = 0) \subset \mathbb{CP}^n$ is simply connected for $n \geq 3$ [Lefschetz24].

2.7.2 Unique factorization in the coordinate ring.

The ring $\mathbb{C}[x, y]/f(x, y)$ is a unique factorization domain iff $\deg f \leq 2$. If $f(x_0, \ldots, x_n)$ defines a smooth hypersurface then $\mathbb{C}[x_1, \ldots, x_n]/f(x_1, \ldots, x_n)$ is a UFD for $n \geq 4$ [Grothendieck68].

2.7.3 Homogeneous spaces.

If $\deg f = 1$ then C is homogeneous under the group $SL(2)$. If $\deg f = 2$ then C is homogeneous under the group $O(\bar{f})$, the 3-variable orthogonal group of $\bar{f}$. In higher dimensions the varieties which are homogeneous under the action of a linear algebraic group give rather special examples of the class that we want.

2.7.4 Number of moduli.

Any two lines in $\mathbb{P}^2$ are equivalent under a change of coordinates, and any two smooth conics in $\mathbb{CP}^2$ are also equivalent. This fails for $\deg f \geq 3$. In all dimensions this property characterizes hypersurfaces of degree at most 2, so does not hold for most of the examples in (1.3). (We need a nondiagonal perturbation to see this.)

The above lists suggest several further possible approaches to low degree polynomials. Below I list some that do not to work, even for curves.

2.8 Noncharacterizations.

2.8.1 Topology over $\mathbb{R}$.

One could study curves such that $C(\mathbb{R})$ is homeomorphic to S^1. If $\deg f \leq 2$ and $C(\mathbb{R})$ is not empty, this is always the case. Unfortunately, there are many other curves with this property. For instance, $(x^{2d} + y^{2d} = 1) \subset \mathbb{RP}^2$ is homeomorphic to S^1.

2.8.2 Solutions modulo p.

If f has integral coefficients, we can ask about solvability modulo p (or modulo any number).

The number of solutions in finite fields are described by the Weil conjectures (see [Freitag-Kiehl88] for a thorough treatment) and the degree of f does not affect the asymptotic behaviour much. (Though the genus can be computed if we know the exact number of solutions modulo p for many values of p.) Low degree equations have solutions in any finite field [Chevalley35], but the same holds for many other cases.

2.8.3 Solutions in p-adic fields.

An equation of degree at most two is not always solvable in p-adic fields. For equations in many variables, solvability in p-adic fields is an interesting question.

The rough picture (which is not quite correct) is that if $f(x_0,\dots,x_n)$ has degree $d\le\sqrt{n}$ then f has a solution in any p-adic field and this fails for larger degree. Thus the answer does not correspond to our class. See [Greenberg69] for a discussion of these topics.

2.9 Other approaches.

2.9.1 Holomorphic maps $h:\mathbb{C}\to C(\mathbb{C})$.

If there is a map $\mathbb{CP}^1\to C(\mathbb{C})$, then we get plenty of holomorphic maps $\mathbb{C}\to C(\mathbb{C})$. If $\deg f\ge 4$ then there are no nonconstant holomorphic maps from $\mathbb{C}$ to $C(\mathbb{C})$. Unfortunately if $\deg f=3$, then there are nonconstant holomorphic maps $\mathbb{C}\to C(\mathbb{C})$. Thus this property characterizes a slightly different class of curves. In higher dimensions the two classes differ substantially. See [Lang86; Vojta91] for various properties of this class.

Vojta pointed out to me that one can consider holomorphic maps $h:\mathbb{C}\to C(\mathbb{C})$ whose Nevanlinna characteristic function grows slowly, to get a characterization of rational curves in the context of the holomorphic theory. The resulting holomorphic maps are rational, so at the end this is equivalent to (2.1).

2.9.2 The Hasse principle.

One way to overcome the difficulties observed in (1.3) is to refine (1.3.1) as follows:

Assume that $f(x_1,\dots,x_n)=0$ has a (nontrivial) solution modulo m for every m and also over $\mathbb{R}$. Does this imply that f has a solution in $\mathbb{Z}$?

(Solvability modulo m for every m is equivalent to solvability in every p-adic field.)

If the answer is yes, one says that the *Hasse principle* holds for f. By the Hasse–Minkowski theorem, this is the case if f is homogeneous of degree 2.

The question for higher dimensions is very difficult. It is still not clear if the Hasse principle is connected with our class in higher dimensions or with some smaller class of varieties. See [Colliot-Thélène86,92] for surveys of this direction.

3. Algebraic Surfaces

The next step is to study zero sets of polynomials in three variables

$$S:=\{(x,y,z)|f(x,y,z)=0\}\subset\mathbb{A}^3.$$

It was noticed in the 19th century that the true measure of complexity of a system of polynomial equations is the dimension of the set of solutions over $\mathbb{C}$. Thus if we have 2 equations in 4 variables, the resulting zero set

$$(f_1(x,y,z,u)=f_2(x,y,z,u)=0)\subset\mathbb{A}^4$$

behaves to a large extent like surfaces in 3-space. Any surface in 4-space can be made into a surface in 3-space by a generic projection. If we generically project a curve in n-space to the plane, the image has only transversal self-intersections. By contrast, if we project a surface to 3-space, the image has complicated self-intersections. According to current view, it is very hard to study a surface this way. (Earlier geometers, being ignorant of this fact, proved rather deep theorems using projections to 3-space.) Thus we are pretty much forced to look at the general case of varieties:

Algebraic varieties. Given polynomials $f_1, \dots, f_k$ in n variables, their common zero set

$$X_{aff} := \{(x_1, \dots, x_n) | f_1(\mathbf{x}) = \dots = f_k(\mathbf{x}) = 0\} \subset \mathbb{A}^n$$

is called an *affine* algebraic variety. Using homogeneous equations $\bar{f}_i$ we obtain *projective* varieties

$$X := \{(x_0, \dots, x_n) | \bar{f}_1(\mathbf{x}) = \dots = \bar{f}_k(\mathbf{x}) = 0\} \subset \mathbb{P}^n.$$

If the coefficients of the f_i are in a field F, we say that X is *defined over* F. X is also defined over every bigger field $E \supset F$, hence $X(E) \subset \mathrm{E}\mathbb{P}^n$, the set of solutions in E, makes sense.

These sets can be very complicated. In order to streamline our discussions, I make two simplifying assumptions:

All varieties will be irreducible and smooth.

Over the complex numbers this means that $X(\mathbb{C})$ is a connected manifold. These assumptions are satisfied if the coefficients of the f_i are chosen at random. The general case can be reduced to this one in various ways.

The dimension of X can be defined in an abstract way. Over $\mathbb{C}$ it is one half of the topological dimension of $X(\mathbb{C})$. This gives the expected value; for instance if $X \subset \mathbb{CP}^n$ is defined by a single equation then it has dimension $n-1$.

In order to decide which varieties are considered equivalent, we look at the example of the Mercator projection from (2.1)

Examples of birational maps.

(i) Let $S = (x^2 + y^2 + z^2 = 1) \subset \mathbb{R}^3$. Project S from the point $(0,0,1)$ to the (x,y)-plane P. This provides a one-to-one map

$$\pi : S \setminus (0,0,1) \xrightarrow{\cong} P \cong \mathbb{R}^2.$$

This looks good, until we notice that projectively there are problems. The plane is usually compactified as $\mathbb{RP}^2$, which is not even homeomorphic to the sphere S.

(ii) $H = (x^2 - y^2 + z^2 = 1) \subset \mathbb{R}^3$ is a hyperboloid. Project H from the point $(0,0,1)$ to the (x,y)-plane P. This provides a one-to-one map

$$\pi : S \setminus \{(x,y,z) | z = 1\} \xrightarrow{\cong} P \setminus \{(x,y) | x^2 - y^2 + 1 = 0\},$$

and π and π^{-1} can not be extended to the removed sets in any reasonable way. Despite this, π is clearly very useful in understanding S. For many problems we can use π to study $S \setminus \{(x,y,z)|z=1\}$. The missing set $\{(x,y,z)|z=1\}$ is isomorphic to the plane curve $\{(x,y)|x^2 - y^2 = 0\}$, which is a pair of lines.

(iii) For $a,b,c \in \mathbb{Q}$, $H_{abc} = (ax^2 + by^2 + cz^2 = 1) \subset \mathbb{A}^3$ is a quadric surface. As above, we would like to find a projection of H to a plane. This can be done over some field, for instance we can project from $(0,0,1/\sqrt{c})$. The formulas for π and π^{-1} involve $\sqrt{c}$, hence they are of little use if we intend to study $H(\mathbb{Q})$.

If $a,b,c<0$ then $H_{abc}(\mathbb{R})$ is empty, thus there is no map $g:\mathbb{R}^2\to H(\mathbb{R})$.

Definition of birational maps. Let $X\subset\mathbb{A}^n$ and $Y\subset\mathbb{A}^m$ be affine varieties. Let x_i (resp. y_j) be coordinates on $\mathbb{A}^n$ (resp. $\mathbb{A}^m$).

A *rational map* $g:\mathbb{A}^n \dashrightarrow \mathbb{A}^m$ is given as

$$g:(x_1,\dots,x_n)\mapsto(g_1(\mathbf{x}),\dots,g_m(\mathbf{x})),$$

where the g_i are rational functions in the variables $x_1,\dots,x_n$. Notice that such maps need not be everywhere defined. If the coefficients of the g_i are in a field F, we say that g is defined over F.

If $g(X)\subset Y$, then g restricts to a map $g:X\dashrightarrow Y$.

We say that $g:X\dashrightarrow Y$ is *birational* if there are subvarieties $A\subsetneq X$ and $B\subsetneq Y$ such that g restricts to a one-to-one map $g:X\setminus A\to Y\setminus B$.

Informally speaking, X and Y are birational if they are isomorphic up to lower dimensional varieties.

Rational maps of projective varieties can be defined similarly. We can mimic the above definitions with projective coordinates (in which case the g_i have to be homogeneous).

A general introduction to algebraic geometry can be found in [Shafarevich94; Hartshorne77]. The analytic theory can be found in [Wells73; Griffiths-Harris78]. The books [Beauville78; BPV84] are devoted to algebraic surfaces. The topological aspects are discussed in [Donaldson-Kronheimer90; Friedman-Morgan94].

Characterizations of "low degree" surfaces.

Let $S\subset\mathbb{P}^n$ be a projective surface defined by homogeneous equations $f_1=\dots=f_k=0$. For simplicity we always assume that S is smooth and connected.

For surfaces, algebraic geometry provides the basic definition. Our task is to see to what extent the other variants (2.2–6) can be generalized to give an equivalent condition.

3.1 Algebraic geometry. *S is rational over* $\mathbb{C}$.

The precise definition of rational is the following:

3.1.1 Definition. Let S be a smooth projective surface defined over $\mathbb{C}$. We say that S is *rational* if there is a birational map $g:\mathbb{CP}^2\dashrightarrow S(\mathbb{C})$.

If S is defined over a subfield $F\subset\mathbb{C}$, we say that S is *rational over F* if there is a birational map $g:\mathbb{P}^2\dashrightarrow S$ defined over F.

Historically this definition appeared as a rather hard theorem. There are three classes of surfaces which are very similar to rational surfaces, but it is not obvious that they are indeed rational. These three classes are:

(3.1.2.1) cubic surfaces $S_3\subset\mathbb{P}^3$;

(3.1.2.2) surfaces S which admit a map $f:S\to\mathbb{P}^1$ whose general fiber is $\mathbb{P}^1$;

(3.1.2.3) surfaces which are images of maps $h:\mathbb{P}^2\dashrightarrow\mathbb{P}^n$.

Cubic surfaces were shown to be rational by [Clebsch1866]. The second case was settled in [Noether1871] and the third class was treated in [Castelnuovo1894].

3.2 Topology. *Homeomorphism versus diffeomorphism.*

Understanding algebraic surfaces in terms of their topology turned out to be extremely difficult.

Some classical questions can be interpreted in topological terms, but this may have been first explicitly done in [Hirzebruch54]. One of the simplest problems is to give a topological characterization of the complex projective plane. This was finally done in [Yau77]:

3.2.1 Theorem. Assume that $S(\mathbb{C})$ is homeomorphic to $\mathbb{CP}^2$. Then S is also isomorphic to $\mathbb{CP}^2$.

The difficulties of this very special case discouraged attempts to move further in this direction.

A fundamental problem in general is that a birational map $g : S_1 \dashrightarrow S_2$ does not induce a homeomorphism. This question can be understood in terms of the connected sum operation as follows:

3.2.2 Proposition. If $S_1(\mathbb{C})$ and $S_2(\mathbb{C})$ are birational then there are natural numbers r, s such that

$$S_1(\mathbb{C})\#(\overline{\mathbb{CP}}^2)^r \quad \text{is diffeomorphic to} \quad S_2(\mathbb{C})\#(\overline{\mathbb{CP}}^2)^s,$$

where $\#$ denotes connected sum and $\overline{\mathbb{CP}}^2$ is $\mathbb{CP}^2$ with reversed orientation. We can assume in addition that $\min\{r,s\} \leq 1$ and even $\min\{r,s\} = 0$ with a few exceptions.

In particular we obtain:

3.2.3 Proposition. If S is rational then $S(\mathbb{C})$ is diffeomorphic to

$$\mathbb{CP}^2\#(\overline{\mathbb{CP}}^2)^r \quad \text{or to} \quad \mathbb{CP}^1 \times \mathbb{CP}^1.$$

(It is not hard to see that $(\mathbb{CP}^1 \times \mathbb{CP}^1)\#\overline{\mathbb{CP}}^2$ is diffeomorphic to $\mathbb{CP}^2\#(\overline{\mathbb{CP}}^2)^2$, that is why we have only one series in (3.2.3).)

By analogy with (2.2) one can ask if the converse is also true. It was noticed some time ago that the answer is no if we use homeomorphism instead of diffeomorphism [Dolgachev66]. As Donaldson theory started to discover the difference between diffeomorphism and homeomorphism in real dimension 4, the hope emerged that the converse of (3.2.3) holds for diffeomorphisms.

This has been one of the motivating questions of the differential topology of algebraic surfaces. After many contributions, the final step was accomplished by [Pidstrigach95; Friedman-Qin95]. With the new methods of Seiberg-Witten theory, the proof is actually quite short [Okonek-Teleman95]:

3.2.4 Theorem. Let S be a smooth, projective algebraic surface over $\mathbb{C}$. Then

$$S \text{ is rational} \quad \Leftrightarrow \quad \begin{array}{c} S(\mathbb{C}) \text{ is diffeomorphic to} \\ \mathbb{CP}^2\#(\overline{\mathbb{CP}}^2)^r \text{ or } \mathbb{CP}^1 \times \mathbb{CP}^1. \end{array}$$

3.3 Hard Arithmetic. *$S(\mathbb{Q})$ is "large".*

Let S be a surface defined over a number field F, most frequently $F = \mathbb{Q}$. As for curves, for any number field $E \supset F$ we define the counting function

$$N_E(S,H) := \#\{P \in S(E) \subset \mathrm{E}\mathbb{P}^n | H(P) \le H\}.$$

We hope that S is rational over $\mathbb{C}$ iff $N_E(S,H)$ grows as a power of H for some E.

Unfortunately this is not quite correct, and there are two related problems.

(3.3.1.1) Look at the surface $T := (x^d + y^d = z^d + u^d) \subset \mathbb{P}^3$. One can check that $T(\mathbb{C})$ is smooth. T has high degree, but $N_{\mathbb{Q}}(T,H)$ grows quadratically with H. A closer inspection reveals that this growth is caused by (finitely many) lines on the surface (for instance $(x - z = y - u = 0)$) which contain many rational points. If we remove these lines, there are very few rational solutions left.

(3.3.1.2) The growth rate of $N_E(T,H)$ is not a birational invariant of T. Here again the problems are caused by finitely many rational curves on T.

The examples suggest that we should refine the hope as follows:

3.3.2 Conjecture. [FMT89; Batyrev-Manin90] If T is rational (over $\mathbb{C}$) then there is a number field E, $0 < \beta \in \mathbb{Q}$ and $r \in \mathbb{N}$ such that

$$N_E(T \setminus A, H) \quad \text{is asymptotic to} \quad \mathrm{const} \cdot H^{\beta} (\log H)^r$$

for every sufficiently large subvariety $A \subsetneq T$.

It is furthermore conjectured that β and r are determined by the geometry of T in a simple way [Batyrev-Manin90]. (For higher dimensions these refinements are problematic, see (4.3).)

A weaker form of (3.3.2) is easy:

3.3.3 Theorem. If T is rational (over $\mathbb{C}$) then there is a number field E and $\epsilon > 0$ such that $N_E(T \setminus A, H) > \mathrm{const} \cdot H^{\epsilon}$ for every subvariety $A \subsetneq T$.

The converse of (3.3.2–3) is not quite true. The conceptually correct formulation will be given in (4.3.2–3). For surfaces the following form suffices (cf. [FMT89]).

3.3.4 Conjecture. Assume that (over $\mathbb{C}$) T is not rational and not birational to $C \times \mathbb{P}^1$ where C is an elliptic curve. Then for every number field E and $0 < \epsilon$, there is a subvariety $A \subsetneq T$ such that

$$N_E(T \setminus A, H) < \mathrm{const} \cdot H^{\epsilon}.$$

Very little is known in this direction since we have no general methods to show that nonrational surfaces have only few rational points.

3.4 Complex manifolds. *Global holomorphic differential forms.*

Global holomorphic differential forms on a complex manifold have been much studied. On a surface we can have 1-forms and 2-forms. These are locally given as

$$f_1 dz_1 + f_2 dz_2, \quad \text{respectively} \quad f dz_1 \wedge dz_2,$$

where z_1, z_2 is a local coordinate system and the f_i are holomorphic. In this context, they were first considered in [Clebsch1868] and systematically studied in [Picard-Simart1897].

As in the curve case, the integrals of these forms over 1- and 2-cycles give basic invariants of a variety [Hodge41]. This approach was developed into a very powerful method of studying complex manifolds, called Hodge theory. If there are no global holomorphic differential forms on a surface, then Hodge theory does not say anything.

It is easy to see that if S is rational then there are no global holomorphic differential forms on $S(\mathbb{C})$. Conversely, one can hope that this property characterizes rational surfaces.

This is close to being true, and there are two ways of developing a complete answer.

(3.4.1.1) It is known that there are only finitely many families of exceptions, though the complete list is not yet known.

(3.4.1.2) The second approach, which is more promising in higher dimensions, is to study multivalued differential forms as well. On a surface a multivalued 2-form is locally written as $f(z_1, z_2)dz_1 \wedge dz_2$ where f is a multivalued analytic function. Thus we may ask about the existence of 2-valued differential forms etc. We have the following:

3.4.2 Theorem. [Castelnuovo1898] S is rational iff there are no global holomorphic 1-forms and no global holomorphic 2-valued 2-forms on $S(\mathbb{C})$.

It is technically easier to talk about global sections of symmetric or tensor powers of the cotangent bundle. In this language the above result reads:

3.4.2' Theorem. S is rational iff $H^0(S, \Omega^1_S) = 0$ and $H^0(S, (\Omega^2_S)^{\otimes 2}) = 0$.

3.5 Easy Arithmetic. *There are many solutions over function fields.*

Let $F = \mathbb{C}(t)$ and $S \subset \mathrm{F}\mathbb{P}^n$ be given by the equations $f_1 = \cdots = f_k = 0$ where the f_i are homogeneous polynomials in $x_0, \dots, x_n$ with coefficients in F. Let $\bar{F}$ denote the algebraic closure of F.

The first good news is that the analog of (2.5.2) holds:

3.5.1 Theorem. [Manin66; Colliot-Thélène86] If S is rational (over $\bar{F}$) then $S(F)$ is not empty.

As for curves, we may want to prove that there are in fact many solutions. In perfect analogy with (2.5) we have:

3.5.2 Theorem. [KoMiMo92b] Assume that S is rational (over $\bar{F}$). There is a finite set $B \subset \mathbb{C}$ such that if $t_1, \dots, t_s \in \mathbb{C} \setminus B$ are arbitrary points and $(x_{0k}, \dots, x_{nk})$ is any solution of $f_1 = \cdots = f_k = 0$ at t_k then there is a solution $(x_0(t), \dots, x_n(t))$ of $f = 0$ such that $(x_0(t_k), \dots, x_n(t_k)) = (x_{0k}, \dots, x_{nk})$ for $k = 1, \dots, s$.

It would be desirable to generalize to the case when we also specify the beginning of the Taylor expansion of $(x_0(t), \dots, x_n(t))$ at certain points. The case when S has a conic bundle structure is quite easy (see [CTSSD87, I.3.9] for a similar hard arithmetic proof). The general case is not known.

All these results hold if $\mathbb{C}(t)$ is replaced with any finite degree extension of $\mathbb{C}(t)$.

3.6 Low degree equations.

First we may ask: is every rational surface defined by low degree equations? The answer is no, there are just too many rational surfaces. It is more reasonable to ask:

Is every rational surface T birational to a surface S which is defined by low degree equations?

By definition, any rational surface is birational to $\mathbb{CP}^2$ over $\mathbb{C}$, but this is rather useless in studying arithmetic properties of S. Thus we should be more precise and ask:

3.6.1 Question. Let T be a rational surface defined over a field F. Is T always birational over F to a surface S which is defined by low degree equations?

In this form the question is very interesting and fruitful. The answer is given in two steps.

3.6.2 Minimal models of surfaces. [Enriques1897]

The first step is to simplify the geometry of an arbitrary smooth projective surface $T(\mathbb{C})$ by birational maps. The classical name for this procedure is "adjunction". Later it was called "contraction of (-1)-curves", and the currently fashionable term is "minimal model program".

For any surface T we aim to find a birational morphism $f : T \to S$ such that S is as simple as possible. (For instance, we may want to make the Betti numbers of $S(\mathbb{C})$ small.) S is called a *minimal model* of T (in general it is not unique).

If T is defined over a field F, then we can choose S so that f and S are also defined over F. (It is remarkable that the original method of Enriques automatically works over any field, while the later variants need additional arguments.)

Next we study the geometry of the minimal models S assuming that S is rational over $\mathbb{C}$. The final result is that there are 4 classes of such surfaces.

3.6.3 Theorem. [Enriques1897; Manin66; Iskovskikh80c] Let T be a surface defined over a field $F \subset \mathbb{C}$ such that T is rational over $\mathbb{C}$. Then any minimal model of T falls in one of four classes. (For simplicity, I use affine coordinates.)

(3.6.3.1) (One low degree equation)
$S = (f(x,y,z) = 0) \subset \mathbb{A}^3$ where f satisfies one of the weighted degree conditions:

$$\begin{aligned}
&\deg(x,y,z) = (1,1,1) \quad \text{and} \quad \deg f \le 3 \quad (\text{e.g.} \quad x^3+y^3+z^3+1);\\
&\deg(x,y,z) = (1,1,2) \quad \text{and} \quad \deg f \le 4 \quad (\text{e.g.} \quad x^4+y^4+z^2+1);\\
&\deg(x,y,z) = (1,2,3) \quad \text{and} \quad \deg f \le 6 \quad (\text{e.g.} \quad x^6+y^3+z^2+1).
\end{aligned}$$

(3.6.3.2) (Two low degree equations)
$S = (f_1(x,y,z,u) = f_2(x,y,z,u) = 0) \subset \mathbb{A}^4$ where $\deg f_i = 2$.

(3.6.3.3) (Two equations with low degree in certain variables)
$S = (f_1(x,y) = f_2(x,y,z,u) = 0) \subset \mathbb{A}^4$ where $\deg f_1 = 2$ and the degree of f_2 in the (z,u) variables is 2. (The degree of f_2 in the (x,y) variables can be high.)

In these three cases a general choice of f, f_1, f_2 always gives a rational surface.

(3.6.3.4) (Miscellaneous)
These are inconvenient to pin down with equations. They are all birational to a surface $S = (f(x,y,z) = 0) \subset \mathbb{A}^3$ where $\deg f \le 9$, but f has to be very special. It is much better to notice that all these remaining cases are birational to a homogeneous space under a linear algebraic group.

These results imply the following arithmetic assertion:

3.6.4 Theorem. Let S be a surface defined over a field $F \subset \mathbb{C}$ which is rational over $\mathbb{C}$. Then there is a field extension $E \supset F$ such that $\deg[E : F] \le 9$ and $S(E)$ is not empty.

4. Higher Dimensional Varieties

After surfaces, the next step is the study of algebraic threefolds. The theory of threefolds is much more complicated than the theory of surfaces, but in the last 20 years a rather satisfactory approach to threefolds was developed. We know much less about higher dimensions, but all the conjectures predict that higher dimensional varieties behave exactly like threefolds, although the proofs are unknown to us.

Of course it may happen that a few examples will completely change this picture, but for the moment there is no point in discussing threefolds and higher dimensional varieties separately.

In the surface case one can always consider only irreducible and smooth surfaces. Starting with dimension three, the smoothness assumption is too strong, but this is a technical question which has very little to do with the essential points of our discussion.

For simplicity, I mostly consider smooth varieties. At a few places, where singularities do cause trouble, I mention this explicitly.

The aspects of higher dimensional algebraic geometry that are discussed here are treated in the books [CKM88; Kollár96a]. Some other works dealing with related topics are [Ueno75; Kollár et al.92]. For symplectic topology see [McDuff-Salamon94,95].

Characterizations of "low degree" varieties.
Let $X \subset \mathbb{P}^n$ be a smooth projective variety defined by homogeneous equations $f_1 = \cdots = f_k = 0$.

As for surfaces, the algebraic geometry condition gives the basic concept, but here it takes some work to establish the correct definition.

4.1 Algebraic geometry. *$X(\mathbb{C})$ is rationally connected.*

Already in the surface case it is not easy to show that all low degree surfaces are rational. Therefore it did not come as a big surprise that in higher dimensions rational varieties are too special. A cubic hypersurface $X_3^n \subset \mathbb{CP}^{n+1}$ certainly has low degree. M. Noether knew that there is a map $p : \mathbb{CP}^n \dashrightarrow X_3^n$ which is generically 2:1, but nobody was able to prove that X_3^n is rational for $n \ge 3$. (And indeed, X_3^3 is not rational [Clemens-Griffiths72].) This leads to the following notion:

4.1.1 Definition. X is *unirational* (over $\mathbb{C}$) if there is a rational map $p : \mathbb{CP}^n \dashrightarrow X(\mathbb{C})$ with dense image, where $n = \dim X$.

Very low degree hypersurfaces in $\mathbb{CP}^n$ are unirational [Morin40b]. Unfortunately, it seems that the class of unirational varieties is still too restrictive.

A new concept was proposed in [KoMiMo92b]. Instead of trying to emulate global properties of $\mathbb{CP}^n$, we concentrate on rational curves. $\mathbb{CP}^n$ has lots of rational curves (lines, conics and many higher degree ones). These are images of maps $\mathbb{CP}^1 \to \mathbb{CP}^n$. The defining property of the new class should be the existence of lots of maps $\mathbb{CP}^1 \to \mathbb{CP}^n$. There are several a priori ways of making this precise. Fortunately, many of these are equivalent:

4.1.2 Theorem. [KoMiMo92b] Let X be a smooth projective variety over $\mathbb{C}$. The following are equivalent:

(4.1.2.1) There is an open subset $\emptyset \neq U \subset X(\mathbb{C})$ such that for every $x_1, x_2 \in U$ there is a morphism $f : \mathbb{CP}^1 \to X$ satisfying $x_1, x_2 \in f(\mathbb{CP}^1)$.

(4.1.2.2) For every $x_1, x_2 \in X(\mathbb{C})$ there is a morphism $f : \mathbb{CP}^1 \to X$ satisfying $x_1, x_2 \in f(\mathbb{CP}^1)$.

(4.1.2.3) For every $x_1, \dots, x_n \in X(\mathbb{C})$ there is a morphism $f : \mathbb{CP}^1 \to X$ satisfying $x_1, \dots, x_n \in f(\mathbb{CP}^1)$.

(4.1.2.4) Let $p_1, \dots, p_n \in \mathbb{CP}^1$ be distinct points. For each i let $f_i : D(p_i) \to X(\mathbb{C})$ be a holomorphic map from a small disc around p_i to $X(\mathbb{C})$. Let n_i be natural numbers. Then there is a morphism $f : \mathbb{CP}^1 \to X$ such that the Taylor series of f_i and of $f|D(p_i)$ coincide up to order n_i for every i.

(4.1.2.5) There is a morphism $f : \mathbb{CP}^1 \to X$ such that f^*T_X is ample (see [ibid] for a definition of ample).

4.1.3 Definition. A smooth projective variety X is called *rationally connected* if it satisfies the equivalent properties in (4.1.2).

Thus among n-dimensional varieties we have 3 classes, with the following easy containment relations:

$$\{\text{rational}\} \subset \{\text{unirational}\} \subset \{\text{rationally connected}\}.$$

Much effort went into understanding the precise relationship between these classes. Since 1910, several authors claimed to have produced examples of rationally connected but nonrational threefolds, but the first correct proofs appeared only around 1970. By now the situation is quite satisfactory:

4.1.4 Examples of rationally connected varieties which are not rational.

(4.1.4.1) Dimension three.

The first examples were quartic 3-folds $X_4 \subset \mathbb{CP}^4$ [Iskovskikh-Manin71] and cubic 3-folds $X_3 \subset \mathbb{CP}^4$ [Clemens-Griffiths72]. Further development by [Beauville77; Iskovskikh80b; Bardelli84] gave a quite complete picture in dimension three.

(4.1.4.2) Conic bundles.

After some very special examples [Artin-Mumford72], a general theory was developed in [Sarkisov81,82]. This shows that $X_{d,2} \subset \mathbb{CP}^n \times \mathbb{CP}^2$ is not rational for $d \gg 1$. Further examples are in [Kollár96b].

(4.1.4.3) Quadric bundles.

Only some special examples are known [CTO89; Peyre93].

(4.1.4.4) Hypersurfaces

$X_5 \subset \mathbb{CP}^5$ is considered in [Pukhlikov87]; the method should give all $X_n \subset \mathbb{CP}^n$. These techniques also give many more examples as in (1.3), see [CPR96]. Very general hypersurfaces $X_d \subset \mathbb{CP}^{n+1}$ for $2n/3+2 \leq d \leq n+1$ are treated in [Kollár95].

(4.1.4.5) Hypersurface bundles

$X_{c,d} \subset \mathbb{CP}^m \times \mathbb{CP}^{n+1}$ where $c \geq 2m$ and $2n/3+2 \leq d \leq n+1$ are considered in [Kollár96b].

As this list suggests, most rationally connected varieties are not rational. Some of the varieties on the above list are unirational, thus rational and unirational are indeed different notions. Despite the long list of settled cases, there are many open problems. I mention two about hypersurfaces; they indicate how little is known.

4.1.5 Some unsolved cases.

(4.1.5.1) Is the general cubic n-fold $X_3^n \subset \mathbb{CP}^{n+1}$ rational for $n \geq 4$? The case of cubic 4-folds has received a lot of attention. It is known that some special ones are rational [Morin40a; Tregub93]. In particular this would show that rationality is not deformation invariant.

(4.1.5.2) Is there any rational (smooth) hypersurface of degree at least 4? There is very little evidence either way.

The biggest unsolved question in this picture is the following:

4.1.6 Conjecture. Most rationally connected varieties are not unirational.

At the moment, there is not a single example known. The simplest case to study may be general quartic threefolds $X_4 \subset \mathbb{CP}^4$.

Assume that X is unirational, that is, there is a map $p : \mathbb{CP}^n \dashrightarrow X$. The images of linear subspaces show that through a general point of $x \in X$ there are unirational subvarieties of every dimension. Even this weaker property may fail in general:

4.1.7 Question.

Let $X_d \subset \mathbb{CP}^n$ be a hypersurface of degree $d \leq n$ (thus X is rationally connected). Is it true that for every point $x \in X$ there is a rational surface $x \in S_x \subset X$?

It is easy to see that this is the case if $\binom{d+1}{2} \leq n$, and probably also for slightly larger values of d.

I do not see any obvious way to construct rational surfaces when d is close to n.

Finally I mention another problem concerning rationally connected varieties.

4.1.8 Conjecture. Let $f : X \to Z$ be a morphism between smooth projective varieties. Assume that Z and the general fiber F are rationally connected. Then X is rationally connected.

It is easy to see that the special case when $Z = \mathbb{P}^1$ implies the general one, thus (4.5.1) implies (4.1.8).

4.2 Topology. *Diffeomorphism versus symplectomorphism.*

Guided by the results of the surface case, one can look for three types of theorems in higher dimension:

4.2.1 Basic Questions.

(4.2.1.1) Determine all algebraic varieties of a given topological type.

(4.2.1.2) Relate the topological properties of birationally equivalent varieties.
(4.2.1.3) Characterize rationally connected varieties in terms of their topology.

As in (3.2), the best example in the first direction is the following result of [Hirzebruch-Kodaira57; Yau77]

4.2.2 Theorem. If $X(\mathbb{C})$ is homeomorphic to $\mathbb{CP}^n$ then X is isomorphic to $\mathbb{CP}^n$.

There are very few such results known, and the proofs use rather lucky coincidences. One may want to have a more modest aim in mind, and try to show that the topological structure of $X(\mathbb{C})$ determines X up to finite ambiguity. I noticed the following special case some time ago (a proof is given in (5.3)):

4.2.3 Theorem. Let M be a compact differentiable manifold with $\dim H_2(M, \mathbb{Q}) = 1$. Then there are only finitely many families of algebraic varieties X such that $X(\mathbb{C})$ is diffeomorphic to M.

For M arbitrary this no longer holds. This is already shown by the example of minimal ruled surfaces, but a more convincing negative result was observed by [Friedman-Morgan88b]. This shows that diffeomorphism of algebraic 3-folds is not as strong as for surfaces:

4.2.4 Example. Let S_i be smooth projective surfaces such that $S_i(\mathbb{C})$ is simply connected. Set $X_i := S_i \times \mathbb{CP}^1$.

For differentiable manifolds of real dimension 6, homeomorphism frequently implies diffeomorphism [Wall66; Sullivan77; Zubr80]. We find that if $S_i(\mathbb{C})$ and $S_j(\mathbb{C})$ are homeomorphic, then $X_1(\mathbb{C})$ and $X_2(\mathbb{C})$ are even diffeomorphic. This gives several unpleasant examples:

(4.2.4.1) Let S_1 be a rational surface which is homeomorphic to a nonrational surface S_2 (3.2). Then X_1 is rational, hence also rationally connected and X_2 is not even rationally connected.

(4.2.4.2) One can construct infinitely many surfaces S_i such that the $S_i(\mathbb{C})$ are all homeomorphic, but the S_i are quite different as algebraic surfaces [Okonek-V.d.Ven86; Friedman-Morgan88a]. Thus the manifolds $X_i(\mathbb{C})$ are all diffeomorphic, but the varieties X_i do not fit into finitely many families.

4.2.5 The Topology of Birational Maps.

Let X_1 and X_2 be smooth projective varieties, birational to each other. In contrast with the surface case, it is not known how the manifolds $X_1(\mathbb{C})$ and $X_2(\mathbb{C})$ are related. There are certain surgery type operations, called blow-ups, that take the role of connected sum with $\overline{\mathbb{CP}}^2$. Unfortunately it is not known whether one can go from $X_1(\mathbb{C})$ to $X_2(\mathbb{C})$ by repeated application of blow-ups. This is a hard problem.

The minimal model program establishes a class of surgery type operations that can be used to go from $X_1(\mathbb{C})$ to $X_2(\mathbb{C})$. At the moment these operations are not well understood from the topological point of view. Furthermore, the intermediate stages involve singular topological spaces. In dimension three they are all rational homology manifolds [Kollár91, 2.1.7], but even this fails in higher dimensions.

As example (4.2.4) shows, the diffeomorphism type alone does not characterize rationally connected varieties. In order to obtain a suitable analog of (3.2.4), it is necessary to study an additional structure on $X(\mathbb{C})$:

4.2.6 Symplectic manifolds.

A *symplectic* manifold is a pair (M^{2n}, ω) where M is a differentiable manifold of dimension $2n$ and ω is a 2-form $\omega \in \Gamma(M, \wedge^2 T^*)$ which is d-closed and nondegenerate. That is, $d\omega = 0$ and ω^n is nowhere zero.

Any smooth projective variety admits a symplectic structure. This can be constructed as follows. On $\mathbb{C}^{n+1}$ consider the Fubini–Study 2-form

$$\omega' := \frac{\sqrt{-1}}{2\pi}\left[\frac{\sum dz_i \wedge d\bar{z}_i}{\sum |z_i|^2} - \frac{(\sum \bar{z}_i dz_i) \wedge (\sum z_i d\bar{z}_i)}{(\sum |z_i|^2)^2}\right].$$

It is closed, nondegenerate on $\mathbb{C}^{n+1} \setminus \{0\}$ and invariant under scalar multiplication. Thus ω' descends to a symplectic 2-form ω on $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$.

If $X \subset \mathbb{CP}^n$ is any smooth variety, then the restriction $\omega|X$ makes $X(\mathbb{C})$ into a symplectic manifold.

The resulting symplectic manifold $(X(\mathbb{C}), \omega|X)$ depends on the embedding $X \hookrightarrow \mathbb{CP}^n$, but the dependence is rather easy to understand:

We say that two symplectic manifolds (M, ω_0) and (M, ω_1) are *symplectic deformation equivalent* if there is a continuous family of symplectic manifolds (M, ω_t) starting with (M, ω_0) and ending with (M, ω_1).

To every smooth projective variety the above construction associates a symplectic manifold $(X(\mathbb{C}), \omega|X)$ which is unique up to symplectic deformation equivalence.

This allows us to formulate the proper generalization of (3.2.4):

4.2.7 Conjecture. Let X_0 and X_1 be smooth projective varieties defined over $\mathbb{C}$ such that $(X_0(\mathbb{C}), \omega_0)$ is symplectic deformation equivalent to $(X_1(\mathbb{C}), \omega_1)$. Then X_0 is rationally connected iff X_1 is.

The evidence for this conjecture comes from three sources:

The first thing to check is that (4.2.7) holds if there is a continuous family of algebraic varieties $\{X_t, t \in [0,1]\}$. This case is settled:

4.2.8 Theorem. [KoMiMo92b, 2.4] Let $\{X_t, t \in [0,1]\}$ be a continuous family of smooth projective varieties. Then X_0 is rationally connected iff X_1 is.

Second, one should try to analyze the examples (4.2.4). This was studied in detail by [Ruan94] who showed that the symplectic structure of $S \times \mathbb{CP}^1$ can be used to study the differentiable structure of S in many cases.

The third piece of evidence is given by the following closely related result, whose formulation requires a definition.

4.2.9 Definition. A smooth projective variety X over $\mathbb{C}$ is called *uniruled*, if it satisfies the following equivalent conditions:

(4.2.9.1) There is an open subset $\emptyset \neq U \subset X(\mathbb{C})$ such that for every $x \in U$ there is a morphism $f : \mathbb{CP}^1 \to X$ satisfying $x \in f(\mathbb{CP}^1)$.

(4.2.9.2) For every $x \in X(\mathbb{C})$ there is a morphism $f : \mathbb{CP}^1 \to X$ satisfying $x \in f(\mathbb{CP}^1)$.

The proof of the next result is outlined in (5.4):

4.2.10 Theorem. Let X_0, X_1 be smooth projective varieties defined over $\mathbb{C}$ such that $(X_0(\mathbb{C}), \omega_0)$ is symplectic deformation equivalent to $(X_1(\mathbb{C}), \omega_1)$. Then X_0 is uniruled iff X_1 is.

(4.2.7) holds if $\dim H_2(X_0, \mathbb{Q}) = 1$, since then X is rationally connected iff it is uniruled [KoMiMo92a].

It should be noted that if X_0 is Fano (4.6.2.1), X_1 need not be Fano, as shown by the examples of rational ruled surfaces.

It would also be interesting to find some topological properties of rationally connected varieties. The only general result is the following:

4.2.11 Theorem. [Campana91b; KoMiMo92b] Let X be a rationally connected variety. Then $X(\mathbb{C})$ is simply connected.

4.3 Hard Arithmetic. $X(\mathbb{Q})$ *is "large".*

As for surfaces, the guiding principle is the following conjecture, which is a natural generalization of a problem of [Batyrev-Manin90].

4.3.1 Conjecture. If X is rationally connected (over $\mathbb{C}$) then there are $r \in \mathbb{N}$, $0 < \beta \in \mathbb{Q}$ and a number field $F' \supset F$ such that

$$N_E(X \setminus A, H) \quad \text{is asymptotic to} \quad \text{const} \cdot H^{\beta} (\log H)^r$$

for every sufficiently large subvariety $A \subsetneq X$, and for every number field $E \supset F'$.

The key point is that β is positive. Even the following weaker form is completely open:

4.3.1' Conjecture. If X is rationally connected then there is an $\epsilon > 0$ such that

$$N_E(X \setminus A, H) > H^{\epsilon} \quad (\text{for } H \gg 1),$$

for every subvariety $A \subsetneq X$, and for every sufficiently large number field E.

There are many special cases where (4.3.1) holds [FMT89; Batyrev-Manin90; Batyrev-Tschinkel95]. There is a more precise version of the conjecture [Batyrev-Manin90] asserting that the numbers β, r are computable from the geometry of T. This has been checked in certain cases, but a recent example of [Batyrev-Tschinkel96] shows that the conjecture for the value of r is incorrect.

A precise computation of the growth of the number of integral solutions of the equations

$$\begin{aligned} x_1^3 + x_2^3 + x_3^3 &= y_1^3 + y_2^3 + y_3^3 \\ x_1 + x_2 + x_3 &= y_1 + y_2 + y_3 \end{aligned}$$

is contained in [Vaughan-Wooley95]. This corresponds to (4.3.1) for a certain singular cubic threefold. The results confirm (4.3.1), but they also seem to contradict the more refined conjecture about r. Further special cases are treated in [EMS96].

The converse of (4.3.1) again fails, but not by much:

4.3.2 Conjecture. Assume that X is not uniruled (over $\mathbb{C}$). Then for every number field E and $0 < \epsilon$, there is a subvariety $A \subsetneq X$ such that

$$N_E(X \setminus A, H) < \text{const} \cdot H^\epsilon.$$

4.3.3 The general case. The problem for a general variety X can be reduced to the above two cases as follows.

Assuming (4.1.8), there is a map $f : X \dashrightarrow Z$ such that Z is not uniruled and the fibers of f are rationally connected [KoMoMi92b].

Thus we can study the points of X in E in two steps. First we have to find the E-points of Z using (4.3.2). Then for every $P \in Z(E)$ we study the E-points in the fiber $f^{-1}(P)$, which is rationally connected.

4.4 Complex manifolds. *Global holomorphic differential forms.*

As in the surface case, one can study multivalued global holomorphic differential forms on $X(\mathbb{C})$. It is easy to see that if X is rationally connected, then there are no such forms:

4.4.1 Proposition. Let X be a smooth projective variety over $\mathbb{C}$. Assume that X is rationally connected. Then

$$H^0\big(X, (\Omega^1_X)^{\otimes m}\big) = 0 \quad \text{for every } m > 0.$$

The converse is conjectured to be true, but it is known only in dimension three:

4.4.2 Theorem. [KoMiMo92b] Let X be a smooth projective threefold over $\mathbb{C}$. The following are equivalent:

(4.4.2.1) X is rationally connected;

(4.4.2.2) $H^0\big(X, (\Omega^1_X)^{\otimes m}\big) = 0$ for every $m > 0$.

In contrast with (3.5), the current proofs of (4.4.2) require the vanishing for all values of m. It is quite likely that finitely many of these values are sufficient, but there is no conjecture for the precise bound. [KoMiMo92b] contains further results in this direction.

4.5 Easy Arithmetic. *There are many solutions over function fields.*

Let $F = \mathbb{C}(t)$ and $X \subset F\mathbb{P}^n$ be a subvariety. Let $\bar{F}$ denote the algebraic closure of F.

The higher dimensional analog of (3.5.1) is open:

4.5.1 Conjecture. If X is rationally connected then $X(F)$ is not empty.

This is known in many special instances (see, e.g. [Kollár96a, IV.6]), but these results give very few hints about the general case.

This of course means that we are also unable to prove that X has many points in F. Surprisingly, one can prove that if $X(F)$ is not empty, then it is very large. I formulate the result in the geometric version, which is more precise.

4.5.2 Theorem. [KoMiMo92b, 2.13] Let X be a projective variety over $\mathbb{C}$ and $f : X \to C$ a morphism onto a smooth curve. Assume that f has a section $\sigma : C \to X$.

Let $c_1, \dots, c_k \in C$ be closed points such that $f^{-1}(c_i)$ are smooth and rationally connected. Pick arbitrary points $p_i \in f^{-1}(c_i)$.

Then f has a section $s = s_{p_1,\dots,p_k} : C \to X$ such that $s(c_i) = p_i$ for every i.

The following more general version is open. In analogy with the number theoretic terminology (cf. [Mazur92]), it should be called "weak approximation for rationally connected varieties over function fields".

4.5.3 Conjecture. Let X be a smooth projective variety over $\mathbb{C}$ and $f : X \to C$ a morphism onto a smooth curve whose general fibers are rationally connected. Let $c_1, \dots, c_k \in C$ be closed points and $c_i \in D(c_i) \subset C$ small discs around c_i. Pick local sections $s_i : D(c_i) \to X$ and natural numbers n_i.

Then f has a section $s : C \to X$ such that the Taylor series of $s|D(c_i)$ agrees with the Taylor series of s_i up to order n_i, for every i.

In the special case when $X = C \times Y$, this follows from (4.1.2.4).

4.6 Low degree equations.

As in the surface case, the principal question is the following:

4.6.1 Question. Let X be a rationally connected variety defined over a field F. Is X always birational over F to a variety Y which is defined by low degree equations?

In contrast with the surface case, this is interesting even for $F = \mathbb{C}$.

In analogy with (3.6), first we need:

4.6.2. Minimal model program.

This is a general method to simplify the structure of an arbitrary smooth projective variety. Already in dimension 3 it is rather complicated (cf. [Mori82,88]), and in higher dimensions remains conjectural. See [Kollár87,90] for introductions. The program can be performed over any field F with minor modifications.

For rationally connected varieties we end up with a variety Y (birational to X) satisfying one of the following conditions:

(4.6.2.1) Y is a Fano variety, that is, $-K_Y$ is ample. Unfortunately, Y may be singular. The singularities are rather mild (terminal and $\mathbb{Q}$-factorial), but they do cause certain problems.

(4.6.2.2) There is a morphism $p : Y \to Z$ such that Z and the fibers of p are rationally connected.

In the second case we hope to reduce problems about X to questions about Z and about the fibers of f. Thus we mainly concentrate on the first case. Some of the basic questions are settled:

4.6.3 Theorem. (4.6.3.1) [Nadel91; Campana91a; KoMiMo92a,c] For any n there are only finitely many families of smooth Fano varieties of dimension n.

(4.6.3.2) [Kawamata92] There are only finitely many families of singular Fano threefolds arising in (4.6.2.1).

In both cases the proof yields explicit (though huge) bounds on the number of families and also on the degrees of the defining equations of the Fano varieties.

In dimension three there is a complete list of all smooth Fano varieties, but no such list exists in the singular case. In any case, classifying Fano threefolds up to isomorphism may not be the sensible thing to do. Our original variety X is determined by Y only up to birational equivalence; thus it makes sense to classify rationally connected threefolds up to birational equivalence. [Alexev94; Corti96] contain significant steps in this direction.

4.6.4 Listing by low degree equations.

Smooth Fano threefolds were studied by G. Fano in a series of articles spanning four decades starting in 1908. A modern account of these works was given in [Iskovskikh80a,b]. The results of [Mukai89] give a better description, especially over nonclosed fields. For singular Fano threefolds there is no general theory; a series of examples can be found in [Fletcher89].

If there is a morphism $p : X \to Z$ as in (4.6.2.2), then the results of (3.6) give us defining equations as in (3.6.3). Instead of listing all cases, I just give two examples:

(4.6.4.1) $S = (f_1(u,v) = f_2(x,y,z,u,v) = 0) \subset \mathbb{A}^5$,

where $\deg f_1 = 2$ and the degree of f_2 in the (x,y,z) variables satsifies one of the conditions of (3.6.3.1) (The degree of f_2 in the (u,v) variables can be high.)

(4.6.4.2) $S = (f_1(x_1,x_2) = f_2(x_1,\dots,x_4) = f_3(x_1,\dots,x_6) = 0) \subset \mathbb{A}^6$,

where $\deg f_1 = 2$, the degree of f_2 in the (x_3,x_4) variables is 2 and the degree of f_3 in the (x_5,x_6) variables is 2. (The degrees in the other variables can be high.)

In both cases a general choice of the f_i gives a rationally connected variety.

These results imply the following arithmetic consequence:

4.6.5 Theorem. There is a constant $D(3)$ with the following property:

Let X be a rationally connected threefold defined over a field $F \subset \mathbb{C}$. Then there is a field extension $E \supset F$ such that $\deg[E:F] \le D(3)$ and $X(E)$ is not empty.

One can write down an explicit bound for $D(3)$, though I have not done it. Conjecturally, a similar result holds in any dimension.

5. Appendix

The aim of this appendix is to outline the proofs of some statements which are new or for which I could not find a suitable reference.

5.1 Proposition. Let B be a smooth proper curve over $\mathbb{C}$ and $f : S \to B$ a proper ruled surface. Let $b_i \in B$ be different points and $D(b_i)$ a small disc around b_i. Let $s_i : D(b_i) \to S$ be holomorphic (or formal) sections and n_i natural numbers.

Then there is a section $s : B \to S$ such that $s|D(b_i)$ agrees with s_i up to order n_i for every i.

Proof. S is birationally trivial; that is, there is a birational map $\pi : \mathbb{P}^1 \times B \dashrightarrow S$. We obtain local sections

$$s_i' := \pi^{-1} \circ s_i : D(b_i) \to \mathbb{P}^1 \times B.$$

Assume that it takes k blow-ups to resolve the indeterminacies of π. Let $s' : B \to \mathbb{P}^1 \times B$ be a section such that $s'|D(b_i)$ agrees with s_i' up to order $n_i + k$ for every i. Then we can take $s := \pi \circ s'$.

Thus it is sufficient to find s'. Equivalently, we need to find a map $\bar{s} : B \to \mathbb{P}^1$ with prescribed local behavior $\bar{s}_i : D(b_i) \to \mathbb{P}^1$. By a generic coordinate change in $\mathbb{P}^1$ we can assume that $\bar{s}_i(b_i) \in \mathbb{C}$ for every i.

Choose another point b_0. One can always find regular functions on the affine curve $B \setminus \{b_0\}$ with prescribed local behaviour at the points b_i. $\square$

5.2 Proof of (4.1.2.4). We need to show that (4.1.2.4) is implied by (4.1.2.3). As a first step, I prove the following weaker version:

(5.2.1) Let $p_1, \dots, p_n \in \mathbb{CP}^1$ be disctinct points. For each i let $f_i : D(p_i) \to X(\mathbb{C})$ be a holomorphic map from a small disc around p_i to $X(\mathbb{C})$. Let n_i be natural numbers. Then there is a morphism $g : \mathbb{CP}^1 \to X$ and holomorphic maps $h_i : D(p_i) \to \mathbb{CP}^1$ such that the Taylor series of f_i and of $g \circ h_i | D(p_i)$ coincide up to order n_i for every i.

To see this, let $D \subset \mathbb{C}$ be the unit disc and $f, g : D \to \mathbb{C}^n$ two holomorphic maps with coordinate functions f^j, g^j. Assume that $f(0) = g(0) = 0 \in \mathbb{C}^n$. Let $B_0\mathbb{C}^n \to \mathbb{C}^n$ be the blow-up of $0 \in \mathbb{C}^n$. f and g lift to holomorphic maps $\bar f, \bar g : D \to B_0\mathbb{C}^n$. Explicit local computation shows the following:

(5.2.2.1) If $\bar f$ and $\bar g$ agree up to order n, then so do f and g.

(5.2.2.2) If $f^1(t) = g^1(t) = t$ and $\bar f$ and $\bar g$ agree up to order $n-1$, then f and g agree up to order n.

Using (5.2.2.1) for repeated blow-ups, we first reduce (5.2.1) to the case when the f_i are immersions. Then up to a local coordinate change we may assume that $f_i^1(t) = t$ for every i. We can now prove (5.2.1) by induction on $\sum n_i$, since (4.1.2.3) gives it for $\sum n_i = 0$.

The only subtle point is the reduction step from order 1 to order 0. Let $p \in D \subset \mathbb{CP}^1$ be a disc. Given an immersion $f : D \to X$, let $x = f(p)$ and $\pi : B_xX \to X$ be the blow-up with exceptional divisor $E \subset B_xX$. Assume that we have $\bar g : \mathbb{CP}^1 \to B_xX$ such that $\bar f$ and $\bar g$ agree up to order 0 at p. We would like to conclude that f and $g := \bar g \circ \pi$ agree up to order 1 at p. (5.2.2.2) gives this, if g is an immersion. Thus we have to choose $\bar g : \mathbb{CP}^1 \to B_xX$ to be transversal to E. This is slightly stronger than (4.1.2.3), but can easily be arranged (see the proofs of II.3.14 and IV.3.9 in [Kollár96a]).

Once we have (5.2.1), we just need to find a map $h : \mathbb{CP}^1 \to \mathbb{CP}^1$ which approximates every h_i up to order n_i and set $f := g \circ h$

The f we found is a multiple cover of a curve in X. As in [Kollár96a, IV.3.9] we can perturb f to obtain another solution of (4.1.2.4) where $f | \mathbb{CP}^1 \setminus \{p_1, \dots, p_n\}$ is an embedding. $\square$

5.3 Proof of (4.2.3). Assume that $X(\mathbb{C})$ is diffeomorphic to M. We use the formula [Hirzebruch66,20.3.6*]

$$\chi(\mathcal{O}_X) = \sum_{s \geq 0} \frac{1}{2^{n+2s}(n-2s)!} c_1(X)^{n-2s} A_s(p_1, \dots, p_s)[M], \tag{5.3.1}$$

where the A_s are certain polynomials of the Pontrjagin classes of M and $A_0 = 1$. From Hodge theory we know that

$$|\chi(\mathcal{O}_X)| \leq \sum \dim_{\mathbb{C}} H^i(X, \mathcal{O}_X) \leq \sum \dim_{\mathbb{C}} H^i(M, \mathbb{C}),$$

and so $\chi(\mathcal{O}_X)$ is bounded in terms of M. Since $b_2(M) = 1$, we can fix an ample divisor H in $\operatorname{Pic}(X)$ and then $c_1(X) \equiv rH$ for some rational number r. (5.3.1) becomes a polynomial equation for r. As $\chi(\mathcal{O}_X)$ runs through all the possible values, we get only finitely many possible values for r. Therefore the self-intersection number (H^n) and the intersection number $(c_1(X)\cdot H^{n-1})$ are bounded depending on M only. The result now follows from Matsusaka's Big Theorem (in the form given in [Kollár-Matsusaka83]). $\square$

The proof provides an effective bound on the number of families of algebraic structures on a given manifold M, but the bound is enormous even in the simplest cases.

5.4 Proof of (4.2.10). The proof is an application of the theory of Gromov–Witten invariants. I recall the main concepts in the needed special case. See [McDuff-Salamon94,95] for details of the general theory.

Let X be a smooth projective variety over $\mathbb{C}$. Fix a point $x\in X$, a homology class $A\in H_2(X(\mathbb{C}),\mathbb{Z})$ and very ample divisors in general position $H_i\subset X$, $i=1,\dots,k$.

Let $y_0,\dots,y_k\in\mathbb{CP}^1$ be general points. For suitable k, there may be only finitely many maps

$$f:\mathbb{CP}^1\to X \quad \text{such that} \quad f_*[\mathbb{CP}^1]=A, f(y_0)=x, \text{ and } f(y_i)\in H_i,\ i=1,\dots,k.$$

We define an invariant

$$\tilde F_{A,X}(x,H_1,\dots,H_k;y_0,\dots,y_k) := \text{ the number of such maps.} \tag{5.4.1}$$

Gromov's theory of pseudo-holomorphic curves shows that one can make a similar definition where X is replaced by a symplectic manifold (M,ω) endowed with a general almost complex structure. The corresponding invariant is denoted by

$$\tilde\Phi_{A,M,\omega}(x,H_1,\dots,H_k;y_0,\dots,y_k). \tag{5.4.2}$$

It is one of the Gromov–Witten invariants of (M,ω). In fact, this is an invariant of the symplectic deformation equivalence class.

In general the algebraic number (5.4.1) and the symplectic number (5.4.2) are different. Under suitable conditions they are equal, and this means that we can get information about rational curves on X from the symplectic structure $(X(\mathbb{C}),\omega_X)$ (4.2.6). This idea was used by [Ruan93] to show that the extremal rays of Mori theory can be described using the symplectic structure. We need the following two results. (In [Ruan93] they are proved under the extra assumption that the symplectic structure is semi-positive. This is no longer necessary.)

5.4.3 Theorem. Let X be a smooth projective variety over $\mathbb{C}$ and (M,ω) the corresponding symplectic manifold.

(5.4.3.1) If $\tilde\Phi_{A,M,\omega}(x,H_1,\dots,H_k;y_0,\dots,y_k)\neq 0$, then there is a rational map $f:\mathbb{CP}^1\to X$ such that $f_*[\mathbb{CP}^1]=A$, $f(y_0)=x$ and $f(y_i)\in H_i$, $i=1,\dots,k$.

(5.4.3.2) $\tilde F_{A,X}(x,H_1,\dots,H_k;y_0,\dots,y_k)=\tilde\Phi_{A,X(\mathbb{C}),\omega_X}(x,H_1,\dots,H_k;y_0,\dots,y_k)$ if the following conditions are satisfied:

(5.4.3.2.1) If $g:\mathbb{CP}^1\to X$ is any map such that $g_*[\mathbb{CP}^1]=A$ and $g(y_0)=x$, then $H^1(\mathbb{CP}^1,g^*T_X)=0$.

(5.4.3.2.2) If $C_1, \dots, C_m \subset X$ are rational curves such that $\sum [C_i] = A$ and $x \in C_1$, then $m = 1$.

We can now prove (4.2.10).

Let (M, ω) be the common symplectic structure of X_0 and of X_1. Fix a very general point $x \in X_0$. Fix a very ample divisor $H \subset X_0$ and let $x \in C \subset X$ be a rational curve such that $(C \cdot H)$ is minimal (C exists since X_0 is uniruled). Set $A := [C]$. By [KoMiMo92c,1.1], the condition (5.4.3.2.1) holds and (5.4.3.2.2) follows from the minimality of $(C \cdot H)$. Let k be the dimension of the space of maps $g : \mathbb{CP}^1 \to X$ such that $g_*[\mathbb{CP}^1] = A$ and $g(y_0) = x$. Let $H_1, \dots, H_k \subset X_0$ be general divisors linearly equivalent to H. By construction, $\tilde{F}_{A,X}(x, H_1, \dots, H_k; y_0, \dots, y_k)$ is defined and is nonzero. Thus $\tilde{\Phi}_{A,M,\omega}(x, H_1, \dots, H_k; y_0, \dots, y_k) \neq 0$.

By (5.4.3.1) this implies that there is a rational curve through any very general point of X_1, and thus X_1 is also uniruled. $\square$

Finally we prove that condition (1.2) correctly identifies the class of rationally connected varieties among diagonal hypersurfaces.

5.5 Proposition. Let X be any smooth compactification of the affine hypersurface

$$(\sum_{i=1}^{n} c_i x_i^{d_i} + c_0 = 0) \subset \mathbb{C}^n.$$

(5.5.1) X is rationally connected iff $\sum 1/d_i \geq 1$.

(5.5.2) The Kodaira dimension of X is nonnegative iff $\sum 1/d_i < 1$.

Proof. Consider first the case $n = 2$, assuming $d_1 \leq d_2$. View X as a d_1-sheeted cover of the line ramified along $c_2 x_2^{d_2} + c_0 = 0$. The Hurwitz formula gives that

$$2g(X) = (d_1 - 1)(d_2 - 2) + (\text{ramification at infinity}).$$

This implies (5.5) for $n = 2$.

If $n \geq 3$ then as in (1.3), we view these as hypersurfaces in weighted projective spaces. Let $d = lcm(d_i)$, $d = d_i a_i$ and set $a_0 = 1$, $d_0 = d$. A (nonsmooth) compactification is given by the projective weighted hypersurface

$$Y := \sum_{i=0}^{n} c_i x_i^{d_i} \subset \mathbb{P}(a_0, \dots, a_n).$$

As long as $\prod c_i \neq 0$, these hypersurfaces are isomorphic (over $\mathbb{C}$), thus Y can be viewed as a general member of the linear system $|x_0^{d_0}, \dots, x_n^{d_n}|$. This implies that Y has only quotient singularities and Picard number 1 for $n \geq 4$.

Assume that $d < \sum a_i$. $K_Y = \mathcal{O}(d - \sum a_i)$, thus Y is $\mathbb{Q}$-Fano. Therefore Y is uniruled by [Miyaoka-Mori86]. Let $p : \bar{Y} \to Y$ be a desingularization and $\bar{f} : \bar{Y}^0 \to Z$ the MRC fibration [KoMiMo92b]. The fibers of p are all rationally connected (cf. [Kollár96a, VI.1.6.2]), thus $\bar{f}$ descends to $f : Y^0 \to Z$. If $n \geq 4$, then as in [Kollár96a,IV.4.14], we obtain that Z is a point, hence Y is rationally connected. If $n = 3$ then we use that $h^1(X, \mathcal{O}_X) = h^1(Y, \mathcal{O}_Y) = 0$. A smooth uniruled surface S with $h^1(S, \mathcal{O}_S) = 0$ is rational, hence X is rational.

Next assume that $d \geq \sum a_i$. Let $e = d - \sum a_i$ and introduce e new coordinates $x_{n+1}, \dots, x_{n+e}$ of weight $a_{n+1} = \cdots = a_{n+e} = 1$. Consider the hypersurface

$$Z := \sum_{i=0}^{n} c_i x_i^{d_i} + \sum_{j=1}^{e} c_{n+j} x_{n+j}^{d} \subset \mathbb{P}(a_0, \dots, a_{n+e}).$$

Here every a_i divides $d = \sum a_i$, thus $\mathbb{P}(a_0, \dots, a_{n+e})$ has only index one canonical singularities. Therefore the same holds for Z. But $\omega_Z \cong \mathcal{O}_Z$, and this implies that $\kappa(Z) = 0$.

General fibers of the projection map

$$Z \dashrightarrow \mathbb{P}^e \quad \text{given by} \quad (x_0, \dots, x_{n+e}) \mapsto (x_0, x_{n+1}, \dots, x_{n+e})$$

are isomorphic to Y. This shows that $\kappa(Y) \geq 0$.

Since a variety can not be rationally connected and have nonnegative Kodaira dimension at the same time, this proves (5.5). $\square$

5.5.3 Remark. It is not true that X is of general type if $d > \sum a_i$. For instance, $x_1^2 + x_2^3 + \sum c_i x_i^{d_i} + c_0 = 0$ has an elliptic fiberspace structure (projection to the $(x_3, \dots, x_n)$-subspace) for every value of the d_i.

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